TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 358, Number 1, Pages 359–371
S 0002-9947(05)03676-7
Article electronically published on March 31, 2005

UNRAMIFIED COHOMOLOGY OF CLASSIFYING VARIETIES FOR EXCEPTIONAL SIMPLY CONNECTED GROUPS

SKIP GARIBALDI

Abstract. Let BG be a classifying variety for an exceptional simple simply connected algebraic group G. We compute the degree 3 unramified Galois cohomology of BG with values in $(\mathbb{Q}/\mathbb{Z})'(2)$ over an arbitrary field F. Combined with a paper by Merkurjev, this completes the computation of these cohomology groups for G semisimple simply connected over all fields.

These computations provide another family of examples of simple simply connected groups G such that BG is not stably rational.

Let G be an algebraic group over a field F with an embedding $\rho\colon G \hookrightarrow SL_n$ over F. We call the quotient variety SL_n/G a *classifying space* of G and denote it by BG. The goal of this paper is to compute the *unramified cohomology* of BG, defined as follows. Write $H^d(F)$ for the Galois cohomology group $H^d(\mathrm{Gal}(F), (\mathbb{Q}/\mathbb{Z})'(d-1))$, where $(\mathbb{Q}/\mathbb{Z})'(d-1) = \varinjlim_n \boldsymbol{\mu}_n^{\otimes(d-1)}$ for n not divisible by the characteristic of F. For each $d \geq 2$, define $H^d_{\mathrm{nr}}(BG/F)$ (or simply $H^d_{\mathrm{nr}}(BG)$) to be the intersection of the kernels of the residue homomorphisms

$$\partial_v\colon H^d(F(BG)) \to H^{d-1}(F(v)) \tag{0.1}$$

as v ranges over the discrete valuations of $F(BG)$ over F. The natural homomorphism $H^d(F) \to H^d_{\mathrm{nr}}(BG/F)$ is split by evaluation at the distinguished point of BG; this gives a direct sum decomposition of $H^d_{\mathrm{nr}}(BG)$, and we denote the complement of $H^d(F)$ by $H^d_{\mathrm{nr}}(BG)_{\mathrm{norm}}$. This group depends only on G and F, and not on ρ or n [M02, 2.3].

This paper completes the computation of $H^3_{\mathrm{nr}}(BG/F)_{\mathrm{norm}}$ for G semisimple simply connected and F arbitrary. The computation is quickly reduced to the case where G is simple simply connected [M02, §4]. In [M02], $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}}$ was computed for G simple and classical. We compute it for the remaining cases, where G is *exceptional*, that is, where G is of type G_2, ${}^3D_4$, ${}^6D_4$, F_4, E_6, E_7, or E_8.

Main Theorem 0.2. *Let G be a simple simply connected* exceptional *algebraic group defined over a field F. Then*

$$H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}} = \begin{cases} \mathbb{Z}/2 & \begin{cases} \textit{if } \operatorname{char} F \neq 2,\ G \textit{ is of type } {}^3D_4, \textit{ and } G \textit{ has} \\ \textit{a nontrivial Tits algebra,} \end{cases} \\ 0 & \textit{otherwise.} \end{cases}$$

(See §5.3 for an explanation of the characteristic $\neq 2$ hypothesis.)

The general motivation for studying $H^d_{\mathrm{nr}}(X)$ is that it can sometimes detect if X is not stably rational; see [C, pp. 35–39]. It was an open question whether BG is

Received by the editors August 15, 2003 and, in revised form, March 21, 2004.
2000 *Mathematics Subject Classification.* Primary 11E76; Secondary 17B25, 20G10.

stably rational for G semisimple simply connected. The first counterexamples were provided in [M02], where Merkurjev exhibited groups G of type ${}^2A_n$, ${}^2D_3$, and ${}^1D_4$ with $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}} \neq 0$, hence with BG not stably rational. The results here give another class of such G's; see Corollary 7.2.

Our basic tool is that one can compute $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}}$ by inspecting the ramification of the Rost invariant of G; see [M02] or Lemma 5.2. Many questions are settled by hopping along the chain of inclusions

$$G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8 \tag{0.3}$$

of split groups; see §§6 and 5.5.

The most interesting part of the proof is where we show that the mod 4 portion of the Rost invariant is ramified for groups of type ${}^2E_6$. We prove (in Theorem 3.1) that every isotropic trialitarian group embeds in a group of type ${}^2E_6$ with trivial Tits algebras. This settles the question, since the mod 4 portion of the Rost invariant for groups of type ${}^6D_4$ is easily shown to be ramified (Lemma 6.3). The proof of Theorem 3.1 uses Galois descent and interpretations of exceptional groups as acting on nonassociative algebras.

Remark 0.4. Computations of $H^d_{\mathrm{nr}}(X/F)$ in the literature for X a smooth variety (e.g., a classifying variety) typically assume that F is algebraically closed. The examples of nonrational classifying varieties BG provided here and in [M02] require that F is *not* algebraically closed (e.g., $F = \mathbb{Q}$).

1. Vocabulary

An (affine) algebraic group is *simple* if it is $\neq 1$, is connected, and has no noncentral connected normal subgroups over an algebraic closure. (These groups are often called "absolutely almost simple".) Simple groups are classified in, e.g., [KMRT, Ch. VI]. We say that a group is *of type* T_n if it is simple with root system of type T_n and *of type* ${}^tT_n$ if additionally the absolute Galois group of F acts as a group of automorphisms of order t on the Dynkin diagram.

Let V be a finite-dimensional irreducible representation of an algebraic group G over F. The F-algebra $\mathrm{End}_G(V)$ is a skew field by Schur's Lemma, and it is finite dimensional over F; it is called a *Tits algebra* for G. If it is a (commutative) field, we say that it is *trivial*.

The *Dynkin index* n_G of a simple simply connected algebraic group G is a natural number which depends only upon the type of G and the (Schur) indices of its Tits algebras. The value of n_G can be found in [M03, App. B], for example:

type of G	G_2	${}^{3,6}D_4$, all Tits alg's trivial	${}^{3,6}D_4$, some Tits alg's nontrivial	F_4	${}^1E_6$	${}^2E_6$	E_7	E_8
n_G	2	6	12	6	6	12	12	60

We have functors $H^1(*, G)$ and $H^3(*)$ which take a field extension of F and give a pointed set and abelian group, respectively. A *degree* 3 *invariant of* G *with values in* $(\mathbb{Q}/\mathbb{Z})'(2)$ is a morphism of functors

$$H^1(*, G) \longrightarrow H^3(*)$$

which takes base points to base points. Such invariants are often called "normalized". We write $\mathrm{Inv}^3(G)$ for the abelian group of such invariants. (Clearly, this definition makes sense for every algebraic group G over F.)

Let $p = \operatorname{char} F$ if the characteristic is prime and $p = 1$ otherwise. Write $n_G = p^k n'_G$, where n'_G is a natural number prime to p. The group $\operatorname{Inv}^3(G)$ is cyclic of order n'_G. It has a canonical generator r_G which we call the *Rost invariant* of G. (This is the prime-to-p part of what is called the Rost invariant in [M03]. I do not know how to define a residue map for the p-primary part.)

For $\alpha\colon H \to G$ a map between simple simply connected algebraic groups over F, there is a positive integer n_α called the *Rost multiplier* or "Dynkin index" of α; see [M03, §7]. It has the properties: n_H divides $n_\alpha n_G$ and for every extension E of F, the composition

$$H^1(E,H) \xrightarrow{\alpha} H^1(E,G) \xrightarrow{r_G} H^3(E)$$

is $n_\alpha r_H$.

The standard inclusions in (0.3) all have Rost multiplier 1. This observation goes back to [Dyn, p. 192]. Excepting G_2, this can be seen as follows: The inclusion of D_4 into E_8 arises from the natural inclusion of root systems. It is easy to check from the definition that the Rost multiplier is 1. Since multipliers are nonnegative integers and the Rost multiplier of a composition is the product of the Rost multipliers [M03, 7.9], the intervening Rost multipliers are all 1. An alternative, explicit argument for the inclusion $F_4 \subset E_6$ can be found in [Ga01, 2.4].

All inclusions of groups of type G_2, D_4, F_4, E_n occurring in this paper are conjugates of the standard inclusions of split groups over an algebraic closure of the base field. Since Rost multipliers are unchanged by field extensions [M03, 7.9.4], the multipliers for these inclusions are all 1.

2. $A_2 \subset D_4$

2.1. In this section, we assume that F contains a primitive cube root of unity and hence has characteristic $\neq 3$. Let L be a cubic Galois extension of F; by Kummer theory it is obtained by adjoining a cube root of some element $\lambda \in F^*$. We write (λ) for the corresponding class in $F^*/F^{*3} = H^1(F, \boldsymbol{\mu}_3)$, where $\boldsymbol{\mu}_3$ is the algebraic group of cube roots of unity.

The short exact sequence $1 \to \boldsymbol{\mu}_3 \to SL_3 \to PGL_3 \to 1$ induces a connecting homomorphism $\delta\colon H^1(F, PGL_3) \to H^2(F, \boldsymbol{\mu}_3)$.

Lemma 2.2. *Continue the hypotheses of §2.1. Let G be the quasi-split simply connected group of type ${}^3D_4$ associated with the extension L/F. Then G contains a subgroup isomorphic to PGL_3 such that for every extension E of F the diagram*

$$\begin{array}{ccc} H^1(E, PGL_3) & \xrightarrow{x \mapsto \delta(x) \cup (\lambda)} & H^3(E, \boldsymbol{\mu}_3^{\otimes 2}) \\ \downarrow & & \downarrow \\ H^1(E,G) & \xrightarrow{r_G} & H^3(E) \end{array}$$

commutes up to sign, where the arrow on the right comes from the natural map $\boldsymbol{\mu}_3^{\otimes 2} \to (\mathbb{Q}/\mathbb{Z})'(2)$.

Proof. We have maps

$$PGL_3 \times \boldsymbol{\mu}_3 \longrightarrow \operatorname{Spin}_8 \rtimes \mathbb{Z}/3 \longrightarrow F_4, \tag{2.3}$$

where F_4 denotes the split algebraic group of that type and we identify $\boldsymbol{\mu}_3$ with $\mathbb{Z}/3$ using the primitive cube root of unity in F. The first map comes from the fact that $PGL_3 = \operatorname{Aut}(M_3(F))$ preserves the subspace of trace zero elements in $M_3(F)$; see

[KMRT, pp. 504, 505]. The second map comes from the Springer decomposition of Albert algebras; see [KMRT, 38.7]. The group $\mathbb{Z}/3$ acts on Spin_8 in a manner which cyclically permutes the vector and half-spin representations and fixes PGL_3 elementwise. The map $\mathrm{Spin}_8 \to F_4$ has Rost multiplier 1.

When we twist the groups in (2.3) by (λ) and restrict to connected components, we obtain the sequence

$$PGL_3 \xrightarrow{\iota} G \xrightarrow{\sigma} F_4. \tag{2.4}$$

The map $H^1(\sigma\iota)$ sends a class $[A] \in H^1(E, PGL_3)$ of a central simple E-algebra of degree 3 to the class of the first Tits construction $[J(A,\lambda)] \in H^1(F, F_4)$; see [KMRT, 39.9]. The composition $r_{F_4} \circ H^1(\sigma\iota)$ is, up to sign, the composition of δ with the cup product $\cdot \cup (\lambda)$ by [KMRT, p. 537]. Since σ has Rost multiplier 1, $r_{F_4} \circ H^1(\sigma) = r_G$, and the lemma is proved. □

3. ${}^3D_4, {}^6D_4 \subset {}^2E_6$

In this section, we assume that F has characteristic $\neq 2$. A simple algebraic group is said to be *trialitarian* if it is of type ${}^3D_4$ or ${}^6D_4$. It is well known that every quasi-split simply connected trialitarian group is a subgroup of the split F_4, hence of every simply connected quasi-split E_6. In this section, we prove:

Theorem 3.1 (char $F \neq 2$)**.** *Let T be a trialitarian simply connected group over F which is F-isotropic but not F-quasi-split. Let K be a quadratic extension of F such that T is K-quasi-split. Then there exists a simply connected group G of type ${}^2E_6$ over F such that*

1. *all of G's Tits algebras are trivial;*
2. *G is of type ${}^1E_6$ over K; and*
3. *T is a subgroup of G with Rost multiplier* 1.

Given a T as in the first sentence of Theorem 3.1, such a K always exists by [Ga98, 0.1]. We postpone the proof of the theorem until the end of this section.

3.2. Let $\mathfrak{C}$ be the split Cayley algebra over F with canonical involution (also known as conjugation) $\pi_{\mathfrak{C}}$. Fix the basis $u_1, \ldots, u_8$ of $\mathfrak{C}$ as in [Ga98] and [Ga01] so that the bilinear norm form $\mathfrak{n}(x,y) = x\pi_{\mathfrak{C}}(y) + y\pi_{\mathfrak{C}}(x)$ is given by $\mathfrak{n}(u_i, u_j) = \delta_{(i+j),9}$ (Kronecker delta). Write σ for the involution on $GL(\mathfrak{C})$ which is adjoint for $\mathfrak{n}$.

Let R denote the subgroup of $GL(\mathfrak{C})^{\times 3}$ consisting of so-called *related triples* of proper similitudes of $\mathfrak{n}$; see [Ga98, §1] or [KMRT, §35] for a definition. This group is reductive with center of rank 2; its derived subgroup consists of triples $\underline{t} = (t_0, t_1, t_2)$ with $t_i \in SO(\mathfrak{n})$ for all i and is isomorphic to Spin_8 [KMRT, 35.7].

The group $S_3 = \langle r, \pi \mid r^3 = \pi^2 = 1, \pi r = r^2\pi \rangle$ acts on R via

$${}^r\underline{t} = (t_1, t_2, t_0) \quad \text{and} \quad {}^\pi(t_0, t_1, t_2) = (\pi_{\mathfrak{C}} t_0 \pi_{\mathfrak{C}}, \pi_{\mathfrak{C}} t_2 \pi_{\mathfrak{C}}, \pi_{\mathfrak{C}} t_1 \pi_{\mathfrak{C}}).$$

Define $R \rtimes S_3$ to be the Cartesian product $R \times S_3$ with multiplication

$$(\underline{t}, \alpha) \cdot (\underline{t}', \beta) = (\underline{t} \cdot {}^\alpha\underline{t}', \alpha\beta).$$

The split Albert algebra J has as an underlying vector space the matrices in $M_3(\mathfrak{C})$ fixed by the conjugate transpose. With that in mind, we may write a general element of J as in (3.4) below where $\varepsilon_i \in F$, $c_i \in \mathfrak{C}$, and the entries given as $\cdot$ are forced by symmetry. The algebra J has a canonically determined norm form; write $\mathrm{Inv}(J)$ for the group of norm isometries.

There is an injection $g\colon R \rtimes S_3 \hookrightarrow \mathrm{Inv}(J)$ defined by

$$g_{\underline{t}}\begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix} := \begin{pmatrix} \mu(t_0)^{-1}\varepsilon_0 & t_2(c_2) & \cdot \\ \cdot & \mu(t_1)^{-1}\varepsilon_1 & t_0(c_0) \\ t_1(c_1) & \cdot & \mu(t_2)^{-1}\varepsilon_2 \end{pmatrix},$$

$$g_r\begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & c_0 & \cdot \\ \cdot & \varepsilon_2 & c_1 \\ c_2 & \cdot & \varepsilon_0 \end{pmatrix}, \text{ and } g_\pi\begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix} = \begin{pmatrix} \varepsilon_0 & \pi_{\mathfrak{C}}c_1 & \cdot \\ \cdot & \varepsilon_2 & \pi_{\mathfrak{C}}c_0 \\ \pi_{\mathfrak{C}}c_2 & \cdot & \varepsilon_1 \end{pmatrix}.$$

Here $\mu(t_i)$ denotes the factor by which t_i scales the norm $\mathfrak{n}$. That is, $\mathfrak{n}(t_ix, t_iy) = \mu(t_i)\mathfrak{n}(x, y)$ for all x, $y \in \mathfrak{C}$.

3.3. Construction of a quasi-split $^2E_6$. The algebra J is also endowed with a nondegenerate symmetric bilinear form s defined by

$$s(x, y) = \mathrm{Tr}_J(xy) = \sum_{i=0}^{2} [\varepsilon_i\nu_i + \mathfrak{n}(c_i, d_i)]$$

for

$$x = \begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix}, \qquad y = \begin{pmatrix} \nu_0 & d_2 & \cdot \\ \cdot & \nu_1 & d_0 \\ d_1 & \cdot & \nu_2 \end{pmatrix}. \tag{3.4}$$

For each $f \in GL(J)$, there is a unique $f^\dagger \in GL(J)$ such that $s(f(x), f^\dagger(y)) = s(x, y)$ for all $x, y \in J$.

The map $f \mapsto f^\dagger$ restricts to automorphisms of $\mathrm{Inv}(J)$ and $R \rtimes S_3$ defined over F. We have

$$r^\dagger = r, \quad \pi^\dagger = \pi, \quad \text{and} \quad \underline{t}^\dagger = (\sigma(t_0)^{-1}, \sigma(t_1)^{-1}, \sigma(t_2)^{-1}).$$

Let ι be a generator for $\mathrm{Gal}(K/F)$. We define the groups E_6^K and H to be the groups $\mathrm{Inv}(J)$ and R with twisted ι-actions. For f a K-point, we set ${}^\iota f = \iota f^\dagger \iota$ where the action on the left is the new action and juxtaposition denotes the usual action. The group E_6^K is quasi-split of type $^2E_6$.

We consider E_6^K as a closed subgroup of $GL(V)$ as follows. Let V be the F-subspace of $(J \otimes K) \oplus (J \otimes K)$ fixed by the ι-semilinear automorphism

$$(j \otimes k, j' \otimes k') \mapsto (j' \otimes \iota(k'), j \otimes \iota(k)).$$

The homomorphism

$$\mathrm{Inv}(J) \to GL(J \oplus J) \quad \text{given by } f \mapsto f \oplus f^\dagger$$

defines an injective F-homomorphism $E_6^K \to GL(V)$. It is the usual 54-dimensional representation of E_6^K; it is irreducible over F.

3.5. *Proof of Theorem* 3.1. Since T is isotropic and not quasi-split, it has a Tits algebra which is a nonsplit quaternion algebra Q over a cubic extension L of F [Ga98, 0.1]. Put $K = F(\sqrt{b})$. Since T is K-quasi-split, Q is split by $L(\sqrt{b})$, hence Q is of the form $(a, b)_L$ for some $a \in L^*$ such that $N_{L/F}(a) = 1$ [KMRT, 43.9]. Since Q is not split over L, it is not split over the normal closure L^c of L/F [Ga98, 3.2]. In particular, L^c does not contain a square root of b, so $P = K \otimes_F L^c$ is a quadratic field extension of L^c.

To simplify our argument, we assume that L is not Galois over F, so $\mathrm{Gal}(L^c/F)$ is isomorphic to S_3. (This is the case that will be used in the rest of the paper. The other case — where L is Galois over F — is only easier.) Then, the group $\mathrm{Gal}(P/F)$ is isomorphic to $S_3 \times \boldsymbol{\mu}_2 \cong \mathbb{Z}/6 \rtimes \boldsymbol{\mu}_2$, where the factor of $\mathbb{Z}/6$ corresponds to the subgroup $\mathrm{Gal}(P/\Delta)$ for Δ the unique quadratic extension of F in L^c. We fix

generators $\zeta := (r^{-1}, -1) \in S_3 \times \boldsymbol{\mu}_2$ (which generates the copy of $\mathbb{Z}/6$) and $\tau = (\pi, 1)$ (which generates the copy of $\boldsymbol{\mu}_2$ in $\mathbb{Z}/6 \rtimes \boldsymbol{\mu}_2$ corresponding to $\mathrm{Gal}(P/L(\sqrt{b}))$).

We construct the group G by descent as follows. We identify E_6^K — and hence $H \rtimes S_3$ — with a closed subgroup of $GL(V)$ as in §3.3. We call an additive homomorphism $f\colon V \otimes P \to V \otimes P$ *ϖ-semilinear* if there is some $\varpi \in \mathrm{Gal}(P/F)$ such that

$$f(pv) = \varpi(p) f(v) \quad \text{for all } p \in P \text{ and } v \in V \otimes P.$$

Let $\widetilde{GL}(V)$ denote the (abstract) group of such maps f. We define a group homomorphism $\phi\colon \mathrm{Gal}(P/F) \to \widetilde{GL}(V)$ such that $\phi(\varpi)$ is ϖ-semilinear for all ϖ and $T(F)$ and $G(F)$ are the subgroups of $\mathrm{Spin}_8(P)$ and $E_6^K(P)$ commuting with $\phi(\varpi)$ for all ϖ.

Define $\underline{t} = (t_0, t_1, t_2) \in GL(\mathfrak{C})^{\times 3}$ by setting $t_i = m_i P$ for

$$m_i = \mathrm{diag}\,(1, \rho^i(a), -\rho^i(a), \rho^{i+2}(a)^{-1}, \rho^{i+1}(a)^{-1}, -1, 1, \rho^i(a))$$

with $\rho := \zeta^2$, and P the matrix permuting the basis vectors as $(1\,2)(3\,6)(4\,5)(7\,8)$, for the basis of $\mathfrak{C}$ fixed in §3.2 above. Since $N_{L/F}(a) = 1$, $\underline{t}$ is a related triple by [Ga98, 1.5(3), 1.6, 1.8]. Set

$$\phi(\zeta) = \underline{t} r \zeta \quad \text{and} \quad \phi(\tau) = \pi\tau. \tag{3.6}$$

We have

$$\zeta \underline{t} \zeta^{-1} = (\sigma(\zeta(t_0)), \sigma(\zeta(t_1)), \sigma(\zeta(t_2)))^{-1} = (\sigma(t_2), \sigma(t_0), \sigma(t_1))^{-1} = r^{-1} \underline{t}^{-1} r,$$

since $\sigma(t_i) = t_i$ for all i. Hence

$$\phi(\zeta)^2 = r^2 \zeta^2, \quad \phi(\zeta)^3 = \underline{t}\zeta^3, \quad \text{and}$$
$$\phi(\zeta)^6 = \mathrm{Id}_{V \otimes P}\,. \tag{3.7}$$

Since π and τ commute, we have

$$\phi(\tau)^2 = \mathrm{Id}_{V \otimes P}\,. \tag{3.8}$$

Since

$$\phi(\zeta)^5 = \phi(\zeta)^2 \cdot \phi(\zeta)^3 = \underline{t} r^2 \zeta^5,$$

and $\pi\tau$ and $\underline{t}$ commute, it is easy to verify that

$$\phi(\tau)\phi(\zeta) = \phi(\zeta)^5 \phi(\tau). \tag{3.9}$$

Equations (3.7)–(3.9) give that (3.6) defines a homorphism $\phi : \mathrm{Gal}(P/F) \to \widetilde{GL}(V)$. Then the set map $z\colon \mathrm{Gal}(P/F) \to H \rtimes S_3$ defined by $z_\varpi := \phi(\varpi)\varpi^{-1}$ is in fact a 1-cocycle. (This correspondence between groups of semilinear transformations and 1-cocycles is well-explained in [Jac, §3].) Set G to be the twisted group $(E_6^K)_z$; it automatically satisfies (2). Since z takes values in the simply connected group E_6^K, (1) holds.

Since the values of z normalize the subgroup Spin_8 of E_6^K, the twisted group $(\mathrm{Spin}_8)_z$ is a subgroup of G. The inclusion has Rost multiplier 1 since the inclusion $\mathrm{Spin}_8 \subset E_6^K$ over an algebraic closure arises from the natural inclusion of root systems $D_4 \subset E_6$. The restriction of z to Spin_8 is the descent given in [Ga98, 4.7] to construct T, i.e., $(\mathrm{Spin}_8)_z$ is isomorphic to T, hence (3). □

Remark 3.10. The isotropic group G occurring in Theorem 3.1 is typically not quasi-split, even over L. This can be seen by examining the mod 2 portion of the Rost invariant for $(z) \in H^1(P/L, E_6^K)$, which is typically nontrivial by [Ga01, 6.7].

4. A construction

The purpose of this section is to construct a suitable extension of F over which we may apply Theorem 3.1:

Proposition 4.1. *Let F be a field of characteristic $\neq 2$, and let K be a quadratic extension of F. There is a regular extension E of F and a group T of type ${}^6D_4$ over E such that T is E-isotropic and $(E \otimes_F K)$-quasi-split, but not E-quasi-split.*

Presumably one could prove Proposition 4.1 by applying [M96] to produce a group T_0 of type ${}^6D_4$ whose Tits algebras have index 8 and then extending scalars to function fields of transfers of generalized Severi-Brauer varieties so that the Tits algebras of T_0 have index 2 and are split by a quadratic extension of the base field. Then by [Ga98] there is a group T as in Proposition 4.1 with the same Tits algebras as in T_0. We give a low-tech argument here.

Lemma 4.2 (char $F \neq 2$)**.** *For $p, q \in F^*$, the ring $L = F(t)[x]/(x^3 + px + qt)$ is a separable cubic field extension of $F(t)$ which is regular over F and not Galois over $F(t)$. There is a prolongation of the t-adic valuation on $F(t)$ to L which is unramified with residue degree 1 and with respect to which x has value 1.*

Proof. If L is not a field, then there is some $a \in F(t)$ such that $a^3 + pa + qt = 0$. Since a is integral over the unique factorization domain $F[t]$, it belongs to $F[t]$, so it makes sense to speak of the degree of a. In particular, at least two of the terms a^3, pa, and qt must have the same degree, which is also the maximum of the degrees. This implies that a cannot have positive degree. But then qt, with degree 1, is the unique term of maximal degree, which is a contradiction.

Since p, q are in F^*, the discriminant $-4p^3 - 27q^2t^2$ of L is not 0, hence L is separable over F. An argument similar to the one in the preceding paragraph shows that the discriminant is not a square in $F(t)$: Any square root $b \in F(t)$ of the discriminant would belong to $F[t]$ and have degree 1. Then the coefficient of t in b^2 would be nonzero. Thus L is not Galois over $F(t)$. If $\ell \in L \setminus F(t)$ is algebraic over F, then L is generated by ℓ as an $F(t)$-algebra and the discriminant of the extension $L/F(t)$ comes from F^*/F^{*2}, which is a contradiction. Hence F is algebraically closed in L and L is regular over F.

Hensel's Lemma gives that $x^3 + px + tq$ has a linear factor of the form $x - \pi$ in $F((t))[x]$, where π has t-adic value 1. The map $x \mapsto \pi$ gives an isomorphism of L with the subfield $F(t)(\pi)$ of $F((t))$, and the t-adic valuation obviously extends to L so that x has value 1. Since $F((t))$ is the completion of $F(t)$ with respect to the t-adic valuation and hence is unramified with residue degree 1, the claims about ramification and residue degree of our prolongation to L follow. □

Lemma 4.3 (char $F \neq 2$)**.** *Let $p, b \in F^*$ be such that the quaternion algebra $(p, b)_F$ is nonsplit. Let L be as in Lemma 4.2. Then the quaternion algebra $(x, b)_L$ is nonsplit and is not isomorphic to $(-qt, b)_L$.*

Proof. Since $N_{L/F(t)}(x) = -qt$, the corestriction of $(x, b)_L$ down to $F(t)$ is Brauer-equivalent to $(-qt, b)_{F(t)}$. This algebra is split if and only if the quadratic form $\langle 1, -b, qt\rangle$ is isotropic over $F(t)$. Over the completion $F((t))$, this form has residue forms $\langle 1, -b\rangle$ and $\langle q\rangle$. Since the algebra $(p, b)_F$ is nonsplit, the first form is anisotropic, hence $\langle 1, -b, qt\rangle$ is anisotropic over $F((t))$ by Springer's Theorem. Thus $(-qt, b)_{F(t)}$ is nonsplit, and hence so is $(x, b)_L$.

For the sake of contradiction, suppose that $(x,b)_L$ is isomorphic to $(-qt,b)_L$, i.e., the algebra $(-xqt,b)_L$ is split. Since

$$-x(qt) = -x(-x^3 - px) = x^4 + px^2 \equiv x^2 + p \mod L^{*2},$$

the algebra $(x^2+p,b)_L$ is split.

Let $\widehat{L}$ be a completion of L with respect to the prolongation of the t-adic valuation on $F(t)$ given by Lemma 4.2. The norm of $(x^2+p,b)_L$ is the form $\langle 1, -(x^2+p), -b, b(x^2+p)\rangle$ over L. Since x has value 1, over $\widehat{L}$ this form has one residue form $\langle 1, -p, -b, bp\rangle$ over the residue field F. This is the norm of the algebra $(p,b)_F$, which is anisotropic because the algebra is nonsplit. By Springer's Theorem, the norm of $(x^2+p,b)_L$ is anisotropic over $\widehat{L}$, hence the algebra is not L-split, which contradicts our assumption that $(x,b)_L$ is isomorphic to $(-qt,b)_L$. □

4.4. *Proof of Proposition* 4.1. Write K as $F(\sqrt{b})$. Let $F_0 := F(p,t)$ for p, t indeterminates. Set $L_0 := F_0[x]/(x^3+px+t)$ as in Lemma 4.2. Set E to be the function field of the Severi-Brauer variety of the quaternion algebra $(-t,b)_{F_0}$. Since E is regular over F_0, $L := L_0 \otimes_{F_0} E$ is a field which is cubic and not Galois over E; it is the function field of the Severi-Brauer variety of $(-t,b)_{L_0}$.

Since b is not a square in F, the quaternion algebra $(p,b)_{F_0}$ is not split. By Lemma 4.3, $(x,b)_{L_0}$ is nonsplit and is not isomorphic to $(-t,b)_{L_0}$, hence $(x,b)_L \cong (x,b)_{L_0} \otimes_{L_0} L$ is not split by a well-known theorem of Amitsur. The corestriction $\mathrm{cor}_{L/E}(x,b)_L$ is $(-t,b)_E$, which is split. Thus there is a simply connected E-isotropic — but not E-quasi-split — trialitarian group T over E with nontrivial Tits algebra $(x,b)_L$ [Ga98, 4.7]. It is of type ${}^6D_4$ since L is not Galois over E. It is $E(\sqrt{b})$-quasi-split since (x,b) is split over $L(\sqrt{b})$ [Ga98, 5.6]. □

5. Ramification

Let G be an algebraic group over F. We say that an invariant $a \in \mathrm{Inv}^3(G)$ is *unramified* if the composition

$$H^1(E((t)),G) \xrightarrow{\ a\ } H^3(E((t))) \xrightarrow{\ \partial\ } H^2(E)$$

is 0 for every field extension E of F. Otherwise we say that a is *ramified*. The following example is typical:

Example 5.1. Let F be a field with a primitive cube root of unity, and let $L = F(\lambda^{1/3})$ be a cubic Galois extension of F. We claim that the invariant $a \in \mathrm{Inv}^3(PGL_3)$ from §2 given by $a(x) = \delta(x) \cup (\lambda)$ is ramified.

The set $H^1(F, PGL_3)$ classifies degree 3 cyclic central simple algebras (C,d) for C a cubic Galois extension of F and $d \in F^*$. Let $[C] \in H^1(F,\mathbb{Z}/3)$ denote the class corresponding to C. Then $\delta(C,d) \cup (\lambda)$ is $\pm[C] \cup (d) \cup (\lambda)$ in $H^3(F, \boldsymbol{\mu}_3^{\otimes 2})$. Taking $E = F(u)$ for u an indeterminate and $C = E((t))(t^{1/3})$ a cubic Galois extension of $E((t))$, we have

$$\partial\,[\delta(C,u) \cup (\lambda)] = \pm(u) \cup (\lambda) \quad \in H^2(E).$$

This is nonzero in $H^2(E)$ since u is not a norm from the extension $E(\lambda^{1/3})/E$. Hence a is ramified, as claimed. □

For the rest of the section, we assume that G is simple and simply connected. We write $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ for the subset of unramified invariants in $\mathrm{Inv}^3(G)$. It is a subgroup since ∂ is a group homomorphism.

Lemma 5.2 ([M02]). $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}} \cong \mathrm{Inv}^3_{\mathrm{nr}}(G)$. □

In particular, $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}}$ is necessarily finite; see §1.

5.3. If F has prime characteristic p, then multiplication by p is an isomorphism of $(\mathbb{Q}/\mathbb{Z})'(2)$. Hence $H^3(F)$, $\mathrm{Inv}^3(G)$, and — by Lemma 5.2 — $H^3_{\mathrm{nr}}(BG/F)_{\mathrm{norm}}$ have no nontrivial p-torsion.

This explains the hypothesis "$\mathrm{char}\, F \neq 2$" in the Main Theorem: a simply connected group G of type ${}^3D_4$ with nontrivial Tits algebras "should" have $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}}$ equal to $\mathbb{Z}/2$, but this is impossible in characteristic 2.

Strongly Inner Lemma 5.4. *Let G be a simple simply connected group over F, and fix $z \in Z^1(F, G)$. The canonical identification* $\mathrm{Inv}^3(G) = \mathrm{Inv}^3(G_z)$ *defined by $r_G \leftrightarrow r_{G_z}$ restricts to an identification* $\mathrm{Inv}^3_{\mathrm{nr}}(G) = \mathrm{Inv}^3_{\mathrm{nr}}(G_z)$.

Groups G_z obtained by twisting G by a 1-cocycle z as in the lemma are called *strongly inner forms* of G. Such groups have the same type and Tits algebras as G, hence also the same Dynkin index as G. In particular, $n'_G = n'_{G_z}$.

Proof. Let E be an extension of F, and consider the diagram

$$\begin{array}{ccccc} H^1(E((t)), G) & \xrightarrow{mr_G} & H^3(E((t))) & \xrightarrow{\partial} & H^2(E) \\ {\scriptstyle \tau_z}\downarrow\cong & & {\scriptstyle \cdot - mr_G(z)}\downarrow & & \| \\ H^1(E((t)), G_z) & \xrightarrow{mr_{G_z}} & H^3(E((t))) & \xrightarrow{\partial} & H^2(E), \end{array}$$

where τ_z is the twisting isomorphism and m is an integer. The left box commutes by [Gi00, p. 76, Lem. 7] or [MPT, 1.7]. The right box commutes because ∂ is a group homomorphism and $\partial(r_G(z)) = 0$. Hence mr_G is ramified if and only if mr_{G_z} is. □

5.5. Functoriality (homomorphisms). Let $\alpha : H \to G$ be a morphism of algebraic groups. Then α induces natural homomorphisms

$$\alpha^* \colon \mathrm{Inv}^3(G) \to \mathrm{Inv}^3(H) \quad \text{and} \quad \alpha^*_{\mathrm{nr}} \colon \mathrm{Inv}^3_{\mathrm{nr}}(G) \to \mathrm{Inv}^3_{\mathrm{nr}}(H).$$

Now suppose that H and G are simple simply connected. If the Rost multiplier of α is 1, then n_H divides n_G, hence n'_G/n'_H divides n_G/n_H. (See §1 for definitions.) Also, α^* is a surjection with kernel of order n'_G/n'_H. Then we have: *If* $\mathrm{Inv}^3_{\mathrm{nr}}(H)$ *is trivial, then* $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ *is* (n_G/n_H)*-torsion.*

5.6. Functoriality (scalar extension). Let K be an extension field of F, and write G_K for $G \times_F K$. The restriction homomorphism

$$\mathrm{res}_{K/F} \colon \mathrm{Inv}^3(G/F) \to \mathrm{Inv}^3(G/K)$$

is the natural surjection $\mathbb{Z}/n'_G \to \mathbb{Z}/n'_{G_K}$; its kernel is the (n'_G/n'_{G_K})-torsion in $\mathrm{Inv}^3(G/F)$. It restricts to a homomorphism

$$(\mathrm{res}_{K/F})_{\mathrm{nr}} \colon \mathrm{Inv}^3_{\mathrm{nr}}(G/F) \to \mathrm{Inv}^3_{\mathrm{nr}}(G/K).$$

The kernel of this map is killed by n'_G/n'_{G_K}, hence by n_G/n_{G_K}. We have: *If* $\mathrm{Inv}^3_{\mathrm{nr}}(G/K)$ *is trivial, then* $\mathrm{Inv}^3_{\mathrm{nr}}(G/F)$ *is* (n_G/n_{G_K})*-torsion.*

6. The case where G has trivial Tits algebras

In this section, we prove:

Proposition 6.1. *Let G be a simple simply connected exceptional algebraic group. If G has only trivial Tits algebras, then* $\mathrm{Inv}^3_{\mathrm{nr}}(G) = 0$.

That is, the Main Theorem holds for groups with only trivial Tits algebras by Lemma 5.2. In proving the proposition, we may assume that G is quasi-split by the Strongly Inner Lemma 5.4.

The proposition still holds if the hypothesis "exceptional" is dropped; the classical groups are treated in [M02].

6.2. Type ${}^3D_4$. Let L be a separable cubic extension of F, such that the quasi-split group G is of type ${}^1D_4$ over L. Since G is quasi-split, $n_{G_L} = 2$ and $\mathrm{Inv}^3_{\mathrm{nr}}(G/L) = 0$ by [M02, 8.5]. Hence $\mathrm{Inv}^3_{\mathrm{nr}}(G/F)$ is 3-torsion by §5.6. By §5.3 we may assume that $\operatorname{char} F \neq 3$.

Let F' be the extension obtained from F by adjoining (if not already in F) a primitive cube root of unity. Then G is still of type ${}^3D_4$ over F' and the invariants $2r_G$ and $4r_G$ are ramified over F' by Lemma 2.2 and Example 5.1. Since $2r_G$ and $4r_G$ are the only nontrivial 3-torsion elements of $\mathrm{Inv}^3(G/F)$, we have shown that $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ is 0.

Lemma 6.3. *Let G be simple simply connected of type ${}^6D_4$ (no restriction on the Tits algebras). Then* $\mathrm{Inv}^3_{\mathrm{nr}}(G) = 0$.

Proof. Let Δ be the unique quadratic extension of the base field F over which G is of type ${}^3D_4$. Let K be a generic quasi-splitting field for G over Δ as in [KR]. Then G is quasi-split of type ${}^3D_4$ over K, hence $n_{G_K} = 6$ and $\mathrm{Inv}^3_{\mathrm{nr}}(G/K) = 0$ by the ${}^3D_4$ case (6.2). We have that $\mathrm{Inv}^3_{\mathrm{nr}}(G/F)$ is 2-torsion by §5.6.

Let L be a cubic extension of F over which G is of type ${}^2D_4$. Then n_{G_L} is 2 or 4 and $\mathrm{Inv}^3_{\mathrm{nr}}(G/L) = 0$ by [M02, 8.5]. Since $n_G = 6$ or 12 (as $n_{G_L} = 2$ or 4), $\mathrm{Inv}^3_{\mathrm{nr}}(G/F)$ is 3-torsion.

Combining the two previous paragraphs, we find that $\mathrm{Inv}^3_{\mathrm{nr}}(G/F) = 0$. □

6.4. Type G_2. Here $n_G = 2$, so we may assume that $\operatorname{char} F \neq 2$. The Rost invariant for the split group of type G_2 is given explicitly in [KMRT, p. 441]. It is the Elman-Lam invariant for 3-Pfister quadratic forms, which is clearly ramified.

6.5. Type F_4. The split G_2 is contained in our split group G of type F_4 with Rost multiplier 1. (See the end of §1. We omit this sort of observation below.) Since $n_{F_4} = 6$ and $n_{G_2} = 2$, the group $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ is 3-torsion by §5.5 and the G_2 case (§6.4). In characteristic $\neq 3$, the mod 3 part of the Rost invariant is described in [PR, 3.2], and it is clearly ramified. So $\mathrm{Inv}^3_{\mathrm{nr}}(G) = 0$.

6.6. Type ${}^1E_6$. The split group G of type E_6 contains a subgroup which is split of type F_4 [Sp, 14.20, 14.24], and we have $n_{F_4} = n_G = 6$. Hence $\mathrm{Inv}^3_{\mathrm{nr}}(G) = 0$ by §5.5 and the F_4 case (§6.5).

6.7. Type ${}^2E_6$. The group G is split by a quadratic extension and $n_G = 12$. By §5.6 and the ${}^1E_6$ case (§6.6), $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ is 2-torsion, hence we may assume that $\operatorname{char} F \neq 2$.

Let T and E be as in Proposition 4.1. By Theorem 3.1, T is contained in a strongly inner form G' of G over E. By Lemma 6.3, $\mathrm{Inv}^3_{\mathrm{nr}}(T/E) = 0$. Since T has

a nontrivial Tits algebra over E, the Dynkin index n_T is 12. Since $n_{G'}$ is also 12, we have $\mathrm{Inv}^3_{\mathrm{nr}}(G'/E) = 0$. By the Strongly Inner Lemma, $\mathrm{Inv}^3_{\mathrm{nr}}(G/E) = 0$. Since G is of type ${}^2E_6$ over F and E, we have $n_G = n_{G_E}$, hence $\mathrm{Inv}^3_{\mathrm{nr}}(G/F) = 0$ by §5.6.

6.8. Type E_7. The natural inclusion of root systems gives a split simply connected subgroup of type E_6 inside the split group G of type E_7. Since $n_G = 12$ and $n_{E_6} = 6$, $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ is 2-torsion by §5.5 and the ${}^1E_6$ case (§6.6). Hence we may assume that $\mathrm{char}\, F \neq 2$.

Set $F' = F(x)$ and $K = F'(\sqrt{x})$. There is a quasi-split simply connected group E_6^K over F of type ${}^2E_6$ associated with the extension K/F'; it injects into $G_{F'}$ with Rost multiplier 1 [Ga01, §3]. Since $n_{E_6^K}$ and $n_{G_{F'}}$ are both 12, $\mathrm{Inv}^3_{\mathrm{nr}}(G/F') = 0$ by §5.5 and the ${}^2E_6$ case (§6.7). Since $n_G = 12$, we have $\mathrm{Inv}^3_{\mathrm{nr}}(G/F) = 0$.

6.9. Type E_8. As in the previous cases, we may assume that our group G of type E_8 is actually split. The natural inclusion of root systems gives an embedding of a split simply connected group of type E_7 in G, so $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ is 5-torsion by §5.5 and the E_7 case (§6.8). In particular, we may assume that $\mathrm{char}\, F \neq 5$ by §5.3.

Let F' be the extension obtained from F by adjoining two indeterminates and (if necessary) a primitive 5th root of unity. There is an F'-central division algebra D of dimension 5^2, namely the symbol algebra determined by the two indeterminates. There are obvious copies of SL_5 in G arising from inclusions of sets of simple roots, and these inclusions have Rost multiplier 1. Arguing in a manner similar to [Gi02, §1], one can twist such an inclusion to find a strongly inner form G' of G over F' and an injection $SL_1(D) \hookrightarrow G'$. Now $n_{SL_1(D)} = 5$ [M03, 11.5] and $\mathrm{Inv}^3_{\mathrm{nr}}(SL_1(D)/F') = 0$ (as can be seen from the explicit formula for the Rost invariant in [M03, p. 138]), hence $\mathrm{Inv}^3_{\mathrm{nr}}(G'/F')$ is 12-torsion by §5.5. By the Strongly Inner Lemma, $\mathrm{Inv}^3_{\mathrm{nr}}(G/F')$ is 12-torsion. Since the Dynkin index of G is 60 over F and F', $\mathrm{Inv}^3_{\mathrm{nr}}(G/F)$ is 12-torsion by §5.6.

Combining the two preceding paragraphs gives that $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ is 0. This completes the proof of Proposition 6.1. □

7. Proof of the Main Theorem

Let G be as in the Main Theorem 0.2. If G has only trivial Tits algebras (e.g., G is of type G_2, F_4, or E_8), then the Main Theorem holds for G by Proposition 6.1 and Lemma 5.2.

If G is of type E_6 or E_7, then we pick a generic quasi-splitting field K of G over F. We have $n_{G_K} = n_G$ and $\mathrm{Inv}^3_{\mathrm{nr}}(G/K) = 0$ (by Proposition 6.1), hence $\mathrm{Inv}^3_{\mathrm{nr}}(G/F) = 0$ by §5.6.

If G is of type ${}^6D_4$, the Main Theorem holds by Lemma 6.3. The remaining case is where G is of type ${}^3D_4$ with nontrivial Tits algebras. We have $n_G = 12$ and as in the proof of Lemma 6.3, $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ is 2-torsion. Hence we may assume that $\mathrm{char}\, F \neq 2$.

The only nontrivial 2-torsion element of $\mathrm{Inv}^3(G)$ is $6r_G$, so we will complete the proof of the Main Theorem if we show that $6r_G$ is unramified. That is, if we show that for every extension E of F, the composition

$$H^1(E((t)), G) \xrightarrow{6r_{G_{E((t))}}} H^3(E((t))) \xrightarrow{\partial} H^2(E) \tag{7.1}$$

is 0.

If G is of type ${}^1D_4$ over E, then it is of type ${}^1D_4$ over $E((t))$ and $n_{G_{E((t))}}$ is 2 or 4 [M03, 15.4]. Hence $6r_{G_{E((t))}} = 2r_{G_{E((t))}}$ and the composition (7.1) is 0 by [M02, 8.2].

Otherwise, G is of type ${}^3D_4$ over E. That is, if L is a cubic Galois extension of F over which G is of type ${}^1D_4$, the tensor product $L' := L \otimes_F E$ is a cubic field extension of E. We have a diagram

$$\begin{array}{ccccc} H^1(E((t)),G) & \xrightarrow{6r_{G_{E((t))}}} & H^3(E((t))) & \xrightarrow{\partial} & H^2(E) \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}_{L'/E}} \\ H^1(L'((t)),G) & \xrightarrow{6r_{G_{L'((t))}}} & H^3(L'((t))) & \xrightarrow{\partial} & H^2(L'). \end{array}$$

The left box commutes because the Rost invariant is compatible with restriction, and the right box commutes because the extension $E((t)) \subset L'((t))$ is unramified, hence the whole diagram commutes. Fix a class α in $H^1(E((t)),G)$. Since $G_{L'((t))}$ is of type ${}^1D_4$, the composition of the two bottom arrows is 0 by the preceding paragraph, and the image of α in $H^2(L')$ is 0. Let $\beta \in H^2(E)$ be the image of α; it is 2-torsion because $6r_G$ is 2-torsion. Hence

$$\beta = 3\beta = \mathrm{cor}_{L'/E}\,\mathrm{res}_{L'/E}(\beta),$$

which is 0 by the commutativity of the diagram. This shows that the composition (7.1) — which is the top row of the diagram — is 0 in this case.

Thus $\mathrm{Inv}^3_{\mathrm{nr}}(G) = \mathbb{Z}/2$ for G of type ${}^3D_4$ with nontrivial Tits algebras when $\mathrm{char}\, F \neq 2$. This completes the proof of the Main Theorem 0.2. □

Corollary 7.2. *Let G be a simply connected group of type ${}^3D_4$ or ${}^6D_4$ over a field F of characteristic $\neq 2$. If G has a nontrivial Tits algebra, then BG is not stably rational.*

Proof. If G is of type ${}^3D_4$, this is a direct consequence of the Main Theorem.

If G is of type ${}^6D_4$, let K be the unique quadratic extension of F over which G is of type ${}^3D_4$. It follows from [Ga98, 3.2] that G has nontrivial Tits algebras when G is considered as a K-group. Therefore BG is not stably rational as a K-variety, hence not as an F-variety. □

Acknowledgements

Thanks to Adrian Wadsworth for his comments on §4 and to Alexander Merkurjev for suggesting the problem and for enlightening conversations.

References

[C] J.-L. Colliot-Thélène, *Birational invariants, purity and the Gersten conjecture*, in K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992) (B. Jacob and A. Rosenberg, eds.), Proceedings of Symposia in Pure Mathematics **58.1** (1995), 1–64. MR1327280 (96c:14016)

[Dyn] E.B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Transl. (2) **6** (1957), 111–244 [Russian original: Mat. Sbornik N.S. **30(72)** (1952), 349–462]. MR0047629 (13:904c)

[Ga98] R.S. Garibaldi, *Isotropic trialitarian algebraic groups*, J. Algebra **210** (1998), 385–418 [DOI `10.1006/jabr.1998.7584`]. MR1662339 (99k:20092)

[Ga01] ———, *The Rost invariant has trivial kernel for quasi-split groups of low rank*, Comment. Math. Helv. **76** (2001), no. 4, 684–711. MR1881703 (2003g:20079)

[Gi00] Ph. Gille, *Invariants cohomologiques de Rost en caractéristique positive*, K-Theory **21** (2000), 57–100. MR1802626 (2001k:11064)

[Gi02] ———, *Algèbres simples centrales de degré 5 et E_8*, Canad. Math. Bull. **45** (2002), no. 3, 388–398. MR1937674 (2004b:16017)

[Jac] N. Jacobson, *Triality and Lie algebras of type D_4*, Rend. Circ. Mat. Palermo (2) **13** (1964), 129–153 (= Coll. Math. Papers 71). MR0181705 (31#5932)

[KMRT] M.-A. Knus, A.S. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, Colloquium Publications **44**, American Mathematical Society, 1998. MR1632779 (2000a:16031)

[KR] I. Kersten and U. Rehmann, *Generic splitting of reductive groups*, Tôhoku Math. J. (2) **46** (1994), 35–70. MR1256727 (95a:20048)

[M96] A.S. Merkurjev, *Maximal indexes of Tits algebras*, Doc. Math. **1** (1996), 229–243. MR1405670 (97g:20052)

[M02] ———, *Unramified cohomology of classifying varieties for classical simply connected groups*, Ann. Sci. École Norm. Sup. (4) **35** (2002), 445–476. MR1914006 (2003d:11059)

[M03] ———, *Rost invariants of simply connected algebraic groups*, with a section by S. Garibaldi, in *Cohomological invariants in Galois cohomology*, University Lecture Series **28**, American Mathematical Society, 2003. MR1999385

[MPT] A.S. Merkurjev, R. Parimala, and J.-P. Tignol, *Invariants of quasi-trivial tori and the Rost invariant*, St. Petersburg Math. J. **14** (2003), 791–821. MR1970336 (2004c:11045)

[PR] H.P. Petersson and M.L. Racine, *An elementary approach to the Serre-Rost invariant of Albert algebras*, Indag. Math. (N.S.) **7** (1996), no. 3, 343–365. MR1621373 (99j:17045)

[Sp] T.A. Springer, *Jordan algebras and algebraic groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 75, Springer-Verlag, 1973. MR0379618 (52:523)

Department of Mathematics & Computer Science, Emory University, Atlanta, Georgia 30322

E-mail address: skip@member.ams.org

URL: http://www.mathcs.emory.edu/~skip/