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MULTI-SCALE YOUNG MEASURES

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Abstract. We introduce multi-scale Young measures to deal with problems where multi-scale phenomena are relevant. We prove some interesting representation results that allow the use of these families of measures in practice, and illustrate its applicability by treating, from this perspective, multi-scale convergence and homogenization of multiple integrals.

1. Introduction

Young measures have been a main tool in several fields in applied mathematics. They have proved their power when dealing with singular continuous optimization problems of different types ([15]). It is also known that they suffer from important drawbacks ([18]).

The main property of Young measures that explains their importance and suitability to treat integral cost functionals in optimization problems is its capability of representing weak limits of compositions of the sequence giving rise to the Young measure with any continuous quantity. Specifically, if $\{u_\epsilon\}$ is a certain sequence of functions defined in a domain $\Omega \subset \mathbf{R}^N$ and taking values on $\mathbf{R}^m$, satisfying some mild uniform integrability condition (like being uniformly bounded in $L^p(\Omega)$), and if φ is any continuous function on $\mathbf{R}^m$ so that $\{\varphi(u_\epsilon)\}$ is weakly convergent in $L^1(\Omega)$, then

$$\varphi(u_\epsilon) \rightharpoonup \overline{\varphi} \quad \text{in } L^1(\Omega)$$

as $\epsilon \searrow 0$ where

$$\overline{\varphi}(x) = \langle \varphi, \nu_x \rangle$$

and $\nu = \{\nu_x\}_{x\in\Omega}$ is the Young measure associated with (a subsequence of) $\{u_\epsilon\}$. This is essentially the main general result for Young measures ([5], [7], [20]). As such, each ν_x is just a device to keep record of the relative distribution of the values of the sequence $\{u_\epsilon\}$ around $x \in \Omega$ as $\epsilon \searrow 0$. No further qualitative information about the sequence $\{u_\epsilon\}$ is carried by ν. For example, we cannot see or tell from ν if the sequence oscillates in one or more length-scales, or if it does in a certain direction; we cannot say how close $\{u_\epsilon\}$ is to oscillating periodically, etc.

What we would like to stress in this paper is that we can extract much more information on the sequence $\{u_\epsilon\}$ if we "test" it against sequences tailored to detect a certain concrete property in the given sequence. The basic concept is that of

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joint Young measure which is nothing but the Young measure associated with pairs $\{(u_\epsilon, v_\epsilon)\}$ where $\{v_\epsilon\}$ is a sequence we have built to test $\{u_\epsilon\}$ against. This joint Young measure will tell us much more about the structure of $\{u_\epsilon\}$ than ν in regard to the property we would like to find. The basic tool is the slicing measure decomposition or disintegration ([4], [13]). For the joint Young measure $\nu = \{\nu_x\}_{x\in\Omega}$ associated with the sequence of pairs $\{(u_\epsilon, v_\epsilon)\}$ and the Young measure $\sigma = \{\sigma_x\}_{x\in\Omega}$ corresponding to the test sequence $\{v_\epsilon\}$, we can write for a.e. $x \in \Omega$,

$$\nu_x = \mu_x^{(\lambda)} \otimes \sigma_x$$

for σ_x-a.e. λ in $\mathbf{R}^d$ if this is the target space for the test sequence $\{v_\epsilon\}$. The family of probability measures

$$\left\{\mu_x^{(\lambda)}\right\}_{x\in\Omega,\lambda\in\mathbf{R}^d}$$

carries much more information (relative to the test sequence) than the single Young measure corresponding to our sequence $\{u_\epsilon\}$. Notice that the projection of ν onto the components corresponding to u_ϵ is precisely the Young measure associated with $\{u_\epsilon\}$.

A simple example will clarify what we mean. Let $\chi(x)$ be the characteristic function of $(0, 1/2)$ over $(0, 1)$ extended by periodicity. Let $a \in (0, 1)$ be arbitrary and let $l(\epsilon)$ be a certain length scale (relative to ϵ) so that

$$\lim_{\epsilon\searrow 0} \frac{l(\epsilon)}{\epsilon} = c \in [0, +\infty].$$

Consider the sequence of pairs $\{(u_\epsilon, v_\epsilon)\}$ where

$$u_\epsilon(x) = \chi\left(\frac{x}{\epsilon}\right), \quad v_\epsilon(x) = \chi\left(a + \frac{x}{l(\epsilon)}\right),$$

and let us look at the joint Young measure associated with such a sequence of pairs. This measure is supported in the set of four points $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$. But the structure depends dramatically on the values of c and a. In particular, if $0 < c < +\infty$, the weights of the joint Young measure depends in a relevant manner on a but they are independent of c; but if $c = 0$ or $c = +\infty$, then the joint Young measure is the product measure with weights $1/4$ on each point and does not depend on a. This independence from a is somehow the key feature indicating that both sequences oscillate at different length scales.

Suppose a sequence $\{u_\epsilon\}$ is given. We would like to know if it oscillates at a given length scale $l(\epsilon)$ in a given direction n. How can we test this property? It turns out that if we build the test sequence

$$w_\epsilon^a(x) = \chi\left(a + \frac{x \cdot n}{l(\epsilon)}\right)$$

where $a \in (0, 1)$ is arbitrary, then the dependence on a of the joint Young measure associated with those pairs will tell us if $\{u_\epsilon\}$ does indeed oscillate at that length scale in the direction determined by n just as we have argued before.

In addition to stating this philosophy of looking at joint Young measures, which in fact has been tacitly used in a number of works (see for example [17]), as a means to extract more information on oscillating sequences of functions, we would like to examine more closely one particular important case.

Let $\Omega \subset \mathbf{R}^N$ be a regular, bounded domain and let $Z = (0, 1)^N$ be the unit cube in $\mathbf{R}^N$. Let us use $\langle\cdot\rangle$ to indicate the fractional part of a number or of a vector

componentwise. It is easy to check that the Young measure associated with the sequence

$$\left\{\langle\frac{x}{l(\epsilon)}\rangle\right\}: \quad x\in\Omega,$$

is the Lebesgue measure restricted to Z, homogeneous (see Section 2 for references). Thus if $\{u_\epsilon\}$ is a certain sequence we are interested in, the joint Young measure $\nu=\{\nu_x\}_{x\in\Omega}$ corresponding to pairs

$$\left\{\left(u_\epsilon(x),\langle\frac{x}{l(\epsilon)}\rangle\right)\right\}$$

can be decomposed as

$$\nu_{x,z}\otimes dz.$$

The family of measures

$$\nu=\{\nu_{x,z}\}_{x\in\Omega,z\in Z}$$

is called the Young measure associated with $\{u_\epsilon\}$ at scale $l(\epsilon)$. The aim of this paper is to start the analysis of these families of probability measures.

In a straightforward manner, we may generalize these ideas to incorporate multi-scale Young measures. Consider a finite family of "separated" length scales

$$\{l_1(\epsilon),l_2(\epsilon),\dots,l_n(\epsilon)\}$$

in the sense that

$$\lim_{\epsilon\searrow 0}\frac{l_{i+1}(\epsilon)}{l_i(\epsilon)}=0,\quad i=1,2,\dots,n-1.$$

Because of this hierarchy on the length scales, we have that the Young measure corresponding to the sequence

$$\left\{\left(\langle\frac{x}{l_1(\epsilon)}\rangle,\langle\frac{x}{l_2(\epsilon)}\rangle,\dots,\langle\frac{x}{l_n(\epsilon)}\rangle\right)\right\}$$

is the Lebesgue measure

$$dz=dz_1\otimes dz_2\otimes\cdots\otimes dz_n$$

over $Z\times Z\times\cdots\times Z$ (n times), homogeneous. Again, the joint Young measure associated with the sequence

$$\left\{\left(u_\epsilon(x),\langle\frac{x}{l_1(\epsilon)}\rangle,\langle\frac{x}{l_2(\epsilon)}\rangle,\dots,\langle\frac{x}{l_n(\epsilon)}\rangle\right)\right\}$$

can be decomposed as

$$\nu_{x,z}\otimes dz.$$

The family of probability measures

$$\nu=\{\nu_{x,z}\}_{x\in\Omega,z\in Z^n}$$

is called the multi-scale Young measure associated with $\{u_\epsilon\}$ at the given length scales. The analysis of these families of measures is the main motivation of this paper.

Our main contribution (Theorem 2.9 below) is a formula that permits the use of these families of measures in practice. If $\{u_\epsilon\}$ is a weakly convergent sequence in some Sobolev space, and $\{l_1(\epsilon),\dots,l_n(\epsilon)\}$ is a collection of "separated" length

scales, then the associated multi-scale Young measure $\nu_{x,z_1,z_2,\dots,z_n}$ is determined through the formula

$$\langle \varphi, \nu_{x,z_1,\dots,z_n} \rangle = \lim_{\epsilon \searrow 0} \frac{1}{|r_j(\epsilon)Z|} \int_{r_j(\epsilon)Z} \varphi\left(v_\epsilon^{(j)}(y, z_1, \dots, z_n)\right) dy$$

where

$$v_\epsilon^{(j)}(y, z_1, \dots, z_n) = u_\epsilon(x + l_j(\epsilon)\,[y] + l_1(\epsilon)z_1 + l_2(\epsilon)z_2 + \cdots + l_n(\epsilon)z_n),$$
$$y \in r_j(\epsilon)Z, \quad z_i \in Z,$$

for a certain sequence $r(\epsilon) \searrow 0$ such that $r_j(\epsilon) = r(\epsilon)/l_j(\epsilon) \nearrow +\infty$ for all j. $[\cdot]$ stands for the integer part, so that $y = [y] + \langle y \rangle$ for any vector y.

We will illustrate how this result can be used in two contexts. The first one (Section 3) is the structure of multi-scale limits in the spirit of [2], [6], or even better [3]. The second one (Section 4) corresponds to the analysis of homogenization of multiple integrals ([8]). We will restrict attention here, on this initial work, to typical situations and defer more complicated problems for the future. The idea of analyzing joint or coupled Young measures to derive more information on oscillating sequences has also been specifically studied for various reasons (nearly always related to homogenization issues for differential equations) and in various contexts in [12], [14] and [19]. Yet the full power of the multi-scale Young measure being capable of representing weak limits of compositions of almost any kind and not just of a particular convenient (multiplicative) type has not been explored (see Section 4). From this point of view, the structure of the multi-scale Young measure itself is considered. The paper [1] was also an inspiration to the author. See also [9] and [10].

Another main issue is to understand how special properties of these measures or their structure indicate special features of generating sequences.

2. Main results

The main concept we would like to introduce is stated in Definition 2.6 below. We treat as a preliminary step the one scale concept.

Definition 2.1. A length scale $l(\epsilon)$ is a smooth map

$$l : (0, \epsilon_0) \to (0, +\infty)$$

for some $\epsilon_0 > 0$ such that $l(\epsilon) \to 0$ as $\epsilon \to 0$.

A length scale $l_1(\epsilon)$ is faster (or finer) than $l_2(\epsilon)$ if

$$\lim_{\epsilon \to 0} \frac{l_1(\epsilon)}{l_2(\epsilon)} = 0.$$

Proposition 2.2. *Let Ω be a bounded domain in $\mathbf{R}^N$, and let $l(\epsilon)$ be any length scale. The Young measure associated with $\{\langle x/l(\epsilon)\rangle\}$ is the Lebesgue measure over Z, homogeneous.*

As a matter of fact, this is a direct consequence of the well-known Riemann-Lebesgue lemma ([11], [16]).

As indicated in the Introduction, if $u_\epsilon : \Omega \subset \mathbf{R}^N \to \mathbf{R}^m$, then the Young measure associated with

$$\left\{ \left(u_\epsilon(x), \langle \frac{x}{l(\epsilon)} \rangle \right) \right\}$$

can be decomposed as

$$\nu_{x,z} \otimes dz$$

for $x \in \Omega$ and $z \in Z$. The measure $\nu_{x,z}$ characterizes the oscillatory periodic behavior of $\{u_\epsilon\}$ at length scale $l(\epsilon)$. In particular, its first moment is the two-scale limit of $\{u_\epsilon\}$ at the given length scale.

Proposition 2.3. *The mapping*

$$u : \Omega \times Z \to \mathbf{R}^m$$

defined by

$$u(x,z) = \int_{\mathbf{R}^m} \lambda \, d\nu_{x,z}(\lambda)$$

is the two-scale limit at length scale $l(\epsilon)$.

Proof. If for a test function g, the sequence $\left\{ u_\epsilon(x) \; g\left(x, \langle \frac{x}{l(\epsilon)} \rangle\right)\right\}$ is weakly convergent in $L^1(\Omega)$, then

$$\lim_{\epsilon \to 0} \int_\Omega u_\epsilon(x) \; g\left(x, \langle \frac{x}{l(\epsilon)} \rangle\right) = \int_\Omega \int_Z \int_{\mathbf{R}^m} g(x,z) \lambda \, d\nu_{x,z}(\lambda) \, dz \, dx.$$

But this last integral can also be written as

$$\int_\Omega \int_Z u(x,z) g(x,z) \, dz \, dx$$

where

$$u(x,z) = \int_{\mathbf{R}^m} \lambda \, d\nu_{x,z}(\lambda)$$

is then clearly seen to be the two-scale limit. □

Definition 2.4. The family of probability measures $\{\nu_{x,z}\}_{x \in \Omega, z \in Z}$ is called the Young measure associated with $\{u_\epsilon\}$ at length scale $l(\epsilon)$.

Similarly, we can define the multi-scale Young measure associated with $\{u_\epsilon\}$ and several length scales $\{l_1(\epsilon), l_2(\epsilon), \ldots, l_n(\epsilon)\}$ where each $l_i(\epsilon)$ is faster than its predecessor according to Definition 2.1. Once again, because of this hierarchy of length scales, we have the following.

Proposition 2.5. *Under the above conditions, the Young measure associated with*

$$\left\{ \left(\langle \frac{x}{l_1(\epsilon)} \rangle, \langle \frac{x}{l_2(\epsilon)} \rangle, \ldots, \langle \frac{x}{l_n(\epsilon)} \rangle \right) \right\}$$

is the Lebesgue measure

$$dz = dz_1 \otimes dz_2 \otimes \cdots \otimes dz_n$$

over Z^n, homogeneous.

The separation of scales is intimately connected to the product structure of the underlying Young measure.

As before, we now look at the Young measure corresponding to

$$\left\{ \left(u_\epsilon(x), \langle \frac{x}{l_1(\epsilon)} \rangle, \langle \frac{x}{l_2(\epsilon)} \rangle, \ldots, \langle \frac{x}{l_n(\epsilon)} \rangle \right) \right\},$$

which can be decomposed as

$$\nu_{x,z} \otimes dz, \quad x \in \Omega, z \in Z^n.$$

Definition 2.6. The family of probability measures $\{\nu_{x,z}\}_{x\in\Omega, z\in Z^n}$ is called the multi-scale Young measure associated with the sequence $\{u_\epsilon\}$ and the family of length scales $\{l_1(\epsilon), l_2(\epsilon), \ldots, l_n(\epsilon)\}$. In particular, its first moment

$$u : \Omega \times Z^n \to \mathbf{R}^m$$

defined by

$$u(x,z) = \int_{\mathbf{R}^m} \lambda \, d\nu_{x,z}(\lambda)$$

is called the multi-scale limit of $\{u_\epsilon\}$ at the given family of length scales.

The main result that enables an efficient use of multi-scale Young measures in practice is a fact that tells how these families of measures are determined in terms of $\{u_\epsilon\}$ and the corresponding family of length scales $\{l_1(\epsilon), l_2(\epsilon), \ldots, l_n(\epsilon)\}$. It is a key tool in the understanding of these families of measures.

Theorem 2.7. *Let $\{u_\epsilon\}$ be a sequence of functions defined over $\Omega \subset \mathbf{R}^N$ taking values in $\mathbf{R}^m$. Given the length scale $l(\epsilon)$, let $\nu = \{\nu_{x,z}\}_{x\in\Omega, z\in Z}$ be the Young measure associated with the given length scale so that*

$$\mu_x = \nu_{x,z} \otimes dz \tag{2.1}$$

is the Young measure associated with pairs $\{(u_\epsilon(x), \langle x/l(\epsilon)\rangle)\}$ as $\epsilon \searrow 0$ (no subsequence). For a.e. $x \in \Omega$, there is $r(\epsilon) \searrow 0$ such that $s(\epsilon) = r(\epsilon)/l(\epsilon) \nearrow +\infty$ and the sequence of functions

$$v_\epsilon(y,z) = u_\epsilon\left(x + l(\epsilon)\,[y] + l(\epsilon)z\right), \quad y \in s(\epsilon)Z, z \in Z,$$

determines the measure $\nu_{x,z}$ in the following sense:

$$\langle \varphi, \nu_{x,z}\rangle = \lim_{\epsilon\searrow 0} \frac{1}{|s(\epsilon)Z|} \int_{s(\epsilon)Z} \varphi\left(v_\epsilon(y,z)\right) dy$$

for a.e. $z \in Z$ and all continuous φ.

Notice that we are NOT saying that $\{\nu_{x,z}\}_{z\in Z}$ is the Young measure associated with the family of functions

$$w_\epsilon(z) = \lim_{\epsilon\searrow 0} \frac{1}{|s(\epsilon)Z|} \int_{s(\epsilon)Z} v_\epsilon(y,z)\, dy.$$

Proof. By the localization property of Young measures (Theorem 7.2 in [16]), for a.e. $x \in \Omega$ there exists $r(\epsilon) \searrow 0$ such that the Young measure associated with the pairs $\{(w_\epsilon, \eta_\epsilon(y))\}$ where

$$w_\epsilon(y) = u_\epsilon(x + r(\epsilon)y), \quad \eta_\epsilon(y) = \langle \frac{x + r(\epsilon)y}{l(\epsilon)}\rangle, \quad y \in Z,$$

is precisely μ_x in (2.1), homogeneous (in y). Then we have

$$\int_Z \int_{\mathbf{R}^m} \varphi(\lambda, z)\, d\nu_{x,z}(\lambda)\, dz = \lim_{\epsilon\searrow 0} \int_Z \varphi(w_\epsilon(y), \eta_\epsilon(y))\, dy$$

for any continuous $\varphi(\lambda, z)$.

When we take φ as an arbitrary Z-periodic function of z, we immediately see that $s(\epsilon) = r(\epsilon)/l(\epsilon) \nearrow +\infty$. If we now take φ as an arbitrary, bounded, continuous function of λ, then

$$\int_Z \langle \varphi, \nu_{x,z}\rangle\, dz = \lim_{\epsilon\searrow 0} \int_Z \varphi(w_\epsilon(y))\, dy.$$

By making the change of variables $s(\epsilon)y = \overline{y}$, we have

$$\int_Z \langle \varphi, \nu_{x,z} \rangle \, dz = \lim_{\epsilon \searrow 0} \frac{1}{|s(\epsilon)Z|} \int_{s(\epsilon)Z} \varphi(u_\epsilon(x + l(\epsilon)\overline{y})) \, d\overline{y}.$$

The next elementary lemma, whose proof is straightforward, is the key to rewrite the last integral in an appropriate way and to finish the proof of Theorem 2.7.

Lemma 2.8. *Let $\Omega \subset \mathbf{R}^N$ be a domain such that*

$$\Omega = \bigcup_i (a_i + rZ), \quad a_i \in r\mathbf{Z}^N.$$

For any continuous ψ, we have

$$\int_\Omega \psi(y) \, dy = \int_Z \int_\Omega \psi \left(r \left[\frac{y}{r} \right] + rz \right) \, dy \, dz.$$

□

There is a similar version for a genuine multi-scale situation.

Theorem 2.9. *Let $\{u_\epsilon\}$ be as in the preceding theorem. Let $\{l_1(\epsilon), \ldots, l_n(\epsilon)\}$ be a finite family of separated scales, and $\nu = \{\nu_{x,z_1,\ldots,z_n}\}_{x \in \Omega, z_i \in Z}$ the corresponding multi-scale Young measure so that*

$$\mu_x = \nu_{x,z_1,\ldots,z_n} \otimes dz, \quad dz = dz_1 \otimes \cdots \otimes dz_n,$$

is the Young measure, as $\epsilon \searrow 0$, associated with

$$\left\{ \left(u_\epsilon(x), \langle \frac{x}{l_1(\epsilon)} \rangle, \ldots, \langle \frac{x}{l_n(\epsilon)} \rangle \right) \right\}.$$

For a.e. $x \in \Omega$, there is $r(\epsilon) \searrow 0$ such that $r_j(\epsilon) = r(\epsilon)/l_j(\epsilon) \nearrow +\infty$ for all j, and the sequence of functions

$$v_\epsilon^{(j)}(y, z_1, \ldots, z_n) = u_\epsilon(x + l_j(\epsilon)\,[y] + l_1(\epsilon)z_1 + l_2(\epsilon)z_2 + \cdots + l_n(\epsilon)z_n),$$

$$y \in r_j(\epsilon)Z, \quad z_i \in Z,$$

determine the measure $\nu_{x,z_1,\ldots,z_n}$ for all j, in the sense that

$$\langle \varphi, \nu_{x,z_1,\ldots,z_n} \rangle = \lim_{\epsilon \searrow 0} \frac{1}{|r_j(\epsilon)Z|} \int_{r_j(\epsilon)Z} \varphi \left(v_\epsilon^{(j)}(y, z_1, \ldots, z_n) \right) \, dy$$

for a.e. $z_i \in Z$ and all continuous φ.

Proof. We will make the proof for the case $n = 2$ when we have just two separated scales. The general case is a straightforward generalization.

As in the proof of Theorem 2.7, there exists a sequence $r(\epsilon) \searrow 0$ such that the blowup sequence

$$w_\epsilon(y) = u_\epsilon(x + r(\epsilon)y), \quad y \in Z,$$

determines the family of measures we are interested in. Moreover, $r_j(\epsilon) = r(\epsilon)/l_j(\epsilon) \nearrow +\infty$, $j = 1, 2$.

Let φ be an arbitrary, bounded, continuous function of λ. Just as before, we have to examine the integrals

$$\int_{Z \times Z} \langle \varphi, \nu_{x,z_1,z_2} \rangle \, dz_1 \, dz_2 = \lim_{\epsilon \searrow 0} \int_Z \varphi(w_\epsilon(y)) \, dy.$$

By the work we have already done in the previous proof, if we let $s(\epsilon) = r_1(\epsilon)$, then

$$\int_{Z\times Z} \langle \varphi, \nu_{x,z_1,z_2}\rangle \, dz_1 \, dz_2 = \lim_{\epsilon\searrow 0} \frac{1}{|s(\epsilon)Z|} \int_Z \int_{s(\epsilon)Z} \varphi(u_\epsilon(x + l_1(\epsilon)\,[y] + l_1(\epsilon)z_1))\, dy\, dz_1.$$

If we set

$$\psi(z_1) = \int_{s(\epsilon)Z} \varphi(u_\epsilon(x + l_1(\epsilon)\,[y] + l_1(\epsilon)z_1))\, dy$$

and apply Lemma 2.8 to this function, to $\Omega = Z$, and to $r = l_2(\epsilon)/l_1(\epsilon)$, we obtain

(2.2)

$$\begin{aligned}
&\int_Z \int_{s(\epsilon)Z} \varphi(u_\epsilon\,(x + l_1(\epsilon)\,[y] + l_1(\epsilon)z_1))\, dy\, dz_1 \\
&\quad = \int_Z \int_Z \int_{s(\epsilon)Z} \varphi\left(u_\epsilon\left(x + l_1(\epsilon)\,[y] + l_1(\epsilon)\left(r\left[\frac{z_1}{r}\right] + rz_2\right)\right)\right) dz_1\, dz_2 \\
&\quad = \int_Z \int_Z \int_{s(\epsilon)Z} \varphi\left(u_\epsilon\left(x + l_1(\epsilon)\,[y] + l_2(\epsilon)\left[\frac{l_1(\epsilon)}{l_2(\epsilon)}z_1\right] + l_2(\epsilon)z_2\right)\right) dz_1\, dz_2.
\end{aligned}$$

Finally, notice that in the limit as $\epsilon \searrow 0$ this last integral yields the same limit as

$$\int_Z \int_Z \int_{s(\epsilon)Z} \varphi\left(u_\epsilon\left(x + l_1(\epsilon)\,[y] + l_1(\epsilon)z_1 + l_2(\epsilon)z_2\right)\right) dz_1\, dz_2.$$

It would suffice to show that the difference tends to zero by using the dominated convergence theorem. This is standard. A suitable, natural change of variables allows us to write the limit in the statement of the theorem as an integral over $r_j(\epsilon)Z$ for any j. □

An interesting corollary is the following.

Corollary 2.10. *Under the same assumptions as in Theorem 2.9,*

$$\lim_{\epsilon\searrow 0} \frac{1}{|r_i(\epsilon)Z|} \int_Z \int_{r_i(\epsilon)Z} u_\epsilon(x+l_1(\epsilon)z_1+l_2(\epsilon)z_2+\cdots+l_{i-1}(\epsilon)z_{i-1}+l_i(\epsilon)w+l_i(\epsilon)[y])\, dy\, dw$$

$$= \lim_{\epsilon\searrow 0} \frac{1}{|r_{i-1}(\epsilon)Z|} \int_{r_{i-1}(\epsilon)Z} u_\epsilon(x + l_1(\epsilon)z_1 + l_2(\epsilon)z_2 + \cdots + l_{i-1}(\epsilon)z_{i-1} + l_{i-1}(\epsilon)[y])\, dy$$

for all $i = 1, 2, \ldots, n$.

This is a direct consequence of Theorem 2.9 (for φ, the identity) and the fact that if $u(x, z_1, z_2, \ldots, z_i)$ is the multi-scale limit at those scales, then

$$\int_Z u(x, z_1, \ldots, z_{i-1}, z_i)\, dz_i = u(x, z_1, \ldots, z_{i-1}).$$

For $i = 1$, the corollary means that

$$\lim_{\epsilon\searrow 0} \frac{1}{|r_1(\epsilon)Z|} \int_Z \int_{r_1(\epsilon)Z} u_\epsilon(x + l_1(\epsilon)[y] + l_1(\epsilon)z)\, dy\, dz,$$

is precisely the weak limit (in the usual sense) u of $\{u_\epsilon\}$.

3. Multi-scale convergence

The defining property of Young measures applied to our context enables us to express the weak limits of non-linear quantities of the type

$$F\left(x, \langle\frac{x}{l_1(\epsilon)}\rangle, \langle\frac{x}{l_2(\epsilon)}\rangle, \dots, \langle\frac{x}{l_n(\epsilon)}\rangle, u_\epsilon(x)\right)$$

whenever these sequences are weakly converging in $L^1(\Omega)$. If this is so, then

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega F\left(x, \langle\frac{x}{l_1(\epsilon)}\rangle, \langle\frac{x}{l_2(\epsilon)}\rangle, \dots, \langle\frac{x}{l_n(\epsilon)}\rangle, u_\epsilon(x)\right)\\ = \int_\Omega\int_{Z^n}\int_{\mathbf{R}^d} F(x,z,\lambda)\,d\nu_{x,z}(\lambda)\,dz\,dx.\end{aligned}$$

In the particular case when F depends linearly on u_ϵ, then the only information we need is the first moment of $\nu_{x,z}$ which is the multi-scale limit of $\{u_\epsilon\}$ at the given family of length scales. In fact,

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega u_\epsilon(x)\; g\left(x, \langle\frac{x}{l_1(\epsilon)}\rangle, \langle\frac{x}{l_2(\epsilon)}\rangle, \dots, \langle\frac{x}{l_n(\epsilon)}\rangle\right)\\ = \int_\Omega\int_{Z^n}\int_{\mathbf{R}^d} g(x,z)\lambda\,d\nu_{x,z}(\lambda)\,dz\,dx.\end{aligned}$$

But this last integral can also be written as

$$\int_\Omega\int_{Z^n} u(x,z)g(x,z)\,dz\,dx$$

where $u(x,z)$ is the corresponding multi-scale limit. This holds true whenever the sequence

$$\left\{u_\epsilon(x)\; g\left(x, \langle\frac{x}{l_1(\epsilon)}\rangle, \langle\frac{x}{l_2(\epsilon)}\rangle, \dots, \langle\frac{x}{l_n(\epsilon)}\rangle\right)\right\}$$

converges weakly in $L^1(\Omega)$. This is the basic theorem of multi-scale convergence as treated in [2] and [6].

We would like to say more about the structure of the multi-scale limit when the sequence we are dealing with is a sequence of gradients $\{\nabla u_\epsilon\}$. Bearing in mind Theorem 2.9, we have to explore the structure of the limits

$$\lim_{\epsilon\searrow 0}\frac{1}{|r_n(\epsilon)Z|}\int_{r_n(\epsilon)Z}\nabla u_\epsilon(x + l_1(\epsilon)z_1 + l_2(\epsilon)z_2 + \dots + l_n(\epsilon)z_n + l_n(\epsilon)[y])\,dy, \tag{3.1}$$

where $r_n(\epsilon) = r(\epsilon)/l_n(\epsilon)$. We can rewrite the limit above in a telescopic form as follows, by taking advantage of Corollary 2.10,

$$\begin{aligned}&\sum_{i=1}^n\Bigg[\lim_{\epsilon\searrow 0}\frac{1}{|r_i(\epsilon)Z|}\int_{r_i(\epsilon)Z}\nabla u_\epsilon(x + l_1(\epsilon)z_1 + l_2(\epsilon)z_2 + \dots + l_i(\epsilon)z_i + l_i(\epsilon)[y])\,dy\\ &- \lim_{\epsilon\searrow 0}\frac{1}{|r_i(\epsilon)Z|}\int_Z\int_{r_i(\epsilon)Z}\nabla u_\epsilon(x + l_1(\epsilon)z_1 + l_2(\epsilon)z_2 + \dots + l_i(\epsilon)w + l_i(\epsilon)[y])\,dy\,dw\Bigg]\\ &+ \lim_{\epsilon\searrow 0}\frac{1}{|r_1(\epsilon)Z|}\int_Z\int_{r_1(\epsilon)Z}\nabla u_\epsilon(x + l_1(\epsilon)w + l_1(\epsilon)[y])\,dy\,dw.\end{aligned} \tag{3.2}$$

We now focus on each term of this sum. The i-th term can be regarded as a Z-periodic function of z_i. In addition, it is easy to check that its curl with respect to

z_i vanishes and, by construction, its average over Z, again with respect to z_i, also vanishes. Therefore, there must exist a Z-periodic function $u^{(i)}(x, z_1, z_2, \ldots, z_i)$ in the variable z_i such that the i-th term of our sum is equal to

$$\nabla_{z_i} u^{(i)}(x, z_1, z_2, \ldots, z_i).$$

Notice also that

$$\lim_{\epsilon \searrow 0} \frac{1}{|s(\epsilon)Z|} \int_Z \int_{s(\epsilon)Z} \nabla u_\epsilon(x + l_1(\epsilon)w + l_1(\epsilon)[y])\, dy\, dw = \nabla u^{(0)}(x)$$

the weak limit (in the usual sense) of $\{\nabla u_\epsilon\}$. We then have that the multi-scale limit (3.1) can also be written as

$$\sum_{i=1}^{n} \nabla_{z_i} u^{(i)}(x, z_1, z_2, \ldots, z_i) + \nabla u^{(0)}(x)$$

for certain Z-periodic function $u^{(i)}(x, z_1, z_2, \ldots, z_i)$ in the last variable.

This is the basic result on two-scale convergence when dealing with sequences of gradients as shown in the two references indicated above.

4. Homogenization of multiple integrals

As remarked earlier, the information that carries multi-scale Young measures is more precise than the one needed for multi-scale convergence. In fact, the representation formula

$$\begin{aligned} \lim_{\epsilon \to 0} \int_\Omega F\left(x, \langle \frac{x}{l_1(\epsilon)} \rangle, \langle \frac{x}{l_2(\epsilon)} \rangle, \ldots, \langle \frac{x}{l_n(\epsilon)} \rangle, \nabla u_\epsilon(x)\right) dx \\ = \int_\Omega \int_{Z^n} \int_{\mathbf{R}^d} F(x, z, \lambda)\, d\nu_{x,z}(\lambda)\, dz\, dx \end{aligned}$$

can be used to analyze, in an alternative way, the homogenization of multiple integrals of this type (see [8]).

As an illustration, let us consider the case of a single length scale $l(\epsilon)$ in the scalar case, so that we consider the functional

$$I_\epsilon(u) = \int_\Omega F\left(x, \frac{x}{l(\epsilon)}, \nabla u(x)\right) dx$$

where we assume, in addition to technical hypotheses, that

$$F(x, z, \lambda) : \Omega \times Z \times \mathbf{R}^N \to \mathbf{R}$$

is Z-periodic in z and convex in λ. These typical, technical hypotheses amount to having growth of order $p > 1$ with respect to λ, uniformly in the other variables, as well as measurable dependence on x and continuity on z and λ.

Theorem 4.1. *The homogenized functional corresponding to $I_\epsilon(u)$ in the sense of Γ-convergence with respect to weak convergence in $W^{1,p}(\Omega)$ is given by*

$$I_h(u) = \int_\Omega F_h(x, \nabla u(x))\, dx$$

where

$$F_h(x, \lambda) = \inf_U \left\{ \int_Z F(x, z, \lambda + \nabla U(z))\, dz : U \in W^{1,p}(Z) \text{ is } Z\text{-periodic} \right\}.$$

Proof. Suppose that $\nabla u_\epsilon \rightharpoonup \nabla u$ in $W^{1,p}(\Omega)$. If $\nu_{x,z} \otimes dz$ is the Young measure associated with pairs

$$\left\{ \left(\nabla u_\epsilon(x), \langle \frac{x}{l(\epsilon)} \rangle \right) \right\},$$

then we have the inequality (Theorem 6.11 in [16])

$$\lim_{\epsilon \searrow 0} \int_\Omega F\left(x, \frac{x}{l(\epsilon)}, \nabla u_\epsilon(x)\right) dx \geq \int_\Omega \int_Z \int_{\mathbf{R}^N} F(x,z,\lambda)\, d\nu_{x,z}(\lambda)\, dz\, dx.$$

Due to the convexity of F with respect to λ and to the fact that each $\nu_{x,z}$ is a probability measure, we conclude that

$$\int_\Omega \int_Z \int_{\mathbf{R}^N} F(x,z,\lambda)\, d\nu_{x,z}(\lambda)\, dz\, dx \geq \int_\Omega \int_Z F(x,z,\nabla u(x) + \nabla U_z(x,z))\, dz\, dx.$$

We have used the decomposition of the first moment of $\nu_{x,z}$ as the sum

$$\nabla u(x) + \nabla_z U(x,z)$$

shown in the preceding section. It is then clear that

$$\lim_{\epsilon \searrow 0} \int_\Omega F\left(x, \frac{x}{l(\epsilon)}, \nabla u_\epsilon(x)\right) dx \geq I_h(u).$$

To show equality, it suffices to take into account that for any mapping $U(x,z)$, Z-periodic in z and belonging to $W^{1,p}(Z)$, the sequence of mappings

$$v_\epsilon(x) = u(x) + l(\epsilon) U\left(x, \frac{x}{l(\epsilon)}\right)$$

converges weakly in $W^{1,p}(\Omega)$ to u. For this sequence of functions

$$\lim_{\epsilon \searrow 0} \int_\Omega F\left(x, \frac{x}{l(\epsilon)}, \nabla v_\epsilon(x)\right) dx = \int_\Omega \int_Z F(x,z,\nabla u(x) + \nabla U_z(x,z))\, dz\, dx.$$

□

The case of several length-scales is formally the same. Keep in mind the more general representation of periodic gradients and first moments of the corresponding multi-scale Young measures in the last section.

Theorem 4.2. *Under the same assumptions of Theorem 4.1, the* Γ*-limit of the sequence of functionals*

$$I_\epsilon(u) = \int_\Omega F\left(x, \langle \frac{x}{l_1(\epsilon)} \rangle, \langle \frac{x}{l_2(\epsilon)} \rangle, \ldots, \langle \frac{x}{l_n(\epsilon)} \rangle, \nabla u(x)\right) dx$$

is given by

$$I_h(u) = \int_\Omega F_h(x, \nabla u(x))\, dx$$

where

$$F_h(x,\lambda) = \inf_{(U_i)} \left\{ \int_{Z^n} F\left(x, z, \lambda + \sum_{i=1}^n \nabla_{z_i} U_i(x, z_1, \ldots, z_i)\right) dz \right\}$$

and each $U_i(x, z_1, \ldots, z_i)$ *is* Z*-periodic in* z_i *and belongs to* $W^{1,p}(Z)$ *in this variable.*

The much more complex, vector situation can also be examined from this perspective, although technicalities are expected to be much more involved. We will pursue this direction in the near future.

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