STEINBERG SYMBOLS MODULO THE TRACE CLASS, HOLOMNY, AND LIMIT THEOREMS FOR TOEPLITZ DETERMINANTS

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Abstract. Suppose that \( \phi = \psi z^\gamma \) where \( \gamma \in \mathbb{Z}_+ \) and \( \psi \in \text{Lip}_\beta \), \( \frac{1}{2} < \beta < 1 \), and the Toeplitz operator \( T_\psi \) is invertible. Let \( D_n(T_\psi) \) be the determinant of the Toeplitz matrix \( (\psi_{i,j}) = ((\hat{\phi}_{i,j})) \), \( 0 \leq i,j \leq n \), where \( \hat{\phi}_k = \frac{1}{2\pi i} \int_0^{2\pi} f(\theta) e^{-ik\theta} \, d\theta \). Let \( P_0 \) be the orthogonal projection onto \( \ker S^{n+1} = V\{1, e^{i\theta}, e^{2i\theta}, \ldots, e^{ni\theta}\} \), where \( S = T_\phi \); set \( Q_n = 1 - P_n \), let \( H_\omega \) denote the Hankel operator associated to \( \omega \), and set \( \omega(t) = \omega(t) \) for \( t \in \mathbb{T} \). For the Wiener-Hopf factorization \( \psi = fg \) where \( f, g \) and \( \frac{1}{2}, \frac{1}{2} \in \text{Lip}_\beta \cap H^\infty(T) \), \( \frac{1}{2} < \beta < 1 \), put \( E(\psi) = \exp \sum_{k=1}^{\infty} k(\log f)k(\log g)k \), \( G(\psi) = \exp(\log \psi) \).

Theorem A. \( D_n(T_\psi) = (-1)^{(n+1)\gamma}(\psi)^{n+1}E(\psi)G\left(\frac{f}{g}\right)^\gamma \cdot \det(T_{\frac{1}{2}}\frac{1}{2})^{n+1} \cdot [1 - H_{f}Q_{n-\gamma}H_{g}]^{-1}2^{-\gamma - 1}2^{-\gamma - 1} \cdot [1 + O(n^{-2})] \). Let \( H^2(T) = \mathcal{X} \downarrow \mathcal{Y} \) be a decomposition into \( T_\phi T_{\phi}^{-1} \) invariant subspaces, \( \mathcal{X} = \bigcap_{n=1}^\infty \text{ran}(T_\phi T_{\phi}^{-1})^n \) and \( \mathcal{Y} = \bigcup_{n=1}^\infty \ker(T_\phi T_{\phi}^{-1})^n \), so that \( T_\phi T_{\phi}^{-1} \) restricted to \( \mathcal{X} \) is invertible, \( \mathcal{Y} \) is finite dimensional, and \( T_\phi T_{\phi}^{-1} \) restricted to \( \mathcal{Y} \) is nilpotent. Let \( \{w_n\} \) be the basis \( \{T_\phi z^n\} \) for the null space of \( T_\phi T_{\phi}^{-1} \), and let \( u_n \) be the top vector in a Jordan root vector chain of length \( m_n + 1 \) lying over \( (-1)^{m_n}w_n \), i.e., \( (T_\phi T_{\phi}^{-1})^{m_n}u_n = (-1)^{m_n}w_n \) where \( m_n = \max\{m \in \mathbb{Z}_+ : \exists x \text{ so that } (T_\phi T_{\phi}^{-1})^m x = w_n\}^{-1} \).

Theorem B. \( E(\psi)G\left(\frac{f}{g}\right)^\gamma \cdot \Pi_{\lambda \in \mathcal{S}(T_\phi T_{\phi}^{-1})} \lambda \cdot \det(\Pi_{\lambda \in \mathcal{S}(T_\phi T_{\phi}^{-1})} \lambda \cdot \det(T_{\frac{1}{2}}\frac{1}{2})) \cdot (T) \), the holonomy of a Deligne bundle with connection defined by the factorization \( \phi = fgz^\gamma \).

Note that the generalizations of the Szegö limit theorem for \( D_n(T_\psi) \) which have appeared in the literature with 1 instead of \( [1 - H_{f}Q_{n-\gamma}H_{g}]^{-1} \) have the defect that the limit of \( D_n(T_\psi) \) does not exist in general. An example is given with \( D_n(T_\psi) \neq 0 \) yet \( D_{\gamma-1}(T_{\frac{1}{2}}\frac{1}{2}) = 0 \) for infinitely many \( n \).
1. Introduction

Jacobi’s theorem on the conjugate minors of the adjugate matrix formed from the cofactors of $D_n(T_\phi)$ has been the main tool of previous attempts to generalize the classical strong Szegő limit theorem. But Toeplitz operators are defined by an algebraic relation with the shift, $S_n^{n+1}T_\phi S^{n+1} = T_\phi$, and it is natural instead to probe the implications of the Toeplitz property algebraically. The result gives, for the first time to our knowledge, a full extension of the Szegő theorem to the case where wind $(\phi, 0) \neq 0$,

$$
\lim_{n \to \infty} \frac{D_n(T_\phi)}{(-1)^{n+1} \gamma G(\psi)^{n+1} \det \left( (T_\phi)^{n+1} \cdot [1 - H_2^* Q_{n-\gamma} H_1^*]^\alpha \bar{z}^{\tau-1} \right)}_{\gamma \times \gamma} = E(\psi) G(\bar{\theta})^\gamma = \prod_{\lambda \in \sigma(T_\phi) \setminus \{0\}} \frac{\lambda}{\det(u_\lambda)}_{\gamma \times \gamma},
$$

and shows that the limit splits into two parts, a “torsion”, $\prod_{\lambda \in \sigma(T_\phi) \setminus \{0\}} \lambda$, the product of the non-zero eigenvalues of $T_\phi T_\phi^{-1}$, and a denominator reflecting the Jordan chain structure of the finite dimensional nilpotent operator $T_\phi T_\phi^{-1}$ acting on $J$. The methods we employ have wider implications. In Orthogonal polynomials and periodic recurrence relations, perturbation vectors are related to linear systems on Riemann surfaces to give the orthogonal polynomials corresponding to periodic Jacobi matrices.

Steinberg symbols. The relative algebraic $K$ group $K_1(L(H), L^1(H))$ may be identified with the quotient group of the invertible elements of $L(H)$ of the form $1+J$ where $J$ is in $L^1(H)$, by the commutator subgroup $\{1 + L^1(H), L(H)\}$ generated by elements of the form $\{1 + J, A\} \equiv (1 + J)A(1 + J)^{-1}A^{-1}$ where $A$ and $1 + J$ are invertible and $J$ is in $L^1(H)$. If $J$ is trace class and $1 + J$ is invertible, then write $[1 + J]_1$ for the corresponding element in $K_1(L(H), L^1(H))$. Every element in $K_1(L(H), L^1(H))$ has such a representation. The map det is a group homomorphism of the invertible operators of the form $1+J$, where $J \in L^1(H)$, onto $C^*$. The homomorphism det is then defined by $\det_+ [1+J]_1 = \det(1+J)$. It is shown in [1] that the map $\det_+$ is the projection onto the first factor in the isomorphism $K_1(L(H), L^1(H)) \cong C^* \oplus V$ in which $V$ has uncountable linear dimension. Furthermore, there is a connecting map $\partial$, so that there is a homomorphism to the non-zero complex numbers $C^*$ given by the composition

$$
\det_+ \circ \partial : K_2(L(H)/L^1(H)) \to C^*;
$$

where, for invertible elements $\alpha, \beta \in L(H)/L^1(H)$ which happen to commute, there is the Steinberg symbol

$$\{\alpha, \beta\} \in K_2(L(H)/L^1(H)) \xrightarrow{\partial} K_1(L(H); L^1(H)) \cong C^* \oplus V.
$$

Let $R, S,$ and $T$ denote regularizers for $A, B$ and $AB$ respectively. Thus $RA - I, AR - I \in L^1(H)$, etc. The elements $a, b \in L(H)/L^1(H)$ are lifted to $A, B$ and $a^{-1}, b^{-1}, (ab)^{-1}$ are lifted to $R, S$ and $T$. 


The definition of Steinberg symbols leads to (see [15])
\[
\partial\{a, b\} = \left[\begin{array}{cc} 1 & A \\ 0 & I \end{array}\right] \cdot \left[\begin{array}{cc} 1 & 0 \\ -R & I \end{array}\right] \cdot \left[\begin{array}{cc} 1 & A \\ 0 & I \end{array}\right] \cdot \left[\begin{array}{cc} 1 & -I \\ 0 & I \end{array}\right] \cdot \left[\begin{array}{cc} 1 & 0 \\ I & I \end{array}\right] \\
\cdot \left[\begin{array}{cc} I & -I \\ 0 & I \end{array}\right] \cdot \left[\begin{array}{cc} I & B \\ 0 & I \end{array}\right] \cdot \left[\begin{array}{cc} I & 0 \\ -S & I \end{array}\right] \cdot \left[\begin{array}{cc} I & B \\ 0 & I \end{array}\right] \\
\cdot \left[\begin{array}{cc} I & -AB \\ 0 & I \end{array}\right] \cdot \left[\begin{array}{cc} I & 0 \\ T & I \end{array}\right] \cdot \left[\begin{array}{cc} I & -AB \\ 0 & I \end{array}\right] \right] \mod L^1(H),
\]
where there is latitude in the choice of regularizers, since different regularizers lead to the same class, and therefore define liftings with the same determinant.

Thus, for example, when \(T_\phi\) is invertible, take \(A = T_\phi, B = S^*\) so that \(AB = T_\phi S^*\) with regularizers \(T = S T_\phi^{-1}, R = T_\phi^{-1}\). Then, by substitution as above,
\[
\partial\{T_\phi + L^1(H), S^* + L^1(H)\}
\]
\[
= \left[\begin{array}{cc} I & T_\phi \\ 0 & I \end{array}\right] \cdot \left[\begin{array}{cc} I & 0 \\ -T_\phi^{-1} & I \end{array}\right] \cdot \left[\begin{array}{cc} I & T_\phi \\ 0 & I \end{array}\right] \cdot \left[\begin{array}{cc} I & -I \\ 0 & I \end{array}\right] \cdot \left[\begin{array}{cc} I & 0 \\ I & I \end{array}\right] \cdot \left[\begin{array}{cc} I & -I \\ 0 & I \end{array}\right] \\
\cdot \left[\begin{array}{cc} I & S^* \\ 0 & I \end{array}\right] \cdot \left[\begin{array}{cc} I & 0 \\ -S & I \end{array}\right] \cdot \left[\begin{array}{cc} I & S^* \\ 0 & I \end{array}\right] \cdot \left[\begin{array}{cc} I & -T_\phi S^* \\ 0 & I \end{array}\right] \cdot \left[\begin{array}{cc} I & 0 \\ ST_\phi^{-1} & I \end{array}\right] \cdot \left[\begin{array}{cc} I & -T_\phi S^* \\ 0 & I \end{array}\right] \right].
\]

If we multiply these twelve factors together and use \(SS^* = 1 - P_0\), we get
\[
\det^* \circ \partial\{T_\phi + L^1(H), S^* + L^1(H)\} = \det(T_\phi^{-1} S T_\phi S^* + T_\phi^{-1} P_0).
\]

Thus,
\[
\frac{\det(P_0 T_\phi P_0)}{\det^* \circ \partial\{T_\phi + L^1(H), S + L^1(H)\}} = \det(P_0 T_\phi P_0) \cdot \det^* \circ \partial\{T_\phi + L^1(H), S^* + L^1(H)\}
\]
\[
= \det(P_0^+ + P_0 T_\phi) \cdot \det(T_\phi^{-1} S T_\phi S^* + T_\phi^{-1} P_0)
\]
\[
= \det(P_0^+ T_\phi^{-1} S T_\phi S^* + P_0 T_\phi^{-1} P_0 + P_0) = \det(P_0^+ T_\phi^{-1} S T_\phi S^* + P_0)
\]
\[
= \det(P_0^+ T_\phi^{-1} S T_\phi S^* + 1 - SS^*) = \det(1 + (P_0^+ T_\phi^{-1} S T_\phi - S) S^*)
\]
\[
= \det(1 + S^*(P_0^+ T_\phi^{-1} S T_\phi - S)) = \det(1 + S^*(T_\phi^{-1} S T_\phi - S)) = \det(S^* T_\phi^{-1} S T_\phi).
\]

The “same” calculation carried out with \(S^*\) replaced by \(S^{n+1}\) and \(P_0\) replaced by \(P_n\) gives
\[
\frac{\det(P_n T_\phi P_n)}{\det^* \circ \partial\{T_\phi + L^1(H), S^{n+1} + L^1(H)\}} = \det(S^{n+1} T_\phi^{-1} S^{n+1} T_\phi).
\]

**Proposition 1.1** (Equation (35) of [15]). If \(\phi \in L^\infty(0, 2\pi)\) with \(T_\phi\) Fredholm, injective, and \(D_n(\phi) \neq 0\), then
\[
\frac{D_n(T_\phi)}{\det^* \circ \partial\{T_\phi + L^1(H), S^{n+1} + L^1(H)\}} = \det(T_\phi S^{n+1} R(T_\phi) S^{n+1} + P(T_\phi) S^{n+1}).
\]

The Steinberg symbol determinant in the denominator makes sense when \(T_\phi\) is Fredholm and not invertible and as already noted can be expressed explicitly. It is shown in §1.2, that if \(T_\phi\) is invertible, then \(\det^* \circ \partial\{T_\psi^{-1} + L^1(H), S^{n+1} + L^1(H)\} = G(\phi)^{n+1} = G(\psi)^{n+1} \cdot (-1)^{(n+1)\gamma}$. 

Thus, when $T_\phi$ is invertible and $\gamma = 0$, since $G(\phi) = G(\psi)$, the geometric mean, the proposition becomes the very special case proved above:

$$
(*) \quad \frac{D_n(T_\phi)}{G(\phi)^{n+1}} = \det(S^{*n+1}T_\phi^{-1}S^{n+1}T_\phi).
$$

This immediately gives the strong Szegő limit theorem

$$
\lim_{n \to \infty} \frac{D_n(T_\phi)}{G(\phi)^{n+1}} = \det(T_\phi T_\phi^{-1}).
$$

Proposition 1.1 was proved in [15] in order to treat the case where $\text{wind}(\phi, 0) \neq 0$. Nevertheless, since there have now been a number of papers (published subsequent to [15]) giving variational proofs of a formula of A. Borodin and A. Okunkov [9]:

$$
\det(P_n T_\phi P_n) = G(\phi)^{n+1} \frac{\det[1 - Q_n H_{\frac{J}{Q}} Q_n]}{\det[1 - H_{\frac{J}{Q}} Q_n]},
$$

for the case of invertible $T_\phi$, $\phi = f\tilde{g}$, we note the following:

**Remark 1.1.** The Borodin-Okunkov formula is equivalent to the special case $(*)$ of Proposition 1.1.

This is immediate.

Since $S_{\tilde{a}b} = T_\tilde{a}T_b + H_a H_b$, and $T_\phi$ is invertible, we have, with $\tilde{b}(t) \equiv b(t)$ for $t \in \mathbb{T}$, and $\langle \alpha, \beta \rangle = \alpha \beta^{*} \alpha^{-1} \beta^{-1}$ denoting the multiplicative commutator of the operators $\alpha$ and $\beta$,

$$
det(T_\phi S^{*n+1}T_\phi^{-1}S^{n+1}) = det(T_\phi T_f S^{*n+1}T_f^{-1}T_{\tilde{g}} S^{n+1}T_{\tilde{g}}^{-1}T_{\tilde{f}} T_{\tilde{g}}^{-1}S^{n+1})
$$

$$
= det(T_f(T_{\tilde{f}}^{-1}, T_{\tilde{g}})T_{\tilde{g}} S^{*n+1}T_f^{-1}T_{\tilde{g}} S^{n+1})
$$

$$
= det((T_{\tilde{f}}^{-1}, T_{\tilde{g}}))(S^{*n+1}T_f T_{\tilde{f}} T_{\tilde{g}} S^{n+1}T_{\tilde{g}})
$$

$$
= \frac{\det(S^{*n+1}T_f T_{\tilde{f}} T_{\tilde{g}} S^{n+1})}{\det(T_{\tilde{f}}(T_{\tilde{f}}^{-1}, T_{\tilde{g}}))} = \frac{\det[1 - Q_n H_{\frac{J}{Q}} Q_n]}{\det[1 - H_{\frac{J}{Q}} Q_n]}.
$$

However, the goal here is to describe a more comprehensive framework for wind $(\phi, 0) \neq 0$. Thus, in [13] we introduced perturbation vectors $\sigma_{A,B}$ associated to a pair of Fredholm operators $A, B$ with $A - B$ in the trace ideal. These non-zero vectors are a substitute for the perturbation determinants studied by Aronszjan, Weinstein, M. G. Krein, and many others. In the special case where $A$ is the identity, $\sigma_{A,B}$ splits and the scalar part of these perturbation vectors amounts to taking the product of the non-zero eigenvalues, while the vector itself comes from a section of an appropriate bundle.

To create this object the authors proved that there exists a trivialization of the pullback $i_M^*(\varphi)$ of a determinant bundle, given by a section, $\sigma$, which they called the perturbation section. Let $\mathcal{L}(H)$ be the algebra of all bounded linear operators on a Hilbert space $H$. Let $\mathcal{L}^1(H)$ denote the ideal of compact operators $T$ with trace $|T| < \infty$. Let $\mathcal{F}$ denote the Fredholm operators on $H$. Let $Q = \mathcal{F}$ be the Quillen det bundle. Form $Q + Q^{*}$ as a bundle over $\mathcal{F} \times \mathcal{F}$. Let $\varphi = det(Q + Q^{*})$ be the determinant bundle. Let $M \equiv \{ (A, B) : (A, B) \in \mathcal{F} \times \mathcal{F} \text{ and } A - B \in \mathcal{L}^1(H) \}$ with $i_M : M \to \mathcal{F} \times \mathcal{F}$ the inclusion map.

The perturbation vector $\sigma_{A,B}$ is the value of the section at $(A, B) \in M$. 

When wind \((\phi, 0)\) is not zero and the normalized Szegö determinant \(\frac{D_n(T_\phi)}{\det(T_\phi S^{n+1} + P(T_\phi) S^{n+1})}\) converges to zero, the perturbation vector \(\sigma_{1,T_\phi} S^{n+1} R(T_\phi) S^{n+1}\) gives a replacement because

\[
\sigma_{1,T_\phi} S^{n+1} R(T_\phi) S^{n+1} \rightarrow \sigma_{1,T_\phi} T_\phi^{-1}.
\]

The analysis of the convergence of this sequence of perturbation vectors in combination with Proposition 1.1 will be our basic study. A review of the perturbation vector construction and some of their properties is given in §2.

We will see in §2 that if \(1 - X \in \mathcal{L}^1\), then

\[
\|\sigma_{1,X}\| = \prod_{\lambda \in \sigma(X) \setminus \{0\}} \frac{|\lambda|}{\det(u_i, y_j)} \prod_{s_n(X) \neq 0} s_n(X),
\]

where \(\{y_j\}\) is any orthonormal basis for \(\ker X^*\), \(\{u_i\}\) is any set of maximal root vectors relative to any orthonormal basis of \(\ker X\) and \(\{s_n(X)\}\) are the singular values of \(X\). For example this implies that \(\frac{dd^c}{\sqrt{-1}} \prod_{s_n \neq 0} s_n(T_{z-\lambda} T_{(z-\lambda)^{-1}}) = \frac{dd^c}{\sqrt{-1}} \prod_{s_n \neq 0} s_n(T_{z-\lambda} T_{(z-\lambda)^{-1}}) = \frac{1}{(1 - |\lambda|^2)} d\lambda \wedge d\bar{\lambda}\) is the Gaussian curvature of the Hermitian bundle whose fibre at the point \(\lambda\) is \(\ker T_{z-\lambda}^*\).

**The perturbation vector version of Proposition 1.1.** Proposition 1.1 is the scalar part of a vector result proved in [15].

The fact that the ratio  
\[
\frac{D_n(T_\phi)}{\det(T_\phi S^{n+1} + R(T_\phi) S^{n+1} + P(T_\phi) S^{n+1})} = \det_* \circ \partial(T_\phi + \mathcal{L}^1(H), S^{n+1} + \mathcal{L}^1(H))
\]
depends only on the cosets of \(T_\phi\) and the shift \(S\) modulo the trace ideal is part of a statement made in terms of perturbation vectors:

For \(\gamma = 0\),

\[
\sigma_{S^{n+1}, T_\phi S^{n+1} T_\phi^{-1}} = \frac{\det(T_\phi S^{n+1} T_\phi^{-1} S^{n+1})}{D_n(T_\phi)} \rho_n^* \otimes T_\phi \rho_n = \frac{\rho_n^* \otimes T_\phi \rho_n}{\det_* \circ \partial(T_\phi + \mathcal{L}^1(H), S^{n+1} + \mathcal{L}^1(H))};
\]

where \(\rho_n\) is any non-zero vector in the exterior product space \(\det P_n H \equiv \bigwedge^{n+1} P_n H\), i.e., the Grassmann product of any set of basis vectors in \(S^{n+1}\). And when \(\gamma \neq 0\),

\[
\sigma_{S^{n+1}, T_\phi S^{n+1} R(T_\phi)} = \frac{\det(T_\phi S^{n+1} R(T_\phi) S^{n+1} + P(T_\phi) S^{n+1})}{D_n(T_\phi)} \rho_n^* \otimes T_\phi \rho_n \wedge x \otimes (\pi_{T_\phi} x)^*;
\]

where \(0 \neq x \in \ker T_\phi\) and \(\pi_{T_\phi} : H \rightarrow H/T_\phi(H)\) is the quotient map.

**The Szegö Problem.** For \(\phi \in \mathcal{L}^\infty(T)\), the Toeplitz operator \(T_\phi\) is defined on \(H^2(T)\) by \(T_\phi f = P\phi \cdot f\) where \(P\) is the orthogonal projection of \(L^2(T)\) onto \(H^2(L(T))\). Let \(P_n\) be the orthogonal projection onto

\[
\ker S^{n+1} = \bigvee \{1, e^{i\theta}, e^{2i\theta}, \ldots, e^{ni\theta}\} = \bigvee \{e_j\},
\]

\footnote{For definiteness we henceforth take \(R(T_\phi) = (T_\phi T_\phi)^{-1} T_\phi\), the Moore-Penrose inverse of \(T_\phi\).}
where $S = T_2$; set $Q_n = 1 - P_n$ with $D_n(T_0) \equiv \det(P_n T_0 P_n)$. Let $G(\phi) \equiv \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log |\phi(e^{i\theta})| \, d\theta\right)$.

The study of the asymptotics of $D_n(T_0)$ originated with the discovery in 1915 by G. Szegő that powers of the geometric mean serve as normalizing factors for the determinants $D_n(T_0) \equiv \det(P_n T_0 P_n)$. Finally, in 1952 he proved the following statement:

**The sharp Szegő Theorem** ([30]). If $\phi(e^{i\theta}) > 0$ in $[0, 2\pi]$ and $\phi'$ satisfies a Lipschitz condition with exponent $\beta$, $0 < \beta \leq 1$, then

$$\lim_{n \to \infty} \frac{D_n(T_0)}{[G(\phi)]^{n+1}} = \exp\left\{ \frac{1}{2} \sum_{k=1}^{\infty} n |k_n|^2 \right\},$$

where $\sum_{k=1}^{\infty} k_n z^n = \frac{1}{2\pi} \int_0^{2\pi} \log(\phi(\theta)) \frac{1 + z e^{-i\theta}}{1 - z e^{-i\theta}} \, d\theta$, and $D_n(T_0)$ is the determinant of the Toeplitz matrix $((\hat{\phi}_{i,j}))$, $0 \leq i, j \leq n$, with $\hat{\phi}_k = \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) e^{-ik\theta} \, d\theta$.

Szegő's theorem was extended by Baxter [5], [6], Devinatz [20], Hirschman [25], Ibragimov [26], Kac [28], Widom [31], [32] and others with fewer restrictive smoothness conditions, and finally with complex symbols but originally with wind $(\phi, 0) = 0$.

However, something different is required when $\text{wind} \left(\phi, 0\right) \neq 0$. For then the normalized Szegő determinant $\frac{|D_n(T_0)|}{G(\phi)^{n+1}} \to 0$ for smooth $\phi$. Thus, to solve the problem of finding a full analog of the Szegő theorem when $\text{wind}(\phi, 0) \neq 0$ it is necessary to

a) produce a normalization divisor $A_n$ so that the sequence $\frac{D_n(T_0)}{A_n}$ has a natural limit,

b) explain the limit in terms of topological and analytic data implicit in the symbol.

The main results below are contained in Theorems 1.1, 1.2, 1.3, and 1.6.

If $\gamma = \psi z^\gamma$, wind $(\psi, 0) = 0$, put

$$\tilde{g} \equiv \exp\left(\sum_{n=1}^{\infty} (\log \psi)_n t^{-n}\right), \quad f \equiv \exp\left(\sum_{n=0}^{\infty} (\log \psi)_n t^n\right)$$

so that $\psi = f \tilde{g}$. Let $H_{\omega}$ be the Hankel operator with symbol $\omega$. Then, with $E(\psi) \equiv \exp \sum_{k=1}^{\infty} k (\log \psi)_k (\log \psi)_{-k}$, and $G(\phi) \equiv \det \partial \{T_{\psi z^\gamma} + L^1(H), S + L^1(H)\}$, a Steinberg symbol determinant defined in terms of the cosets relative to the trace ideal of the Toeplitz operator and the shift:

a) $A_n \sim G(\phi)^{n+1} \det \left(\left[T_{\frac{\psi z^{\gamma+1}}{\psi}}, 1 - H_{\frac{\psi z^{\gamma+1}}{\psi}} Q_{n-\gamma} H_{\frac{\psi z^{\gamma+1}}{\psi}}^{-1} z^{\alpha-1}, z^{\tau-1}\right]\right)$, and

$$\beta) \lim_{n \to \infty} \frac{D_n(T_0)}{A_n} = E(\psi) G(\frac{\tilde{g}}{f})^{\gamma} = \prod_{\lambda \varepsilon \sigma(T_{\psi z^\gamma} + L^1(H), S^{n+1} + L^1(H))} \frac{\lambda^{\gamma}}{\det(u_\alpha, T_{\frac{\psi z^{\gamma+1}}{\psi}} z^{\alpha-1}, z^{\tau-1})} = \left(\tilde{g} \cup f = \frac{\tilde{g}}{f} \cup z^\gamma\right) (T).$$

When $\phi = \psi z^\gamma$ and wind $(\psi, 0) = 0$, we will prove that

$$G(\phi)^{n+1} = \det \partial \{T_{\psi z^\gamma} + L^1(H), S^{n+1} + L^1(H)\} = (-1)^{(n+1)\gamma} G(\psi)^{n+1}.$$

Although it is known (see §1.0.1 and §5) that

$$\lim_{n \to \infty} \frac{D_n(T_0)}{-(-1)^{(n+1)\gamma} G(\psi)^{n+1} \det(T_{\frac{\psi z^{\gamma+1}}{\psi}} z^{\alpha-1}, z^{\tau-1})} = E(\psi) G(\frac{\tilde{g}}{f})^{\gamma}$$
for many symbols, the limit on the left does not exist in general. There are smooth symbols \( \phi \) so that \( D_n(T_\phi) \neq 0 \), yet \( D_{\gamma-1}(T_{\psi}^* z^{n+1}) = \det(T_{\phi}^* z^{n+1}, z^{\alpha-1}, z^{\gamma-1}) = 0 \) for infinitely many \( n \). See §6 below. Thus the corrected normization provided by \( \alpha \) is necessary to find the leading term in the asymptotics of \( D_n(T_\phi) \) and thereby get the general limit theorem in \( \beta \).

The topological formula \( (\bar{g} \cup f \times \bar{f} \cup z^\gamma) (T) \) denotes the holonomy of a flat bundle taken over the unit circle \( T \). Both the bundle and its connection are defined by the factorization \( \phi = z^\gamma f \bar{g} \). This will be discussed in §1.2.

**Theorem 1.1.** Suppose \( \phi \in \mathcal{L}^\infty(T) \) where \( T_\phi \) is Fredholm, injective, \( T_\phi T_{\phi-1} - 1 \in \mathcal{L}^1(H^2(T)) \). Let \( \{u_\alpha\}_1^\gamma \) be any basis for \( \ker T_{\phi-1} \) and let \( \{u_\alpha\}_1^\gamma \) be any set of maximal root vectors relative to \( \{(1)^{m_\alpha}u_\alpha\}_1^\gamma \) and \( T_\phi T_{\phi-1} \), i.e., \( (T_\phi T_{\phi-1})^{m_\alpha}u_\alpha = (-1)^{m_\alpha}w_\alpha \) where \( m_\alpha = \max\{m \in \mathbb{Z}_{+} : \exists x \text{ so that } (T_\phi T_{\phi-1})^{m}x = w_\alpha\} \). Then, if \( \{w_\alpha^{(n)}\}_1^\gamma \) is any sequence of bases in \( \ker S^{n+1}R(T_\phi)S^{n+1} \) for which \( w_\alpha^{(n)} \) converges to \( u_\alpha \) for \( 1 \leq \alpha \leq \gamma \) and if \( \{t_\beta\}_1^\gamma \) is any basis of \( \ker T_\phi \), then for \( n \gg 0 \),

\[
\text{i) } D_n(T_\phi) = \mathbf{G}(\phi)^{n+1} \cdot \det(w_\alpha^{(n)}; S^{n+1}T_{\tau})_{\gamma \times \gamma} \cdot \left[ \frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi-1}) \{0\}} \lambda}{\det(u_\alpha, T_{\tau})} \right] \left[ 1 + o(1) \right].
\]

If in addition, \( \phi \in \text{Lip}_\beta, \frac{1}{2} < \beta < 1 \), then for \( n \gg 0 \),

\[
\text{ii) } D_n(T_\phi) = \mathbf{G}(\phi)^{n+1} \cdot \det(w_\alpha^{(n)}; S^{n+1}T_{\tau})_{\gamma \times \gamma} \cdot \left[ \frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi-1}) \{0\}} \lambda}{\det(u_\alpha, T_{\tau})} \right] \left[ 1 + O(n^{1-2\beta}) \right].
\]

**Theorem 1.2.** Let \( \phi = \psi z^\gamma \) for integer-valued \( \gamma > 0 \) with normalized Wiener-Hopf factorization \( \psi = f \bar{g} \) and \( f \) and \( g \) be outer functions in \( \mathcal{K}_{\infty}^{1,2} = \{ \phi : \phi \in L^\infty, \sum_{k=1}^\infty |\phi(k)|^2 |k| < \infty \} \). Let \( \{e_\alpha\}_1^\gamma = \{1, z, \ldots, z^{\gamma-1}\} \) denote the standard basis for \( \ker T_{\phi z} \), and let \( u_1, u_2, \ldots, u_\gamma \) be associated root vectors of the nilpotent part of the operator \( T_\phi T_{\phi-1} \) so that \( u_\alpha \) has maximal order with respect to \( (-1)^{m_\alpha}T_{\psi}e_\alpha \), for \( \alpha = 1, 2, \ldots, \gamma \), i.e., \( (T_\phi T_{\phi-1})^{m_\alpha}u_\alpha = (-1)^{m_\alpha}T_\psi e_\alpha \) where \( m_\alpha = \max\{m : \exists x \text{ so that } (T_\phi T_{\phi-1})^{m}x = T_\psi e_\alpha\} \). Then with \( E(\psi) = \det(T_\psi T_{\psi-1}) = \exp \sum_{k=1}^\infty k(\log f)_k(\log \bar{g})_{-k} \), we have

\[
\text{a) } \sum_{\alpha=1}^\gamma m_\alpha = \text{dim root space } [Q_{\gamma-1}(I - H_\psi H_{\frac{1}{\gamma}}|\ker(Q_{\gamma-1}))],
\]

\[
\text{b) } E(\psi)G\left(\frac{g}{f}\right)^\gamma = \frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi-1}) \{0\}} \lambda}{\det(u_\alpha, T_{\frac{1}{\gamma}}e_\tau)},
\]

\[
\text{c) when } m_\alpha = 0 \text{ for } \alpha = 1, \ldots, \gamma, \text{ then } E(\psi)G\left(\frac{g}{f}\right)^\gamma = \frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi-1}) \{0\}} \lambda}{D_{\gamma-1}(T_{\frac{1}{\gamma}})}.
\]

Note that \( m_\alpha = 0 \) if \( \phi \) is unimodular.

**Theorem 1.3.** Suppose \( \phi = \psi z^\gamma \) where \( \gamma \in \mathbb{Z}_{+} \) and \( \psi \in \text{Lip}_\beta, \frac{1}{2} < \beta < 1 \) and \( T_{\psi} \) is invertible. Then we have \( \psi = f \bar{g} \) where \( f, g \) and \( \frac{f}{g}, \frac{g}{f} \in \text{Lip}_\beta \cap H^\infty(T), \frac{1}{2} < \beta < 1, \)
If \( e_n = z^n \), \( \alpha = 1, 2, \ldots, \gamma \), and \( b = \frac{2}{\gamma} \),

\[
D_n(T_\phi) = G(\phi)^{n+1} E(\psi) G(b)^\gamma \\
\cdot \det \left( (T_{\frac{\phi}{\phi} z^{n+1}} \cdot [1 - H_{\frac{1}{\gamma}} Q_{n-\gamma} H_{\frac{1}{\gamma}}]^{-1} e_\alpha, e_\tau) \right)_{\gamma \times \gamma} [1 + O(n^{1-2\beta})].
\]

If \( \phi \in K^{1,1}_{2,2} \), then

\[
D_n(T_\phi) = G(\phi)^{n+1} E(\psi) G(b)^\gamma \\
\cdot \det \left( (T_{\frac{\phi}{\phi} z^{n+1}} \cdot [1 - H_{\frac{1}{\gamma}} Q_{n-\gamma} H_{\frac{1}{\gamma}}]^{-1} e_\alpha, e_\tau) \right)_{\gamma \times \gamma} [1 + o(1)].
\]

The proof uses Theorem 1.1 and the identity

\[
\det \left( (S^{n+1} [S^{n+1-\gamma} T^{-1} S^{n+1-\gamma}]^{-1} e_\alpha, T_\phi e_\tau) \right) = \det \left( (T_{\frac{\phi}{\phi} z^{n+1}} \cdot [1 - H_{\frac{1}{\gamma}} Q_{n-\gamma} H_{\frac{1}{\gamma}}]^{-1} e_\alpha, e_\tau) \right)_{\gamma \times \gamma}.
\]

1.0.1. Relation to other results. Undefined factors in the Böttcher-Silbermann approximation. An important step towards the non-zero index case, was made when M. Fisher and E. Hartwig [21] used Jacobi’s theorem on the conjugate minors of the adjugate matrix formed from the cofactors of \( D_n(T_\phi) \) to give asymptotic information even when \( \phi \) is piecewise continuous and \( \text{wind} (\phi, 0) \neq 0 \).

The pioneering Fisher-Hartwig results were refined by Böttcher and Silbermann [10] where it was assumed that \( \phi = az^\gamma \), with \( a \in \text{Lip}_0 \), \( \beta > \frac{1}{2} \), with the Wiener-Hopf factorization \( a = a_+ a_- \), and invertible \( T_\alpha \). For integer \( \gamma \geq 0 \) and \( n \gg 0 \), the Böttcher-Silbermann approximation is:

\[
D_n(T_{az^\gamma}) = (-1)^{(n+1)\gamma} G(a)^{n+1} E(a) G(b)^\gamma \\
\cdot \det \left[ \begin{array}{c} \hat{c}_{-n-1} & \cdots & \hat{c}_{-n+\gamma-2} \\ \vdots & \ddots & \vdots \\ \hat{c}_{-n-\gamma} & \cdots & \hat{c}_{-n-1} \end{array} \right] + O(n^{-3\beta}) \right] [1 + O(n^{1-2\beta})],
\]

with \( c = \frac{a_+}{a_-} \), \( b = \frac{a_-}{a_+} \), and \( \hat{c}_j \) denoting the \( j \)th Fourier coefficient, and \( E(a) = \exp \sum_{k=1}^{\infty} k (\log a_+) k (\log a_-) \). As noted above, this approximation fails to give a limit theorem. In §6 an example is given of a rational symbol for which the factor \( \text{det}(C_{n,\gamma}) = 0 \) for infinitely many \( n \) and yet \( D_n(\phi) \neq 0 \) for \( n \gg 0 \). In this case, for a subsequence \( \{n_j\} \) for which \( \text{det}(C_{n_j,\gamma}) = 0 \), the Fisher-Hartwig, Böttcher-Silbermann assertion becomes the statement that there is an unspecified denominator, \( O(n_j^{-3\beta}) \), so that

\[
\frac{D_{n_j}(T_{az^\gamma})}{(-1)^{(n_j+\gamma)\gamma} G(a)^{n_j+1} O(n_j^{-3\beta})} = E(a) G(b)^\gamma [1 + O(n_j^{1-2\beta})], \quad j = 1, \ldots, \infty.
\]

Note that the geometric series for

\[
[1 - H_{\frac{1}{\gamma}} Q_{n-\gamma} H_{\frac{1}{\gamma}}]^{-1} = 1 + \sum_{k=1}^{\infty} (H_{\frac{1}{\gamma}} Q_{n-\gamma} H_{\frac{1}{\gamma}})^k
\]

of Theorem 1.3, which is valid for large \( n \), provides a correction for \( \text{det} C_{n,\gamma} \) since we have the following remark.
Remark 1.2. If \(a = a_+a_- \in L^\infty(T)\) with \(a_+\) and \(a_-\) outer in \(H^2(T)\) and with \(T_a\) invertible, then

\[
\ker T_a z^n = \sqrt{\left\{ \frac{1}{a_-}, \frac{z}{a_-}, \cdots, \frac{z^{\gamma-1}}{a_-} \right\}},
\]

\[
\ker T_{(a z^n)^{-1}} = \sqrt{\left\{ a_+, a^2 a_+, \cdots, a^{\gamma-1} a_+ \right\}},
\]

and if we set \(w_\alpha = z^{\alpha-1} a_+\) and \(y_\tau = \frac{\bar{c}_\tau - 1}{\bar{a}_\tau}\) for \(\alpha, \tau = 1, \ldots, \gamma\), then

\[
det(C_{n,\gamma}) = det(w_\alpha, S^{*n+1} y_\tau) = det\left(\begin{array}{ccc}
\hat{c}_{-n-1} & \cdots & \hat{c}_{-n+\gamma-2} \\
\vdots & \ddots & \vdots \\
\hat{c}_{-n-\gamma} & \cdots & \hat{c}_{-n-1}
\end{array}\right) = det(T_{\frac{a}{z_{n+1}} z^\alpha, z^\tau}).
\]

Identification of factors in the Widom approximation. Harold Widom [32] treats symbols having the form \(\phi = fgz^\tau\) where \(f\) is piecewise \(C^\infty\) but \(g\) is not in \(C^\infty\), i.e., there are finitely many points at which certain higher order derivatives are piecewise continuous. The aim of his work is to produce an asymptotic formula for \(D_n(T_\phi)\) in the presence of such higher order singularities and non-vanishing index. His main result is

\[
D_n(T_\phi) = G(f \overline{g})^{n+1} n^{-\epsilon} (c_n + o(1)),
\]

where \(\epsilon\) is a non-negative exponent depending on the nature of the singularities of \(\phi\) (these are points with no neighborhoods in which \(\phi \in C^\infty\)) and \(c_n\) is a bounded sequence given by “a rather complicated but (in principle) perfectly explicit formula.”

Comparison of the Widom result and Theorem 1.1 gives

\[
n^{-\epsilon}[c_n + o(1)] = (-1)^{(n+1)\gamma} \frac{\prod_{\lambda \in \sigma(T_{T_a}) \setminus \{0\}} \lambda}{\det(w_\alpha, t_\tau)}.
\]

For a certain class of generating functions \(\phi\) Widom succeeds in showing that for almost all \(x\) not in the range of \(\phi\), the sequence \(c_n = c_n(x)\) corresponding to the symbol \(\phi - x\) remains bounded away from zero and he says that “we have no doubt that this is always true”—but “we have not been able to prove that this is always the case”.

Note that for fixed \(x\) the sequence \(c_n(x)\) may approach zero very rapidly. For example, suppose that \(f \in \text{Lip}(\beta)\) with \(\beta > \frac{1}{2}\) and is piecewise in \(C^\infty\), while \(g \in C^\infty\). Since \(\ker T_\phi = T_{\frac{f}{g}} \{1, z, z^2, \ldots, z^{\gamma-1}\}\), it follows that \(\| S^{*n+1} y_\tau \| = o(n^{-p})\), \(\forall p > 0\), and therefore \(c_n + o(1) = o(n^{-p})\) for any \(p > 0\).

On the other hand, if \(c_n\) is bounded away from zero, Theorem 1.3 says

\[
c_n \sim (-1)^{(n+1)\gamma} n^p E(f \overline{g}) G(\frac{\bar{g}}{f})^{\gamma} \cdot \det \left( (T_{\frac{f}{g}}^{*n+1} \cdot \left[1 - H_{\frac{f}{g}} Q_{n-\gamma} H_{\frac{f}{g}}^{-1} e_n, e_\tau \right]^{-1} e_n, e_\tau \right) \right) ^{\gamma \times \gamma}.
\]

Some key relations. The vectors \(\sigma_{S^{*n+1} T_a S^{*n+1} T_a^{-1}}\) and \(\sigma_{1, T_a S^{*n+1} B(T_a) S^{*n+1}}\) were calculated in [15]. The following is one consequence of these calculations that is fundamental for the present investigation.
The appearance of root vectors. The limit in Theorem 1.1 above is obtained from the convergence of the sequence of perturbation vectors

$$
\lim_{n \to \infty} \sigma_1, T_\theta S^{n+1} R(T_\theta) S^{n+1} = \sigma_1, T_\theta T_\theta^{-1} = \frac{\prod_{\lambda \in \sigma(T_\theta T_\theta^{-1}) \setminus \{0\}} \lambda}{\det(u_\alpha, y_\beta)} \left( \bigwedge_{k=1}^\gamma w_k \right) \otimes \left( \bigwedge_{k=1}^\gamma [y_k + \text{ran } T_\theta]^* \right).
$$

This form for the right-hand side of the limiting perturbation vector $\sigma_1, T_\theta T_\theta^{-1}$ follows from an apparently new observation about finite matrices given in Proposition 2.1 below:

If $X$ is an operator on Hilbert space so that $X = \text{identity} + \text{trace class}$, and $\{w_\alpha\}_1^\gamma$ is any basis for ker $X$, $\{\alpha_1\}_1^\gamma$ is any orthonormal basis for ker $X^*$, and if $\{u_\alpha\}_1^\gamma$ is a corresponding set of root vectors of maximal height for $\{w_\alpha\}_1^\gamma$, i.e., $(X)^{\text{max}} u_\alpha = w_\alpha$, $\alpha = 1, 2, \ldots, \gamma$. Then with $m_\alpha + 1$ denoting the algebraic multiplicity of $w_\alpha$,

$$
(-1)^{\sum_{\alpha=1}^\gamma m_\alpha} \prod_{\lambda \in \sigma(X) \setminus \{0\}} \frac{||\bigwedge w_j||^2}{\det(u_j, y_j)} = \det(X + \sum_{j=1}^\gamma y_j \otimes w_j).
$$

Three examples. Consider the matrix given by $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ with minimal polynomial $x(x - 1)$. Then an orthonormal basis for ker $A$ is $w_1 = (0, 0, 0, 1)$ and an orthonormal basis for ker $A^*$ is $y_1 = \left( \frac{1}{\sqrt{31}}, -\frac{2}{\sqrt{31}}, \frac{3}{\sqrt{31}}, -\frac{1}{\sqrt{31}} \right)$. Since $m_\alpha = 0$, $u_1 = w_1$ is a root vector of maximal height lying over the kernel vector of $A$, i.e., $(A)^0 u_1 = w_1$.

Then, $\det(w_1, y_1) = -\frac{1}{\sqrt{31}}$ so that

$$
\frac{1}{\det(u_1, y_1)} = -\sqrt{31} = \det(A + y_1 \otimes w_1) = \det(1 2 1 \frac{1}{\sqrt{31}} \\ 0 1 3 \frac{2}{\sqrt{31}} \\ 0 0 1 \frac{3}{\sqrt{31}} \\ 1 0 0 -\frac{1}{\sqrt{31}}).
$$

For another example, take $B = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$. The minimal polynomial is $x^3(x + 1)$, an orthonormal basis for ker $B$ is $w_1 = (\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}})$ while $y_1 = (0, 0, 1, 0)$ is an orthonormal basis for ker $X^*$. $u_1 = (0, 0, \frac{\sqrt{2}}{2}, 0)$ is a root vector of maximal height lying over $w_1$, i.e., $B^2 u_1 = w_1$. The length $m_\alpha + 1$ of the corresponding Jordan chain is 3, but $\prod_{\lambda \in \sigma(X) \setminus \{0\}} \lambda = -1$. Hence we have

$$
y_1 \otimes w_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (1/2) \cdot \sqrt{2} & 0 & 0 & (1/2) \cdot \sqrt{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$
and

\[
\det(B + y_1 \otimes w_1) = \det \begin{pmatrix}
0 & 2 & 1 & 0 \\
0 & 0 & 3 & 0 \\
(1/2) \cdot \sqrt{2} & 0 & 0 & (1/2) \cdot \sqrt{2} \\
1 & 0 & 0 & -1
\end{pmatrix} = -6\sqrt{2} = \frac{(-1)^3}{\det(u_1, y_1)} = \frac{(-1)^3}{\frac{\sqrt{2}}{12}}.
\]

For a Hilbert space example, let us consider \( B \equiv 1 - a \otimes b \) where \((a, b) = 1\). Then \( \ker B \) has basis \( \{a\} \) and \( a \) is a maximal root vector lying over itself. Hence, since the non-zero eigenvalues of \( B \) are all equal to 1, we get \( \sigma_{1,B} = a \otimes (a + \text{ran } B)^* \). Thus, \( \|a + \text{ran } B\| = \|P(B^*)a\| \). Now \( P(B^*) = \frac{b}{\|b\|} \otimes \frac{b}{\|b\|} \), and \((a, b) = 1\), so we have \( \|P(B^*)a\| = \frac{1}{\|b\|} \). Consequently, \( \|\sigma_{1,B}\| = \|a\| \cdot \|b\| \).

Now let \( \phi = z - \lambda, \|\lambda\| < 1 \). Then \( T_\phi T_{\phi^{-1}} = 1 - e_0 \otimes e_\lambda \), where \( e_\lambda = \sum_{k=0}^\infty \bar{\lambda}^k z^k \).

But \( \|e_0\| = 1 \) and \( \|e_\lambda\| = \frac{1}{\sqrt{1-|\lambda|^2}} \). Thus, by the paragraph above, \( \|\sigma_{1,T_\phi T_{\phi^{-1}}}\| = \frac{1}{\sqrt{1-|\lambda|^2}} \).

**The invariant vector.** Suppose that for a Fredholm operator \( B \) we have \( 1 - B \in \mathcal{L}^1 \). We observe now that if we pick any basis \( \{(-1)^{m_\alpha}w_\alpha\} \) for \( \ker B \), and any corresponding basis of top root vectors \( \{u_\alpha\} \) for Jordan chains lying over the kernel vectors, we may form the vector

\[
\mathcal{I}(B) \equiv (\bigwedge u_\alpha)^* \otimes (\bigwedge u_\alpha + H/BH).
\]

a) \( \mathcal{I}(B) \) is invariant under the choice of the root vectors \( \{u_\alpha\} \) for a fixed choice of \( \{w_\alpha\} \).

b) \( \mathcal{I}(B) \) is invariant under the choice of null vectors \( \{w_\alpha\} \).

Pairing the invariant vector with the perturbation vector gives \( \prod_{\lambda \in \sigma(B) \setminus \{0\}} \lambda \cdot \mathcal{I}(B) \) so that

\[
\sigma_{1,B} = \prod_{\lambda \in \sigma(B) \setminus \{0\}} \lambda \cdot \mathcal{I}(B).
\]

But Theorem 1.1 exploits the fact that we have the freedom to choose a different basis \( \{t_\tau\} \) for the cokernel space. We obtain then the same vector \( \sigma_{1,B} \), but another numerical coefficient multiplying the corresponding tensor product. In the preceding theorems and Theorem 1.6 below we exploit this freedom to choose bases for the cokernel space corresponding to the factorization \( \phi = \psi z^\gamma \), and in Theorem A we obtain an additional meaning for the resulting coefficient in terms of holonomy.

**The separation property for finite matrices or for operators of the form 1+ trace class.** Suppose that \( A \) and \( B \) are stationary\(^2\) Fredholm operators. Then we have the Riesz reduction \( H^2(\mathbb{T}) = R_\infty(A) + N_\infty(A) = R_\infty(B) + N_\infty(B) \), where now we use the notation \( N_\infty(T) = \bigcup_{n=1}^\infty \ker T^n \) and \( R_\infty(T) = \bigcap_{n=1}^\infty \text{ran } T^n \).

\(^2\)Recall that \( T \) is stationary if there exist a finite \( p \) so that \( N_p \equiv \ker T^p = \ker T^{p+1} = \cdots = N_\infty(T) \) and \( R_p(T) \equiv \text{ran } T^p = R_{p+1}(T) = \cdots = R_\infty(T) \).
Select a basis $\bigwedge_{j=1}^{\gamma} (-1)^{m_j} w^A_j$ for $\det A$ with associated maximal root vectors $\{u^A_j\}_{j=1}^{\gamma}$ and form the invariant vector 

$$I(A) = \left( \bigwedge_{j=1}^{\gamma} w^A_j \right) \otimes \left[ \bigwedge_{j=1}^{\gamma} u^A_j + \text{ran } A \right].$$

Similarly, select a basis $\bigwedge_{j=1}^{\mu} (-1)^{m_j} w^B_j$ for $\det B$ and associated maximal root vectors $\{u^B_j\}_{j=1}^{\mu}$ lying over these basis vectors and form the invariant vector 

$$I(B) = \left( \bigwedge_{j=1}^{\mu} w^B_j \right) \otimes \left[ \bigwedge_{j=1}^{\mu} u^B_j + \text{ran } B \right].$$

Let $M$ be the linear map on $H$ so that $M = 0$ on $R_\infty(A)$ and $M$ maps $w^A_j$ to $u^A_j$ for all $j = 1, \ldots, \gamma$.

Let $L$ be the linear map on $H$ so that $L = 0$ on $R_\infty(B)$ and $L$ maps $w^B_\tau$ to $u^B_\tau$ for all $\tau = 1, \ldots, \mu$. Then

$$\sigma_{A,B} = \det[(A + M)^{-1}(B + L)] \cdot I(A) \otimes I(B)^*.$$

**Factorization.** Suppose $A$ and $B$ are finite matrices (alternately, $A$ and $B$ are of the form $1 + \mathcal{L}$). The indices are both zero and the given description of the vector $\sigma_{A,B}$ applies. The coefficient factors as 

$$\sigma_{A,B} = \sigma_{1,A}^* \otimes \sigma_{1,B} = \sigma_{A,1} \otimes \sigma_{1,B}.$$

**1.1. Perturbation vectors and the Szegö limit theorem.**

**Lemma 1.1.** Suppose $T_\phi$ is injective and $T_\phi T_{\phi^{-1}} - 1 \in \mathcal{L}^1(\mathbb{T})$. Let $w_1, \ldots, w_\gamma$ be any basis for $\ker T_{\phi^{-1}}$. Then there is a basis $\{w_\ell^{(n)}\}$ in $\ker[S^{n+1}R(T_\phi)S^{n+1}]$ for $n \gg 0$ so that $w_k^{(n)}$ converges to $w_k$ for $1 \leq k \leq \gamma$.

**Lemma 1.2.** Suppose $D_n(T_\phi) \neq 0$, and $\{t_\tau\}^\gamma$ is a basis for $\ker T_{\phi^{-1}}$, while $\{w_\ell^{(n)}\}_{k=1}^{\gamma}$ is any basis for $\ker[S^{n+1}R(T_\phi)S^{n+1}]$. Then

$$\sigma_{1,T_\phi S^{n+1}R(T_\phi)S^{n+1}} = \frac{D_n(T_\phi)}{G(\phi)^{n+1}} \cdot \det \left( \left( w_\ell^{(n)}, S^{n+1}t_\tau \right) \right)$$

$$\cdot \left( \bigwedge_{k=1}^{\gamma} w_k^{(n)} \right) \otimes \left( \bigwedge_{k=1}^{\gamma} [t_k + \text{ran } T_{\phi}]^* \right)^*,$$

and if top root vectors $v_\alpha$ of $T_{\phi^{-1}}$, are chosen to lie over the kernel vectors $(-1)^{m_\alpha} w_\alpha \in \ker T_{\phi^{-1}}$, i.e., $(T_{\phi^{-1}})^{m_\alpha} w_\alpha = (-1)^{m_\alpha} w_\alpha$ where $m_\alpha = \max \{ m \in \mathbb{Z}_+ : \exists x \text{ so that } (T_{\phi^{-1}})^{m} x = w_\alpha \}$, we have

$$\sigma_{1,T_\phi T_{\phi^{-1}}} = \frac{\prod_{\lambda \in \sigma} (T_{\phi} T_{\phi^{-1}})^{\lambda}(0)}{\det(u_\alpha, t_\tau)} \left( \bigwedge_{k=1}^{\gamma} w_k \right) \otimes \left( \bigwedge_{k=1}^{\gamma} [t_k + \text{ran } T_{\phi}]^* \right)^*.$$

If $w_k^{(n)} \to w_k$ it is clear that this representation of $\sigma_{1,T_\phi T_{\phi^{-1}}}$ immediately gives a limit theorem for the coefficients of the two perturbation vectors 

$$\lim_{n \to \infty} \frac{D_n(T_\phi)}{G(\phi)^{n+1} \cdot \det(w_\alpha^{(n)}, S^{n+1}t_\tau)} = \frac{\prod_{\lambda \in \sigma} (T_{\phi} T_{\phi^{-1}})^{\lambda}(0)}{\det(u_\alpha, t_\tau)}.$$
The factor $G(\phi)^{n+1}$ is explicitly evaluated using Theorem 1.5 below. If $\phi = \psi z^\gamma$, then $G(\phi)^{n+1} = (-1)^{(n+1)} \cdot G(\psi)^{n+1} = \det(\partial T_\phi + L^1(H), T_{n+1} + L^1(H)) = \phi \cup z^{n+1}(\Sigma)$, the holonomy of a bundle $\phi \cup z^{n+1}$ over the unit circle $T$ which is computed by an integral.

**The form of the perturbation vector in Lemma 1.2.** Using Proposition 1.1 it was shown in [15], equation (73), that when $T_\phi$ is injective, Fredholm, $\det(P_n T_\phi P_n) \neq 0$, and $\{v_\alpha^{(n)}\}_1^\gamma$ is a basis for $\ker S^{n+1}R(T_\phi)S^{n+1}$, then

$$\sigma_{1,T_\phi S^{n+1}} R(T_\phi) S^{n+1} = \frac{D_n(T_\phi)}{\det(v_\alpha^{(n)}, S^{n+1} y_\beta)} \cdot \left( \bigwedge_{\alpha=1}^\gamma v_\alpha^{(n)} \right) \times \left( \bigwedge_{\alpha=1}^\gamma [S^{n+1}v_\alpha^{(n)} + \text{ran } T_\phi]^* \right).$$

Since the cosets $S^{n+1}v_\alpha^{(n)} + \text{ran } T_\phi = \sum_1^\gamma (S^{n+1}v_\alpha^{(n)} y_\beta) y_\beta + \text{ran } T_\phi$, for any orthonormal basis $\{y_\alpha\}_1^\gamma \subset \ker T_\phi$, it follows that $(\bigwedge_1^{\gamma} [S^{n+1}v_\alpha^{(n)} + \text{ran } T_\phi]^*)^* = (\bigwedge_1^{\gamma} [S^{n+1}v_\alpha^{(n)} + \text{ran } T_\phi])^*$, and therefore (1.1) becomes the stated result in Lemma 1.2.

$$\sigma_{1,T_\phi S^{n+1}} R(T_\phi) S^{n+1} = \frac{D_n(T_\phi)}{G(\phi)^{n+1} \cdot \det(v_\alpha^{(n)}, S^{n+1} y_\beta)} \cdot \left( \bigwedge_{\alpha=1}^\gamma v_\alpha^{(n)} \right) \times \left( \bigwedge_{\alpha=1}^\gamma [y_\alpha + \text{ran } T_\phi]^* \right).$$

The second equation in Lemma 1.2 follows by substitution of the formulas of Proposition 2.1 into the definition of the perturbation vector.

1.2. Bundles, holonomy, Steinberg symbols. Recall the definition of the tame symbol: if $x \in X$, a complex curve in $C^n$, and $f$ and $h$ are meromorphic functions at $x$, the function $(\frac{\partial x}{\partial x})^h f |_{x=1}$ is of order 0 at $x$ and we can say it has a value at $x$. The value

$$c_x(f, h) = (-1)^{\text{order}_x h \cdot \text{order}_x f} \cdot \left( \frac{f |_{x=1}}{h |_{x=1}} \right)$$

is called the tame symbol at $x$.

Beilinson [8] and Deligne [19] investigated the “universal” bundle $(m, \nabla)$ with connection on $C^* \times C^*$, and $f \cup g$ the pull-back $(f, g)^*(m, \nabla)$. It is known that for any Riemann surface $Y$ there is an element of the group $H^1(Y, \mathcal{O}_Y)$ obtained from the universal line bundle by the pull-back

$$r_0(f, g) = (f, g)^*(m, \nabla) \text{ where } (f, g) : Y \to C^* \times C^*, \ x \to ((f(x), g(x))).$$

The regulator map $r_0$ is associated with the residue map, so that if $S = Z(f) \cup Z(g)$ is a finite subset of $Y$ there is the long exact Gysin sequence:

$$0 \to H^1(Y, C*) \to H^1(Y - S, C^*) \to \bigoplus_{\lambda \in S} \partial_\lambda \to \prod_{\lambda \in S} C^* \to H^2(Y, C^*) \to \cdots$$

where $\partial_\lambda$ is the residue map at $\lambda$, $\prod_{\lambda \in S} C^*$ denotes the direct product and $\partial_\lambda r_0(f, g) = c_\lambda(f, g)$. 

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The bundle construction. For smooth complex valued \( \phi \) and \( \psi \) defined and non-vanishing on \( T \), the line bundle \( \phi \cup \psi \) over the loop \( T \) is defined by specifying transition functions

\[
\{ \psi \frac{1}{2\pi i} (\log_\alpha \phi - \log_\beta \phi) \} \quad \text{together with the connection form} \quad \{ -\frac{1}{2\pi i} \log_\alpha \phi \cdot \frac{d\psi}{\psi} \}
\]

for a Čech cover \( \{ U_\alpha \} \) such that \( \log \phi|_{U_\alpha} \) is defined (denoted by \( \log_\alpha \phi \)). In other words, the relation between local sections \( s_\mu = \{ \log_\mu \phi, \psi \} \) defined by different branches of the logarithm on \( U_\mu \cap U_\nu \) is

\[
\{ \psi \frac{1}{2\pi i} (\log_\mu \phi - \log_\mu \phi) \} \cdot \{ \log_\mu \phi, \psi \} = \{ \log_\mu \phi, \psi \}
\]

and the connection \( \nabla \) on the bundle \( E \) defined by these transition functions is characterized by

\[
\nabla(\{ \log(\phi), \psi \}) = \frac{1}{2\pi i} \log \phi \cdot \frac{d\psi}{\psi} \oplus \{ \log(\phi), \psi \}.
\]

There is an identification \( H^1(T, C^*) \cong \text{Hom}(\pi_1(T), C^*) \cong C^* \) determined by lifting a generator \( \rho \) of \( \pi_1(T) \) to the bundle \( E \) defined by these transition functions.

Fix base points \( p \in T \) and \( \zeta \in C^* \) so that \( [(p, \zeta)] \in E \). Let \( \rho \) be a closed path in \( T \) starting at \( p \) and winding counterclockwise to \( p \). Let \( \tilde{\rho} \) be a horizontal lift of \( \rho \) to \( E \) starting at \( [(p, \zeta)] \). Parallel transporting via the connection around a loop leads to \( [(p, \tau \cdot \zeta)] \), where \( \tau \in C^* \). The map \( \rho \mapsto \tau \) defines an element of \( \text{Hom}(\pi_1(T), C^*) \).

The holonomy \( \tau \) is computed by choosing a disjoint union of open arcs obtained by decomposing \( \{ \phi^{-1}(U_\alpha) \cap \psi^{-1}(U_\beta) \}_{\alpha, \beta} \) into a disjoint union of open arcs \( \{ I_j^\alpha \}_{j=1}^m \) on \( T \) so that each arc \( I_j \) has a branch \( \log \phi \) of the logarithm defined, and then further refining this union into a covering by open arcs \( \{ J_j \}_{j=1}^n \) of \( T \), so that \( p \in J_1 \) and \( p_\mu \in J_\mu \cap J_{\mu+1} \) for \( 2 \leq \mu \leq n-1 \) while \( p_n \in J_1 \cap J_n \). Then as we proceed around the circle counterclockwise we first hit \( J_2 \), then \( J_3 \), etc. Thus for \( 1 \leq \mu \leq n-1 \) we have \( J_\mu \cap J_{\mu+1} = \emptyset \), and \( J_1 \cap J_n = \emptyset \).

The horizontal lift on \( J_\mu \) is

\[
\psi(p_{\mu+1}) \frac{1}{2\pi i} \log_{p_{\mu+1}}(\phi(p_{\mu+1}) - \log_\mu \phi(p_{\mu+1})) \cdot \exp \frac{1}{2\pi i} \int_{p_\mu}^{p_{\mu+1}} \log_\mu \phi \cdot \frac{d\psi}{\psi},
\]

so that the total lift around \( T \) is

\[
\tau = \prod_{\mu=1}^{n-1} \left( \left( \psi(p_{\mu+1}) \frac{1}{2\pi i} \log_{p_{\mu+1}}(\phi(p_{\mu+1}) - \log_\mu \phi(p_{\mu+1})) \cdot \exp \frac{1}{2\pi i} \int_{p_\mu}^{p_{\mu+1}} \log_\mu \phi \cdot \frac{d\psi}{\psi} \right) \times \psi(p_n) \frac{1}{2\pi i} \log_{p_n}(\phi(p_n) - \log_\mu \phi(p_n)) \cdot \exp \frac{1}{2\pi i} \int_{p_n}^{p_1} \log_1 \phi \cdot \frac{d\psi}{\psi} \right).
\]

Keeping track of telescoping factors \([8],[10]\) shows that this product can be rewritten:

**Theorem 1.4.** If the unit circle taken positively is a generator \( \Sigma \) of \( \pi_1(T) \), then for functions \( \phi \) and \( \psi \) smooth on \( T \), for example in the Krein algebra \( K_{*}^{2,2} \), the element \( \phi \cup \psi \) of \( H^1(T, C^*) \cong \text{Hom}(\pi_1(T), C^*) \) is given by

\[
\phi \cup \psi(\Sigma) = \exp \frac{1}{2\pi i} \left( \int_{|z|=1} \log \phi \cdot \frac{d\psi}{\psi} - \log \psi(p) \int_{|z|=1} \frac{d\phi}{\phi} \right).
\]
Here $p$ is a base point of the unit circle $T$, the branches of the logarithms are continuous except possibly at $p$, and the integrals are taken positively over the circle $T$ starting at $p$.

The present authors proved an Index Theorem \[14\], \[16\] which deals with the Steinberg symbols of equivalence classes of Toeplitz operators modulo the trace ideal and smooth functions $\phi, \psi$ defined on the unit circle $T$:

**Theorem 1.5.** For $\phi, \psi \in K^2_{*,2}$, Fredholm $T_\psi$ and $T_\phi$,

$$\det_* \partial \{T_\phi + \mathcal{L}^1(H), T_\psi + \mathcal{L}^1(H)\} = \phi \cup \psi(\Sigma),$$

where $\Sigma$ is the unit circle taken positively.

Theorem 1.5 is applied to give $\det_* \partial \{T_\phi + \mathcal{L}^1(H), T_{z^{n+1}} + \mathcal{L}^1(H)\} = \phi \cup z^{n+1}(\Sigma)$.

**Remark 1.3.** Taking $p = 1$ and using Theorem 1.4 and the Index Theorem 1.5 above we get

$$G(\phi)^{n+1} = \det_* \partial \{T_{\phi z^n} + \mathcal{L}^1(H), S^{n+1} + \mathcal{L}^1(H)\} = \psi z^\gamma \cup z^{n+1}(\Sigma).$$

Accordingly, since

$$\frac{1}{2\pi i} \int_T \log z^\gamma d\log z = \frac{\gamma}{2\pi i} \int_0^{2\pi} i\theta d\theta = \pi \gamma i,$$

we have

$$G(\phi)^{n+1} = \phi \cup z^{n+1}(\Sigma) = \det_* \partial \{T_\phi + \mathcal{L}^1(H), T_{z^{n+1}} + \mathcal{L}^1(H)\} = G(\psi)^{n+1} \cdot \exp[-(n+1)\pi \cdot \gamma i].$$

The connection of the Steinberg symbols of operator cosets and the geometric mean of an essentially bounded symbol $\phi$ is intrinsic and does not depend on the smoothness of the symbol. A relationship persists even when the smoothness assumptions used here to define the bundle $\phi \cup z^{n+1}$ are not satisfied.

Recall the following result \[17\], \[16\]. If $\phi \in L^\infty(T)$ and $T_\phi$ is Fredholm, then

$$|\det_* \{T_\phi + \mathcal{L}^1(H), S^{n+1} + \mathcal{L}^1(H)\}| = \exp \frac{n+1}{2\pi} \int_0^{2\pi} \log |\phi(e^{i\theta})| d\theta.$$  

**The classical Szegő Theorem as a holonomy result.** It is of some interest to see now that the original Szegő result may be viewed geometrically:

**Remark 1.4.** Suppose $\phi \in K^*_{*,2} \cap \mathcal{W}$, where $\mathcal{W}$ is the Wiener algebra, and wind $(\phi, 0) = 0$. Then with the Wiener- Hopf factorization $\phi = \phi_+ \phi_-$, the foregoing considerations give

$$\lim_{n \to \infty} \frac{D_n(T_\phi)}{\phi \cup z^{n+1}(\Sigma)} = \phi_- \cup \phi_+(\Sigma) = \det_* \partial \{T_{\phi_-} + \mathcal{L}^1(H), T_{\phi_+} + \mathcal{L}^1(H)\} = \exp \sum_{k=1}^{\infty} k(\log \phi_+)_k(\log \phi_-)_{-k}.$$
The limit is holonomy when the index is non-zero. Note that for $\gamma \neq 0$, $E(\psi) = \psi \cup f(\Sigma)$ and by definition $G(b) = \frac{\psi}{f} \cup z(\Sigma)$ is a holonomy. The bundles over $\mathbb{T}$ form a group. Thus with $\times$ the group multiplication, the holonomy of the product bundle is

$$E(\psi)G(b) = \left(\psi \cup f \times \frac{\psi}{f} \cup z\right)(\Sigma).$$

There is a relationship between the Jordan chains lying over null vectors, and the holonomy of a product bundle.

Proposition 1.2. Let $\phi = \psi z^\gamma$ for integer-valued $\gamma > 0$ with normalized Wiener-Hopf factorization $\psi = fg$, and let $f$ and $g$ be outer functions in $K_z\mathbb{C}$ $\cap W$. Then

$$(-1)^{\sum_{a=1}^{m_{\alpha}} \{\sum_{\lambda \in \sigma(T_\alpha T_\beta^{-1}) \setminus \{0\}} \lambda \}} \frac{\det(u_\alpha, T_\beta e_\tau)}{f_{\leq a, \tau \leq \gamma}} = E(\psi)G\left(\frac{\psi}{f}\right)^\gamma = \left(\psi \cup f \times \psi \cup z\right)(\Sigma).$$

1.3. Inner outer factorization and normalization. We are free to take any factorization for the symbol. There are several reasons for considering the factorization in the form $\phi = \Theta_1 e_\tau \Theta_2 f \psi$ with $\Theta_1 = \prod_{k=1}^{\nu} \frac{1}{\mu_k}$ and $\Theta_2 = \prod_{k=1}^{\nu} \frac{1}{\mu_k}$ coprime finite Blaschke products, and $f$ and $g$ outer functions. Among these we note:

- Any $\phi \in L^\infty$ with $\inf |\phi| > 0$ factors as $\Theta_1 \Theta_2 h k$ where $h$ is outer and $k$ is continuous; $T_\alpha$ is Fredholm iff $T_{\Theta_1, \Theta_2}$ is Fredholm and index $T_\phi = \text{index} T_{\Theta_1, \Theta_2}$. This factorization is unique (up to constants) when $k = 1$.

- The factorization is unique if $\phi$ is rational.

- Factoring $\phi$ into $\Theta_1 \Theta_2 f \psi$ defines a generator of $\det H(T_\phi T_{\phi^{-1}})$ as follows: pick a basis $\{x_j\}$ of $\ker T_{\Theta_1, \Theta_2}$. The tensor product $\wedge T_j x_j \otimes (T_\frac{1}{2} x_j + \text{ran} T_\phi)^*$ is a non-zero element of $\det(T_\phi T_{\phi^{-1}})$ which does not depend on the choice of $\{x_j\}$. Pairing with the perturbation vector produces the coefficient

$$\left(\frac{-1}{\sum_{a=1}^{m_{\alpha}} \{\sum_{\lambda \in \sigma(T_\alpha T_\beta^{-1}) \setminus \{0\}} \lambda \}} \frac{\det(u_\alpha, T_\beta e_\tau)}{f_{\leq a, \tau \leq \gamma}} \right) \frac{\wedge T_\frac{1}{2} \ker T_{\Theta_1, \Theta_2}}{\wedge T_\frac{1}{2} \ker T_{\Theta_1, \Theta_2}}^2,$$

where $\{u_\alpha\}$ are the top root vectors associated with $\{T_j x_\alpha\}$.

There are two prominent generators for $\det H(T_\phi S^{n+1} R(T_\phi) S^{n+1})$, if $D_\alpha(T_\phi) \neq 0$. One is the perturbation vector $\sigma_1, T_\phi S^{n+1} R(T_\phi) S^{n+1}$ and the other is $\wedge w_j \otimes S^{n+1} \wedge (S^{n+1} w_j + \text{ran} T_\phi)^*$, where $w_j$ is any basis for $\ker S^{n+1} R(T_\phi) S^{n+1}$. Again, we have independence of the choice of basis. Pairing these two yields the Szegö sequence $D_\alpha(T_\phi)$, with the generalized geometric mean in the denominator.

- When $\phi$ is both unimodular and rational we have previously [17] explored a relation between the geometry of subspaces in Hilbert space and hyperbolic plane geometry by evaluating the Szegö limit in terms of the Bolyai-Lobachevsky angles of parallelism associated with pairs of zeros of the Blaschke factors. Recall that these angles are defined by limiting directions.

\[^3\text{N.K. Nikol’skii, Treatise on the shift operator, Springer-Verlag, Grundlehren 273, Berlin (1986).}\]

\[^4\text{It is natural to consider the Szegö limit problem for more general inner functions. For example, when $\Theta_1$ is singular and the supports of $\Theta_1$ and $\Theta_2$ are identical, as in the work of Lee and Sarason (The spectra of some Toeplitz operators, J. Anal. Appl. 33 (1971), 529–543) where $\Theta_1$ is singular with mass one and $\Theta_2$ is a Blaschke product with zeros clustering at 1.}\]
The limit result for the factorization \( \phi = \Theta_1 \Theta_2 f \tilde{g} \) clearly displays the way in which the perturbation vector is related to algebraic K-theory, Steinberg symbols, and the tame symbols formed from the zeros of the symbols.

Steinberg symbols were originally introduced for the study of arithmetic because of their natural connection to tame symbols. We are interested in another relationship to geometry in Hilbert space involving the notion of separable kernel spaces (see \[15\]).

\[ \chi^2(P(T_{\Theta_1}), P(T_{\Theta_2})) \equiv \det(P(T_{\Theta_1})P(T_{\Theta_2})|_{\ker T_{\Theta_1}}) \] can be computed in terms of tame symbols and was found to express angular relationships between the indicated kernel spaces (see \[15\]).

**Theorem 1.6.** Let \( \phi = \Theta_1 \cdot \Theta_2 f \cdot \tilde{g} \) with \( \Theta_1 = \prod_{i=1}^{q} \frac{z-\mu_i}{1-\mu_i} \) and \( \Theta_2 = \prod_{k=1}^{q} \frac{z-\mu_k}{1-\mu_k} \) coprime finite Blaschke products, and let \( f \) and \( g \) be outer functions in \( K_{\mathbb{Z},2} \) \( \cap \mathcal{W} \), the intersection of the Krein algebra \( \{ a \in L^\infty(\mathbb{T}) : \sum_{n \in \mathbb{Z}} |(n+1)| |a_n|^2 < \infty \) and the Wiener algebra, \( \mathcal{W} \). Let \( \gamma = q - \ell = \text{wind}(\phi, 0) \geq 0 \). Let \( w_\alpha = T_j x_\alpha \) and let \( u_\alpha \) be the associated top root vectors lying over \((-1)^m x_\alpha\) where \( x_\alpha \) is an orthonormal basis for \( \ker T_{\Theta_1} \Theta_2 = T_{\Theta_2} \). Then for any sequence of vectors \( \{ v_{n, \alpha} \} \) in \( S^{n+1} R(T_{\phi})S^{n+1} \) for which \( \lim_{n \to \infty} v_{n, \alpha} = T_j x_\alpha \), for \( n \gg 0 \),

\[
\begin{align*}
\text{i) } D_n(T_\phi) &= G(\phi)^{n+1} \cdot \det(u_{n, \alpha}, S^{n+1} T_j x_\alpha)_{\gamma \times \gamma} \\
&= \left[ \prod_{\lambda \in \sigma(T_{\Theta_1} T_{\Theta_2}^{-1}) \setminus \{0\}} \lambda \right] \left[ 1 + O(n^{1-2\beta}) \right],
\end{align*}
\]

\[
\begin{align*}
\text{ii) } \lim_{n \to \infty} \frac{D_n(T_\phi)}{G(\phi)^{n+1} \cdot \det(u_{n, \alpha}, S^{n+1} T_j x_\alpha)} &= \chi^2(P(T_{\Theta_1}), P(T_{\Theta_2})) \cdot \prod_{k=1}^{\ell} \left[ \frac{f(\mu_k)}{g(\mu_k)} \right] \prod_{i=1}^{q} \left[ \frac{g(\nu_i)}{f(\nu_i)} \right] \\
&= \chi^2(P(T_{\Theta_1}), P(T_{\Theta_2})) \cdot \exp \sum_{k=1}^{\infty} k(\log f)(\log g)_{-k} \cdot \det(P(T_{\Theta_1}) \land \text{rp}(T_{\Theta_2})) \\
&\cdot T_{\left[f, K_{\Theta_1} \cap K_{\Theta_2}^\perp\right]^{\frac{1}{2}}} \cdot \det(P(T_{\Theta_1}) \land \text{rp}(T_{\Theta_2}) \cdot T_{\left[g, K_{\Theta_1} \cap K_{\Theta_2}^\perp\right]^{\frac{1}{2}}},
\end{align*}
\]

where explicitly for simple zeros \( \{ \mu_\alpha \} \) and \( \{ \nu_\tau \} \),

\[
\chi^2(P(T_{\Theta_1}), P(T_{\Theta_2})) = \prod_{1 \leq \alpha, \tau \leq \ell} \left| 1 - \bar{\mu}_\alpha \mu_\tau \right| \prod_{1 \leq \beta, \gamma \leq \eta} \left| 1 - \bar{\mu}_\alpha \nu_\tau \right| \\
\cdot \left| \sum_{\mathcal{N}} \prod_{\mu_\alpha \in \mathcal{N}} \prod_{\mu_\tau \in \mathcal{N}} \frac{(\mu_\alpha - \nu_\tau)}{1 - \bar{\mu}_\alpha \nu_\tau} \cdot \frac{\bar{\mu}_\alpha - \bar{\nu}_\tau}{\bar{\mu}_\alpha - \nu_\tau} \cdot \frac{\mu_\alpha - \nu_\tau}{\mu_\alpha - \bar{\nu}_\tau} \cdot \frac{\bar{\mu}_\alpha - \nu_\tau}{\bar{\mu}_\alpha - \bar{\nu}_\tau} \\
\cdot \prod_{1 \leq \tau \leq \eta} \left( 1 - \bar{\mu}_\alpha \nu_\tau \right) \cdot \prod_{\mu_\tau \in \mathcal{N}} \left( 1 - \bar{\mu}_\alpha \bar{\nu}_\tau \right) \cdot \prod_{\nu_\tau \in \mathcal{N}} \left( 1 - \mu_\alpha \bar{\nu}_\tau \right) \cdot \prod_{\bar{\mu}_\alpha \in \mathcal{N}} \left( 1 - \mu_\alpha \nu_\tau \right) \cdot \prod_{\bar{\nu}_\tau \in \mathcal{N}} \left( 1 - \bar{\mu}_\alpha \bar{\nu}_\tau \right) \cdot \prod_{\nu_\tau \in \mathcal{N}} \left( 1 - \mu_\alpha \nu_\tau \right) \cdot \prod_{\bar{\mu}_\alpha \in \mathcal{N}} \left( 1 - \bar{\mu}_\alpha \bar{\nu}_\tau \right) \cdot \prod_{\bar{\nu}_\tau \in \mathcal{N}} \left( 1 - \nu_\tau \right) \cdot \prod_{\nu_\tau \in \mathcal{N}} \left( 1 - \bar{\nu}_\tau \right).}
\]
The sum is extended over all subsets $N \subset \{\mu_1, \ldots, \mu_t; \frac{1}{p_1}, \ldots, \frac{1}{p_t}\}$ of cardinality $t$, and $\bar{N} = \{\mu_1, \ldots, \mu_t; \frac{1}{p_1}, \ldots, \frac{1}{p_t}\} \setminus N$.

For an inner function $\Theta$ we use the notation $K_\Theta = H^2 \ominus \Theta H^2$. The numbers
\[
\det(P(T_\Theta_1) \wedge \text{rp}(T_\Theta_2)T_{T_\Theta^2}) K_{\Theta_1} \cap K_{\Theta_2}^{\frac{1}{2}} \text{ and } \det(P(T_\Theta_1) \wedge \text{rp}(T_\Theta_2)T_{T_\Theta^2}) K_{\Theta_1} \cap K_{\Theta_2}^{\frac{1}{2}}
\]
are fixed finite order determinants. These normalization factors are computed from the eigenvectors associated with the zeros.

2. Perturbation vectors

Let $H$ be an $n$-dimensional vector space over a field $F$. Let $T$ be an endomorphism on $H$. Let $p = \dim \ker T$. Then we have the exact sequence
\[
T : \quad 0 \longrightarrow \ker T \xrightarrow{i} H \xrightarrow{T} H \xrightarrow{\pi_T} \coker T \longrightarrow 0.
\]
The torsion vector of this complex is the element in $\ker T$ defined as follows: Pick non-zero vectors $s_1 \in \ker T$, $s_2 \in \bigwedge^{n-p} H$ and $s_3 \in \bigwedge^p H$, where $s_1 \wedge s_2 \neq 0$ and $Ts_2 \wedge s_3 \neq 0$. Then
\[
(-1)^p(n-p) s_3^* \otimes (s_1 \wedge s_2) \otimes (Ts_2 \wedge s_3)^* \otimes \pi_T s_3
\]
defines a generator $\sigma(T)$ of $\ker T$ that is independent of the choice of vectors $s_1, s_2, s_3$.

In the same way, a pair of maps $S$ and $T$ acting between finite dimensional spaces $H_1$ and $H_2$ produces torsion vectors $\sigma(S)$ and $\sigma(T)$. The product
\[
\sigma_{S,T} \equiv \sigma(S) \otimes \sigma(T)^*
\]
is then a well-defined element in $\det S \otimes (\det T)^*$.

In infinite dimensions, $\sigma(S)$ and $\sigma(T)$ are not defined, so the coupling represented by $\sigma_{S,T}$ may not factor. But in [13] it was shown how the construction extends to the case where $H_1$ and $H_2$ are finite dimensional, with $S$ and $T$ algebraically Fredholm and $S - T$ finite rank, or in the case of Banach spaces where $S$ and $T$ are Fredholm in the usual sense and $S - T$ is nuclear.

The following properties of the norms of perturbation vectors were proved in [13]:

Let $H_1$ and $H_2$ be Hilbert spaces. Then
i) $\|\sigma_{T,S}\| = \|\sigma_{S,T}\|^{-1}$.
ii) If index $T \geq 0$, then $\|\sigma_{T,S}\|^2 = \det \left( (TT^* + P(T^*))^{-1} \cdot (SS^* + P(S^*)) \right)$.
iii) If index $T \leq 0$, then $\|\sigma_{T,S}\|^2 = \det \left( (T^*T + P(T))^{-1} (S^*S + P(S)) \right)$.
iv) Suppose $H_1 = H_2 = H$ is a complex Hilbert space and $T^*T - TT^* \in L^1(H)$. Then $\|\sigma_{T,S}\| = \|\sigma_{T^*,S^*}\|$.
\[
\sigma_{S,T} = \sigma_{T,S}^*; \quad \sigma_{S,T} \otimes \sigma_{S,R} = \sigma_{T,R}.
\]

The equality in (v) is to be understood in the sense that the vectors on the right and left-hand sides are images of each other under the canonical isomorphism $H \otimes H^* \cong \mathcal{C}$.

Example. Suppose $H = H_1 = H_2$ is a separable Hilbert space and $1 - X$ is a trace class operator. Let $x$ be any vector in $\ker X$. Then $x$ is decomposable and there is a basis $\{x_j\}$ for $\ker X$ so that $x = \bigwedge x_j$. Let $\{u_j\}$ be a set of root vectors of maximal height lying over $((-1)^{m_j} x_j)$ relative to $X \big|_{\bigcup_{n=1}^\infty \ker(X)^n}$ with algebraic
multiplicity \( m_j + 1 \). Recall that the Riesz-Schauder theorem implies there is a direct sum decomposition of the Hilbert space \( H = \mathcal{X} + \mathcal{Y} \) into \( \mathcal{X} \) invariant subspaces, \( \mathcal{X} = \bigcup_{n=1}^{\infty} \text{ran}(X)^{n} \) and \( \mathcal{Y} = \bigcup_{n=1}^{\infty} \text{ker}(X)^{n} \), of \( X \) so that \( X \) restricted to \( \mathcal{X} \) is invertible, \( \mathcal{Y} \) is finite dimensional, and \( X \) restricted to \( \mathcal{Y} \) is nilpotent. Furthermore,

\[
\sigma_{1,X} = \prod_{\lambda \in \sigma(X) \setminus \{0\}} \lambda \cdot \left( \wedge_{j} x_{j} \otimes \left( \wedge_{1} (u_{j} + \text{ran } X) \right) \right)^{\ast}
\]

and

\[
\|\sigma_{1,X}\| = \prod_{\lambda \in \sigma(X) \setminus \{0\}} \frac{|\lambda|}{|\det(u_{j}, y_{j})|} = \prod_{s_{\alpha}(X) \neq 0} s_{\alpha}(X),
\]

where \( \{y_{j}\} \) is any orthonormal basis for \( \text{ker } X^{\ast} \), \( \{u_{i}\} \) is any set of maximal root vectors relative to any orthonormal basis for \( \text{ker } X \) and \( \{s_{\alpha}(X)\} \) are the singular values of \( X \). If \( \sigma(X) \setminus \{0\} = \emptyset \), the product is understood to be 1.

**Proposition 2.1.** Let \( X \) be an operator on Hilbert space \( H \) so that \( X = \text{identity} + \text{trace class} \). Suppose \( \{w_{\alpha}\}_{\gamma} \) is any basis for \( \text{ker } X \) and \( \{y_{\alpha}\}_{\gamma} \) is any orthonormal basis for \( \text{ker } X^{\ast} \). Let \( \{u_{\alpha}\}_{\gamma} \) be a corresponding set of maximal root vectors for \( \{w_{\alpha}\}_{\gamma} \), i.e., \( (X)^{\max} u_{\alpha} = w_{\alpha}, \alpha = 1, \ldots, \gamma. \) Then

i) \( \sigma_{1,X} = (-1)^{\sum_{\alpha=1}^{m_{\alpha}}} \prod_{\lambda \in \sigma(X) \setminus \{0\}} \lambda \cdot \left( \wedge_{k=1}^{\gamma} w_{k} \right) \otimes \left( \wedge_{\tau=1}^{\gamma} [u_{\tau} + \text{ran } X] \right)^{\ast}, \)

ii) \( (-1)^{\sum_{\alpha=1}^{m_{\alpha}}} \prod_{\lambda \in \sigma(X) \setminus \{0\}} \lambda \cdot \frac{\|\wedge_{\alpha} w_{\lambda}\|^{2}}{\det(u_{j}, y_{j})} = \det(X + \sum_{j=1}^{\gamma} y_{j} \otimes w_{j}). \)

iii) Let \( \{u'_{\alpha}\}_{\gamma} \) be the maximal root vectors corresponding to \( \{w'_{\alpha}\}_{\gamma} \), the orthonormal vectors obtained from \( \{w_{\alpha}\}_{\gamma} \) by the Gram-Schmidt process. Then \( \|\wedge_{\alpha=1}^{\gamma} w_{\alpha}\|, \)

\[
\det(u'_{\alpha}, y_{\tau}) = \det(u_{\alpha}, y_{\tau}).
\]

**Proof.** By definition of the perturbation vector \( \sigma_{1,X} \) (see §2 in [15]), we have

\[
\sigma_{1,X} = \det \left( X + \sum_{j=1}^{\gamma} u_{j} \otimes w_{j} \right) \left( \wedge_{1} t_{j} \right) \otimes \left( \wedge_{1} (u_{j} + \text{ran } X) \right)^{\ast},
\]

where \( \ast \) denotes the dual vector and \( \{t_{j}\}_{\gamma} \) is a dual basis to \( \{w_{j}\}_{\gamma} \), i.e., \( (t_{j}, w_{k}) = \delta_{jk} \) and \( \{t_{j}\}_{\gamma} \) spans \( \text{ker } X \). Note that \( \wedge_{1} t_{j} = \|\wedge_{1} w_{j}\|^{-2} \wedge_{1} w_{j} \).

But

\[
\det(X + \sum_{j=1}^{\gamma} u_{j} \otimes w_{j}) = \det(X \big|_{\mathcal{X}}) \det(X + \sum_{j=1}^{\gamma} u_{j} \otimes w_{j} \big|_{\mathcal{Y}}),
\]

and

\[
\det(X \big|_{\mathcal{X}}) = \prod_{\lambda \in \sigma(X) \setminus \{0\}} \lambda,
\]

\[
\det(X + \sum_{j=1}^{\gamma} u_{j} \otimes w_{j} \big|_{\mathcal{Y}}) = (-1)^{\sum_{j=1}^{m_{\gamma}}} \|\wedge_{1} w_{j}\|^{2}.
\]

Thus i) follows by substitution.

Now we prove ii). Again by definition of the perturbation vector \( \sigma_{1,X} \), we have

\[
\sigma_{1,X} = \det \left( X + \sum_{j=1}^{\gamma} y_{j} \otimes w_{j} \right) \left( \wedge_{1} t_{j} \right) \otimes \left( \wedge_{1} (y_{j} + \text{ran } X) \right)^{\ast}.
\]
With \( u_j = \sum_{k=1}^{\gamma} (u_j, y_k)y_k \mod \text{ran } X \) we have
\[
\left( \bigwedge_{1}^{\gamma}(u_j + \text{ran } X)^{\ast} \right) = \frac{\left( \bigwedge_{1}^{\gamma}(y_j + \text{ran } X)^{\ast} \right)}{\det(u_j, y_k)}.
\]
Therefore,
\[
\frac{\det(X + \sum_{j=1}^{\gamma} u_j \otimes w_j)}{\det(w_j, y_k)} = \det(X + \sum_{j=1}^{\gamma} y_j \otimes w_j),
\]
so that
\[
\left( -1 \right)^{\sum_{\alpha=1}^{m_{\alpha}}} \prod_{\lambda \in \sigma(X) \setminus \{0\}} \lambda \left\| \bigwedge_{1}^{\gamma} w_j \right\|^{2} \right/ \det(w_{j}, y_{k}) = \det(X + \sum_{j=1}^{\gamma} y_{j} \otimes w_{j}).
\]

Next we prove iii). Replacing \( \{w_j\}_{1}^{\gamma} \) by \( \{w'_j\}_{1}^{\gamma} \) and \( \{u_j\}_{1}^{\gamma} \) by \( \{u'_j\}_{1}^{\gamma} \) in part ii) gives
\[
\frac{\left( -1 \right)^{\sum_{\alpha=1}^{m_{\alpha}}} \prod_{\lambda \in \sigma(X) \setminus \{0\}} \lambda \left\| \bigwedge_{1}^{\gamma} w_{j} \right\|^{2}}{\det(w'_{j}, y_{k})} = \det(X + \sum_{j=1}^{\gamma} y_{j} \otimes w'_{j}).
\]

By equating expressions for the perturbation vector, we get
\[
\det \left( X + \sum_{1}^{\gamma} y_{j} \otimes w'_{j} \right) \left( \bigwedge_{1}^{\gamma} w'_{j} \right) \otimes \left( \bigwedge_{1}^{\gamma}(y_{j} + \text{ran } X)^{\ast} \right) = \det \left( X + \sum_{1}^{\gamma} y_{j} \otimes w_{j} \right) \left( \bigwedge_{1}^{\gamma} t_{j} \right) \otimes \left( \bigwedge_{1}^{\gamma}(y_{j} + \text{ran } X)^{\ast} \right).
\]
Again \( \bigwedge_{1}^{\gamma} t_{j} = \left\| \bigwedge_{1}^{\gamma} w_{j} \right\|^{-2} \bigwedge_{1}^{\gamma} w_{j} \) and \( \bigwedge_{1}^{\gamma} w'_{j} = \left\| \bigwedge_{1}^{\gamma} w_{j} \right\|^{-1} \bigwedge_{1}^{\gamma} w_{j} \). Therefore,
\[
\det \left( X + \sum_{1}^{\gamma} y_{j} \otimes w'_{j} \right) = \left\| \bigwedge_{1}^{\gamma} w_{j} \right\|^{-1} \det \left( X + \sum_{1}^{\gamma} y_{j} \otimes w_{j} \right).
\]
Consequently,
\[
\left( -1 \right)^{\sum_{\alpha=1}^{m_{\alpha}}} \prod_{\lambda \in \sigma(X) \setminus \{0\}} \lambda \left\| \bigwedge_{1}^{\gamma} w_{j} \right\|^{-1} \left[ \frac{1}{\det(w'_{j}, y_{k})} \right] = \left\| \bigwedge_{1}^{\gamma} w_{j} \right\|^{-1} \left[ \left( -1 \right)^{\sum_{\alpha=1}^{m_{\alpha}}} \prod_{\lambda \in \sigma(X) \setminus \{0\}} \lambda \left\| \bigwedge_{1}^{\gamma} w_{j} \right\|^{2} \right].
\]

Canceling like factors gives iii). \( \square \)

3. **Proof of Theorem 1.1, part i)**

The proof of Theorems 1.1, 1.2, and 1.3 are centered around equation (35) of [15] cited earlier:
\[
\frac{D_{\alpha}(T_{\bar{\phi}})}{G_{(\bar{\phi})}^{n+1}} = \det(T_{\bar{\phi}})S^{n+1}R(T_{\bar{\phi}})S^{n+1} + P(T_{\bar{\phi}})S^{n+1}).
\]

We proceed under the assumptions of Theorem 1.1, part i). In particular, since \( T_{\bar{\phi}}T_{\phi^{-1}} \) is of the form \( 1 + J \), where \( J \) is in the trace ideal, \( T_{\phi^{-1}} \) is a regularizer for \( T_{\phi} \) and \( R(T_{\phi}) - T_{\phi^{-1}} \) is in \( L^{1}(H) \). Let \( \gamma = \text{Index } T_{\bar{\phi}} = \dim \text{ker } T_{\bar{\phi}} \geq 1 \). Thus \( P(T_{\bar{\phi}}) \) has rank \( \gamma \).
Lemma 3.1 of §1.1 is contained in the following lemma.

**Lemma 3.1.** Suppose φ ∈ \( \mathcal{L}^\infty(\mathbb{T}) \) and \( T_\phi \) is injective and Fredholm with \( T_\phi T_\phi^{-1} - 1 \in \mathcal{L}^1(H^2(\mathbb{T})) \). Then as \( n \to \infty \), \( P(S^{n+1} R(T_\phi) S^{n+1}) \) converges to \( P(T_\phi) \) in trace norm and \( D_n(T_\phi) = 0 \iff \det(S^{n+1} v_\alpha^{(n)} y) = 0 \), where \( \{v_\alpha^{(n)}\} \) (respectively \( \{y\} \) ) is a basis for \( \ker(S^{n+1} R(T_\phi) S^{n+1}) \) (respectively \( \ker(T_\phi) \) ).

**Proof.** Since \( T_\phi T_\phi^{-1} - 1 \) is compact, \( R(T_\phi) \) has the form \( T_\phi^{-1} + F \) where \( F \) is also compact. Consequently, \( S^{n+1} R(T_\phi) S^{n+1} \) converges in the uniform norm to \( T_\phi^{-1} \). Since \( T_\phi^{-1} \) is surjective, the same is true for \( S^{n+1} R(T_\phi) S^{n+1} \) provided that \( n \gg 0 \). The first assertion now follows since \( P(S^{n+1} R(T_\phi) S^{n+1}) = 1 - S^{n+1} R(T_\phi) S^{n+1} S^{n+1} R(T_\phi) S^{n+1} [S^{n+1} R(T_\phi) S^{n+1} R(T_\phi) S^{n+1}]^{-1} S^{n+1} R(T_\phi) S^{n+1} \).

Now index \( S^{n+1} R(T_\phi) S^{n+1} = \gamma \) and therefore \( \dim \ker(S^{n+1} R(T_\phi) S^{n+1}) = \gamma \). Now fix \( n \gg 0 \). When \( D_n(\phi) \neq 0 \) we have from Proposition 1.1

\[
\frac{D_n(T_\phi)}{G(\phi)^{n+1}} = \det(T_\phi S^{n+1} R(T_\phi) S^{n+1} + P(T_\phi) S^{n+1}).
\]

If we replace \( \phi \) by \( \phi - \lambda \), then continuity in the generating symbol implies that this equation holds also when \( D_n(T_\phi) = 0 \). Thus \( D_n(T_\phi) = 0 \) if and only if there is a non-zero vector \( x \in \ker(S^{n+1} R_\phi S^{n+1}) \) and \( P(T_\phi) S^{n+1} x = 0 \). For \( n \gg 0 \) this means \( x \) is a linear combination \( x = \sum_{\alpha=1}^{\gamma} \lambda_\alpha v_\alpha^{(n)} \) and \( (S^{n+1} x, y) = 0 \) for all \( \tau \). By Cramer’s rule this is the same as \( \det((S^{n+1} v_\alpha^{(n)}, y)) = 0 \). □

**Lemma 3.2.** Suppose that \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) and \( A \) are invertible, and are of the form 1+ trace class. Then

\[
\det\left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \frac{\det(\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1})}{\det(\begin{pmatrix} A & B \\ C & D \end{pmatrix})} \det(A) \det(-CA^{-1}B + D).
\]

**Proof.** We have \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} \). Taking determinants we have

\[
\det\left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \det(A) \det(-CA^{-1}B + D).
\]

Now (cf. Lemma 2.9 in [12]), \( D - CA^{-1}B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \). Hence

\[
\det\left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \frac{\det(A) \det(-CA^{-1}B + D)}{\det(\begin{pmatrix} A & B \\ C & D \end{pmatrix}).}
\]

The Riesz Schauder decomposition applies to \( X = T_\phi T_\phi^{-1} \) so that \( H^2(\mathbb{T}) = X + \mathcal{Y} \) where \( \mathcal{X} = \bigcap_{n=1}^{\infty} \operatorname{ran}(T_\phi T_\phi^{-1}) \) and \( \mathcal{Y} = \bigcup_{n=1}^{\infty} \ker(T_\phi T_\phi^{-1}) \). Let \( P_X \) be the projection onto \( \mathcal{X} = \bigcap_{n=1}^{\infty} \operatorname{ran}(X) \) parallel to \( \mathcal{Y} \), and set \( P_Y = 1 - P_X \).

Then we have a block matrix decomposition

\[
X_n = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = T_\phi S^{n+1} R(T_\phi) S^{n+1} + P(T_\phi) S^{n+1}.
\]

Since \( R(T_\phi) = T_\phi^{-1} - E \) where \( E \) is trace class, \( X_n \) converges to \( T_\phi T_\phi^{-1} \) in the uniform norm. It then follows that \( P(\mathcal{X}) X_n|_X \) is invertible provided that \( n \gg 0 \).
Now suppose that \( n \gg 0 \) is chosen so that \( X_n \) is invertible, e.g., \( D_n(T_\phi) \neq 0 \). Then by Proposition 1.1 and Lemma 3.2

\[
\frac{D_n(T_\phi)}{\mathbb{G}(\phi)^{n+1}} = \frac{\det(P_{X_n}|_x)}{\det(P_{Y_n}|_{\phi^{-1}})}.
\]  

(3.1)

Since the numerator of this quotient converges to

\[
\det(P_{X_n T_\phi T_{\phi^{-1}}}|_{\phi^{-1}}) = \prod_{\lambda \in \sigma(T_{\phi}|_{\phi^{-1}}) \setminus \{0\}} \lambda,
\]

the product of the non-zero eigenvalues—using the well-known theorem of V.B. Lidskii—it suffices to consider the denominator \( \det(P_{Y_n}|_{\phi^{-1}}) \).

Choose a basis \( \{w_\alpha\}_1 \) for the kernel of \( T_{\phi^{-1}} \). Let \( u_1, u_2, \ldots, u_\gamma \) be associated vectors so that \( u_\alpha \) has maximal order with respect to \( \{w_\alpha\} \), i.e., \( (T_{\phi} T_{\phi^{-1}})^{m_\alpha} u_\alpha = w_\alpha \) where \( m_\alpha = \max\{m : \exists x \text{ so that } (T_{\phi} T_{\phi^{-1}})^m x = w_\alpha\} \). Then the collection \( \{(T_{\phi} T_{\phi^{-1}})^{m_\alpha} u_\alpha : 0 \leq r \leq m_\alpha, 1 \leq \alpha \leq \gamma\} \) is a basis for \( \mathcal{Y} \). The set of algebraic multiplicities \( \{m_\alpha + 1\} \) is independent of the choice of basis for \( \ker T_{\phi} T_{\phi^{-1}} \) and \( \sum_1 m_\alpha + \gamma = \dim \mathcal{Y} \).

In what follows we take the collection \( \{u_\alpha\}_1 \) so that for \( j \geq \alpha \geq 1 \) we have \( m_\alpha \geq 1 \) and for \( \gamma \geq \alpha \geq j + 1 \) we have \( m_\alpha = 0 \). Let \( \mathcal{Z} \) be obtained from \( \mathcal{Y} \) by replacing \( \bigvee_{\alpha=1}^\gamma u_\alpha \) with the kernel of \( T_{\phi} = \ker(T_{\phi} T_{\phi^{-1}})^* \). Then we have the direct sum decomposition \( H^2(\mathbb{T}) = \mathcal{X} \oplus \mathcal{Z} \).

Moreover, \( \det(P_{Y_n}|_{\phi^{-1}}) = \det(P_{Z_n}|_{\phi^{-1}}) \). So we can work with \( \mathcal{Z} \) instead of \( \mathcal{Y} \). The presence of the projection \( P(T_{\phi}) \) in the right-hand side of Proposition 1.1 makes this change desirable.

For Fredholm operators \( T \) let \( R(T) \) denote the regularizer of \( T \) with respect to the trace ideal so that \( TR(T) = 1 - P(T^*) \), \( R(T)T = 1 - P(T) \), with \( P(T) \) denoting the orthogonal projection to the kernel of \( T \). The Moore-Penrose inverse is such a regularizer.

**Lemma 3.3.** Suppose \( \{B_n\}_1^\infty \) is a sequence of Fredholm operators which converges in the uniform norm to an operator \( B \), and suppose also that \( P(B_n) \) converges in the uniform norm to \( P(B) \).

**Proof.** Recall that for any operator \( A \) with closed range the Moore-Penrose inverse \( A^1 \) has the form \( R(A) = [A^\ast A + P(A)]^{-1} A^\ast \). Thus the assertion follows by continuity of the inverse map in the uniform topology. \( \square \)

**Corollary 3.1.**

\[
\lim_{n \to \infty} R(T_{\phi} S_n^{n+1} R(T_{\phi}) S_n^{n+1}) = R(T_{\phi} T_{\phi^{-1}}) \quad \text{uniformly.}
\]

**Proof.** Since \( T_{\phi} \) is injective \( \ker T_{\phi} S_n^{n+1} R(T_{\phi}) S_n^{n+1} = \ker S_n^{n+1} R(T_{\phi}) S_n^{n+1} \). Also \( S_n^{n+1} R(T_{\phi}) S_n^{n+1} \) converges uniformly to \( T_{\phi^{-1}} \). So the result follows by Lemmas 3.1 and 3.3. \( \square \)

Now we will prove Lemma 1.1 which asserts that if \( w_1, \ldots, w_\gamma \) is a basis for \( \ker T_{\phi^{-1}} \), there is a basis \( \{w_\nu^{(n)}\} \) in \( \ker[S_n^{n+1} R(T_{\phi}) S_n^{n+1}] \) so that \( w_\nu^{(n)} \) converges to \( w_\tau \) for \( 1 \leq \tau \leq \gamma \).
Proof. Recall that by Lemma 3.1 \( P(S^{*n+1}R(T\phi)S^{n+1}) \) converges to \( P(T\phi^{-1}) \). Hence we can take \( w^{(n)}_{\alpha} = P(S^{*n+1}R(T\phi)S^{n+1})w_{\alpha} \).

Note that if \( \{v^{(n)}_{\beta}\} \) are the vectors obtained by applying the Gram-Schmidt procedure to \( \{w^{(n)}_{\alpha}\} \), then \( v^{(n)}_{\alpha} = \frac{w^{n}_{\alpha} - \text{proj}_{1 \leq j \leq \alpha} w^{n}_{\beta} w^{n}_{\beta}}{\|w^{n}_{\alpha} - \text{proj}_{1 \leq j \leq \alpha} w^{n}_{\beta} w^{n}_{\beta}\|} \) and \( \|w^{n}_{\alpha} - \text{proj}_{1 \leq j \leq \alpha} w^{n}_{\beta} w^{n}_{\beta}\| = \frac{\det(w^{n}_{\alpha}, w^{n}_{\beta})}{\det(w^{n}_{\alpha}, w^{n}_{\alpha})} \rightarrow 1 \) lim \( v^{(n)}_{\alpha} = v_{\alpha} \), where the \( v_{\alpha} \) are obtained by applying the Gram-Schmidt procedure to \( w_{\alpha} \).

The action of \( P_{Z}X_{n}^{-1}|_{Z} \). We consider first the action of \( P_{Z}X_{n}^{-1} \) on the vectors \( \{(T\phi T\phi^{-1})^{r}u_{\alpha} \} \). For \( m_{\alpha} \geq r \geq 1 \), define \( z_{r\alpha}(n) = X_{n}^{-1}(T\phi T\phi^{-1})^{r}u_{\alpha} \). Then \( (T\phi T\phi^{-1})^{r}u_{\alpha} = X_{n}z_{r\alpha}(n) \). Since \( r \geq 1 \), we have \( P(T\phi)S^{*n+1}z_{r\alpha}(n) = 0 \) so that

\[
(T\phi T\phi^{-1})^{r}u_{\alpha} = T\phi S^{*n+1}R(T\phi)S^{n+1}z_{r\alpha}(n).
\]

and

\[
R(T\phi S^{*n+1}R(T\phi)S^{n+1})(T\phi T\phi^{-1})^{r-1}u_{\alpha} = [1 - P(S^{*n+1}R(T\phi)S^{n+1})]z_{r\alpha}(n).
\]

By Lemma 3.1, \( P(S^{*n+1}R(T\phi)S^{n+1}) \) converges in the trace norm to \( P(T\phi^{-1}) = P(T\phi T\phi^{-1}) \).

Therefore, by Corollary 3.1 it follows that

\[
[1 - P(S^{*n+1}R(T\phi)S^{n+1})]z_{r\alpha}(n) \text{ converges to } [1 - P(T\phi T\phi^{-1})](T\phi T\phi^{-1})^{r-1}u_{\alpha},\]

and \( z_{r\alpha}(n) = X_{n}[1 - P(S^{*n+1}R(T\phi)S^{n+1})]z_{r\alpha}(n) \) converges to the basis vector \( (T\phi T\phi^{-1})^{r}u_{\alpha} \). Note that \( Z = \bigvee_{1 \leq r \leq m_{\alpha}, 1 \leq \alpha \leq \gamma} \{z_{r\alpha}(n)\} + \ker T\phi \). Furthermore,

\[
P_{Z}X_{n}^{-1}z_{r\alpha}(n) = P_{Z}[1 - P(S^{*n+1}R(T\phi)S^{n+1})]z_{r\alpha}(n) \rightarrow P_{Z}[1 - P(T\phi T\phi^{-1})](T\phi T\phi^{-1})^{r-1}u_{\alpha}.
\]

Next we consider \( P_{Z}X_{n}^{-1} \) acting on \( \ker T\phi \). Let \( y_{1}, \ldots, y_{\gamma} \) be an orthonormal basis for \( \ker T\phi \). Set \( z_{\alpha}(n) = X_{n}^{-1}y_{\alpha} \). Then \( X_{n}z_{\alpha}(n) = y_{\alpha} \) so that

\[
S^{*n+1}R(T\phi)S^{n+1}z_{\alpha}(n) = 0 \quad \text{and} \quad (S^{n+1}z_{\alpha}(n), y_{\gamma}) = \delta_{\alpha\gamma}.
\]

Let \( \{v^{(n)}_{\alpha}\} \) be any basis for \( \ker(S^{*n+1}T\phi S^{n+1}) \), and suppose that \( \{v^{(n)}_{\gamma}\} \) denotes the associated orthonormal basis derived by the Gram-Schmidt procedure. Then, since \( z_{\alpha}(n) = \sum_{\alpha=1}^{\gamma} (z_{\alpha}(n), v^{(n)}_{\gamma})v^{(n)}_{\gamma} \), we have

\[
(3.2) \quad \delta_{\alpha\mu} = (S^{n+1}z_{\alpha}(n), y_{\mu}) = \sum_{\gamma=1}^{\gamma} (S^{n+1}v^{(n)}_{\gamma}, y_{\mu})(z_{\alpha}(n), v^{(n)}_{\gamma}),
\]

and it follows that

\[
\det((z_{\alpha}(n), v^{(n)})) = \frac{1}{\det((S^{n+1}v^{(n)}_{\gamma}, y_{\gamma}))}.
\]

Let \( C_{n} \) and \( K_{n} \) denote the linear maps defined on \( Z \) so that for \( \alpha = 1, 2, \ldots, \gamma \),

1) \( C_{n}y_{\alpha} = \sum_{\gamma=1}^{\gamma} (S^{n+1}v^{(n)}_{\gamma}, y_{\gamma})y_{\gamma} \),
2) \( C_{n}(T\phi T\phi^{-1})^{r}u_{\alpha} = z_{r\alpha}(n) = (T\phi T\phi^{-1})^{r}u_{\alpha} - d_{r\alpha}(n) \), where \( d_{r\alpha}(n) \in \ker T\phi \),
3) \( K_{n}y_{\alpha} = P_{Z}v^{(n)}_{\alpha} \),
4) \( K_{n}(T\phi T\phi^{-1})^{r}u_{\alpha} = P_{Z}R(T\phi S^{*n+1}R\phi S^{n+1})(T\phi T\phi^{-1})^{r}u_{\alpha} \) for \( m_{\alpha} \geq r \geq 1 \).
Then

\[ K_n = P_Z X_n^{-1} P_Z C_n. \]

To check this, first consider the action on \( y_\alpha \). We have

\[
X_n^{-1} P_Z C_n (y_\alpha) = X_n^{-1} \sum_{\ell=1}^{\gamma} (S^{n+1} v_\alpha^{(n)}, y_\ell) y_\ell = \sum_{\ell=1}^{\gamma} (S^{n+1} v_\alpha^{(n)}, y_\ell) z_\ell(n) \\
= \sum_{\ell=1}^{\gamma} (S^{n+1} v_\alpha^{(n)}, y_\ell) \left( \sum_{\tau=1}^{\gamma} (z_\tau(n), v_\tau^{(n)}) v_\tau^{(n)} \right) \\
= \sum_{\ell=1}^{\gamma} \left[ \sum_{\tau=1}^{\gamma} (S^{n+1} v_\alpha^{(n)}, y_\ell) \cdot (z_\tau(n), v_\tau^{(n)}) \right] v_\tau^{(n)}.
\]

After taking the transpose of equation (3.2), the inner sum becomes \( \delta_{\alpha\tau} \) and therefore \( P_Z X_n^{-1} C_n y_\alpha = P_Z v_\alpha^{(n)} \).

Next we consider the action on vectors \((T_\phi T_{\phi^{-1}})^{r} u_\alpha\). It follows that

\[
X_n^{-1} P_Z C_n (T_\phi T_{\phi^{-1}})^{r} u_\alpha = [1 - P(S^{n+1} R(T_\phi) S^{n+1})] z_{r,\alpha}(n) \\
= R(T_\phi S^{n+1} R(T_\phi) S^{n+1})(T_\phi T_{\phi^{-1}})^{r} u_\alpha;
\]

thus \( K_n = P_Z X_n^{-1} P_Z C_n \).

Now, since \( \det(C_n) = \det(S^{n+1} v_\alpha^{(n)}, y_\tau) \), it follows that

\[
det(P_Z X_n^{-1} y) = det(P_Z X_n^{-1} Z) = \frac{\det(K_n)}{\det(C_n)} = \frac{\det(K_n)}{\det(S^{n+1} v_\alpha^{(n)}, y_\tau)};
\]

and therefore,

\[
D_n(T_\phi) = G(\phi)^{n+1} \frac{\det(P_Z X_n|x)}{\det(K_n)} \cdot \det(S^{n+1} v_\alpha^{(n)}, y_\tau) \quad \text{provided } D_n(\phi) \neq 0.
\]

On the other hand, since \( \dim \ker(S^{n+1} R(T_\phi) S^{n+1}) = \gamma \) when \( n \gg 0 \), the operator \( K_n \) is defined even when \( X_n \) is singular. By Gram-Schmidt, if \( w_\alpha^{(n)} \rightarrow w_\alpha \), then \( v_\alpha^{(n)} \rightarrow v_\alpha \) where \( \{v_\alpha\}_\gamma \) is derived from \( \{w_\alpha\}_\gamma \) by the Gram-Schmidt procedure. In that case, \( K_n \) converges to the operator \( K \) defined by

\[
K(y_\alpha) = P_Z v_\alpha,
\]

\[
K(T_\phi T_{\phi^{-1}})^{r} u_\alpha = P_Z [1 - P(T_\phi T_{\phi^{-1}})] (T_\phi T_{\phi^{-1}})^{r-1} u_\alpha \quad \text{for } m_\alpha \geq r \geq 1.
\]

Since \( Z \) is finite dimensional, \( K_n \) will be invertible provided that \( K \) is invertible.

**Lemma 3.4.** \( \det(K) = (-1)^{\sum_{\alpha=1}^{\gamma} m_\alpha} \det(b_\alpha, y_\ell) \) where \( \{b_\alpha\}_1^\gamma \) is any set of \( T_\phi T_{\phi^{-1}} \) maximal vectors for \( \{v_\alpha\}_1^\gamma \). In particular, \( K \) is invertible.

**Proof.** Let \( L : Z \rightarrow Y \) be the linear map where for \( \alpha = 1, 2, \ldots, \gamma \), \( L y_\alpha = v_\alpha \) and \( L(T_\phi T_{\phi^{-1}})^{r} u_\alpha = [1 - P(T_\phi T_{\phi^{-1}})] (T_\phi T_{\phi^{-1}})^{r-1} u_\alpha \), for \( 1 \leq r \leq m_\alpha \). Then \( K = P_Z L \).

Hence, \( \det(K) = \det(P_Z L = \det L \cdot P_Z \cdot Z|_Y \).

Let

\[
Y_1 = \bigcap_{\alpha=1}^{\gamma} \{u_\alpha\} \quad \text{and} \quad Y_2 = \bigcup_{\alpha=1}^{\gamma} \bigcup_{r=1}^{m_\alpha} \{(T_\phi T_{\phi^{-1}})^{r} u_\alpha\},
\]

and let

\[
Y_3 = \bigcap_{\alpha=1}^{\gamma} \{w_\alpha\} \quad \text{and} \quad Y_4 = \operatorname{ran}(1 - P(T_\phi T_{\phi^{-1}})) Y.
\]
Then
\[ \mathcal{Y} = \mathcal{Y}_1 + \mathcal{Y}_2 = \mathcal{Y}_3 + \mathcal{Y}_4. \]

Note that \( \mathcal{Y}_4 = rp [(T_\phi T_{\phi^{-1}})^r] \mathcal{Y} \). Let \( P(\mathcal{Y}_3) \) be the linear map on \( \mathcal{Y} \) which acts as the projection onto \( \mathcal{Y}_3 \) parallel to \( \mathcal{Y}_4 \). Define the linear map \( M : \mathcal{Y} \to \mathcal{Y} \) by setting
\[ M = (P(\mathcal{Y}_3)LP_Z)_{\gamma_1}^{-1}P(\mathcal{Y}_3) + (LP_Z)_{\gamma_2}^{-1}(1 - P(\mathcal{Y}_3)). \]

Computation shows that \( MLP_Z |_\mathcal{Y} = I + N \) where \( N \mathcal{Y}_2 = 0 \), and \( N : \mathcal{Y}_1 \to \mathcal{Y}_2 \). Therefore, \( \det LP_Z |_\mathcal{Y} = \det M^{-1} \). Further computation shows \( M^{-1} u_\alpha = \sum_{\rho=1}^\gamma a_{\alpha \rho} u_\rho \) where \( a_{\alpha \rho} = \sum_{\tau=1}^\gamma (u_\alpha, y_\tau) \Gamma_{\tau \rho} \) with \( (\Gamma_{\tau \rho}) \) the matrix relative to \( \{w_\tau\}_1^\gamma \) for the map defined by \( \Gamma(w_\tau) = v_\tau \), and we also have
\[ M^{-1}(T_\phi T_{\phi^{-1}})^r u_\alpha = [1 - P(T_\phi T_{\phi^{-1}})](T_\phi T_{\phi^{-1}})^{r-1} u_\alpha. \]

Recall that the \( \{u_\alpha\}_1^\gamma \) have been so ordered that \( 1 \leq m_\alpha \leq j \) and \( m_\alpha = 0 \) for \( j + 1 \leq \alpha \leq \gamma \). Thus, if \( X = T_\phi T_{\phi^{-1}} \), then \( M^{-1} \) has the following matrix representation:

\[
\begin{array}{cccccccccccc}
& x_1 & x_2^{\ast} & \cdots & x_m u_1 & x_2 u_2 & \cdots & x_m u_j & x_1 x_2 u_1 & \cdots & x_1 x_j u_j & u_1 & u_2 & \cdots & u_j \\
1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{array}
\]

Now \( \det \Gamma = \frac{1}{\|A\|} \). Therefore by ii) of Proposition 2.1 we have
\[ \det K = \det M^{-1} = (-1)^{\sum_{\alpha=1}^m m_\alpha} \det(u_\alpha, y_\tau) \cdot \det(\Gamma) = (-1)^{\sum_{\alpha=1}^m m_\alpha \gamma} \det(b_\alpha, y_\tau), \]
where the \( \{b_\alpha\}_1^\gamma \) are maximal vectors for \( \{v_\alpha\}_1^\gamma \). \( \square \)

Now, since \( K_n \) is invertible, and \( D_n(T_\phi) = 0 \) if and only if \( \det(S^{n+1} v_\alpha^{(n)}, y_\tau) = 0 \), we have shown that for all \( n \gg 0 \),
\[
D_n(\phi) = G(\phi)^{n+1} \frac{\det(P_X X_n|X)}{\det(K_n)} \cdot \det(S^{n+1} v_\alpha^{(n)}, y_\tau). \tag{3.3}
\]
Let $\Gamma(n)$ be the linear map on $\ker(S^{n+1}R(T_\phi)S^{n+1})$ so that $\Gamma(n)(w^{(n)}_\alpha) = v^{(n)}_\alpha$. Then $\det(\Gamma(n)) \to \det(\Gamma) = \frac{1}{\|w_\alpha\|}$, where as above $\Gamma(w_\alpha) = v_\alpha$.

Let $\{t_\gamma\}$ be any basis for $\ker T_\phi$. Then upon substitution into (3.3), Proposition 2.1 gives

$$D_n(\phi) = \mathbf{G}(\phi)^{n+1} \cdot \det(S^{n+1}w^{(n)}_\alpha, t_\gamma) \cdot \left[ (-1)^n \sum_i m_i \prod_{\lambda \in \sigma(T_\phi T^{-1}_\phi) \setminus \{0\}} \lambda^i \cdot [1 + o(1)] \right],$$

where the eigenvalues $\lambda$ are repeated according to their multiplicity. Again the equality $\det T_\phi T^{-1}_\phi |_{\mathcal{X}} = \prod_{\lambda \in \sigma(T_\phi T^{-1}_\phi) \setminus \{0\}} \lambda$ follows from the Riesz-Schauder decomposition and the well-known result of V.B. Lidskii. This completes the proof of part i) of Theorem 1.1.

4. PROOFS OF THEOREM 1.2 AND THEOREM 1.6

Assertion a) of Theorem 1.2 is contained in Proposition 4.1 below. Assertion c) follows from b) while b) is a special case of Proposition 4.3 with $\Theta_1 = z^\gamma$ and $\Theta_2 = 1$.

Let $\phi = \Theta_1 f \cdot \Theta_2 g$ with $\Theta_1$ and $\Theta_2$ coprime finite Blaschke products, and let $f$ and $g$ be outer functions in $K_2^{1 \frac{1}{2}} \cap W$, the intersection of the Krein algebra $\{a \in \mathcal{L}(\mathbb{T})^\infty : \sum_{n \in \mathbb{Z}} |a_n| < \infty \}$ and the Wiener algebra, $W$. Note that if $a \in K_2^{1 \frac{1}{2}} \cap W$ and has no zeros on $\mathbb{T}$, then $a^{-1} \in K_2^{1 \frac{1}{2}} \cap W$. In this case the Hankel operators $H(a)$ and $H(a^{-1})$ are in Hilbert-Schmidt class; cf. [12]. Let $\gamma = \text{wind}(\phi, 0) = \deg \Theta_1 - \deg \Theta_2 > 0$. Note that $\ker T_\phi = T_\phi^{1} \ker T_{\Theta_1 \Theta_2}$, $\ker T_\phi^{-1} = T_\phi \ker T_{\Theta_1 \Theta_2}$ and $\ker T_{\Theta_1 \Theta_2} = T_{\Theta_1} [K_{\Theta_1} \cap K_{\Theta_2}^\perp]$.

**Lemma 4.1.** For $h \in \mathcal{L}^\infty(\mathbb{T})$ and the multiplication operator $M_h$ on $L^2(\mathbb{T})$

$$K_{\Theta_1} \cap K_{\Theta_2}^\perp \xrightarrow{P(T_{\Theta_1}) \wedge \text{rp}(T_{\Theta_2}) M_h} K_{\Theta_1} \cap K_{\Theta_2}^\perp,$$

$$\ker T_{\Theta_1 \Theta_2} \xrightarrow{P(T_{\Theta_1 \Theta_2}) M_h} \ker T_{\Theta_1 \Theta_2}^\perp.$$

**Proof.** The result is a consequence of the following two facts:

1) $P(T_{\Theta_1 \Theta_2}) = T_{\Theta_2} P(T_{\Theta_1}) \wedge \text{rp}(T_{\Theta_2}) T_{\Theta_2}^\perp$,

2) $P(T_{\Theta_1}) \wedge \text{rp}(T_{\Theta_2}) \cdot M_{\Theta_2} P = P(T_{\Theta_1}) \wedge \text{rp}(T_{\Theta_2}) \cdot M_{\Theta_2}$.

As a consequence of the lemma we see that

$$\det P(T_{\Theta_1 \Theta_2}) M_h |_{\ker T_{\Theta_1 \Theta_2}} = \det P(T_{\Theta_1}) \wedge \text{rp}(T_{\Theta_2}) M_h |_{K_{\Theta_1} \cap K_{\Theta_2}^\perp},$$

$$\dim \ker P(T_{\Theta_1 \Theta_2}) M_h |_{\ker T_{\Theta_1 \Theta_2}} = \dim \ker P(T_{\Theta_1}) \wedge \text{rp}(T_{\Theta_2}) M_h |_{K_{\Theta_1} \cap K_{\Theta_2}^\perp}.$$

Fix orthonormal bases $\{w_\alpha\}$ and $\{y_\alpha\}$ of $\ker T_{\phi^{-1}}$ and $\ker T_\phi$. Pick $\{s_\alpha\}$ to be orthonormal eigenvectors of $P(T_{\Theta_1 \Theta_2}) T_{/f^2} |_{\ker T_{\Theta_1 \Theta_2}}$ so that

$$P(T_{\Theta_1 \Theta_2}) T_{/f^2} |_{\ker T_{\Theta_1 \Theta_2}} s_\alpha = \mu_\alpha s_\alpha.$$
Then \( \{ \frac{1}{\sqrt{\lambda_\beta}} T_j s_\beta \} \) is a complete orthonormal set in \( \ker T_{\phi^{-1}} \). Similarly, pick \( \{ r_\alpha \} \) to be orthonormal eigenvectors of \( P(T_{\bar{\phi}_1}\bar{\phi}_2) \big| \ker T_{\bar{\phi}_1,\bar{\phi}_2} \big| \) so that

\[
P(T_{\bar{\phi}_1}\bar{\phi}_2) \big| \ker T_{\bar{\phi}_1,\bar{\phi}_2} \big| r_\alpha = \lambda_\alpha r_\alpha.
\]

Then \( \{ \frac{1}{\sqrt{\lambda_\beta}} T_j s_\beta \} \) is a complete orthonormal set in \( \ker \bar{T}_\phi \). Let \( W : \ker P(T_{\bar{\phi}_1}\bar{\phi}_2) \to \ker P(T_{\bar{\phi}_1}\bar{\phi}_2) \) be the unitary map so that \( WR_{\alpha} = s_\alpha \). Similarly, let \( V : \ker T_{\phi^{-1}} \to \ker T_{\bar{\phi}^{-1}} \) be the unitary map such that \( VT_j s_\beta = w_\beta \) and let \( U : \ker T_{\bar{\phi}} \to \ker \bar{T}_\phi \) be the unitary map so that \( UT_j s_\beta = \bar{y}_\beta \).

**Proposition 4.1.** a) \( \# \{ \alpha : m_\alpha \neq 0 \} = \dim \ker (P(T_{\bar{\phi}_1}\bar{\phi}_2) \cap \ker T_{\phi^{-1}}) \).

In particular, if \( \phi \) is unimodular, then \( m_\alpha = 0 \) for all \( \alpha \).

b) \( \sum \gamma m_\alpha = \dim \ker (P(T_{\bar{\phi}_1}\bar{\phi}_2) \cap \ker T_{\phi^{-1}}) \).

**Proof.** a) Since \( T_{\phi^{-1}} \) is surjective,

\[
j = \text{cardinality } \{ \alpha : m_\alpha \neq 0 \} = \dim \ker T_{\phi^{-1}} \big| \ker T_{\phi^{-1}} \big|.
\]

Now we may pick the basis \( \{ w_\alpha \} \) so that \( w_1, \ldots, w_j \) is a basis for \( \ker T_{\phi^{-1}} \big| \ker T_{\phi^{-1}} \big| \) and then \( w_{j+1}, \ldots, w_{\gamma} \) is a basis for \( \ker T_{\phi^{-1}} \big| \ker T_{\phi^{-1}} \big| \). Since the inner product \( (w_\alpha, w_\beta) = (u_\alpha, u_\beta) \) for \( j + 1 \leq \alpha \leq \gamma \) and \( (w_\alpha, y_\beta) = 0 \) for \( 1 \leq \alpha \leq j \), it follows that

\[
\gamma - j = \# \{ \alpha : m_\alpha = 0 \} = \text{column rank } (T_j s_\alpha, T_j s_\beta) = \text{column rank } (T_j s_\alpha, T_j s_\beta) = \dim \ker (P(T_{\bar{\phi}_1}\bar{\phi}_2) T_j s_\beta) = \gamma - \dim \ker (P(T_{\bar{\phi}_1}\bar{\phi}_2) T_j s_\beta)
\]

If \( \phi \) is unimodular, then \( f = c \cdot \frac{1}{g} \) for some \( c \in C^* \), and then \( j = 0 \).

b) With \( \phi = \bar{g}z^\gamma \), and \( \langle , \rangle \) denoting multiplicative commutator we have

\[
T_{\bar{\phi}} T_{\phi^{-1}} = T_{\bar{g}z} Q_{\gamma^{-1}} T_{\bar{\phi}} = T_{\bar{g}z} Q_{\gamma^{-1}} T_{\bar{\phi}}.
\]

Therefore, \( T_{\bar{\phi}} T_{\phi^{-1}} \) is similar to

\[
T_{\bar{g}z} T_{\bar{\phi}} Q_{\gamma^{-1}} = T_{\bar{g}z} T_{\bar{\phi}} Q_{\gamma^{-1}} = [1 - H_{\bar{g}z} T_{\bar{g}z}] Q_{\gamma^{-1}}.
\]

This last operator has the block matrix form \( \begin{pmatrix} M & 0 \\ L & 0 \end{pmatrix} \) relative to the decomposition \( H^2(T) = \text{ran } S^\gamma \oplus \ker S^\gamma \). Since \( H_{\bar{g}z} H_{\bar{g}z} \) is trace class, we have for sufficiently
small $\delta > 0$

$$\gamma + \sum_{1}^{\gamma} m_{\alpha} = \dim \text{root space } T_{\bar{\psi}} T_{\bar{\psi}}^{-1} = \dim \text{root space } \begin{pmatrix} M & 0 \\ L & 0 \end{pmatrix} = \frac{1}{2\pi i} \int_{|z|=\delta} \left( z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} M & 0 \\ L & 0 \end{pmatrix} \right)^{-1} dz \bigg|_{z=\delta}
$$

$$= \frac{1}{2\pi i} \int_{|z|=\delta} \left( \frac{L(z-M)^{-1}}{z} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) dz \bigg|_{z=\delta}
$$

$$= \frac{1}{2\pi i} \int_{|z|=\delta} \left( \frac{(z-M)^{-1} - (z-1)^{-1}}{z} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) dz \bigg|_{z=\delta}
$$

$$= \gamma + \frac{1}{2\pi i} \int_{|z|=\delta} (z-M)^{-1} dz = \gamma + \dim \text{root space } [Q_{\gamma-1}(I - H_{\psi} H_{\bar{\psi}}^{\dagger} \text{ran } S_{\gamma})].$$

\[\square\]

**Proposition 4.2.** Suppose that $m_{\alpha} = 0$, $\forall \alpha$. Then

i) $\sigma_{1, T_{\psi} T_{\bar{\psi}}^{-1}} = \frac{\prod_{1}^{\lambda \in \sigma(T_{\bar{\psi}} T_{\psi}^{-1}) \setminus \{0\}}}{\det(v_{\psi}, y_{\bar{\psi}})} \lambda \bigwedge \left( \bigwedge y_{\bar{\psi}} + \text{ran } T_{\bar{\psi}} \right)^{*}$

where $\{v_{\alpha}\}_{1}$ and $\{y_{\alpha}\}_{1}$ are respectively orthonormal bases for $\ker T_{\psi} T_{\bar{\psi}}^{-1}$ and $\ker T_{\bar{\psi}}$.

In particular,

$$\|\sigma_{1, T_{\psi} T_{\bar{\psi}}^{-1}}\| = \left| \frac{\prod_{1}^{\lambda \in \sigma(T_{\bar{\psi}} T_{\psi}^{-1}) \setminus \{0\}}}{\lambda(P(T_{\bar{\psi}}^{-1}), P(T_{\bar{\psi}}))} \right|.$$

ii) $\prod_{1}^{\lambda \in \sigma(T_{\bar{\psi}} T_{\psi}^{-1}) \setminus \{0\}} \lambda = \lambda^{2}(P(T_{\bar{\psi}}), P(T_{\bar{\psi}}^{*})), \lambda \left( P_{1} T_{\bar{\psi}}^{*} \right)$

$$\times \prod_{1}^{c_{z}(\frac{f}{g} \Theta_{1}) \det(P(T_{\bar{\psi}}) \wedge \text{ran } T_{\bar{\psi}} T_{\bar{\psi}}^{-1}) \frac{1}{f^{2}} \det(P_{1} T_{\bar{\psi}}^{*} \text{ran } T_{\bar{\psi}}^{*} \text{ran } T_{\bar{\psi}}^{*}) \text{det}_{*}(T_{g} + T_{f} + T_{1}).$$

iii) With $e^{ip} = \det U^{*} \det V \det W$,

$$\det(v_{j}, y_{k}) = \det(P(T_{\bar{\psi}}) \wedge \text{ran } T_{\bar{\psi}} T_{\bar{\psi}}^{-1}) \frac{1}{f^{2}} \det(P_{1} T_{\bar{\psi}}^{*} \text{ran } T_{\bar{\psi}}^{*} \text{ran } T_{\bar{\psi}}^{*}) \text{det}_{*}(T_{g} + T_{f} + T_{1}).$$

iv) $\prod_{1}^{\lambda \in \sigma(T_{\bar{\psi}} T_{\psi}^{-1}) \setminus \{0\}} \lambda = \lambda^{2}(P(T_{\bar{\psi}}), P(T_{\bar{\psi}})) \prod_{1}^{c_{z}(\frac{f}{g} \Theta_{1}) \det(v_{j}, y_{k})} \lambda \left( P_{1} T_{\bar{\psi}}^{*} \right)$

$$\cdot \text{det}_{*}(T_{g} + T_{f} + T_{1}) e^{-ip} \det(P(T_{\bar{\psi}}) \wedge \text{ran } T_{\bar{\psi}} T_{\bar{\psi}}^{-1}) \frac{1}{f^{2}} \det(P_{1} T_{\bar{\psi}}^{*} \text{ran } T_{\bar{\psi}}^{*} \text{ran } T_{\bar{\psi}}^{*}) \text{det}_{*}(T_{g} + T_{f} + T_{1}).$$

In particular, if $\Theta_{1} = z^{\gamma}, \Theta_{2} = 1$, then

$$E(\psi) G_{\frac{g}{f}}^{\gamma} = \frac{\prod_{1}^{\lambda \in \sigma(T_{\bar{\psi}} T_{\psi}^{-1}) \setminus \{0\}} \lambda}{D_{\gamma-1}(T_{g})}.$$
v) Suppose \( \{x_\alpha\}_1^\gamma \) is an orthonormal basis for \( \ker T_{\Theta_1 \Theta_2} \). Let \( \{w_\alpha\}_1^\gamma \) be obtained from the vectors \( \{T_j x_\alpha\}_1^\gamma \) via the Gram-Schmidt orthogonalization procedure. Similarly, let \( \{y_\alpha\}_1^\gamma \) be the result of orthogonalizing the vectors \( \{T_j^* x_\alpha\}_1^\gamma \). Then, with this choice of bases for \( \ker T_{\phi^{-1}} \) and \( \ker T_{\phi} \) respectively, statements iii) and iv) hold with \( \rho = 0 \).

**Proof.** Part i) follows immediately from Proposition 2.1 by taking \( X = T_{\phi} T_{\phi^{-1}} \). To prove ii) recall that since \( T_{\phi} \) is injective, \( T_{\phi^{-1}} \) is surjective and so \( \ran T_{\phi} T_{\phi^{-1}} = \ran T_{\phi} \). Consequently, \( X = \ran T_{\phi} \) and so

\[
\prod_{\lambda \in \sigma(T_{\phi} T_{\phi^{-1}}) \setminus \{0\}} \lambda = \det(T_{\phi} T_{\phi^{-1}}|_X) = \det(T_{\phi} T_{\phi^{-1}}|_{\ran T_{\phi}}) = \det(T_{\phi^{-1}} T_{\phi}).
\]

Since \( \phi^{-1} = \Theta_2 \Theta_1 j \Theta_2 \), assertion ii) is obtained by the arguments given in the proofs of Theorems 3.6 and Theorem 3.7 of [17].

To prove ii) note that \( u_j = w_j \) for \( j = 1, \ldots, \gamma \).

Recall that \( \{s_\alpha\} \) are orthonormal eigenvectors of \( P(T_{\Theta_1 \Theta_2}) T_{[f]} |_{\ker T_{\Theta_1 \Theta_2}} \) so that

\[
P(T_{\Theta_1 \Theta_2}) T_{[f]} |_{\ker T_{\Theta_1 \Theta_2}} s_\alpha = \mu_\alpha s_\alpha,
\]

and \( \{\frac{1}{\sqrt{\mu_\alpha}} T_j s_\alpha\} \) is a complete orthonormal set in \( \ker T_{\Theta^{-1}} \). Also, the \( \{r_\alpha\} \) are an orthonormal set of eigenvectors of \( P(T_{\Theta_1 \Theta_2}) T_{[f]} |_{\ker T_{\Theta_1 \Theta_2}} \) so that

\[
P(T_{\Theta_1 \Theta_2}) T_{[f]} |_{\ker T_{\Theta_1 \Theta_2}} r_\alpha = \lambda_\alpha r_\alpha,
\]

and \( \{\frac{1}{\sqrt{\lambda_\alpha}} T_j r_\alpha\} \) is a complete orthonormal set in \( \ker T_{\phi} \). Let \( W : \ker P(T_{\Theta_1 \Theta_2}) \rightarrow \ker P(T_{\Theta_1 \Theta_2}) \) be the unitary map so that \( W r_\alpha = s_\alpha \) and \( V : \ker T_{\phi^{-1}} \rightarrow \ker T_{\phi^{-1}} \) is the unitary map such that \( VT_j \frac{1}{\sqrt{\lambda_\alpha}} = w_\gamma \) while \( U : \ker T_{\phi} \rightarrow \ker T_{\phi} \) is the unitary map so that \( UT_j \frac{1}{\sqrt{\lambda_\alpha}} = y_\gamma \).

Then

\[
det(w_j, y_k) = \det(VT_j W \frac{1}{\sqrt{\lambda_\alpha}} T_{[f]} |_{\ker T_{\Theta_1 \Theta_2}} T_k \frac{1}{\sqrt{\lambda_\alpha}}) = \det(U^* \det V \det W) \times \det(P(T_{\Theta_1}) \wedge \Rp(T_{\Theta_2}) T_{[f]} |_{K_{\Theta_1 \cap K_{\Theta_2}}}) \cdot \det(P(T_{\Theta_1}) \wedge \Rp(T_{\Theta_2}) T_{[f]} |_{K_{\Theta_1 \cap K_{\Theta_2}}}) \cdot \det(P(T_{\Theta_1}) \wedge \Rp(T_{\Theta_2}) T_{[f]} |_{K_{\Theta_1 \cap K_{\Theta_2}}}) \cdot \frac{1}{\sqrt{\lambda_\alpha}}.
\]

Finally, set \( \det(U^* \det V \det W) = e^{i\rho} \).

Combining ii) and iii) we arrive at iv).

**Proof of v.** It suffices to prove iii) with \( \rho = 0 \). We have

(4.3) \[
u_j = w_j = \frac{|T_j x_j - \Proj_{1 \leq i < j} T_j x_i|}{|T_j x_j - \Proj_{1 \leq i < j} T_j x_i|}.
\]

Therefore,

\[
\prod_{j=1}^{\gamma} u_j = \prod_{j=1}^{\gamma} \frac{|T_j x_j|}{|T_j x_j - \Proj_{1 \leq i < j} T_j x_i|}.
\]
Since \( \| \gamma u_j \| = 1 \),
\[
\prod_{j=1}^{\gamma} \| T_j x_j - \text{Proj}_{1 \leq i \leq j} T_j x_j \|^2 = \| \prod_{j=1}^{\gamma} T_j x_j \|^2 = \det(T_j x_j, T_j x_j)_{1 \leq \alpha, \tau \leq \gamma}.
\]
Consequently,
\[
\prod_{j=1}^{\gamma} u_j = \frac{\prod_{j=1}^{\gamma} T_j x_j}{\det(P(T_{\theta_1}) \wedge \text{rp}(T_{\theta_2}) \cdot T_{\gamma}|_{K_{\theta_1} \cap K_{\theta_2}})}
\]
and similarly,
\[
\prod_{j=1}^{\gamma} y_j = \frac{\prod_{j=1}^{\gamma} T_{\gamma} x_j}{\det(P(T_{\theta_1}) \wedge \text{rp}(T_{\theta_2}) \cdot T_{\gamma}|_{K_{\theta_1} \cap K_{\theta_2}})}
\]
Therefore,
\[
\det(u_j, y_j) = (\prod_{j=1}^{\gamma} u_j, \prod_{j=1}^{\gamma} y_j) = \frac{\left(\prod_{j=1}^{\gamma} T_j x_j, \prod_{j=1}^{\gamma} T_{\gamma} x_j\right)}{\det(P(T_{\theta_1}) \wedge \text{rp}(T_{\theta_2}) \cdot T_{\gamma}|_{K_{\theta_1} \cap K_{\theta_2}})}.
\]
On the other hand,
\[
(\prod_{j=1}^{\gamma} (T_j x_j, \prod_{j=1}^{\gamma} T_{\gamma} x_j) = \det(T_{\gamma} x_j, x_j) = \det(P(T_{\theta_1}) \wedge \text{rp}(T_{\theta_2}) \cdot T_{\gamma}|_{K_{\theta_1} \cap K_{\theta_2}}).
\]
Combining the last two equations gives the desired result.

In order to compute \( \sigma_{1,T_{\theta_1},T_{\theta_2}} \) in general we must remove the restriction that \( m_\alpha = 0 \), for all \( \alpha \).

**Lemma 4.2.** For small enough \( t \neq 0 \in C \), the operator \( P(T_{\theta_1}, \theta_2) T_{\frac{1}{\gamma-t}} \big|_{\ker T_{\theta_1}} \) is invertible.

**Proof.** \( \sigma(T_{\frac{1}{\gamma-t}} |_{K_{\theta_1}}) = \left\{ \frac{1}{g(z)} : z \in \Theta_1^{-1}(0) \right\} \); moreover for small \( t \neq \sigma(T_{\frac{1}{\gamma-t}} |_{K_{\theta_1}}) \), we have \( P(T_{\theta_1}, \theta_2) T_{\frac{1}{\gamma-t}} |_{\ker T_{\theta_1}, \theta_2} = P(T_{\theta_1}, \theta_2) g(P(T_{\theta_1}|_{K_{\theta_1}})^* - t^{-1} |_{\ker T_{\theta_1}, \theta_2} \).
Consequently, \( \det (P(T_{\theta_1}, \theta_2) T_{\frac{1}{\gamma-t}} |_{\ker T_{\theta_1}, \theta_2}) \) is a non-zero function of \( t \) analytic in a neighborhood of the origin. Thus it has at most an isolated zero at the origin.

**Proposition 4.3.** Suppose that \( \{x_j\}_{\gamma} \) is any orthonormal basis for \( \ker T_{\theta_1}, \theta_2 \). Suppose that \( \{x_j\}_{\gamma} \) is obtained from \( \{T_j x_j\}_{\gamma} \) by the Gram-Schmidt procedure. Suppose also that \( \{y_j\}_{\gamma} \) is obtained from \( \{T_{\gamma} x_j\}_{\gamma} \) by the Gram-Schmidt orthogonalization procedure. Then \( \{b_j\}_{\gamma} \) are the maximal root vectors associated to the orthonormal
basis \( \{ v_j \}_1^\gamma \),
\[
(-1)^{\sum_{n=1}^\gamma m_n} \prod_{\lambda \in \sigma(T_{a_n}T_{-\alpha_n-1}) \setminus \{ \emptyset \}} \lambda 
\]
\[
= \chi^2(P(T_{\bar{\Theta}_1}), P(T_{\bar{\Theta}_2})) \cdot \prod_{|z|<1} c_z \left( \frac{f}{g} \Theta_2 \Theta_1 \right) \cdot \det \delta(T_2 + L^1(H), T_f + L^1(H))
\]
\[
\times \det (P(T_{\bar{\Theta}_1}) \wedge \operatorname{rp}(T_{\bar{\Theta}_2}) \cdot T_{\phi_2})^{\frac{1}{2}}
\]
\[
\cdot \det (P(T_{\bar{\Theta}_1}) \wedge \operatorname{rp}(T_{\bar{\Theta}_2}) \cdot T_{\phi_2})^{\frac{1}{2}}.
\]

Proof. Set \( q(t) = \det P(T_{\bar{\Theta}_1}T_{\bar{\Theta}_2})T_{\frac{f}{g-r} \Phi_2} \mid_{\ker T_{\bar{\Theta}_1} \cap \ker T_{\bar{\Theta}_2}} \). By Lemma 4.2 for small \( t \) the operator \( P(T_{\bar{\Theta}_1}T_{\bar{\Theta}_2})T_{\frac{f}{g-r} \Phi_2} \mid_{\ker T_{\bar{\Theta}_1} \cap \ker T_{\bar{\Theta}_2}} \) is invertible. For small \( t \) and \( a \in C \) consider the two parameter family of operators

\[
P(T_{\bar{\Theta}_1}T_{\bar{\Theta}_2})T_{\frac{f}{g-r} \Phi_2} \mid_{\ker T_{\bar{\Theta}_1} \cap \ker T_{\bar{\Theta}_2}} = P(T_{\bar{\Theta}_1}T_{\bar{\Theta}_2})T_{\frac{f}{g-r} \Phi_2} \mid_{\ker T_{\bar{\Theta}_1} \cap \ker T_{\bar{\Theta}_2}} - a P(T_{\bar{\Theta}_1}T_{\bar{\Theta}_2})T_{\frac{f}{g-r} \Phi_2} \mid_{\ker T_{\bar{\Theta}_1} \cap \ker T_{\bar{\Theta}_2}}^{-1}
\]
\[
\times P(T_{\bar{\Theta}_1}T_{\bar{\Theta}_2})T_{\frac{f}{g-r} \Phi_2} \mid_{\ker T_{\bar{\Theta}_1} \cap \ker T_{\bar{\Theta}_2}} - q(t) a],
\]

provided \( q(t) \neq 0 \).

Now \( [q(t)](P(T_{\bar{\Theta}_1}T_{\bar{\Theta}_2})T_{\frac{f}{g-r} \Phi_2} \mid_{\ker T_{\bar{\Theta}_1} \cap \ker T_{\bar{\Theta}_2}})^{-1}(P(T_{\bar{\Theta}_1}T_{\bar{\Theta}_2})T_{\frac{f}{g-r} \Phi_2} \mid_{\ker T_{\bar{\Theta}_1} \cap \ker T_{\bar{\Theta}_2}} - q(t) a] \) is analytic in \( t \) in a neighborhood \( N \) of zero. Therefore,
\[
\det P(T_{\bar{\Theta}_1}T_{\bar{\Theta}_2})T_{\frac{f}{g-r} \Phi_2} \mid_{\ker T_{\bar{\Theta}_1} \cap \ker T_{\bar{\Theta}_2}} = \frac{p(a, t)}{q(t)^{\gamma-1}}
\]

where \( p(a, t) \) is a polynomial in \( a \) of degree \( \gamma \) with coefficients analytic in \( t \in N \). Since the divisor of the right-hand side is supported on a one-dimensional analytic variety, there exists a sequence of points \( \{ a_n, t_n \} \subset C^2 \) converging to \( (0, 0) \) where \( P(T_{\bar{\Theta}_1}T_{\bar{\Theta}_2})T_{\frac{f}{g-r} \Phi_2} \mid_{\ker T_{\bar{\Theta}_1} \cap \ker T_{\bar{\Theta}_2}} \) is invertible.

For \( t_n \in N \) and small \( |a_n| \), let \( \phi_n = \Theta_1 \Theta_2 (f - a_n) (g - t_n) \). The corresponding \( m_n = 0 \). Therefore, Proposition 4.2 iv) holds for \( \phi_n \). Since the right-hand side of Proposition 4.2 iv) is continuous in the spectral parameters \((t, a)\), it suffices to check the continuity of the left-hand side. To that end we shall use Proposition 2.1. So suppose \( \{ v_j^{(n)} \}_1^\gamma \) is obtained from \( \{ T_{f-a_n} x_j \}_1^\gamma \) by the Gram-Schmidt procedure and suppose also that \( \{ y_j^{(n)} \}_1^\gamma \) is similarly obtained from \( \{ T_{\frac{1}{g-t_n}} x_j \}_1^\gamma \). By Propositions
4.3 and 4.2 part iv) we have for \( n = 1, 2, \ldots \),

\[
\det(T_{\phi_n} T_{\phi_n}^{-1} + \sum_{j=1}^{\gamma} y_j^{(n)} \otimes v_j^{(n)})
\]

\[
= \chi^2(P(T_{\phi_1}), P(T_{\phi_2})) \cdot \prod_{|s|<1} c_s \left( \frac{f - a_n}{g - t_n} \right) \Theta_1 \Theta_2.
\]

So in general,

\[
\left\{ h_j^{(n)} \right\}
\]

are a sequence of linearly independent sets of vectors so that \( h_j^{(n)} \to h_j \) for each \( j = 1, 2, \ldots, \gamma \). Suppose also that \( \left\{ h_j \right\} \) is linearly independent with associated orthonormal vectors \( \left\{ z_j \right\} \). For each \( n = 1, 2, \ldots \), we can apply the Gram-Schmidt procedure to \( \left\{ h_j^{(n)} \right\} \) to get vectors \( \left\{ z_j^{(n)} \right\} \).

We claim that \( z_j^{(n)} \to z_j \), for \( j = 1, 2, \ldots, \gamma \). To check this, note that

\[
\begin{align*}
h_1^{(n)} & = \frac{h_1^{(n)}}{\| h_1^{(n)} \|} \to h_1 = z_1, \\
\end{align*}
\]

\[
\begin{align*}
\left( h_2^{(n)} - (h_2^{(n)} \cdot z_1^{(n)}) z_1^{(n)} \right) & = \frac{h_2^{(n)} - (h_2, z_1) z_1}{\| h_2^{(n)} \|} \to h_2 - (h_2, z_1) z_1 = z_2.
\end{align*}
\]

So in general,

\[
\begin{align*}
\begin{aligned}
z_j^{(n)} & = \frac{h_j^{(n)} - \sum_{1 \leq i < j} (h_j^{(n)} \cdot z_i^{(n)}) z_i^{(n)}}{\| h_j^{(n)} - \sum_{1 \leq i < j} (h_j, z_i) z_i \|} \to h_j - \sum_{1 \leq i < j} (h_j, z_i) z_i = z_j.
\end{aligned}
\end{align*}
\]

Now with \( h_j^{(n)} = T_{j-a_n} x_j \), we get \( v_j^{(n)} = z_j^{(n)} \to z_j = v_j \). Similarly, if \( h_j^{(n)} = T_{\frac{y}{y-t_n}} x_j \), then \( y_j^{(n)} = z_j^{(n)} \to z_j = y_j \). Therefore,

\[
\sum_{j=1}^{\gamma} y_j^{(n)} \otimes v_j^{(n)} \to \sum_{j=1}^{\gamma} y_j \otimes v_j \quad \text{in trace norm.}
\]

**Lemma 4.3.** If \( \left\{ b_j \right\} \) are maximal vectors associated to the orthonormal basis \( \left\{ v_j \right\} \), then

\[
\lim_{n \to \infty} \frac{\prod_{\lambda \in \sigma(T_{\phi_n} T_{\phi_n}^{-1}) \setminus \{0\}} \lambda^{\mu}}{\det(v_j^{(n)}, v_k^{(n)})} = (-1)^{\sum_{\alpha=1}^{m_a}} \frac{\prod_{\lambda \in \sigma(T_{\phi_n} T_{\phi_n}^{-1}) \setminus \{0\}} \lambda^{\mu}}{\det(b_j, y_k)}.
\]

**Proof.** By Proposition 2.1 it suffices to show that

\[
\lim_{n \to \infty} \det(T_{\phi_n} T_{\phi_n}^{-1} + \sum_{j=1}^{\gamma} y_j^{(n)} \otimes v_j^{(n)}) = \det(T_{\phi_1} T_{\phi_1}^{-1} + \sum_{j=1}^{\gamma} y_j \otimes v_j).
\]

Since the dyadic sums converge in \( L^1(H) \) it remains to check that \( 1 - T_{\phi_n} T_{\phi_n}^{-1} \) converges to \( 1 - T_{\tilde{\phi}_1} T_{\tilde{\phi}_1}^{-1} \) in trace norm.

Note that the Hankel operators \( H_{\tilde{\phi}_1, \tilde{\phi}_2} \) and \( H_{\tilde{\phi}_1, \tilde{\phi}_2} \) are trace class.
Let \( \psi = \Theta_1 \Theta_2 \cdot \omega \eta \), where \( \omega, \eta \) are outer in \( K_{\frac{1}{\omega}} \cap \mathcal{W} \). Then
\[
T_\psi T_\psi^{-1} = [T_{\Theta_1 \Theta_2} T_{\omega \eta} + H_{\Theta_1 \Theta_2} H_{\omega \eta}] [T_{\omega^{-1} \eta^{-1}} T_{\Theta_1 \Theta_2} + H_{\omega^{-1} \eta^{-1}} H_{\Theta_1 \Theta_2}].
\]
With \( \omega = f-a_n \) and \( \eta = g^{-i_n} \) we have \( H_{\omega \eta} \rightarrow H_{f g^{-i}} \) and \( H_{\omega^{-1} \eta^{-1}} \rightarrow H_{f^{-1} g^{-1}} \) in the uniform norm. Also, \( T_{\Theta_1 \Theta_2} T_{\omega \eta} T_{\omega^{-1} \eta^{-1}} T_{\Theta_1 \Theta_2} = T_{\Theta_1 \Theta_2} [1 + T_{\omega \eta} T_{\omega^{-1} \eta^{-1}}] T_{\Theta_1 \Theta_2} \). But \( [T_{\overline{\eta}}, T_{\omega}] = [T_{\overline{g}}, T_{f}] \in L^1(H^2(T)) \). Since \( T_{\omega^{-1} \eta^{-1}} T_{\omega^{-1} \eta^{-1}} \rightarrow T_{\overline{g}} T_{f}^{-1} \) in the uniform norm it follows that \( 1 - T_{\phi_1} T_{\phi_1}^{-1} \) converges to \( 1 - T_{\overline{g}} T_{f}^{-1} \) in trace norm. \( \square \)

To complete the proof of Theorem 1.6 we note that when the zeros of \( \Theta_1 \) and \( \Theta_2 \) are simple, the equality
\[
\chi^2(P(T_{\Theta_1}), P(T_{\Theta_2})) = \prod_{1 \leq \alpha, \beta < \ell} \left| \frac{1 - \bar{\mu}_\alpha \mu_\beta}{1 - \mu_\alpha \mu_\beta} \right|^2 \sum_N \prod_{\mu_\alpha \in N} \frac{(\mu_\alpha - \nu_\beta)}{\mu_\beta} \prod_{\mu_\beta \in N} \frac{(1 - \mu_\alpha \nu_\beta)}{1 - \mu_\beta \nu_\beta}
\]
with the sum extended over all subsets \( N \subset \{ \mu_1, \ldots, \mu; \frac{1}{\mu_1}, \ldots, \frac{1}{\mu} \} \) of cardinality \( \ell \) and \( \bar{N} = \{ \mu_1, \ldots, \mu; \frac{1}{\mu_1}, \ldots, \frac{1}{\mu} \} \backslash N \) is proved in Proposition 3.4 of [17].

**Corollary 4.1.** Let \( \{x_j\}_{j=1}^\infty \) be an orthonormal basis for \( \ker T_{\Theta_1 \Theta_2} \). Let \( \{u_j\}_{j=1}^\infty \) be the top root vectors of the associated Jordan chain lying over \( T_{f \overline{g}} x_j \). Then
\[
\begin{align*}
s_{T_{\overline{g}}} T_{\overline{g}}^{-1} & = (-1)^{\sum_{\alpha=1}^\infty m_\alpha} \prod_{\lambda \in \sigma(T_{\overline{g}} T_{\overline{g}}^{-1}) \backslash \{0\}} \lambda \cdot \frac{\| \bigwedge T_{\overline{g}} T_{\overline{g}}^{-1} \|_{\ker T_{\Theta_1 \Theta_2}}}{\det(u_j, T_{\overline{g}} x_j)}^2 \cdot [\bigwedge T_{f \overline{g}} x_j \otimes \bigwedge T_{\overline{g}} x_j + \ker T_{\Theta_1}]^*, \\
(\alpha) & = (-1)^{\sum_{\alpha=1}^\infty m_\alpha} \prod_{\lambda \in \sigma(T_{\overline{g}} T_{\overline{g}}^{-1}) \backslash \{0\}} \lambda \cdot \frac{\chi^2(P(T_{\Theta_1}), P(T_{\Theta_2}))}{\det(u_j, T_{\overline{g}} x_j)} \cdot \det_4 \big( f \overline{g} \Theta_2 \Theta_1 \big) \cdot \det_4 \big( \overline{g} L^1, T_f + L^1 \big).
\end{align*}
\]

5. **Proofs of Theorem 1.1 part ii) and Theorem 1.3**

It is of interest to estimate the speed of convergence in our limit theorems. The proofs of Theorem 1.1 part ii) and Theorem 1.3 are based on a series of lemmas with numerical estimates. In what follows \( \| \cdot \|_2 \) denotes the Hilbert-Schmidt norm and \( \| \cdot \|_1 \) denotes the trace norm.

**Lemma 5.1.** Suppose \( f \in \operatorname{Lip}_\beta, \beta > \frac{1}{2} \). Then
\[
\left\| S^{n+1} Pf Q \right\|_2 \leq \left( \frac{4\pi}{3} \right)^{\beta} \left( \frac{1}{1 - (\frac{1}{2})^{2\beta}} \right)^{\frac{1}{2}} \left( \frac{1}{2\beta - 1} \right)^{\frac{1}{2}} \left\| f \right\|_{\operatorname{Lip}_\beta} \cdot (n + 1)^{\frac{1}{2} - \frac{2\beta}{1}}.
\]
Proof: The proof ultimately depends upon the Bernstein inequality; cf. p. 32, Katznelson [27]. It is known\footnote{See for example equation (6.4), Katznelson.} that

\[
\sum_{2^m \leq k \leq 2^{m+1}} |\hat{f}(k)|^2 \leq \left( \frac{2\pi}{3 \cdot 2^m} \right)^{2\beta} \|f\|_{\text{Lip}}^2.
\]

So, if \(2^m \leq n \leq 2^{m+1}\), we have

\[
\sum_{n \leq k} |\hat{f}(k)|^2 \leq \left( \frac{2 \pi}{3} \right)^{2\beta} \|f\|_{\text{Lip}}^2 \sum_{r \geq m} \left( \frac{1}{2^r} \right)^{2\beta}
\]

\[
= \left( \frac{2 \pi}{3} \right)^{2\beta} \|f\|_{\text{Lip}}^2 \left( \frac{1}{2^m} \right)^{2\beta} \frac{1}{1 - \left( \frac{1}{2} \right)^{2\beta}}.
\]

Hence \(\sum_{n \leq k} |\hat{f}(k)|^2 \leq \frac{\Omega}{n^{2\beta}}\), where \(\Omega = \left( \frac{4 \pi}{12 \pi^2} \right)^{2\beta} \|f\|_{\text{Lip}}^2 \). Now,

\[
\|S^{n+1} P M f Q\|_2^2 = \sum_{k \geq 1} k |\hat{f}(k + n + 1)|^2
\]

\[
= \sum_{k \geq 1} |\hat{f}(k + n + 1)|^2 + \sum_{k \geq 1} |\hat{f}(k + n + 2)|^2 + \sum_{k \geq 1} |\hat{f}(k + n + 3)|^2 + \cdots
\]

\[
\leq \Omega \left[ \frac{1}{(n+2)^{2\beta}} + \frac{1}{(n+3)^{2\beta}} + \frac{1}{(n+4)^{2\beta}} + \cdots \right] \leq \frac{\Omega}{(2\beta - 1)} (n+1)^{1-2\beta}.
\]

\(\square\)

Since \(S^n T_\psi T_\psi [1 - P(T_\psi)] = R(T_\phi)\), it follows that for \(n \gg 0\),

\[
\ker[S^{n+1} R(T_\phi)] = \text{ran}[S^{n+1} T_\psi T_\psi [1 - P(T_\psi)] S^{n+1}]^{-1} P_\gamma.
\]

Moreover, \(\{S^{n+1} T_\psi T_\psi [1 - P(T_\psi)] S^{n+1}\}^{-1}\) converges uniformly to \(T_\psi T_\psi \).

Lemma 5.2. Suppose \(\phi = z^n \psi\) with \(\psi \in \text{Lip}_\beta\), \(\frac{1}{2} < \beta < 1\), and \(T_\psi\) is invertible. Set \(J_n = S^{n+1} T_\psi^{-1} [1 - P(T_\psi)] S^{n+1}\). Then

\((i)\) \(\|S^{n+1} R(T_\phi) S^{n+1} - T_\psi^{-1}\| = O(n^{-2\beta})\).

\((ii)\) \(\|J_n - T_\psi^{-1}\| = O(n^{-2\beta})\) and \(\|J_n - T_\psi^{-1}\|_1 = O(n^{1-2\beta})\).

In particular, for \(n \gg 0\), \(J_n\) is invertible and

\[
\ker S^{n+1} R(T_\phi) S^{n+1} = J_n^{-1} P_{\gamma-1}(H^2(T)).
\]

\((iii)\) \(\|J_n^{-1} T_\psi e_\alpha - T_\psi e_\alpha\| = O(n^{-2\beta})\) for \(\alpha = 0, 1, 2, \ldots, \gamma - 1\).

Proof. We have \(R(T_\phi) = S^n T_\phi^{-1} [1 - P(T_\phi)]\), and \(\ker P(T_\phi) = \{ V_{\alpha=0}^\gamma \frac{1}{\gamma} e_\alpha \}\) where \(\psi = f_\phi \), \(f, g \in H^\infty \cap \text{Lip}_\beta\). In particular, \(\|P(T_\phi) S^{n+1}\| = O(n^{-\beta})\).

The fact that \(f, g \in \text{Lip}_\beta \cap H^\infty\) follows because \(T_\psi\) is invertible and the conjugate function is bounded on \(\text{Lip}_\beta, 0 < \beta < 1\); cf. [28]. Accordingly, there are polynomials
\[ p_n \text{ and } q_n \text{ of degree } \leq n \text{ so that } \|\frac{1}{f} - p_n\|_{\infty} \text{ and } \|\frac{1}{g} - q_n\|_{\infty} \text{ are } O(n^{-\beta}). \] Also, since 
\[ S^{n+1}T_{\psi}^{-1}S^{n+1} = T_{\psi}^{-1} - S^{n+1}P_{\frac{1}{f}Q_{\psi}^{\frac{1}{g}}}S^{n+1}, \]
we have 
\[ J_n = S^{n+1}T_{\psi}^{-1}[1 - P(T_{\psi})]S^{n+1} = S^{n+1}T_{\psi}^{-1}S^{n+1} - S^{n+1}T_{\psi}^{-1}P(T_{\psi})S^{n+1} \]
\[ = T_{\psi}^{-1} - S^{n+1}P_{\frac{1}{f}Q_{\psi}^{\frac{1}{g}}}S^{n+1} - S^{n+1}T_{\psi}^{-1}P(T_{\psi})S^{n+1} + S^{n+1}P_{\frac{1}{f}Q_{\psi}^{\frac{1}{g}}}P(T_{\psi})S^{n+1}. \]

Also 
\[ \|S^{n+1}P_{\frac{1}{f}Q_{\psi}^{\frac{1}{g}}}S^{n+1}\| \leq \|S^{n+1}P_{\frac{1}{f}Q}\| \|Q_{\psi}^{\frac{1}{g}}\| \|S^{n+1}\| \leq \|\frac{1}{f} - p_n\| \|\frac{1}{g} - q_n\|_{\infty} = O(n^{-2\beta}), \]
and 
\[ \|S^{n+1}P_{\frac{1}{f}Q_{\psi}^{\frac{1}{g}}}S^{n+1}\|_2 \leq \|S^{n+1}P_{\frac{1}{f}Q}\|_2 \|S^{n+1}\|_2 = O(n^{1-2\beta}). \]

Now, 
\[ S^{n+1}T_{\psi}^{-1} = T_{\psi}^{-1}S^{n+1}T_{\psi}, \text{ and ker } T_{\psi} = \bigvee_{\alpha=0}^{\gamma-1} \{ \frac{1}{g} e_{\alpha} \}, \text{ ran } T_{\psi}^* = \bigvee_{\alpha=0}^{\gamma-1} \{ \frac{1}{g} e_{\alpha} \}. \]
\[ \text{Since } \frac{1}{f}, \frac{1}{g} \in \text{ Lip}_{\beta}, \text{ it follows that} \]
\[ \|S^{n+1}T_{\psi}^{-1}P(T_{\psi})S^{n+1}\|_1 = O(n^{-2\beta}). \]
Therefore, 
\[ \|J_n - T_{\psi}^{-1}\| = O(n^{-2\beta}), \text{ and } \|J_n - T_{\psi}^{-1}\|_1 = O(n^{1-2\beta}). \]

The fact that ker \( S^{n+1}R(T_{\psi})S^{n+1} = J_n^{-1}P_{k-1}(H^2(T)) \) follows immediately since \( S^{n+1}R(T_{\psi})S^{n+1} = S^k J_n. \) Because \( \|J_n - T_{\psi}^{-1}\| = O(n^{-2\beta}) \) and \( T_{\psi}^{-1} \) is invertible, we have \( \|J_n^{1-1} - T_{\psi}^{-1}\|_1 = O(n^{-2\beta}). \) Furthermore, since for all \( \alpha \) we have \( T_{\psi}^{-1}T_{\psi}^* e_{\alpha} = T_{\psi} e_{\alpha}, \) we have \( \|J_n^{1-1}T_{\psi}^* e_{\alpha} - T_{\psi} e_{\alpha}\| = O(n^{-2\beta}). \]

**Lemma 5.3.** 
\[ \|R(T_{\psi}S^{n+1}R(T_{\psi})S^{n+1}) - R(T_{\psi}T_{\psi}^{-1})\| = O(n^{-2\beta}). \]

**Proof.** Set \( Y_n = T_{\psi}S^{n+1}R(T_{\psi})S^{n+1} \) and \( Y = T_{\psi}T_{\psi}^{-1}. \) For \( n \gg 0, S^{n+1}R(T_{\psi})S^{n+1} \) is surjective. Hence \( \text{rp}(Y_n) = 1 - P(T_{\psi}). \) Therefore, since \( R(Y_n^*) = [Y_n^*Y_n + P(T_{\psi})]^{-1}Y_n, \) we have \( \|R(Y_n^*) - R(Y^*)\| = O(n^{-2\beta}). \) Consequently, since \( R(A)^* = R(A^*), \) we have 
\[ \|R(Y_n) - R(Y)\| = \|R(Y_n^*) - R(Y^*)\| = \|R(Y_n^*)^* - R(Y^*)\| = O(n^{-2\beta}). \]

Fix an orthonormal basis \( \{y_\tau\}_{\tau=1}^{\gamma} \) of ker \( (T_{\psi})^*. \) Suppose \( n \gg 0 \) so that \( \dim \ker(S^{n+1}R(T_{\psi})S^{n+1} = \gamma \) and \( J_n \) is invertible. For \( \alpha = 1, 2, \ldots, \gamma \) set \( u_{\alpha}^{(n)} = J_n^{-1}T_{\psi}^* e_{\alpha} \) and let \( \{v_{\alpha}^{(1)}\}_{\alpha=1}^{\gamma} \) be obtained from \( \{u_{\alpha}^{(n)}\}_{\alpha=1}^{\gamma} \) by the Gram-Schmidt orthogonalization procedure. Note that \( \|u_{\alpha}^{(n)} - f e_{\alpha}\| = O(n^{-2\beta}). \)

As above let \( Z \) denote the subspace of \( H^2(T) \) given by ker \( (T_{\psi})^* \oplus \bigvee \{ (T_{\psi}T_{\psi}^{-1})^* u_{\alpha} : m_{\alpha} \geq r \geq 1, \alpha = 1, 2, \ldots, \gamma \}, \) where \( \{u_{\alpha}\} \) is any set of maximal root vectors for \( \{T_{\psi}^* e_{\alpha}\}_{\alpha=1}^{\gamma}. \) Let \( K_n \) denote the operator on \( Z \) where 
\[ K_n(y_r) = P_Z (u_{\alpha}^{(n)}), \]
\[ K_n(T_{\psi}T_{\psi}^{-1})^* u_{\alpha} = P_Z R(T_{\psi}S^{n+1}R(T_{\psi})S^{n+1}) (T_{\psi}T_{\psi}^{-1})^* u_{\alpha}. \]
By (ii) of Lemma 5.2 and the continuity of the Gram-Schmidt procedure, \( v_{\alpha}^{(n)} \to v_{\alpha} \) where \( \{v_{\alpha}^{(n)}\} \) is obtained from \( \{T\phi e_{\alpha}\} \) by applying the Gram-Schmidt orthogonalization. Let \( K \) denote the limit operator on \( Z \) so that

\[
K(y_{\tau}) = P_{Z}v_{\tau},
\]

\[
K(T_{\phi T_{\phi^{-1}}})u_{\alpha} = P_{Z}[1 - P(T_{\phi T_{\phi^{-1}}})](T_{\phi T_{\phi^{-1}}})^{-1}u_{\alpha}.
\]

\[\square\]

**Lemma 5.4.** \( \|K_{n} - K\| = O(n^{-2\beta}) \).

**Proof.** Since \( \|J_{n}^{-1}T_{\phi}e_{\alpha} - T_{\phi}e_{\alpha}\| = O(n^{-2\beta}) \), computation shows that \( \|v_{\alpha}^{(n)} - v_{\alpha}\| = O(n^{-2\beta}) \). Now, comparing \( K_{n} \) and \( K \) on basis elements of \( Z \), we have with \( z_{r_{\alpha}}(n) = X_{n}^{-1}(T_{\phi T_{\phi^{-1}}})^{r}u_{\alpha} \) as in §3, for \( j = 1, 2, \ldots, \gamma \),

\[
\|(K_{n} - K)y_{j}\| = \|P_{Z}(v_{\alpha}^{(n)} - v_{\alpha})\| = O(n^{-2\beta}).
\]

Also, on \( (T_{\phi T_{\phi^{-1}}})^{r}u_{\alpha} \) we have

\[
[K_{n} - K](T_{\phi T_{\phi^{-1}}})^{r}u_{\alpha} = P_{Z}[1 - P(S^{s+1}R(T_{\phi})S^{n+1})]z_{r_{\alpha}}(n)
\]

\[
- P_{Z}[1 - P(T_{\phi^{-1}})](T_{\phi T_{\phi^{-1}}})^{-1}u_{\alpha}.
\]

Now,

\[
[1 - P(S^{s+1}R(T_{\phi})S^{n+1})]z_{r_{\alpha}(n)} = R(T_{\phi}S^{s+1}R(T_{\phi})S^{n+1})(T_{\phi T_{\phi^{-1}}})^{r}u_{\alpha}.
\]

Hence, by Lemma 5.3 we have

\[
\|[K_{n} - K](T_{\phi T_{\phi^{-1}}})^{r}u_{\alpha}\| = \|P_{Z} \{ R(T_{\phi}S^{s+1}R(T_{\phi})S^{n+1})(T_{\phi T_{\phi^{-1}}})^{r}
\]

\[
- [1 - P(T_{\phi^{-1}})] \} (T_{\phi T_{\phi^{-1}}})^{-1}u_{\alpha}\|
\]

\[
= \|P_{Z} \{ R(T_{\phi}S^{s+1}R(T_{\phi})S^{n+1}) - R(T_{\phi}T_{\phi^{-1}}) \} (T_{\phi T_{\phi^{-1}}})^{r}u_{\alpha}\| = O(n^{-2\beta}).
\]

Consequently,

\[
\|K_{n} - K\| = O(n^{-2\beta}).
\]

\[\square\]

Recall that for \( n \gg 0 \),

\[
D_{n}(T_{\phi}) = G(\phi)^{n+1} \frac{\det(P_{X}X_{n}^{1})}{\det K_{n}} \cdot \det(S^{n+1}v_{\alpha}^{(n)}, y_{\tau}).
\]

Now

\[
\|X_{n} - T_{\phi T_{\phi^{-1}}}\|_{1} = \|T_{\phi}S^{s}J_{n} - P(T_{\phi})S^{n+1} - T_{\phi}S^{s}T_{\phi^{-1}}\|_{1}
\]

\[
\leq \|T_{\phi}S^{s}J_{n} - T_{\phi T_{\phi^{-1}}}\|_{1} + \|P(T_{\phi})S^{n+1}\|_{1} = O(n^{1-2\beta}).
\]

Using the inequality

\[
|\det[1 + M] - \det[1 + N]| \leq \|M - N\| \exp(\|M\|_{1} + \|N\|_{1} + 1),
\]

we see that

\[
\det P_{X}X_{n}P_{X} = \det[P_{X}T_{\phi T_{\phi^{-1}}}X][1 + O(n^{1-2\beta})]
\]

(5.1)

\[
= \prod_{\lambda \in \sigma(T_{\phi T_{\phi^{-1}}}) \setminus \{0\}} \lambda \ [1 + O(n^{1-2\beta})].
\]
Also by Lemmas 5.2 and 5.4
\[
\frac{\det(S^{n+1}w_\alpha^{(n)}, y_\tau)}{\det K_n} = (-1)^{\sum_\gamma m_\alpha} \frac{\det(S^{n+1}w_\alpha^{(n)}, y_\tau)}{\det(u_\alpha, y_\tau)} \cdot [1 + O(n^{-2\beta})].
\]

Consequently, switching vectors \(\{y_\tau\}\) to an arbitrary basis \(\{t_\tau\}\) of \(\ker T_\phi\), we have
\[
D_n(T_\phi) = G(\phi)^{n+1} \det(S^{n+1}w_\alpha^{(n)}, t_\tau) \left(\frac{-1}{\det(u_\alpha, t_\tau)} \sum_{\lambda \in \sigma(T_\alpha T_\phi^{-1}) \setminus \{0\}} \lambda \right) [1 + O(n^{-2\beta})].
\]

This proves statement ii) of Theorem 1.1. We now complete the proof of Theorem 1.3.

**Lemma 5.5.**

\[
\det(S^{n+1}w_\alpha^{(n)}, T_\phi e_\tau) = \frac{1}{g(0)\gamma} \det(S^{n+1}(S^{n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma})^{-1}e_\alpha, T_\phi e_\tau) [1 + O(n^{-2\beta})].
\]

**Proof.** It is enough to prove the statement with \(T_\phi e_\alpha\) replaced by any basis of \(\ker T_\phi\).

Let \(\{y_\tau\}\) be an orthonormal basis for \(\ker T_\phi\). Since \(S^{n+1}w_\alpha^{(n)} = S^{n+1}J_n^{-1}T_\psi^{-1}e_\alpha \in \ker[T_\phi \oplus \operatorname{ran} T_\psi P_n] \cap \operatorname{ran} S^{n+1}\), there are complex numbers \(\lambda_{n,\alpha,\tau}\) such that \(\gamma_{n,\alpha,\tau} \leq \gamma\) and a vector \(g_\alpha \in \operatorname{ran}(T_\phi P_n)\) so that
\[
S^{n+1}w_\alpha^{(n)} = \sum_{\tau=1}^\gamma \lambda_{n,\alpha,\tau} y_\tau + T_\phi g_\alpha,
\]
where \(\lambda_{n,\alpha,\tau} = (S^{n+1}w_\alpha^{(n)}, y_\tau)\). Since \(T_\phi = T_\psi S^\gamma\) and \(S^{*\gamma}T_\psi S^{\gamma} = T_\psi\), we have for \(n \gg 0\)
\[
S^{n+1}T_\psi^{-1}S^{n+1-\gamma}w_\alpha^{(n)} = \sum_{\tau=1}^\gamma \lambda_{n,\alpha,\tau} S^{n+1}T_\psi^{-1}S^{*\gamma}y_\tau.
\]

Multiplying by \(S^\gamma\) and rewriting gives
\[
S^{n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma}w_\alpha^{(n)} = \sum_{\tau=1}^\gamma (S^{n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma}w_\alpha^{(n)}, e_\tau) e_\tau
\]
\[
+ \sum_{\tau=1}^\gamma \lambda_{n,\alpha,\tau} S^\gamma S^{n+1}T_\psi^{-1}S^{*\gamma}y_\tau.
\]

Applying the inverse of \(S^{n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma}\) to both sides gives
\[
w_\alpha^{(n)} = \sum_{\tau=1}^\gamma (S^{n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma}w_\alpha^{(n)}), e_\tau)(S^{n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma})^{-1}e_\tau
\]
\[
+ \sum_{\tau=1}^\gamma \lambda_{n,\alpha,\tau} (S^{n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma})^{-1}S^\gamma S^{n+1-\gamma}T_\psi^{-1}S^{*\gamma}y_\tau.
\]
Taking the inner product on both sides with $S^{n+1}y_k$ we get

$$(w^{(n)}_\alpha, S^{n+1}y_k) = \sum_{\tau=1}^{\gamma} \left( S^{n+1} - \gamma T^{-1}_\psi S^{n+1} T^{-1}_\psi \right) \left( w^{(n)}_\alpha, e_\tau \right)$$

$$\cdot \left( [S^{n+1} - \gamma T^{-1}_\psi S^{n+1} - \gamma e_\tau], S^{n+1}y_k \right)$$

$$+ \sum_{\ell=1}^{\gamma} (w^{(n)}_\alpha, S^{n+1}y_k) \left( [S^{n+1} - \gamma T^{-1}_\psi S^{n+1} - \gamma e_\tau], S^{n+1}y_k \right).$$

Now define $\gamma \times \gamma$ matrices $A_n, B_n, E_n, F_n$ as follows:

$$A_n = ((S^{n+1}w^{(n)}(\alpha), y_\tau)),$$

$$B_n = ((S^{n+1} - \gamma T^{-1}_\psi S^{n+1} - \gamma e_\tau), e_\tau),$$

$$E_n = ((S^{n+1}(S^{n+1} - \gamma T^{-1}_\psi S^{n+1} - \gamma)^{-1} e_\alpha, y_\tau),$$

$$F_n = ((S^{n+1}(S^{n+1} - \gamma T^{-1}_\psi S^{n+1} - \gamma)^{-1} \cdot S^{n+1}S^{n+1} - \gamma T^{-1}_\psi S^{n+1} - \gamma e_\alpha, y_\tau).$$

The preceding system of equations becomes $A_n = B_n E_n + A_n F_n$. Accordingly, we have $A_n = \det B_n \det E_n$. By Lemma 5.2, $B_n = ((T_\psi^1 e_\alpha, e_\tau)) + O(n^{-1/3})$, and since $(T_\psi)^{-1}$ maps Lip to Lip we have $F_n = O(n^{-2/3})$, so that

$$\det A_n = \det(T_\psi^1 e_\alpha, e_\tau) \cdot \det E_n [1 + O(n^{-2/3})] = \frac{1}{\tilde{g}(0)^\gamma} \det E_n [1 + O(n^{-2/3})].$$

Therefore,

$$\det((S^{n+1}w^{(n)}(\alpha), y_\tau)) = \frac{1}{\tilde{g}(0)^\gamma} \det(S^{n+1}(S^{n+1} - \gamma T^{-1}_\psi S^{n+1} - \gamma)^{-1} e_\alpha, y_\tau).$$

**Lemma 5.6.**

$$\det((S^{n+1}(S^{n+1} - \gamma T^{-1}_\psi S^{n+1} - \gamma)^{-1} e_\alpha, T_\psi e_\tau)) = \tilde{g}(0)^\gamma \cdot \det(T_\psi^1 z_{n+1} \cdot [1 - H_\psi Q_{n-\gamma} H_{(\psi)}^{-1}]^{-1} e_\alpha, e_\tau).$$

**Proof.** We have

$$S^{n+1} - \gamma T^{-1}_\psi S^{n+1} - \gamma = S^{n+1} - \gamma T^{-1}_\psi S^{n+1} - \gamma = S^{n+1} - \gamma T_\psi T_\psi^1 S^{n+1} - \gamma$$

$$= S^{n+1} - \gamma T_\psi (T_\psi, T_\psi^1) T_\psi S^{n+1} - \gamma = T_\psi S^{n+1} - \gamma T_\psi^1 S^{n+1} - \gamma T_\psi$$

$$= T_\psi (1 - S^{n+1} - \gamma H_{\psi} H_{(\psi)}^1) S^{n+1} - \gamma T_\psi = T_\psi [1 - H_\psi Q_{n-\gamma} H_{(\psi)}] T_\psi.$$

Thus

$$[S^{n+1} - \gamma T^{-1}_\psi S^{n+1} - \gamma]^{-1} = T_\psi [1 - H_\psi Q_{n-\gamma} H_{(\psi)}]^{-1} T_\psi,$$

and so,

$$\det((S^{n+1}[S^{n+1} - \gamma T^{-1}_\psi S^{n+1} - \gamma]^{-1} e_\alpha, T_\psi e_\tau))$$

$$= \det(T_\psi^1 z_{n+1} \cdot [1 - H_\psi Q_{n-\gamma} H_{(\psi)}]^{-1} T_\psi e_\alpha, e_\tau)$$

$$= \tilde{g}(0)^\gamma \cdot \det(T_\psi^1 z_{n+1} \cdot [1 - H_\psi Q_{n-\gamma} H_{(\psi)}]^{-1} e_\alpha, e_\tau).$$
Collecting the results: by equation (3.3) we have for \( n \gg 0 \),
\[
D_n(T_\phi) = G(\phi)^{n+1} \frac{\det P_X X_n P_X}{\det(K_n)} \cdot \det(S^{n+1} u^{(n)}_\alpha, y_\tau).
\]
By equation (5.1) we have
\[
\det P_X X_n P_X = \prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}})} \lambda \ [1 + O(n^{1-2\beta})],
\]
and by Lemmas 3.5 and 5.4
\[
\det(K_n) = (-1)^{\sum_1^m} \det(b_\alpha, y_\tau) \cdot [1 + O(n^{-2\beta})],
\]
while Lemmas 5.5 and 5.6 give us
\[
(S^{n+1} u^{(n)}_\alpha, y_\tau) = \det \Gamma^{(n)} \cdot \det(S^{n+1} w^{(n)}_\alpha, y_\tau) = \frac{1}{\| \bigwedge T_I e_\alpha \|} \det(S^{n+1} T_I [1 - H_{\frac{2}{3}} Q_{n-\gamma} H_{\frac{2}{3}}]^{-1} e_\alpha, y_\tau) \cdot [1 + O(n^{-2\beta})].
\]
On the other hand, Proposition 2.1 ii) gives
\[
\| \bigwedge T_I e_\alpha \| \det(b_\alpha, y_\tau) = \det(u_\alpha, y_\tau),
\]
and we have “the ratio invariance property”
\[
\frac{\det(S^{n+1} w^{(n)}_\alpha, y_\tau)}{\det(u_\alpha, y_\tau)} = \frac{\det(S^{n+1} w^{(n)}_\alpha, T_{\frac{2}{3}} e_\tau)}{\det(u_\alpha, T_{\frac{2}{3}} e_\tau)},
\]
while Theorem 1.2 gives us
\[
(-1)^{\sum_1^n} \prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}}) \setminus \{0\}} \lambda \frac{\det(u_\alpha, T_{\frac{2}{3}} e_\alpha)}{\det(u_\alpha, T_{\frac{2}{3}} e_\alpha)} = E(\psi) G(\bar{\theta}/T)^n.
\]
Putting these pieces together yields the approximation theorem
\[
D_n(T_\phi) = G(\phi)^{n+1} E(\psi) G(\bar{\theta}/T)^n \cdot \det \left( (T_{\frac{2}{3}} z^{n+1} : [1 - H_{\frac{2}{3}} Q_{n-\gamma} H_{\frac{2}{3}}]^{-1} e_\alpha, e_\tau) \right)_{\gamma \times \gamma} [1 + O(n^{-2\beta})].
\]
Note that if \( \phi = z^\gamma f \bar{g} \) where \( f, g \in K_{\frac{1}{2}, \frac{1}{2}} \cap H^\infty(\mathbb{T}) \), the same formula holds with \( O(n^{-2\beta}) \) replaced by \( o(1) \). This concludes the proof of Theorem 1.3. \( \square \)

6. Rational symbol \( \phi \) with \( D_n(T_\phi) \neq 0 \) and \( D_{n-1}(T_{\frac{2}{3}} T_\phi z^{n+1}) = 0 \)

Now we construct a rational symbol \( \phi \) with \( \phi \neq 0 \) on \( \mathbb{T} \) so that \( T_\phi \) is injective and Fredholm and \( D_n(T_\phi) \neq 0 \) for \( n \gg 0 \), yet for an \( r \neq 0 \), we have the factorization \( \det(P(T_\phi) S^{n+1} P(T_{\phi^{-1}}) S^{n+1}) = a_n r^{n+1} \) where \( a_n \) is periodic in \( n \), one of the values it assumes is zero. Let \( r \in (0, 1), \theta \in (0, \pi) \). Set \( \lambda_1 = r e^{i \theta}, \lambda_2 = r e^{-i \theta}, \tilde{\lambda}_1 \neq \lambda_1, \mu = r = |\lambda_1| \neq \lambda_1, \lambda_2 \). Set \( \phi = \Theta_\lambda \Theta \Theta \Theta \Theta \Theta \). Then \( \phi^{-1} = \ldots \).
\( \phi, T_\phi \) is injective and Fredholm. Thus \( \det(P(T_\phi)S^{n+1}P(T_{\phi^{-1}})S^{n+1}) \mid \ker T_\phi = \det(P(T_\phi)S^{n+1} \mid \ker T_\phi)^2 \). In Lemma 52 of [15] we showed that
\[
\det(P(T_\phi)S^{n+1}P(T_{\phi^{-1}}) \mid \ker T_\phi) = \frac{1}{1 + r^2 e^{i\theta} |2 + r^2 |e^{i\theta} - 1|^2 (e^{-i\theta} - e^{i\theta})} \left[(e^{i\theta} - 1)(1 - r^2 e^{i\theta})(1 - r^2 e^{-2i\theta}) e^{i(n+1)\theta} - (e^{-i\theta} - 1)(1 - r^2 e^{-i\theta})(1 - r^2 e^{2i\theta}) e^{-i(n+1)\theta}\right]\]
\( r + n + 1 \).

whenever \( B_n(\theta, r) \) is real valued. Suppose that \( n = 0 \). For fixed \( \theta \in (0, \frac{\pi}{2}) \), consider \( \Im B_0(\theta, r) \) as a function of \( r \in [0, 1] \). We have
\[
\Im B_0(\theta, 0) = \sin 2\theta - \sin \theta > 0,
\]
\[
\Im B_0(\theta, 1) = -|1 - e^{i\theta}|^2 \sin 2\theta < 0.
\]
Hence, by the intermediate value theorem, we get \( r(\theta) \in (0, 1) \) so that \( \Im B_0(\theta, r(\theta)) = 0 \).

Now pick \( \theta \in (0, \frac{\pi}{2}) \) so that \( \frac{\theta}{\pi} \) is rational. Then with \( r = r(\frac{\theta}{\pi}) \) the sequence \( \{\Im B_n(\theta, r)\}_{n=1}^\infty \) is periodic with some period \( P \) and has zero for one of its values. Hence \( \det(P(T_\phi)S^{kP+1}P(T_{\phi^{-1}}) \mid \ker T_\phi) = 0 \) for \( k = 0, 1, 2, \ldots \).

Now we must check that for the indicated choices of \( r \) and \( \theta \) the determinant \( D_n(T_\phi) \neq 0 \) for \( n \gg 0 \).

For this purpose we use results of K.M. Day [13].

**Theorem 6.1.** Let \( \phi = \frac{c_0 \prod_{j=1}^{r_p}(z-r_j)}{\prod_{j=1}^{n}(1-z-r_j) \prod_{j=1}^{h}(z-\rho_j)} \) on the circle \( |z| = 1 \) with \( |\delta_j| < 1 \) \((j = 1, \ldots, \ell)\), and \( |\rho_j| > 1 \) for \( j = 1, \ldots, h \), \( c_0 \) a constant and \( r_1, \ldots, r_p \) pairwise distinct. Then,

\[
(i) \quad \text{For } p \geq \ell + h \text{ and } n \geq 0, \quad D_n(\phi) = (-1)^{(p-\ell)(n+1)} \sum_M A_M r_M^{n+1}
\]
\[
(ii) \quad \text{For } n \geq \ell, \quad D_n(\phi) = 0 \text{ if } p < \ell,
\]
\[
\text{and } D_n(\phi) = (-1)^{(p-\ell)(n+1)} \sum_M A_M r_M^{n+1} \text{ if } p \geq \ell.
\]
where the sum is taken over all \( \binom{p}{\ell} \) subsets \( M \subset \{1, \ldots, p\} \) of cardinality \( \ell \), and with \( M = \{1, \ldots, p\} \setminus M \) the \( r_M \)'s and the \( A_M \)'s are given as

\[
r_M = c_0 \prod_{j \in \overline{M}} r_j,
\]

\[
A_M = \prod_{\alpha \in \{1, \ldots, r\}} (r_j - \delta_\alpha) \prod_{\beta \in \{1, \ldots, \ell\}} (\rho_{\beta} - r_i) \prod_{\alpha \in \{1, \ldots, \ell\}} (\rho_{\beta} - \delta_\alpha)^{-1} \prod_{j \in \overline{M}} (r_j - r_i)^{-1}.
\]

In order to apply this theorem the \( \phi \) of our example is rewritten as

\[
\phi = c_0 \frac{(z - r_1)(z - r_2)(z - r_3)}{(1 - \frac{z}{\rho_1})(1 - \frac{z}{\rho_2})(z - \delta_1)}
\]

i.e.,

\[
c_0 = -r, r_1 = re^{i\theta}, \quad \rho_2 = \frac{1}{re^{-i\theta}}, \quad \delta_1 = r.
\]

Thus, \( p = 3, h = 2 \) and \( \ell = 1 \). Accordingly, the set \( M \) is either \{1\}, \{2\} or \{3\} and we have

\[
A_{\{1\}} r_{\{1\}}^{n+1} = (r_2 - \delta_1)(r_3 - \delta_1)(\rho_1 - r_1)(\rho_2 - r_1) \\
\cdot (\rho_1 - \delta_1)^{-1}(\rho_2 - \delta_1)^{-1}(r_2 - r_1)^{-1}(r_3 - r_1)^{-1}(r_2 r_3)^{n+1}.
\]

\[
A_{\{2\}} r_{\{2\}}^{n+1} = (r_1 - \delta_1)(r_3 - \delta_1)(\rho_1 - r_2)(\rho_2 - r_2) \\
\cdot (\rho_1 - \delta_1)^{-1}(\rho_2 - \delta_1)^{-1}(r_1 - r_2)^{-1}(r_3 - r_2)^{-1}(r_1 r_3)^{n+1}.
\]

\[
A_{\{3\}} r_{\{3\}}^{n+1} = (r_1 - \delta_1)(r_2 - \delta_1)(\rho_1 - r_3)(\rho_2 - r_3) \\
\cdot (\rho_1 - \delta_1)^{-1}(\rho_2 - \delta_1)^{-1}(r_1 - r_3)^{-1}(r_2 - r_3)^{-1}(r_1 r_2)^{n+1}.
\]

A straightforward substitution in the Day formula now gives

\[
A_{\{1\}} r_{\{1\}}^{n+1} + A_{\{2\}} r_{\{2\}}^{n+1} = (r_3 - \delta_1)(\rho_1 - \delta_1)^{-1}(\rho_2 - \delta_1)^{-1}(r_1 - r_2)^{-1}(r_1 r_2)^{n+1} \\
\cdot \frac{(1 - r^2)}{|1 - r^2 e^{i\theta}|^2} \cdot [\left( e^{i\theta} - 1 \right) (1 - r^2 e^{2i\theta}) (1 - r^2 e^{-2i\theta}) e^{i(n+1)\theta} \\
- \left( e^{-i\theta} - 1 \right) (1 - r^2 e^{-i\theta}) (1 - r^2 e^{i\theta}) e^{i(n+1)\theta} ] \\
= (r_3 - \delta_1)(\rho_1 - \delta_1)^{-1}(\rho_2 - \delta_1)^{-1}(r_1 - r_2)^{-1}(r_1 r_2)^{n+1} \\
2 \cdot \frac{(1 - r^2)}{|1 - r^2 e^{i\theta}|^2} \Im B_n(\theta, r).
\]

Hence there are non-zero constants \( C_1 \) and \( C_2 \) so that \( D_n(T_{\theta}) \) has the form

\[
D_n(T_{\theta}) = [C_1 \Im B_n(\theta, r) + C_2 r^{2(n+1)}] (-r)^{n+1}.
\]

Since \( 0 < r < 1 \) and \( \Im B_n(\theta, r) \) assumes only finitely many values, it follows that \( D_n(T_{\theta}) \neq 0 \) provided that \( n \gg 0 \).

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