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CONSTANT MEAN CURVATURE SURFACES IN $M^2 \times \mathbf{R}$

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Abstract. The subject of this paper is properly embedded $H-$surfaces in Riemannian three manifolds of the form $M^2 \times \mathbf{R}$, where M^2 is a complete Riemannian surface. When $M^2 = \mathbf{R}^2$, we are in the classical domain of $H-$surfaces in $\mathbf{R}^3$. In general, we will make some assumptions about M^2 in order to prove stronger results, or to show the effects of curvature bounds in M^2 on the behavior of $H-$surfaces in $M^2 \times \mathbf{R}$.

1. Introduction

There is an enormous difference between the theory of minimal surfaces ($H = 0$) and nonzero constant-mean curvatures surfaces ($H \neq 0$), and this is most evident in $\mathbf{R}^3$. A properly embedded minimal surface in $\mathbf{R}^3$ cannot be compact; if it is simply connected, it must be the plane or the helicoid; and if it is an annulus, it must be the catenoid [15, 3]. For Σ, a properly embedded surface of nonzero constant mean curvature in $\mathbf{R}^3$: if Σ is compact, it must be a round sphere; if Σ is noncompact, it cannot be simply connected; and, if $\Sigma \sim S^1 \times \mathbf{R}$, it must be rotationally symmetric (a Delaunay surface). For both $H = 0$ and $H \neq 0$, there are many known examples of finite topology with genus greater than zero and more than two ends, all discovered in the last twenty years [6, 7, 27, 26, 28].

The theory of minimal surfaces in $M^2 \times \mathbf{R}$ is now well developed from the point of view of examples and theory [17, 21, 16]. However, the theory of $H-$surfaces in $M^2 \times \mathbf{R}$ is just beginning. (We will assume throughout, unless we say so explicitly, that $H \neq 0$.) With one exception, there are no general theorems in the literature, but there is useful information in some special cases. The exception is when $M^2 = T^2$ with the flat metric. Here we are considering doubly periodic $H-$surfaces in $\mathbf{R}^3$, and there are existence results that come from the theory of triply periodic $H-$surfaces in $\mathbf{R}^3$ [9, 10, 5]. There are many known examples. Additional examples come from the existence of doubly periodic $H-$surfaces in $\mathbf{R}^3$ [12].

When $M^2 = S^2$, there is a class of examples due to Hsiang and Hsiang, [8], and Pedrosa and Ritoré, [20], who studied the isoperimetric problem in $S^n \times \mathbf{R}$. Among

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other things, they analyzed the rotationally-invariant $H-$surfaces in $S^n \times \mathbf{R}$. Let

$$B = \frac{S^n \times \mathbf{R}}{O(n)} = \{(x,y)|\ x \in [0,\pi],\ y \in \mathbf{R}\},$$

and consider the generating curves $\gamma(s) = (x(s), y(s))$ satisfying

$$\gamma'(s) = (\cos\sigma, \sin\sigma),$$

where $\sigma = \sigma(s)$ satisfies

$$\frac{d\sigma}{ds} = H + (n-1)\cot(x)\ \sin(\sigma).$$

Their $O(n)-$orbits are $H-$surfaces. Pedrosa and Ritoré found first integrals and solutions of Delaunay-type, i.e. properly embedded and periodic $H-$hypersurfaces that were annular (topologically $S^{n-1} \times \mathbf{R}$). They also found closed profile curves that give rise to compact tori (topologically $S^1 \times S^{n-1}$) that are unstable, and bounded profile curves that define examples that are hyperspheres.

A motivating question for this paper is the following: *Are there properly embedded annular $H-$surfaces in $S^2 \times \mathbf{R}$ that are not of Delaunay type (i.e., rotational)?* Korevaar, Kusner and Solomon proved that, in $\mathbf{R}^3$, all such surfaces are Delaunay [11].

The authors thank Rick Schoen for useful conversations concerning the rescaling argument used in the proof of Theorem 3 in Section 5.

2. Height estimates in $M^2 \times \mathbf{R}$

For an interval, $I \subset \mathbf{R}$, we define $M_I := M^2 \times I \subset M^2 \times \mathbf{R}$. When $I = \{a\}$, we will write $M^2_a = M^2 \times \{a\}$.

We prove height estimates for compact embedded $H-$surfaces Σ with boundary in some M^2_a. The estimates depend on curvature bounds for M^2 and on the value of H. We also prove that when M^2 is compact and Σ is a noncompact and embedded $H-$surface, that Σ has both a top and a bottom end.

Proposition 1. *Suppose Σ is a compact and embedded $H-$surface in $N = M_{[a,\infty)}$ with boundary in M^2_a. If the Gauss curvature of M^2 satisfies $K_M \geq 2\tau$ $(\tau \leq 0)$ and $H^2 > |\tau|$, then*

$$\Sigma \subset M_{[a,a+2c]}, \tag{1}$$

where $c = \frac{H}{H^2-|\tau|}$. In particular, if the sectional curvature of M^2 is nonnegative and $H \neq 0$, then

$$\Sigma \subset M_{[a,a+\frac{2}{H}]}. \tag{2}$$

Our proof of Proposition 1 (and of Proposition 2 below) uses a height estimate for $H-$surfaces that are graphs in $M^2 \times \mathbf{R}$ with zero boundary values over compact domains in M^2_0. Let $h : \Sigma \to \mathbf{R}$ be the restriction of the projection $t : M^2 \times \mathbf{R} \to \mathbf{R}$, to Σ. We refer to h as the height function.

Lemma 1. *Suppose that the Gauss curvature of M^2 satisfies $K_M \geq 2\tau$ $(\tau \leq 0)$. Let W be an $H-$surface with $H^2 \geq |\tau|$, which is a graph over a compact region in M^2_0, with zero boundary values. Then*

$$h \leq \frac{H}{H^2 - |\tau|} \tag{3}$$

on W. In particular, if the Gauss curvature of M^2 is nonnegative and $H \neq 0$, then

$$h \leq \frac{1}{H}. \tag{4}$$

We will provide the proof of Lemma 1 at the end of the section.

Proof of Proposition 1. Observe that translation by t_0 (i.e. $M^2_t \to M^2_{t+t_0}$) is an isometry of $N = M^2 \times \mathbf{R}$. Therefore, without loss of generality, we may assume that there exists $b > 0$ for which $\Sigma \subset M_{[0,b]}$, with $\partial\Sigma \subset M^2_0$ and $\Sigma \cap M^2_b \neq \emptyset$ and $\Sigma \cap M^2_s = \emptyset$ for $s > b$. To prove (1), we must show that $\frac{b}{2} < c$.

For $|b-t|$ small, $\Sigma \cap M_{[t,b]}$ is a vertical graph over a compact domain in M^2_0. By the Alexandrov reflection technique, the part of Σ in $M_{[\frac{b}{2},b]}$ is also a graph over a compact domain in M^2_0. To see this, first observe that since reflection in M_t cannot be a symmetry of Σ for $t > \frac{b}{2}$, one can do Alexandrov reflection through all the M^2_τ, as τ decreases from b to $\frac{b}{2}$, and no accident occurs. That is, the part of Σ above each M^2_τ is a vertical graph for $\frac{b}{2} \leq \tau \leq b$. By Lemma 1, $\frac{b}{2} < c$, which gives (1). The second statement, (2), follows immediately from the first. □

We will say that a surface in $M^2 \times \mathbf{R}$ *lies in a halfspace* if it is contained in a region of $M^2 \times \mathbf{R}$ of the form

$$M_{[a,\infty)} = \{(p,t) \in M^2 \times \mathbf{R} \mid t \geq a\} \quad \text{or} \quad M_{(-\infty,a]} = \{(p,t) \in M^2 \times \mathbf{R} \mid t \leq a\}.$$

Proposition 2. *Suppose M^2 is a compact surface without boundary whose Gauss curvature is bounded below by 2τ, for some real number τ, and suppose Σ is a noncompact properly embedded H surface in $N = M^2 \times \mathbf{R}$. If $\tau < 0$, assume that $H^2 \geq |\tau|$. Then Σ cannot lie in a halfspace. In particular, Σ must have at least one "top" and one "bottom" end.*

Proof. We first prove that $H \neq 0$ if Σ lies in a halfspace. This is because we can find an M^2_t tangent to Σ at a point, with M^2 lying on one side of Σ. By the maximum principle we would have $M^2_t = \Sigma$, a contradiction since M^2 is compact and Σ is not. This means that we satisfy the conditions of Lemma 1 even when M^2 has nonnegative curvature.

Observe that translation by t_0 (i.e., $M^2_t \to M^2_{t+t_0}$) and reflection R_{t_0} in $M^2_{t_0}$ (i.e., $M^2_t \to M^2_{2t_0-t}$) are isometries of $N = M^2 \times \mathbf{R}$. Therefore, without loss of generality, we may assume that Σ lies in the halfspace $M_{[0,\infty)}$ and has a nonempty intersection with M_0.

Since $\Sigma_a := \{(x,t) \in \Sigma | \ t \leq \alpha\}$ is compact, we may do Alexandrov-reflection coming up from M^2_0. For any value of a, $R_a(\Sigma_a)$ is compact. Since Σ is not compact, R_a is not a symmetry of Σ. This implies that for every $a > 0$, the compact surface Σ_a must be a vertical graph over a domain in M_a. However, the hypotheses of this proposition on the curvature of M^2 and the lower bounds for H allow us to use Lemma 1 for such a vertical graph, providing a height estimate that is violated for $a > 0$ sufficiently large. This contradiction establishes the first statement of the proposition. The second statement is an immediate consequence of the first. □

Proof of Lemma 1. We begin with some necessary general observations. For any immersion $X : \Sigma \to N$ of one Riemannian manifold into another, the Laplacian of

X is the divergence of the vector-valued one form DX

$$\Delta X := \sum_{i=1}^{m} \left[\nabla_{DX(e_i)} DX(e_i) - DX(\hat{\nabla}_{e_i} e_i) \right],$$

where ∇ and $\hat{\nabla}$ are the Riemannian connections on N and Σ, respectively, and the $\{e_i\}$ are an orthonormal framing of $T\Sigma$. If X is an isometric immersion, then $DX(\hat{\nabla}_{e_i} e_i) = [\nabla_{DX(e_i)} DX(e_i)]^t$, where $[\ ]^t$ is the projection of $TN_{|X(\Sigma)}$ onto $X_*(T\Sigma)$. Then

$$\Delta X = \sum_{i=1}^{m} [\nabla_{DX(e_i)} DX(e_i)]^n = \sum_{i=1}^{m} B(e_i, e_i),$$

where $[\ ]^n$ is projection onto the normal bundle of $X(\Sigma) \subset N$ and B is the second fundamental form of X. [13] Therefore, the Laplacian of X is the trace of the second fundamental form of Σ,

$$\Delta X = m\vec{H},$$

where m is the dimension of Σ and $\vec{H}$ is the mean curvature vector field of the isometric immersion X.

In our context, $N = M^2 \times \mathbf{R}$. Let $\vec{n}$ denote the unit normal to Σ and define

$$H := \langle \vec{H}, \vec{n} \rangle \text{ and } n := \langle \vec{n}, \frac{\partial}{\partial t} \rangle.$$

We have

$$\Delta X = 2H\vec{n}.$$

A simple calculation using either the fact that the height function, h, is the projection onto a one-dimensional subspace in a Riemannian splitting, or that $\frac{\partial}{\partial t}$ is a Killing field, gives

$$\Delta h = 2Hn. \tag{5}$$

We will also have need of an equation for the Laplacian of $n = \langle \vec{n}, \frac{\partial}{\partial t} \rangle$:

$$\Delta n = -(|A|^2 + Ric(\vec{n}))n, \tag{6}$$

where A is the shape operator of Σ. (Recall that $A : T\Sigma \to T\Sigma$ with $A(U) = \nabla_U \vec{n}$.) Formula (6) can be derived as in [22] or [18] by looking at the second variation of area of an $H-$surface in the direction of a normal component of a Killing field (in this case $\frac{\partial}{\partial t}$). Alternatively, one can derive this equation, directly using the fact that $\frac{\partial}{\partial t}$ is a Killing field. From general considerations,

$$\Delta n = \Delta \langle \vec{n}, \frac{\partial}{\partial t} \rangle = \langle \Delta \vec{n}, \frac{\partial}{\partial t} \rangle + 2\langle \nabla \vec{n}, \nabla \frac{\partial}{\partial t} \rangle + \langle \vec{n}, \Delta \frac{\partial}{\partial t} \rangle.$$

Because $\frac{\partial}{\partial t}$ is Killing, $\langle \nabla_U \frac{\partial}{\partial t}, V \rangle$ is skew symmetric, which is enough to show that the middle term on the right-hand side of the above expression vanishes. It also allows expression of the last summand in terms if $Ric(\vec{n}, \frac{\partial}{\partial t})$. Simplification leads to (6).

Now we restrict our attention to an $H-$surface W that satisfies the assumptions of the lemma. Define

$$\phi = ch + n, \tag{7}$$

where c is a real-valued positive constant. Since W is a graph we may choose a "downward-pointing" unit normal vector field $\vec{n}$ so that

$$n := \langle \vec{n}, \frac{\partial}{\partial t} \rangle \leq 0$$

on W. Since W is assumed to be a graph with zero boundary values, it follows that

$$\phi = n \leq 0$$

on ∂W. If we can find a value of c for which $\Delta\phi \geq 0$ on W, it will follow from the maximum principle that $\phi = ch + n \leq 0$ on W; hence

$$h \leq -n/c \leq 1/c. \tag{8}$$

We will now find such a value of c. From (5) and (6), we get

$$\Delta\phi = (2cH - |A|^2 - Ric(\vec{n}))n.$$

Since $n \leq 0$, we can assert that $\Delta\phi \geq 0$ if and only if

$$|A|^2 + Ric(\vec{n}) \geq 2cH. \tag{9}$$

Using the simple estimate

$$\begin{aligned} |A|^2 &= tr^2(A) - 2det(A) \\ &= 2(\frac{tr(A)}{2})^2 + (\frac{1}{2}tr^2(A) - 2det(A)) \\ &= 2H^2 + \frac{1}{2}(a_{11} - a_{22})^2 + 2a_{12}^2 \\ &\geq 2H^2, \end{aligned}$$

it suffices to choose c so that

$$H^2 + \frac{1}{2}Ric(\vec{n}) \geq cH.$$

Since $Ric(\vec{n}) = |Pr(\vec{n})|^2 K_M$, where $Pr(\ )$ is the projection from TN onto TM^2, and $|n| = 1$, it follows from our assumption that $K_M \geq 2\tau$, $\tau \leq 0$, and that $Ric(\vec{n}) \geq 2\tau$. Therefore we can satisfy the condition on c above with

$$\frac{H^2 - |\tau|}{H} \geq c. \tag{10}$$

The left-hand side of (10) is positive by assumption and so choosing c to equal the left-hand side is sufficient to force $\Delta\phi \geq 0$. The height estimate (3) follows from (8).

If M^2 has nonnegative sectional curvature, then $Ric(N) \geq 0$, the requirement (10) holds for $c = H$, and (8) gives the height estimate (4). □

Remark 1. The height estimate in Lemma 1 in the case that $M^2 \times \mathbf{R} = H^2 \times \mathbf{R}$ works for $H > \frac{1}{\sqrt{2}}$. However the rotational $H-$surfaces in $H^2 \times \mathbf{R}$ are compact for $H > 1/2$, and noncompact for $H \leq \frac{1}{2}$, [17]. So one suspects our height estimates are not sharp.

3. First variations of area and volume

Let U be bounded domain in a Riemannian three manifold N, whose boundary, ∂U, consists of a smooth connected surface, Σ, and the union Q of finitely many smooth, compact and connected surfaces. The closed surface ∂U is piecewise-smooth and smooth except perhaps on $\partial\Sigma = \partial Q$. Let

$$\begin{aligned} \vec{n} &= \text{ the outward-pointing unit normal vector field on } \partial U = \Sigma \cup Q; \\ \vec{n}_\Sigma &= \text{ the restriction of } \vec{n} \text{ to } \Sigma; \\ \vec{n}_Q &= \text{ the restriction of } \vec{n} \text{ to } Q. \end{aligned}$$

Suppose Y is a vector field defined on a region of N that contains U. The first-variation of the volume, $|U|$, of U is given by

$$\begin{aligned} \delta_Y|U| = \int_U DivY &= \int_{\partial U} Y \cdot \vec{n} \\ &= \int_\Sigma Y \cdot \vec{n}_\Sigma + \int_Q Y \cdot \vec{n}_Q, \end{aligned} \tag{11}$$

where $Div = Div_N$ is the divergence operator on N. To write down $\delta_Y(|\Sigma|)$, the first variation of area of Σ under Y, we introduce the following notation:

$$\begin{aligned} \nu &= \text{ the outward-pointing unit conormal to } \Sigma \text{ along } \partial\Sigma; \\ Y^t &= \text{ the tangential projection of } Y \text{ onto } T\Sigma; \\ Y^n &= Y - Y^t, \text{ the projection of } Y \text{ onto the normal bundle of } \Sigma \text{ in } N. \end{aligned}$$

We may then write

$$\begin{aligned} \delta_Y(|\Sigma|) &= \int_\Sigma divY \\ &= \int_\Sigma divY^t + \int_\Sigma divY^n \\ &= \int_{\partial\Sigma} Y \cdot \nu - \int_\Sigma Y \cdot \vec{H}_\Sigma, \end{aligned}$$

where $div = div_\Sigma$, is the divergence operator on Σ.

If Σ is an $H-$surface, $H := \langle \vec{H}, \vec{n} \rangle$ is a constant and therefore

$$\delta_Y(|\Sigma|) = \int_{\partial\Sigma} Y \cdot \nu - H \int_\Sigma Y \cdot \vec{n}. \tag{12}$$

Putting (11) and(12) together, we have for $H-$surfaces,

$$\delta_Y(|\Sigma| + H|U|) = \int_{\partial\Sigma} Y \cdot \nu + H \int_Q Y \cdot \vec{n}_Q. \tag{13}$$

The following propostion is well known and is immediate from (13).

Proposition 3. *If Y is a Killing vector field on N, and Σ is an $H-$surface, then*

$$\int_{\partial\Sigma} Y \cdot \nu + H \int_Q Y \cdot \vec{n}_Q = 0, \tag{14}$$

where Q, ν,H and $\vec{n}_Q$ are as defined above.

Suppose Σ is a properly embedded $H-$surface in $N = M^2 \times \mathbf{R}/$, and assume that Σ bounds a region $U \subset N$. Define:

$$\begin{aligned} M_t &= M^2 \times \{t\}; \quad (15)\\ \Sigma_t &= \Sigma \cap M_t;\\ \Sigma_{[a,b]} &= \Sigma \cap (M^2 \times [a,b]) = \bigcup_{a \le s \le b} \Sigma_s\,;\\ U_a^b &= U \cap (M^2 \times [a,b]); \quad (16)\\ Q_t &= U \cap M_t \end{aligned}$$

Clearly,

$$\partial U_a^b = \Sigma_{[a,b]} \cup Q_a \cup Q_b, \quad \partial\Sigma = \Sigma_a \cup \Sigma_b, \quad \vec{n}_{Q_b} = +\frac{\partial}{\partial t}, \quad \vec{n}_{Q_a} = -\frac{\partial}{\partial t}.$$

We now apply Proposition 3 to the vector field $\frac{\partial}{\partial t}$. Since vertical translation $M_t \to M_{t+t_0}$ is an isometry, $\frac{\partial}{\partial t}$ is a Killing field and we have from (14)

$$\begin{aligned} 0 &= \int_{\Sigma_0 \cup \Sigma_t} \frac{\partial}{\partial t} \cdot \nu + H \int_Q \frac{\partial}{\partial t} \cdot \vec{n}_Q \quad (17)\\ &= \int_{\Sigma_0} \frac{\partial}{\partial t} \cdot \nu + \int_{\Sigma_t} \frac{\partial}{\partial t} \cdot \nu + H(|Q_t| - |Q_0|). \end{aligned}$$

We will state an important consequence of this computation as a proposition for later use.

Proposition 4. *Suppose Σ is a properly embedded $H-$surface in $N = M^2 \times \mathbf{R}$, M^2 compact. Then the* vertical flux across Σ_t*,*

$$\int_{\Sigma_t} \frac{\partial}{\partial t} \cdot \nu,$$

varies within a bounded range

The proposition follows immediately from (17) and the assumption that M^2 is compact: since $Q_t \subset M_t$, the volume $|Q_t|$ is bounded independent of t.

Remark 2. If Σ is a minimal surface ($H = 0$), then it follows from (17) that the $\int_{\Sigma_t} \frac{\partial}{\partial t} \cdot \nu$, the vertical flux across Σ_t, is a constant, independent of t.

4. Linear area growth

In this section, we prove that a properly embedded $H-$surface in $N = S^2 \times \mathbf{R}$ will have linear area growth provided it has one additional property. For any choice of antipodal points on S^2, consider the rotations of S^2 that fix these two points. These extend to rotations of N whose common fixed point set consists of the two vertical geodesics through these points. Choose a geodesic arc γ joining these antipodal points and let $\mathcal{P}$ be the union of $\gamma \times \mathbf{R}$ and any tubular neigborhoods of the two vertical geodesics.

Theorem 1. *Let Σ be a properly embedded H-surface in $N = S^2 \times \mathbf{R}$. If Σ is disjoint from $\mathcal{P}$, then Σ has linear area growth.*

A generalization of this theorem is given in Section 4.1.

Remark 3. In the complement of $\mathcal{P}$ we may define θ, the angle measured from γ. The gradient of θ is a Killing field whose length is bounded above by 1, and below by a positive constant that depends on the radius of the tubular neigborhoods chosen in the definition of $\mathcal{P}$.

Proof. Given $a > 0$, we will show that

$$|\Sigma_{[-a,a]}| \leq ca, \tag{18}$$

for some constant c that does not depend on a.

Note that $(x,t) \to (x,-t)$ is an isometry of N, and therefore

$$|\Sigma_{[-a,a]}| = |-\Sigma_{[-a,a]}|,$$

where $-\Sigma = \{(x,-t)|\ (x,t) \in \Sigma\}$. Hence, by working with $-\Sigma$ if necessary, we may assume, without loss of generality, that

$$|\Sigma_{[-a,0]}| \leq |\Sigma_{[0,a]}|.$$

Since $(x,t) \to (x, t+a)$ is an isometry of N, the inequality (18) will follow if we establish

$$|\Sigma_{[0,2a]}| \leq ca, \text{ assuming } |\Sigma_{[0,a]}| \leq |\Sigma_{[a,2a]}|. \tag{19}$$

We will now prove (19).

On $S^2 - \gamma$, we may define polar coordinates, (r, θ), with $\pi < \theta < 3\pi$. The function θ is well defined on $N - \mathcal{P}$, a region that contains Σ by assumption.

Let $\frac{\partial}{\partial\theta}$ be the rotational Killing field for the axis defined by the end-points of γ, and let $\frac{\partial}{\partial t}$ be the Killing field defined by vertical translation. Clearly, $|\frac{\partial}{\partial t}| = 1$ and, as noted in Remark 3, $|\frac{\partial}{\partial\theta}|$ is bounded above and also bounded away from zero. Define the vector fields

$$X = t\frac{\partial}{\partial t}, \qquad Y = \theta\frac{\partial}{\partial\theta}, \qquad Z = tY.$$

A direct calculation using

$$divW = DivW - \langle\nabla_n W, n\rangle, \quad Div fW = \langle\nabla f, W\rangle + fDivW,$$

and the fact that for a Killing field V, $\langle\nabla_e V, e\rangle = 0$ and therefore

$$DivV = divV = 0\ , \text{ and } \ \langle\nabla_n V, n\rangle = 0,$$

gives

$$\begin{aligned} DivX &= 1, & divX &= 1 - n_t^2, \\ DivY &= |\tfrac{\partial}{\partial\theta}|^2, & divY &= |\tfrac{\partial}{\partial\theta}|^2 - n_\theta^2, \\ DivZ &= t|\tfrac{\partial}{\partial\theta}|^2, & divZ &= t(|\tfrac{\partial}{\partial\theta}|^2 - n_\theta^2) - \theta n_t n_\theta, \end{aligned} \tag{20}$$

where $n_t = \vec{n_\Sigma}\cdot\frac{\partial}{\partial t}$ and $n_\theta = \vec{n_\Sigma}\cdot\frac{\partial}{\partial\theta}$. Using X as the vector field in the first variation formula (13) on $\Sigma_{[0,2a]}$, we have

$$\begin{aligned} \int_{\Sigma_{[0,2a]}} (1 - n_t^2) + \ H|U_0^{2a}| &= \int_{\Sigma_0\cup\Sigma_{2a}} t\frac{\partial}{\partial t}\cdot\nu + H\int_{Q_0\cup Q_{2a}} t\frac{\partial}{\partial t}\cdot\vec{n_Q} \\ &= 2a\left[\int_{\Sigma_{2a}} \frac{\partial}{\partial t}\cdot\nu + H\int_{Q_{2a}} \frac{\partial}{\partial t}\cdot\vec{n_Q}\right] \\ &= 2a\left[\int_{\Sigma_{2a}} \frac{\partial}{\partial t}\cdot\nu + H|Q_{2a}|\right]. \end{aligned}$$

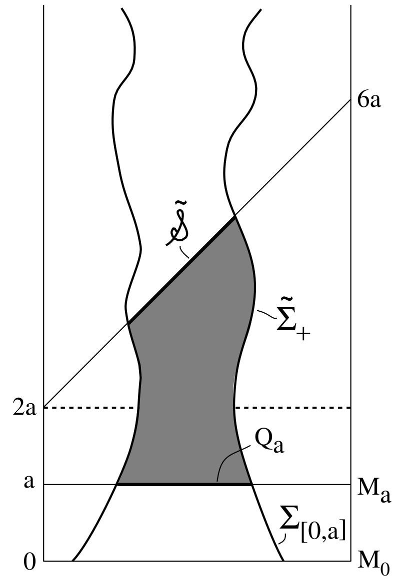

FIGURE 1. $\tilde{\Sigma}$.

By Proposition 3, equation (14), the bracketed term on the right-hand side is a constant. Moreover, since S^2 has finite area, $|U_0^{2a}|$ grows at most linearly in a. Hence

$$\int_{\Sigma_{[0,2a]}} 1 - n_t^2 \leq c_1 a, \tag{21}$$

where c_1 does not depend on a.

We now use (13) with the variation vector field Z. This time we will cut off Σ below by $S_0^2 = S^2 \times \{0\}$, and above by a surface, $\mathcal{S}$, defined by

$$t\theta = 6\pi a.$$

Note that when $t = 2a$, we have $\theta = 3\pi$, and when $t = 6a$, we have $\theta = \pi$. (See Figure 1.) We will denote the parts of Σ and U between S_0^2 and $\mathcal{S}$ by $\tilde{\Sigma}$ and $\tilde{U}$, respectively, and define $\tilde{\mathcal{S}} = \mathcal{S} \cap U$ and $\Sigma_{\mathcal{S}} = \Sigma \cap \mathcal{S}$. (See Figure 1, again.) We get from (13)

$$\int_{\tilde{\Sigma}} \left(t(|\frac{\partial}{\partial\theta}|^2 - n_\theta^2) - \theta n_t n_\theta \right) \ - \ H \int_{\tilde{U}} |\frac{\partial}{\partial\theta}|^2 t$$
$$= \int_{\Sigma_0 \cup \Sigma_{\mathcal{S}}} Z \cdot \nu + H \int_{Q_0 \cup \tilde{\mathcal{S}}} Z \cdot \vec{n_Q} = 6\pi a \left[\int_{\Sigma_{\mathcal{S}}} \frac{\partial}{\partial\theta} \cdot \nu + H \int_{\tilde{\mathcal{S}}} \frac{\partial}{\partial\theta} \cdot \vec{n_Q} \right].$$

Since $\frac{\partial}{\partial\theta}$ is a Killing field, it follows from Proposition 3, equation (14), that the bracketed term on the right is a constant that does not depend on a. Hence

$$\int_{\tilde{\Sigma}} t(|\frac{\partial}{\partial\theta}|^2 - n_\theta^2) = c_2 a + H \int_{\tilde{U}} t|\frac{\partial}{\partial\theta}|^2 + \int_{\tilde{\Sigma}} \theta n_t n_\theta.$$

Now observe that $\pi < \theta < 3\pi$ on $\tilde{\Sigma}$; the vector fields $\vec{n}$ and $\frac{\partial}{\partial t}$ both have length 1. Moreover, the Killing field $\frac{\partial}{\partial\theta}$ has length bounded above and is also bounded away from zero on $\tilde{\Sigma}$:

$$0 < c_* \leq |\frac{\partial}{\partial\theta}| \leq c^*. \tag{22}$$

Therefore

$$|\theta n_t n_\theta| \leq 3\pi n_t = 3\pi\langle n, \frac{\partial}{\partial\theta}\rangle \ \leq 3\pi c^* := c_4.$$

Also note that $0 \leq t \leq 6a$ on $\tilde{U}$. Using these estimates we have

$$\begin{aligned} \int_{\tilde{\Sigma}} t(|\frac{\partial}{\partial\theta}|^2 - n_\theta^2) &\leq c_2 a + c_3 a^2 + \int_{\tilde{\Sigma}} \theta n_t n_\theta \\ &\leq c_2 a + c_3 a^2 + c_4|\tilde{\Sigma}|. \end{aligned} \tag{23}$$

Here we have used the fact that S^2 has finite area and therefore the volume of $|\tilde{U}|$ is linear in a.

Let $\tilde{\Sigma}_+ = \tilde{\Sigma} \cap M_{[a,6a]}$. We are assuming that $|\Sigma_{[0,a]}| \leq |\Sigma_{[a,2a]}|$, and since $\Sigma_{[a,2a]} \subset \tilde{\Sigma}_+$,

$$|\tilde{\Sigma}| = |\Sigma_{[0,a]}| + |\tilde{\Sigma}_+| \leq 2|\tilde{\Sigma}_+|. \tag{24}$$

From (24), (23) and (22), we get

$$\begin{aligned} ac_*^2 \int_{\tilde{\Sigma}_+} (1 - (\frac{n_\theta}{|\frac{\partial}{\partial\theta}|})^2) &\leq \int_{\tilde{\Sigma}_+} t|\frac{\partial}{\partial\theta}|^2(1 - n_\theta^2) \\ &\leq \int_{\tilde{\Sigma}} t|\frac{\partial}{\partial\theta}|^2(1 - n_\theta^2) \leq c_2 a + c_3 a^2 + 2c_4|\tilde{\Sigma}_+|. \end{aligned}$$

Dividing by ac_*^2 gives

$$\int_{\tilde{\Sigma}_+} (1 - (\frac{n_\theta}{|\frac{\partial}{\partial\theta}|})^2) \leq \frac{1}{c_*^2}(c_2 + c_3 a + \frac{2c_4}{a}|\tilde{\Sigma}_+|).$$

In estimate (21), we saw that $\int_{\Sigma_{[0,2\hat{a}]}} 1 - n_t^2 \leq c_1\hat{a}$, where c_1 does not depend on $\hat{a}$. Using $\hat{a} = 3a$ and observing that $\tilde{\Sigma}_+ \subset \Sigma_{[0,6a]}$, we get (using the fact that ${n_t}^2 + (\frac{n_\theta}{|\frac{\partial}{\partial\theta}|})^2 \leq 1$)

$$|\tilde{\Sigma}_+| \leq \int_{\tilde{\Sigma}_+} (2 - ({n_t}^2 + (\frac{n_\theta}{|\frac{\partial}{\partial\theta}|})^2) \leq (\frac{1}{c_*^2}(c_2 + c_3 a) + 3c_1 a) + \frac{2c_4}{ac_*^2}|\tilde{\Sigma}_+| = c_5 + c_6 a + \frac{2c_4}{ac_*^2}|\tilde{\Sigma}_+|.$$

If $a > \frac{4c_4}{c_*^2}$, then

$$|\tilde{\Sigma}_+| \leq 2(c_5 + c_6 a).$$

This estimate together with (24) produces

$$|\tilde{\Sigma}| \leq 4(c_5 + c_6 a),$$

from which (19) follows because $\Sigma_{[0,2a]} \subset \tilde{\Sigma}$. This completes the proof of the theorem. □

4.1. **Linear area growth in $M^2 \times \mathbf{R}$ when M^2 has a pole.** The proof of Theorem 1 will give linear area growth in more general situations. Suppose M^2 has a pole, $\mathbf{p}$, a fixed point for a one-parameter group of rotational isometries. Assume there is a region $\mathcal{R} \subset N$, inside of which the angle of rotation is well defined and the gradient of that angle (a Killing field) has length bounded above and also away from zero. Let $\Sigma \subset \mathcal{R}$ be a properly embedded $H-$surface, and assume that Σ bounds a region $U \subset \mathcal{R}$ with the property that the area of $U_t = U \cap M_t^2$ is uniformly bounded. Then the proof of Theorem 1 is directly generalizable to a proof of the linear area growth of Σ. In particular

Theorem 2. *Suppose M^2 is a complete surface with a pole $\mathbf{p}$. The Killing field $\frac{\partial}{\partial\theta}$ is defined in a punctured neighborhood, say D, of $\mathbf{p}$, and the angle θ is defined in the complement in D of a geodesic, say γ, starting at $\mathbf{p}$ and running to the cut locus $C_{\mathbf{p}}$ of $\mathbf{p}$. Let $\mathcal{R} = (D \setminus \gamma) \times \mathbf{R}$, and suppose that $\Sigma = \partial U$, $U \subset \mathcal{R}$, is a properly embeddded $H-$surface. Suppose further that $|U_t| = |U \cap M_t^2|$ is bounded independent of t and that the Killing field $\frac{\partial}{\partial\theta}$ associated with the pole $\mathbf{p}$ satisfies the boundedness condition* (22)*. Then Σ has linear area growth.*

This theorem has an immediate corollary when $M^2 = H^2$.

Corollary 1. i) *Suppose $\gamma \subset H^2$ is a geodesic ray beginning at $\mathbf{p}$. Let $\Sigma = \partial U \subset \mathcal{R} = (H^2 \setminus \gamma) \times \mathbf{R}$ be a properly embedded $H-$surface with the property that $|U_t|$ is bounded independent of t. Suppose further that the Killing field $\frac{\partial}{\partial\theta}$ associated with the pole $\mathbf{p}$ satisfies the boundedness condition of* (22)*. Then Σ has linear area growth.*

ii) *Suppose $\alpha : \mathbf{R} \to H^2$ is either a constant map or a constant-speed $(= b)$ parametrization of a complete geodesic. Let $\beta(t) = (\alpha(t), ct)$, $c \neq 0$, $b^2 + c^2 = 1$, be a parametrization of a complete unit-speed geodesic $\beta \subset H^2 \times \mathbf{R}$. Suppose $\Sigma \subset H^2 \times \mathbf{R}$ is a properly embedded H-surface which is* cylindrically bounded *in the sense that it stays a bounded distance from β. Then Σ has linear area growth.*

Proof. The first statement follows directly from Theorem 2 and the fact that any point $\mathbf{p} \in H^2$ is a pole whose cut locus is empty.

To prove the second statement, we work in the disk model of H^2. We first observe that without loss of generality, we may assume that either $\alpha(t)$ is indentically equal to zero, or is a constant-speed parametrization (in H^2) of the real axis. Furthermore, we may assume that Σ is within distance one of β.

If $\beta(t) = (0, ct)$, choose $\mathbf{p}$ to be a point on the imaginary axis at distance $d > 3$ from the origin, and let γ be the geodesic ray beginning at $\mathbf{p}$ and diverging along the imaginary axis. Let $\mathcal{R} = B_2(0) \times \mathbf{R} \subset H^2 \times \mathbf{R}$. All the conditions of statement (i) are now satisfied, so Σ has linear area growth.

If α is the real axis and the cylindrical radius is one, we again choose γ as before. Let $\mathcal{B} = B \times \mathbb{R}$, where B is the geodesic disc in $\mathbb{H}^2$ centered at $\beta(0) = O$, and of radius 3. Clearly $\Sigma_{[0,c]} \subset \mathcal{B}$. Our previous calculations in Theorem 1 show that the area of $\Sigma_{[0,c]}$ is bounded by kc, where the constant k depends on the bounds for $|\partial_\theta|$ in $\mathcal{B}$. Observe that the same estimate holds for the area of $\Sigma_{[c,2c]}$, since there is an isometry ϕ of $\mathbb{H}^2 \times \mathbb{R}$ to itself, leaving β invariant, taking $\beta(1)$ to $\beta(0)$; ϕ^{-1} is the composition of the (hyperbolic) translation of $\mathbb{H}^2$, taking O to $\alpha(1)$, with the vertical translation taking $\alpha(1)$ to $\beta(1)$. This map takes $\beta'(0)$ to $\beta'(1)$, and hence leaves β invariant. Then $\phi(\Sigma_{[c,2c]}) \subset \mathcal{B}$, so its area is also at most $k\,c$. Continuing this way, we conclude Σ has linear (vertical) area growth. □

5. Curvature bounds

We will be concerned in this section with properly embedded $H-$surfaces, $\Sigma \subset N = M^2 \times R$, of finite topology with the additional property that the restriction of the height function on $M^2 \times \mathbf{R}$ to Σ is proper. If M^2 is compact, every such Σ has this property. Note that an annular end of such an $H-$surface must be either a top end or a bottom end.

We will assume that M^2 is complete, that M^2 has bounded Gauss curvature, and that the injectivity radius of M^2 is bounded away from zero. Again, all these properties hold if M^2 is compact.

Theorem 3. *Let Σ and M^2 satisfy the above conditions. If Σ has linear area growth, then $|A|$ is bounded on Σ.*

Corollary 2. *Suppose Σ is a properly embedded $H-$surface with finite topology in $S^2 \times \mathbf{R}$, with the additional property that $\Sigma \subset D \times R \subset S^2 \times \mathbf{R}$, where D is a geodesic disk of radius strictly less than π (or, more generally satisfies the conditions of Theorem 1). Then $|A|$ is bounded on Σ.*

The corollary follows immediately from Theorem 3, since S^2 is compact and Theorem 1 gives linear area growth. Similar results for $H-$surfaces $\Sigma \subset H^2 \times R$ on which the height function is proper follow from Theorem 3 and Corollary 1.

Theorem 3 will be proved after we present two preliminary results.

We begin by observing as we did in (10), that $|A|^2 = tr^2(A) - 2det(A) = 4H^2 - 2det(A)$ —a local formula not requiring any of our hypotheses—and that the Gauss curvature $K(p)$ of Σ at a point $p \in \Sigma$ satisfies $det(A_p) = K(p) - K_N(p)$, where $K_N(p)$ is the sectional curvature of N on the plane $T_p\Sigma$. Therefore

$$|A|^2 = 4H^2 + 2K_N - 2K. \tag{25}$$

The geometric quantity $4H^2 + 2K_N$ is bounded on Σ if M^2 has bounded curvature. Therefore

Lemma 2. *Let $\Sigma = \partial U$, $U \subset N = M^2 \times \mathbf{R}$, be a properly embedded $H-$surface. Assume Σ has linear area growth and that M^2 has bounded curvature. Then*

$$\int_\Sigma 4H^2 + 2K_N$$

has linear growth.

Proposition 5. *Let Σ be an $H-$surface in $M^2 \times \mathbf{R}$ satisfying the assumptions of Theorem 3. Then $\int_\Sigma |A|^2$ has linear growth.*

Remark 4. For the proof of this proposition, we do not require that the injectivity radius of M^2 be bounded below.

Proof. Since we are assuming that Σ has finite topology, we may decompose it into a compact piece and a finite number of annular ends. Let E be an annular end of Σ. We may assume, without loss of generality, that E is a top end, i.e., $h(x_n) \to \infty$, as $x_n \in E$ diverges in E, where h is the height function. Then if $h_{|\partial E} < 0$ and $t \geq 0$, E transverse to M_t^2, it is clear that $E \cap M_t^2$ contains at least one Jordan curve α that is a generator of $\pi_1(E)$. We call such an α an essential curve. We can assume $E \pitchfork M_0^2$ and (after replacing E by a subend) that $\partial E \subset M_0^2$. Note that E may intersect M_0^2 in other Jordan curves, but the part of E below height zero is compact and at bounded distance from M_0^2, by our assumption that h, restricted to Σ, is proper.

Let $\phi : \mathbf{S}^1 \times \mathbf{R}^+ \to E$ be a parametrization of E with the property that

$$\phi(\mathbf{S}^1 \times \{0\}) = \partial E \subset M_0.$$

For a, $b \in \mathbf{R}^+$ with $b > a$, we will assume that E is transverse to M_a^2 and M_b^2, an assumption that is true generically. The set $E_a = E \cap M_a^2$ consists of a finite number of loops, and the same is true of E_b.

The parametrization ϕ orders the essential loops by proximity to $\mathbf{S}^1 \times \{0\}$ and, in particular, it orders the essential loops in E_a and E_b. Let α_1 be the first essential loop in E_a and let β be the first essential loop in E_b. The annulus $\hat{E}$ bounded by α_1 and β is not necessarily contained in $E_{[a,b]}$. However, by Proposition 1 in Section 2, there exists a constant C (that depends on H and the bounds on the curvature of M^2, but not on t) such that regions of E bounded by inessential loops in M_t^2 must lie at a distance of at most C from M_t^2. Hence we may assert that

$$\hat{E} \subset E_{[a-C,b+C]}.$$

Using Proposition 1 again, we can assert that if $b - a > 2C$, any essential loop that is past $\beta \subset E_b$ in the ordering cannot lie in E_a. Let β_1 be the last essential loop in E_b. The loops α_1 and β_1 bound an annulus $\tilde{E} \subset E$, and we can use Proposition 1 again to assert that

$$E_{[a+C,b-C]} \subset \tilde{E} \subset E_{[a-C,b+C]}. \tag{26}$$

Since we are assuming that Σ has linear area growth, we know that $E \subset \Sigma$ has linear area growth. This implies that there exists a constant c such that for all $t \geq 0$,

$$|E_{[t,t+1]}| < c.$$

By the co-area formula we know that there exists, for every integer $k \geq 0$, an a_k, $k \leq a_k \leq k+1$, with the property that

$$|E_{a_k}| \leq c,$$

where $|E_{a_k}|$ is the length of E_{a_k}. Choose $k_0 > c + 2C$, let $k > k_0 + 1 + 2c + 2C$, and consider $\tilde{E}_{[a_0,a_k]}$, the annulus bounded by the essential curves $\alpha_1 \subset E_{a_0}$, and $\beta_1 \subset E_{a_k}$. Choose points $x \in \alpha_1$ and $y \in \beta_1$, and let $\overline{\alpha_1}$ (respectively $\overline{\beta_1}$) be the minimizing geodesics on E homotopic to α_1 relative to x (respectively homotopic to β_1 relative to y). Both $\overline{\alpha_1}$ and $\overline{\beta_1}$ cannot have length greater than c. Let $\overline{E}$ be the annulus in E bounded by $\overline{\alpha_1}$ and $\overline{\beta}_1$. The exterior angle of $\overline{\alpha}_1$ (with respect to $\overline{E}$) at x, and the exterior angle of $\overline{\beta}_1$ at y cannot be bigger than π in absolute value. It is an annulus because we have chosen k large enough to keep $\overline{\alpha}_1$ and $\overline{\beta}_1$ disjoint. By Gauss-Bonnet, the total curvature of $\overline{E}$ is at most 2π in absolute value. By the definition of $\overline{E}$,

$$E_* := E_{[a_0+c+C,a_k-c-C]} \subset \overline{E} \subset E_{[a_{k_0}-c-C,a_k+c+C]} =: E^*. \tag{27}$$

Let $c_0 = |4H^2 + sup_{M^2} K_N|$. Using (25), (27) and Lemma 2, we have

$$\begin{aligned}\int_{E_*} |A|^2 \leq \int_{\overline{E}} |A|^2 &= \int_{\overline{E}} (4H^2 + 2K_N) - 2\int_{\overline{E}} K \\ &\leq c_0|\overline{E}| + 4\pi \\ &\leq c_0|E^*| + 4\pi \\ &\leq c_0 c(a_k - a_0 + 2(C+c)) + 4\pi \\ &\leq c_0 c(k + 1 + 2(C+c)) + 4\pi \\ &\leq \mathit{constant} \times k,\end{aligned}$$

for k large. Hence $\int_{E_*} |A|^2$ has linear growth, which implies that $\int_E |A|^2$ also has this property. □

Now we present the

Proof of Theorem 3. Because Σ has finite topology, it suffices to show that $|A|$ is bounded on each annular end E of M. Assume that E is a top annular end of M, $\partial E \subset M_0 = M^2 \times \{0\}$. Recall that $E_{[a,b]}$ denotes $E \cap (M^2 \times [a,b])$ and $E_t = E \cap M_t$. We will assume $|A|$ is unbounded on E and obtain a contradiction.

Let $p_n \in E$ satisfy $|A(p_n)| \geq n$, $n = 1, 2, \ldots$, and let p_n be the lowest point of E with this property. For each integer n, let $m = m(n)$ be the smallest integer larger than $h(p_n) + 1$. For $q \in E_{[0,m]}$ let $d(q) = m - h(q)$.

Consider the continuous function $F = F_n$ defined on $E_{[0,m]}$:

$$F(q) = \begin{cases} |A(q)|, & \text{if} \quad h(q) \leq m-1, \\ |A(q)|\, d(q)^2, & \text{if} \quad h(q) \geq m-1. \end{cases} \tag{28}$$

Let $\tilde{q} = \tilde{q}_n \in E$ be a point where F achieves its maximum value. Clearly $|A(\tilde{q})| \geq n$.

Let $a_n = h(\tilde{q}) - \frac{d(\tilde{q})}{2}$, $\quad b(n) = h(\tilde{q}) + \frac{d(\tilde{q})}{2}$ and consider $\mathcal{E} = \mathcal{E}_n = E_{[a_n, b_n]}$, the part of E between heights a_n and b_n. If $b_n \leq m - 1$, then for each $p \in \mathcal{E}$,

$$|A(p)| \leq |A(\tilde{q})|.$$

If $b_n > m - 1$ and $h(\tilde{q}) < m$, then for each $p \in E_{[a_n,b_n]}$ we have $d(p) \geq \frac{1}{2}$. So, if $h(p) \geq m$, then $|A(p)|\, d(p)^2 \leq |A(\tilde{q})|$, hence

$$|A(p)| \leq 4|A(\tilde{q})|.$$

If $h(p) \leq m$, then $|A(p)| \leq |A(\tilde{q})|$. Similarly, the reader can easily check that this inequality also holds if $h(\tilde{q}) \geq m$; so in all cases, when $p \in \mathcal{E}$, one has

$$|A(p)| \leq 4|A(\tilde{q})|. \tag{29}$$

At this point, we have produced a sequence $\{\tilde{q}_n\}$ whose heights $h(\tilde{q}_n)$ diverge with n, and which have the property that $|A(\tilde{q}_n)| \geq n$. Moreover, on the subset $\mathcal{E}_n = E_{[a_n,b_n]} \subset E$, we have the estimate

$$|A(p)| \leq 4|A(\tilde{q}_n)|.$$

Since $p_n \in E$ was chosen to be a lowest point where $|A(p_n)| \geq n$, it follows that $d(\tilde{q}_n) \leq 2$, which in turn implies that $b_n - a_n = d(\tilde{q}) \leq 2$. Therefore, it follows from

Proposition 5 that

$$\int_{\mathcal{E}_n} |A|^2 \leq c, \tag{30}$$

where c does not depend on n.

For each $n \geq 0$, translate E down by $h(\tilde{q}_n)$. We may consider $\{\tilde{q}_n\}$ to be a sequence of points in M_0^2, and by abuse of notation, we will refer to these points by the same notation. We also consider the subsets $\mathcal{E}_n$ to be similarly translated and we similarly abuse notation. For each n, define $\lambda_n = |A(\tilde{q}_n)|$. Rescale the metric on $M^2 \times \mathbf{R}$ by λ_n and denote by $\mathcal{E}_n^*$ the surface $\mathcal{E}_n$ in the rescaled metric. The distance between the top and the bottom of $\mathcal{E}_n^*$ is

$$\lambda_n d(\tilde{q}_n) = |A(\tilde{q}_n)| d(\tilde{q}_n) \geq n.$$

We also have from (29) the estimate

$$|A^*(p)| \leq 4$$

on $\mathcal{E}_n^*$ and the normalization $|A(\tilde{q}_n)| = 1$, where $|A^*|$ denotes the second fundamental form in the rescaled metric. The integral in (30) is scale invariant:

$$\int_{\mathcal{E}_n^*} |A^*|^2 \leq c, \tag{31}$$

independent of n. Since we are assuming that Σ has linear area growth, it follows that E also has linear area growth, and this remains true in the rescaled metrics.

For each n, denote by U_n the simply connected neighborhood of the origin in $T_{\tilde{q}_n}(M^2 \times \mathbf{R})$, which corresponds via the exponential map to $M^2 \times \mathbf{R}$ minus the cut locus of $\tilde{q}_n$. We consider U_n with the metric of $M^2 \times \mathbf{R}$ rescaled by λ_n, and we denote this metric space by U_n^*. Since the injectivity radius of M^2 is bounded below, and the Gauss curvature of M^2 is bounded in absolute value, the U_n^* converge to $\mathbf{R}^2 \times \mathbf{R} = \mathbf{R}^3$ with the flat metric. (The distance from the origin to ∂U_n^* goes to infinity with n because the of the global lower bound on the injectivity radius and the fact that $\lambda_n \to \infty$. The curvature of U_n^* behaves like $\frac{const.}{\lambda_n^2}$ since there is a global bound on the curvature of M^2, and hence of $M^2 \times \mathbf{R}$.)

For each n, consider the part of $\mathcal{E}_n$ inside the cut locus of $\tilde{q}_n^*$, and let $\mathcal{E}_n^*$ be the pullback by the exponential map to U_n^* of the component of this surface containing the point $\tilde{q}_n^*$. The mean curvature of $\mathcal{E}_n^*$ is $\frac{H}{\lambda_n}$, which goes to zero as $n \to \infty$. We have established in (31) bounds for the integral of $|A^*|^2$. We are assuming linear growth for Σ, and this implies that $\mathcal{E}_n^*$ has linear area growth. Therefore, we may choose a subsequence of the $\mathcal{E}_n^*$ which converges—with finite multiplicity—to an embedded minimal surface, $\mathcal{S}$, in $\mathbf{R}^3$ with the flat metric. Moreover, $\mathcal{S}$ has $|A^*(0)| = 1$ and $\int_{\mathcal{S}} |A^*|^2$ is finite by (31). Since $|A|^2 = -2K$, $\mathcal{S}$ has finite total curvature, but is not the plane because $|A^*(0)| \neq 0$. It is complete because Σ was complete, and the distance from the origin to the boundary of Σ_n^* is not less than the distance from the origin to the boundary of U_n^*, which diverges with n.

In the previous paragraph we have shown that $\mathcal{S} \subset \mathbf{R}^3$ is a complete and nonplanar embedded minimal surface with finite total curvature. Such a surface must have a top and a bottom catenoidal end. Consider a top catenoidal end $\bar{\Sigma}$ of $\mathcal{S}$. Let γ be a Jordan curve on $\bar{\Sigma}$, which is the tranverse intersection of Σ with some plane orthogonal to the axis of the top catenoid end. The flux of $\bar{\Sigma}$ along γ is a vector whose component, a, in the direction of the axis of the catenoidal end, is equal to the logarithmic growth of the end. Since the end is catenoidal, $a \neq 0$.

Now $k\gamma = \lim \gamma_n$, where γ_n are Jordan curves on Σ_n^* and k is the (fixed) multiplicity with which the $\mathcal{E}_n^*$ (a subsequence) are converging to $\mathcal{S}$. The γ_n are coming from loops β_n on $E \subset \Sigma$ by scaling by λ_n with $\lambda_n \to \infty$. We know that this flux is zero if β_n is homologous to zero on E and equals some fixed vector when β_n is a generator of $\pi_1(E)$. Let b be the component of this vector in the direction determined by the top catenoid end. Then one has $\lambda_n b \to ka$, as $\lambda_n \to \infty$. Since b is finite, this is a contradiction. □

Remark 5. We conjecture that a properly embedded H-surface, Σ, of finite topology in $M^2 \times \mathbf{R}$ must have linear area growth if M^2 is compact. (This is true when $H = 0$, [16].) Using Theorem 3 and arguing as in Corollary 2, this conjecture, if true, implies that such a Σ would have bounded second fundamental form.

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