

POISSON STRUCTURES ON COMPLEX FLAG MANIFOLDS ASSOCIATED WITH REAL FORMS

PHILIP FOTH AND JIANG-HUA LU

Dedicated to Alan Weinstein on the occasion of his 60th birthday

ABSTRACT. For a complex semisimple Lie group G and a real form G_0 we define a Poisson structure on the variety of Borel subgroups of G with the property that all G_0 -orbits in X as well as all Bruhat cells (for a suitable choice of a Borel subgroup of G) are Poisson submanifolds. In particular, we show that every non-empty intersection of a G_0 -orbit and a Bruhat cell is a regular Poisson manifold, and we compute the dimension of its symplectic leaves.

1. INTRODUCTION

Let G be a connected and simply-connected complex semisimple Lie group with Lie algebra \mathfrak{g} , and let X be the variety of Borel subalgebras of \mathfrak{g} . In this paper we use a real form \mathfrak{g}_0 of \mathfrak{g} to define a Poisson structure on X . This Poisson structure depends on a choice of a Borel subalgebra \mathfrak{b} of \mathfrak{g} such that $\mathfrak{g}_0 \cap \mathfrak{b}$ is a maximally compact Cartan subalgebra of \mathfrak{g}_0 . Instead of dealing with each real form individually, we fix a Borel subalgebra \mathfrak{b} of \mathfrak{g} and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Then, as is shown in [6], a real form \mathfrak{g}_v of \mathfrak{g} can be constructed from each Vogan diagram v for \mathfrak{g} such that $\mathfrak{g}_v \cap \mathfrak{b}$ is a maximally compact Cartan subalgebra of \mathfrak{g}_v . The corresponding Poisson structure on X is denoted by Π_v .

Let G_v be the real form of G corresponding to \mathfrak{g}_v , and let B be the Borel subgroup of G with Lie algebra \mathfrak{b} . The Poisson structure Π_v has the property that each G_v -orbit as well as each B -orbit in X is a Poisson submanifold. The B -orbits in X will be referred to as the Bruhat cells. We compute the rank of Π_v . In particular, if a G_v -orbit \mathcal{O} meets a Bruhat cell \mathcal{C} , they intersect transversally, and we find that all the symplectic leaves in $\mathcal{O} \cap \mathcal{C}$ have the same dimension, so $\mathcal{O} \cap \mathcal{C}$ is a regular Poisson manifold. Moreover, we show that all symplectic leaves in each connected component of $\mathcal{O} \cap \mathcal{C}$ are translates of each other by elements of a Cartan subgroup of G_v . We also show that the G_v -invariant Poisson cohomology for each open G_v -orbit in X is isomorphic to the de Rham cohomology of X .

Results similar to those presented here for the full flag manifold $X = G/B$ are also valid for a partial flag manifold G/P , where P is a parabolic subgroup of

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G containing B . We will treat these more general cases as well as some further properties of Π_v in a future paper.

Throughout this paper, if V is a set and σ is an involution on V , we will use V^σ to denote the fixed point set of σ in V .

2. REAL FORMS OF \mathfrak{g} AND VOGAN DIAGRAMS

Let \mathfrak{g} be a complex simple Lie algebra. In this section we recall the classification of real forms of \mathfrak{g} by Vogan diagrams. Details can be found in [6, Chapter 6].

Suppose that \mathfrak{g}_0 is a real form of \mathfrak{g} and that τ_0 is the corresponding complex-conjugate linear involution on \mathfrak{g} . Let θ_0 be a Cartan involution of \mathfrak{g}_0 , and let \mathfrak{h}_0 be a θ_0 -stable maximally compact Cartan subalgebra of \mathfrak{g}_0 . Set $\mathfrak{t}_0 = \mathfrak{h}_0^{\theta_0}$ and $\mathfrak{a}_0 = \mathfrak{h}_0^{-\theta_0}$ so that $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$. Let γ_0 be the complexification of θ_0 . Then the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0 + i\mathfrak{h}_0$ of \mathfrak{g} is γ_0 -stable. Let Δ be the root system for $(\mathfrak{g}, \mathfrak{h})$. Since \mathfrak{h}_0 is a maximally compact Cartan subalgebra of \mathfrak{g}_0 , there exists $x_0 \in i\mathfrak{t}_0$ that is regular for Δ . Define the subset Δ^+ of positive roots in Δ by $\alpha \in \Delta^+$ if and only if $\alpha(x_0) > 0$. Then $\gamma_0(\Delta^+) = \Delta^+$. Let $\Sigma \subset \Delta^+$ be the set of simple roots in Δ^+ . Then $\gamma_0(\Sigma) = \Sigma$, so γ_0 gives rise to an involutive automorphism of the Dynkin diagram of \mathfrak{g} . Let \mathcal{I} be the set of non-compact imaginary simple roots. The Vogan diagram of \mathfrak{g}_0 associated to the triple $(\theta_0, \mathfrak{h}_0, \Delta^+)$ is the Dynkin diagram $D(\mathfrak{g})$ of \mathfrak{g} , together with an involutive automorphism γ_0 on $D(\mathfrak{g})$ and the vertices corresponding to the simple roots in \mathcal{I} painted black.

In general, a Vogan diagram for \mathfrak{g} is defined to be a triple $(D(\mathfrak{g}), d, \mathcal{I})$, where $D(\mathfrak{g})$ is the Dynkin diagram of \mathfrak{g} , d is an involutive automorphism of $D(\mathfrak{g})$, and \mathcal{I} is a subset of vertices of $D(\mathfrak{g})$ such that $d(\alpha) = \alpha$ for each $\alpha \in \mathcal{I}$. Every Vogan diagram for \mathfrak{g} comes from a real form of \mathfrak{g} (see below), although two different Vogan diagrams can come from isomorphic real forms. A non-redundant list of Vogan diagrams with the corresponding isomorphism class of real forms for all simple Lie algebras is given in [6]. Every Vogan diagram in the list in [6] is *normalized* in the sense that at most one vertex is painted black.

For the purpose of defining Poisson structures on the variety of Borel subalgebras of \mathfrak{g} , we now recall the explicit construction of a real form of \mathfrak{g} from a Vogan diagram [6, Theorem 6.88]. We need to fix the following data for \mathfrak{g} .

Choose a Cartan subalgebra \mathfrak{h} of \mathfrak{g} and let Δ be the root system for $(\mathfrak{g}, \mathfrak{h})$. Fix a choice of positive roots Δ^+ and let Σ be the basis of simple roots. Let $\langle\langle \cdot, \cdot \rangle\rangle$ be the Killing form of \mathfrak{g} and let root vectors $\{E_\alpha : \alpha \in \Delta\}$ be chosen such that $[E_\alpha, E_{-\alpha}] = H_\alpha$ for each $\alpha \in \Delta^+$, where H_α is the unique element of \mathfrak{h} defined by $\langle\langle H, H_\alpha \rangle\rangle = \alpha(H)$ for all $H \in \mathfrak{h}$, and such that the numbers $m_{\alpha, \beta}$ given by $[E_\alpha, E_\beta] = m_{\alpha, \beta} E_{\alpha+\beta}$ when $\alpha + \beta \in \Delta$ are real. Define a compact real form \mathfrak{k} of \mathfrak{g} as

$$\mathfrak{k} = \text{span}_{\mathbb{R}}\{iH_\alpha, X_\alpha := E_\alpha - E_{-\alpha}, Y_\alpha := i(E_\alpha + E_{-\alpha})\},$$

and let θ be the complex conjugation of \mathfrak{g} defining \mathfrak{k} . If d is an involutive automorphism of the Dynkin diagram of \mathfrak{g} , define γ_d to be the unique automorphism of \mathfrak{g} satisfying $\gamma_d(H_\alpha) = H_{d(\alpha)}$ and $\gamma_d(E_\alpha) = E_{d(\alpha)}$ for each simple root α .

Given a Vogan diagram v for \mathfrak{g} , not necessarily normalized, with the involutive diagram automorphism d , let t_v be the unique element in the adjoint group of \mathfrak{g} such that

$$\text{Ad}_{t_v}(E_\alpha) = \begin{cases} E_\alpha & \text{if } \alpha \text{ is a blank vertex in } v, \\ -E_\alpha & \text{if } \alpha \text{ is a painted vertex in } v. \end{cases}$$

Define a complex conjugate linear involution

$$\tau_v := \text{Ad}_{t_v} \circ \gamma_d \circ \theta.$$

Notation 2.1. We use $\mathfrak{g}_v = \mathfrak{g}^{\tau_v}$ to denote the real form of \mathfrak{g} defined by τ_v . Set $\theta_v = \theta|_{\mathfrak{g}_v}$. Then θ_v is a Cartan involution of \mathfrak{g}_v , and \mathfrak{h}^{τ_v} is a θ_v -stable maximally compact Cartan subalgebra of \mathfrak{g}_v , with $\mathfrak{h} = \mathfrak{h}^{\tau_v} + i\mathfrak{h}^{\tau_v}$. The complexification of τ_v is

$$(2.1) \quad \gamma_v := \tau_v \theta = \theta \tau_v = \text{Ad}_{t_v} \gamma_d.$$

Since $\gamma_v(\Delta^+) = \Delta^+$, the Vogan diagram of \mathfrak{g}_v associated to the triple $(\theta_v, \mathfrak{h}^{\tau_v}, \Delta^+)$ is v .

One of the advantages of introducing the real form \mathfrak{g}_v is as follows. We say that a real subalgebra \mathfrak{l} of \mathfrak{g} is *Lagrangian* if its real dimension is equal to the complex dimension of \mathfrak{g} and if $\text{Im}\langle x_1, x_2 \rangle = 0$ for all $x_1, x_2 \in \mathfrak{l}$. A decomposition $\mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$ is called a *Lagrangian splitting* if both \mathfrak{l}_1 and \mathfrak{l}_2 are Lagrangian. Let \mathfrak{n} be the subalgebra of \mathfrak{g} spanned by the set of all positive root vectors for Δ^+ . The following fact is easy to prove.

Lemma 2.2. *Let $\mathfrak{l}_d := \mathfrak{h}^{-\tau_v} + \mathfrak{n}$. Then $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$ is a Lagrangian splitting of \mathfrak{g} .*

Let $\mathfrak{a} = \text{span}_{\mathbb{R}}\{iH_\alpha : \alpha \in \Sigma\}$, and let $\mathfrak{t} = i\mathfrak{a}$. We note that since

$$\mathfrak{h}^{-\tau_v} = \mathfrak{h}^{-\gamma_d \circ \theta} = \mathfrak{t}^{-\gamma_d} + \mathfrak{a}^{\gamma_d},$$

the Lagrangian complement \mathfrak{l}_d of \mathfrak{g}_v depends only on d , and in the case when $d = 1$, we have $\mathfrak{l}_d = \mathfrak{a} + \mathfrak{n}$. Note that $\mathfrak{h}^{\tau_v} = \mathfrak{h}^{\gamma_d \circ \theta} = \mathfrak{t}^{\gamma_d} + \mathfrak{a}^{-\gamma_d}$ also depends only on d .

Remark 2.3. Recall [2, Definition 6.10] that two real forms τ_1 and τ_2 are said to be in the same *inner class* if there exists $g \in \text{Int}(\mathfrak{g})$, the adjoint group of \mathfrak{g} , such that $\tau_1 = \text{Ad}_g \tau_2$. Inner classes of real forms are in one-to-one correspondence with involutive automorphisms of the Dynkin diagram of \mathfrak{g} [2, Proposition 6.12]. Let d be an involutive automorphism of $D(\mathfrak{g})$. Then as v runs over the collection of all Vogan diagrams with d as the diagram automorphism, the real form \mathfrak{g}_v runs over all $\text{Int}(\mathfrak{g})$ -conjugacy classes of real forms of \mathfrak{g} in the inner class corresponding to d .

3. THE POISSON STRUCTURE Π_v ON X

Let \mathfrak{g} be a complex semi-simple Lie algebra, and let X be the variety of all Borel subalgebras of \mathfrak{g} . We keep the notation from Section 2. Let v be a Vogan diagram for \mathfrak{g} and let $\mathfrak{g}_v = \mathfrak{g}^{\tau_v}$ be the real form of \mathfrak{g} constructed in Section 2. Let G be the connected and simply-connected Lie group with Lie algebra \mathfrak{g} . Without any risk of confusion, we shall also denote by τ_v the lift of τ_v from \mathfrak{g} to G , and we set $G_v = G^{\tau_v}$. It follows from [5, Theorem 8.2, p. 320] that the group G_v is connected.

In this section, we will start with a Vogan diagram v for \mathfrak{g} and define a Poisson structure Π_v on X such that every G_v -orbit in X is a Poisson submanifold. This Poisson structure comes from an identification of X with the G -orbit through $\mathfrak{t} + \mathfrak{n}$ inside the variety \mathcal{L} of Lagrangian subalgebras of \mathfrak{g} , which was studied in [3]. We now recall the relevant details.

Set $n = \dim_{\mathbb{C}} \mathfrak{g}$ and let $\text{Gr}_{\mathbb{R}}(n, \mathfrak{g})$ be the Grassmannian of real n -dimensional subspaces of \mathfrak{g} . The set \mathcal{L} of all Lagrangian subalgebras of \mathfrak{g} is naturally a real subvariety of $\text{Gr}_{\mathbb{R}}(n, \mathfrak{g})$. The natural action of G on $\text{Gr}_{\mathbb{R}}(n, \mathfrak{g})$ gives rise to a Lie algebra anti-homomorphism κ from \mathfrak{g} to the Lie algebra of vector fields on $\text{Gr}_{\mathbb{R}}(n, \mathfrak{g})$,

whose extension from $\wedge^2 \mathfrak{g}$ to the space of bi-vector fields on $\text{Gr}_{\mathbb{R}}(n, \mathfrak{g})$ will also be denoted by κ . Given a Lagrangian splitting $\mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$, we define the element $R_{\mathfrak{l}_1, \mathfrak{l}_2} \in \wedge^2 \mathfrak{g}$ by

$$(3.1) \quad \langle R_{\mathfrak{l}_1, \mathfrak{l}_2}, (x_1 + \xi_1) \wedge (x_2 + \xi_2) \rangle = \langle \xi_2, x_1 \rangle - \langle \xi_1, x_2 \rangle, \quad x_1, x_2 \in \mathfrak{l}_1, \xi_1, \xi_2 \in \mathfrak{l}_2,$$

where $\langle \cdot, \cdot \rangle = \text{Im} \langle \langle \cdot, \cdot \rangle \rangle$. Set $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2} = \frac{1}{2} \kappa(R_{\mathfrak{l}_1, \mathfrak{l}_2})$. Clearly, $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ is tangent to every G -orbit in $\text{Gr}_{\mathbb{R}}(n, \mathfrak{g})$, so it is tangent to \mathcal{L} .

Theorem 3.1 ([3, Theorems 2.14 and 2.18]). *The bi-vector field $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ restricts to a Poisson structure on \mathcal{L} . If L_1 and L_2 are the connected subgroups of G with Lie algebras \mathfrak{l}_1 and \mathfrak{l}_2 respectively, then all the L_1 - as well as L_2 -orbits in \mathcal{L} are Poisson submanifolds with respect to $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$.*

For $\mathfrak{l} \in \mathcal{L}$, let $\mathfrak{n}(\mathfrak{l})$ be the normalizer subalgebra of \mathfrak{l} in \mathfrak{l}_1 . Let $\mathfrak{m}(\mathfrak{l})$ be the annihilator of $\mathfrak{n}(\mathfrak{l})$ in \mathfrak{l} , i.e. $\mathfrak{m}(\mathfrak{l}) = \{x \in \mathfrak{l} : \langle x, y \rangle = 0 \ \forall y \in \mathfrak{n}(\mathfrak{l})\} \subset \mathfrak{l}$, and let $\mathcal{V}(\mathfrak{l}) = \mathfrak{n}(\mathfrak{l}) + \mathfrak{m}(\mathfrak{l})$.

Proposition 3.2 ([3, Theorem 2.21], [9, Corollary 7.3]). *For each $\mathfrak{l} \in \mathcal{L}$, the space $\mathcal{V}(\mathfrak{l})$ is a Lagrangian subalgebra of \mathfrak{g} . The co-dimension of the symplectic leaf of $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ through \mathfrak{l} in the orbit $L_1 \cdot \mathfrak{l}$ is equal to $\dim(\mathcal{V}(\mathfrak{l}) \cap \mathfrak{l}_2)$.*

Notation 3.3. Let v be a Vogan diagram for \mathfrak{g} . We denote by Π_v , the Poisson structure on \mathcal{L} defined by the Lagrangian splitting $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$ in Lemma 2.2. Let H, N , and B be respectively the connected subgroups of G with Lie algebras $\mathfrak{h}, \mathfrak{n}$, and $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, so $B = HN$. Identify the G -orbit through $\mathfrak{t} + \mathfrak{n} \in \mathcal{L}$ with $G/B \cong X$. The induced Poisson structure on X will also be denoted by Π_v . Let $H^{-\gamma_d \circ \theta} = \{h \in H : \gamma_d \circ \theta(h) = h^{-1}\}$ and let $L_d = H^{-\gamma_d \circ \theta} N$. By the Bruhat lemma, orbits of L_d in $X \cong G/B$, which are the same as the N -orbits in X , are labeled by the elements in the Weyl group W of Δ . We refer to these N -orbits as the Bruhat cells in X .

By [3, Theorem 2.18], we have

Proposition 3.4. *Each G_v -orbit in X as well as each Bruhat cell in X is a Poisson submanifold with respect to Π_v .*

When v is the Vogan diagram with $d = 1$ and no vertex painted, we have $\tau_v = \theta$, so $\mathfrak{g}_v = \mathfrak{k}$. The Poisson structure Π_v in this case was first introduced in [11] and [13], and it has the property that its symplectic leaves are precisely the Bruhat cells (hence the name ‘‘Bruhat Poisson structure’’ in [11]). In [3] and [10] this Poisson structure was related to some earlier work of Kostant [7] and of Kostant-Kumar [8] on the Schubert calculus on X .

The splitting $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$ naturally defines a Lie bialgebra structure on \mathfrak{g}_v and therefore a Poisson Lie group structure on G_v [11]. All the G_v -orbits in \mathcal{L} become G_v -Poisson homogeneous spaces [3, 9]. We remark that in [1], Andruskiewitsch and Jancsa classified non-triangular Lie bialgebra structures on \mathfrak{g}_v using Belavin-Drinfeld triples. The one defined by the splitting $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$ comes from the standard Belavin-Drinfeld triple. We refer to [1] for details.

Example. Here we take $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and

$$\mathfrak{g}_v = \mathfrak{su}(1, 1) = \left\{ \begin{pmatrix} ix & y + iz \\ y - iz & -ix \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Then $d = 1$ and $\mathfrak{l}_d = \mathfrak{a} + \mathfrak{n}$ consists of upper triangular matrices in $\mathfrak{sl}(2, \mathbb{C})$ with real diagonal entries. Identify G/B with \mathbb{P}^1 via the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [w_0 : w_1] = [aw_0 + bw_1 : cw_0 + dw_1]$$

of G on \mathbb{P}^1 and by taking $[1 : 0] \in \mathbb{P}^1$ as the basepoint. There are two Bruhat cells: the zero-dimensional basepoint $[1 : 0]$, and the other being the rest:

$$U_1 = \mathbb{P}^1 \setminus \{[1 : 0]\} = \{[w_0 : w_1], w_1 \neq 0\}.$$

In terms of the holomorphic coordinate z on U_1 given by $z = w_0/w_1$, the Poisson structure Π_v , up to a scalar multiple, is given by

$$\Pi_v = i(1 - |z|^2) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}}.$$

Setting $u = 1/z$, we see that in the u -coordinate on the open set

$$U_0 = \{[w_0 : w_1] \in \mathbb{P}^1, w_0 \neq 0\} = \{[1 : u], u \in \mathbb{C}\},$$

we have

$$\Pi_v = i(|u|^2 - 1)|u|^2 \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial \bar{u}}.$$

Thus Π_v vanishes precisely at the basepoint $[1 : 0]$ and at every point of the form $[z : 1]$ with $|z| = 1$. If we identify \mathbb{P}^1 with the unit sphere S^2 in \mathbb{R}^3 via

$$(3.2) \quad \mathbb{P}^1 \longrightarrow S^2 : [w_0, w_1] \longmapsto \left(\frac{2\operatorname{Re}(w_0\bar{w}_1)}{|w_0|^2 + |w_1|^2}, \frac{2\operatorname{Im}(w_0\bar{w}_1)}{|w_0|^2 + |w_1|^2}, \frac{|w_0|^2 - |w_1|^2}{|w_0|^2 + |w_1|^2} \right),$$

then we see that Π_v vanishes at the “North pole” $(0, 0, 1)$ and at every point on the Equator $x_3 = 0$. Under this identification, there are exactly three orbits of $\operatorname{SU}(1, 1)$ on S^2 : the Northern hemisphere, the Equator, and the Southern hemisphere. Each of these three orbits is clearly a Poisson submanifold.

4. SYMPLECTIC LEAVES OF Π_v IN X

Suppose that \mathcal{O} is a G_v -orbit in X and \mathcal{C} is a Bruhat cell such that $\mathcal{O} \cap \mathcal{C} \neq \emptyset$. Since $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$, \mathcal{O} and \mathcal{C} intersect transversally. By Proposition 3.4, $\mathcal{O} \cap \mathcal{C}$ is a Poisson submanifold of Π_v . In this section we show that $(\mathcal{O} \cap \mathcal{C}, \Pi_v)$ is a regular Poisson manifold, and we compute the dimension of its symplectic leaves.

It is well known [14] that there are only finitely many G_v -orbits in X . We first recall from [12, Section 6] some facts about these orbits.

Let $N_G(\mathfrak{h})$ be the normalizer subgroup of \mathfrak{h} in G . Set

$$\mathcal{Z} = \{g \in G : g^{-1}\tau_v(g) \in N_G(\mathfrak{h})\}.$$

Then H acts on \mathcal{Z} from the right by right multiplication, and G_v acts on \mathcal{Z} from the left by left multiplication. Let Z be the double coset space

$$Z = G_v \backslash \mathcal{Z} / H.$$

For each $z \in Z$, choose any $g_z \in \mathcal{Z}$ in the double coset z and define \mathcal{O}_z to be the G_v -orbit in X through $g_z B \in X \cong G/B$. Clearly, \mathcal{O}_z is independent of the choice of g_z . According to [12, Theorem 6.1.4], the map $z \mapsto \mathcal{O}_z$ is a one-to-one correspondence between the set Z and the set of G_v -orbits in X . Let $W = N_G(\mathfrak{h})/H$ be the Weyl group. Thus we also have the map

$$\varphi : Z \longrightarrow W : z = G_v g_z H \longmapsto g_z^{-1} \tau_v(g_z) H \in W.$$

According to [12, Theorem 6.4.2], the codimension of the G_v -orbit \mathcal{O}_z in X equals $l(\varphi(z))$, where l is the length function on the Weyl group W . We also introduce the map

$$\sigma_z = \varphi(z)\tau_v : \mathfrak{h} \longrightarrow \mathfrak{h}.$$

For any g_z in the double coset z , we also have $\sigma_z = \text{Ad}_{g_z}^{-1} \circ \tau_v \circ \text{Ad}_{g_z}$, so σ_z is an involution.

Assume now that $z \in Z$ and $w \in W$ are such that $\mathcal{O}_z \cap \mathcal{C}_w \neq \emptyset$, where \mathcal{C}_w is the Bruhat cell in X corresponding to w , i.e. the N -orbit through $w \in G/B$. Then $\dim_{\mathbb{R}} \mathcal{C}_w = 2l(w)$, and since \mathcal{O}_z and \mathcal{C}_w intersect transversally, we have

$$\dim(\mathcal{O}_z \cap \mathcal{C}_w) = 2l(w) - l(\varphi(z)).$$

Now define

$$\delta_{z,w} = \dim(\mathfrak{h}^{w\sigma_z w^{-1}} \cap \mathfrak{h}^{-\tau_v}).$$

Theorem 4.1. *Each symplectic leaf in the intersection $\mathcal{O}_z \cap \mathcal{C}_w$ has dimension equal to*

$$\dim(\mathcal{O}_z \cap \mathcal{C}_w) - \delta_{z,w} = 2l(w) - l(\varphi(z)) - \delta_{z,w}.$$

Proof. We use Proposition 3.2 to compute dimensions of the symplectic leaves in $\mathcal{O}_z \cap \mathcal{C}_w$. Let $x = g_z B \in X$ be a point in $\mathcal{O}_z \cap \mathcal{C}_w$, where $g_z \in Z$ lies in the double coset z . Let $\mathfrak{l}_x = \text{Ad}_{g_z}(\mathfrak{t} + \mathfrak{n}) \in \mathcal{L}$. Let $\mathfrak{n}(\mathfrak{l}_x) = \mathfrak{g}_v \cap \text{Ad}_{g_z}(\mathfrak{h} + \mathfrak{n})$ be the normalizer subalgebra of \mathfrak{l}_x in \mathfrak{g}_v , let $\mathfrak{m}(\mathfrak{l}_x)$ be the annihilator subspace of $\mathfrak{n}(\mathfrak{l}_x)$ in \mathfrak{l}_x , and let $\mathcal{V}(\mathfrak{l}_x) = \mathfrak{n}(\mathfrak{l}_x) + \mathfrak{m}(\mathfrak{l}_x)$. We claim that $\mathcal{V}(\mathfrak{l}_x) = \text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})$. Indeed, it follows from the definition of σ_z that

$$\text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z}) \subset \mathfrak{g}_v \cap \text{Ad}_{g_z}(\mathfrak{h} + \mathfrak{n}) = \mathfrak{n}(\mathfrak{l}_x).$$

It is also clear that $\text{Ad}_{g_z} \mathfrak{n} \subset \mathfrak{m}(\mathfrak{l}_x)$, so

$$\text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + \mathfrak{n}) \subset \mathfrak{n}(\mathfrak{l}_x) + \mathfrak{m}(\mathfrak{l}_x) = \mathcal{V}(\mathfrak{l}_x).$$

Since both $\text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})$ and $\mathcal{V}(\mathfrak{l}_x)$ have the same dimension, they must coincide.

Now let S_x be the symplectic leaf of Π_v in X through x . By Proposition 3.2, the codimension of S_x in \mathcal{O}_z is equal to $\dim(\mathcal{V}(\mathfrak{l}_x) \cap \mathfrak{l}_d)$. Let $\dot{w} \in N_G(\mathfrak{h})$ be a representative of w in K . Since $x \in \mathcal{C}_w$, there exist $n \in N$ and $b \in B$ such that $g_z = n\dot{w}b$. Then we have

$$\begin{aligned} \mathcal{V}(\mathfrak{l}_x) \cap \mathfrak{l}_d &= (\text{Ad}_{n\dot{w}b}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})) \cap (\mathfrak{h}^{-\tau_v} + \mathfrak{n}) \\ &= \text{Ad}_n((\text{Ad}_{\dot{w}}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})) \cap (\mathfrak{h}^{-\tau_v} + \mathfrak{n})) \\ &= \text{Ad}_n(\mathfrak{h}^{w\sigma_z w^{-1}} \cap \mathfrak{h}^{-\tau_v} + (\text{Ad}_{\dot{w}} \mathfrak{n}) \cap \mathfrak{n}), \end{aligned}$$

where in the last line we have the direct sum of vector spaces. Since

$$\dim(\text{Ad}_{\dot{w}} \mathfrak{n}) \cap \mathfrak{n} = \dim_{\mathbb{R}} X - \dim_{\mathbb{R}} \mathcal{C}_w,$$

we have

$$\dim(\mathcal{V}(\mathfrak{l}_x) \cap \mathfrak{l}_d) = \delta_{z,w} + \dim_{\mathbb{R}} X - \dim_{\mathbb{R}} \mathcal{C}_w,$$

and thus

$$\dim S_x = \dim \mathcal{O}_z - \dim(\mathcal{V}(\mathfrak{l}_x) \cap \mathfrak{l}_d) = \dim(\mathcal{O}_z \cap \mathcal{C}_w) - \delta_{z,w}.$$

□

Note that the number $\delta_{z,w}$ depends only on d and the two Weyl group elements $\varphi(z)$ and w . Define $d : W \rightarrow W$ by $d(w) = \gamma_d w \gamma_d$. Following [12], we say that $w \in W$ is a d -twisted involution if $d(w) = w^{-1}$. Denote by \mathcal{I}_d the set of all d -twisted involutions in W . Since for every $g \in G$ we have $\tau_v(g^{-1}\tau_v(g)) = (g^{-1}\tau_v(g))^{-1}$, every $\varphi(z)$ is in \mathcal{I}_d . The Weyl group W acts on \mathcal{I}_d by

$$w_1 * w = w_1 w d(w_1^{-1}) \text{ for } w_1 \in W, \text{ and } w \in \mathcal{I}_d,$$

and the set $\varphi(Z) \subset \mathcal{I}_d$ is W -invariant. In fact, the W -action on G/H , given by $w \cdot gH = gw^{-1}H$, commutes with the left action of G_v by left multiplication, and thus induces a left action of W on Z , which we denote by $w \cdot z$ for $w \in W$ and $z \in Z$. It is also easy to see that $\varphi : Z \rightarrow W$ is W -equivariant, i.e. $\varphi(w \cdot z) = w * \varphi(z)$ for all $w \in W$ and $z \in Z$. Similarly, the involution $\tau_v : G \rightarrow G$ gives rise to an involution on Z which depends only on d . Denote this involution by $z \rightarrow d(z)$. Then we also have $\varphi(d(z)) = d\varphi(z) = \varphi(z)^{-1}$. As maps on \mathfrak{h} , we see that $w\sigma_z w^{-1} = (w * \varphi(z))\tau_v$. Thus we also have

$$\delta_{z,w} = \dim(\mathfrak{h}^{(w * \varphi(z))\tau_v} \cap \mathfrak{h}^{-\tau_v}).$$

Corollary 4.2. 1) When $w * \varphi(z) = 1$, symplectic leaves of Π_v in $\mathcal{O}_z \cap \mathcal{C}_w$ are precisely its connected components.

2) Every open orbit \mathcal{O}_z has an open symplectic leaf $\mathcal{O}_z \cap \mathcal{C}_{w_0}$, where w_0 is the longest element in W .

3) If $d = 1$, symplectic leaves in an open orbit \mathcal{O}_z are precisely the connected components of intersections of Bruhat cells with \mathcal{O}_z .

Proof. 1) When $w * \varphi(z) = 1$, we have $\delta_{z,w} = 0$, so every symplectic leaf in $\mathcal{O}_z \cap \mathcal{C}_w$ is open in $\mathcal{O}_z \cap \mathcal{C}_w$.

2) Since \mathcal{C}_{w_0} is dense in X , it intersects with every open orbit \mathcal{O}_z . Since an orbit \mathcal{O}_z is open if and only if $\varphi(z) = 1$, statement 2) follows from 1) and the fact that w_0 commutes with d . The fact that $\mathcal{C}_{w_0} \cap \mathcal{O}_z$ is connected follows from the observation that \mathcal{O}_z is a connected open complex submanifold of X and thus $\mathcal{O}_z \cap (X \setminus \mathcal{C}_{w_0})$ is a divisor in \mathcal{O}_z .

3) follows directly from 1). □

Now consider the group $H^{\tau_v} = H \cap G_v$. Since the centralizer of \mathfrak{h}^{τ_v} in G_v also centralizes \mathfrak{h} , we see that H^{τ_v} is the Cartan subgroup of G_v corresponding to the Cartan subalgebra \mathfrak{h}^{τ_v} . Then according to [6, Proposition 7.90] the group H^{τ_v} is connected.

The Poisson structure Π_v on X is H^{τ_v} -invariant. Indeed, let $R \in \wedge^2 \mathfrak{g}$ be the element given in (3.1) for $\mathfrak{l}_1 = \mathfrak{g}_v$ and $\mathfrak{l}_2 = \mathfrak{l}_d$. We can also represent R as $R = \sum_i \xi_i \wedge y_i$, where $\{y_i\}$ is a basis of \mathfrak{g}_v , and $\{\xi_i\}$ is the dual basis of \mathfrak{l}_d with respect to the pairing between \mathfrak{g}_v and \mathfrak{l}_d given by $\langle \cdot, \cdot \rangle$, the imaginary part of the Killing form on \mathfrak{g} . If $h \in H^{\tau_v}$, then $\{\text{Ad}_h y_i\}$ is a basis of \mathfrak{g}_v , and $\{\text{Ad}_h \xi_i\}$ is its dual basis. Thus $\text{Ad}_h R = R$.

Let $z \in Z$ and $w \in W$ be such that \mathcal{O}_z and \mathcal{C}_w have a non-empty intersection, and let $x \in \mathcal{O}_z \cap \mathcal{C}_w$. Clearly, H^{τ_v} leaves $\mathcal{O}_z \cap \mathcal{C}_w$ invariant. Since the Poisson structure Π_v is H^{τ_v} -invariant, if S_x is the symplectic leaf of Π_v through x , then $hS_x := \{hx_1 : x_1 \in S_x\}$ is the symplectic leaf of Π_v through hx . Define

$$F_x := \bigcup_{h \in H^{\tau_v}} hS_x.$$

Proposition 4.3. For any $x \in X$, the set F_x is a connected component of $\mathcal{O}_z \cap \mathcal{C}_w$.

Proof. It is easy to see that if $F_{x_1} \cap F_{x_2} \neq \emptyset$, then $F_{x_1} = F_{x_2}$. The statement would follow once we prove that F_x is an open subset of $\mathcal{O}_z \cap \mathcal{C}_w$ for each x .

Let $x = g_z B \in \mathcal{O}_z \cap \mathcal{C}_w$ with $g_z \in \mathcal{Z}$ in the double coset z . For $y \in \mathfrak{h}^{\tau_v}$, let X_y be the vector field on X generating the action of $\exp(ty) \in H^{\tau_v}$ on X . We claim that $X_y(x) \in T_x S_x$ if and only if $y \in p(\mathfrak{h}^{(w*\varphi(z))\tau_v})$, where $p : \mathfrak{h} \rightarrow \mathfrak{h}^{\tau_v}$ is the projection with respect to the decomposition $\mathfrak{h} = \mathfrak{h}^{\tau_v} + \mathfrak{h}^{-\tau_v}$. Assume the claim. Then since the kernel of the map $p : \mathfrak{h}^{(w*\varphi(z))\tau_v} \rightarrow \mathfrak{h}^{\tau_v}$ has dimension $\dim(\mathfrak{h}^{(w*\varphi(z))\tau_v} \cap \mathfrak{h}^{-\tau_v}) = \delta_{z,w}$, the image of the map

$$J_x : \mathfrak{h}^{\tau_v} \rightarrow T_x \mathcal{O}_z / T_x S_x : y \mapsto X_y(x) + T_x S_x$$

has dimension equal to $\dim(\mathfrak{h}^{\tau_v}) - \dim(\mathfrak{h}^{(w*\varphi(z))\tau_v}) + \delta_{z,w} = \delta_{z,w}$. Thus J_x is onto, and the H^{τ_v} -orbit in $\mathcal{O}_z \cap \mathcal{C}_w$ through x is transversal to the symplectic leaf S_x . It follows that F_x is open in $\mathcal{O}_z \cap \mathcal{C}_w$.

It remains to prove the claim. Also denote by $p : \mathfrak{g} \rightarrow \mathfrak{g}_v$ the projection with respect to the decomposition $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$, and let q be the projection $q : \mathfrak{g}_v \rightarrow \mathfrak{g}_v / \mathfrak{g}_v \cap \text{Ad}_{g_z} \mathfrak{b} \cong T_x \mathcal{O}_z$. Then by [9, Corollary 7.3], we have $T_x S_x = (q \circ p)(\mathcal{V}(\mathfrak{l}_x))$, where, as in the proof of Theorem 4.1, $\mathcal{V}(\mathfrak{l}_x) = \text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})$. Let $y \in \mathfrak{h}^{\tau_v}$. If $X_y(x) \in T_x S_x$, then there exist $y_1 \in \mathfrak{l}_d$ and $y_2 \in \mathfrak{g}_v$ with $y_1 + y_2 \in \mathcal{V}(\mathfrak{l}_x)$ such that $y - y_2 \in \mathfrak{g}_v \cap \text{Ad}_{g_z} \mathfrak{b} \subset \mathcal{V}(\mathfrak{l}_x)$. Thus $y + y_1 = y - y_2 + y_1 + y_2 \in \mathcal{V}(\mathfrak{l}_x)$. Write $y_1 = \xi_1 + u_1$, where $\xi_1 \in \mathfrak{h}^{-\tau_v}$ and $u_1 \in \mathfrak{n}$. Then there exist $\xi_2 \in \mathfrak{h}^{\sigma_z}$ and $u_2 \in \mathfrak{n}$ such that $y + \xi_1 + u_1 = \text{Ad}_{g_z}(\xi_2 + u_2)$. Write $g_z = n\dot{w}b$, where $n \in N, b \in B$, and \dot{w} is a representative of w in K . Write $\text{Ad}_{n^{-1}}(y + \xi_1 + u_1) = y + \xi_1 + u'_1$ and $\text{Ad}_b(\xi_2 + u_2) = \xi_2 + u'_2$, where $u'_1, u'_2 \in \mathfrak{n}$. Then we have

$$y + \xi_1 + u'_1 = \text{Ad}_{\dot{w}}(\xi_2 + u'_2).$$

Since $y + \xi_1, \text{Ad}_{\dot{w}}\xi_2 \in \mathfrak{h}$ and $u'_1, \text{Ad}_{\dot{w}}u'_2 \in \mathfrak{n} + \mathfrak{n}_-$, where $\mathfrak{n}_- = \theta(\mathfrak{n})$, we have $y + \xi_1 = \text{Ad}_{\dot{w}}\xi_2 \in \mathfrak{h}^{(w*\varphi(z))\tau_v}$. Thus $y \in p(\mathfrak{h}^{(w*\varphi(z))\tau_v})$. Conversely, if $y \in \mathfrak{h}^{\tau_v}$ is such that $y + \xi_1 \in \mathfrak{h}^{(w*\varphi(z))\tau_v} = \text{Ad}_{\dot{w}}\mathfrak{h}^{\sigma_z}$ for some $\xi_1 \in \mathfrak{h}^{-\tau_v}$, write $y + \xi_1 = \text{Ad}_{\dot{w}}\xi_2$ for $\xi_2 \in \mathfrak{h}^{\sigma_z}$. Let $\text{Ad}_{b^{-1}}\xi_2 = \xi_2 + u_2$ for some $u_2 \in \mathfrak{n}$. We then have

$$\text{Ad}_n(y + \xi_1) = \text{Ad}_{n\dot{w}b}(\xi_2 + u_2) \in \mathcal{V}(\mathfrak{l}_x).$$

On the other hand, let $\text{Ad}_n(y + \xi_1) = y + \xi_1 + u_1$ with $u_1 \in \mathfrak{n}$. We see that $y = p(\text{Ad}_n(y + \xi_1))$ so $X_y(x) \in T_x S_x$. □

5. INVARIANT POISSON COHOMOLOGY OF OPEN ORBITS

Let \mathcal{O}_z be a G_v -orbit in X equipped with the Poisson structure Π_v . Then (\mathcal{O}_z, Π_v) is a Poisson homogeneous space for the Poisson Lie group G_v . The G_v -invariant Poisson cohomology of (\mathcal{O}_z, Π_v) , denoted by $H_{\Pi_v, G_v}^\bullet(\mathcal{O}_z)$, is defined as the cohomology of the cochain complex $(\chi^\bullet(\mathcal{O}_z)^{G_v}, \partial_{\Pi_v})$, where $\chi^\bullet(\mathcal{O}_z)^{G_v}$ is the space of all G_v -invariant complex multi-vector fields on \mathcal{O}_z , $d_{\Pi_v}(V) = [\Pi_v, V]$, and $[\cdot, \cdot]$ is the Schouten bracket of the multi-vector fields.

Proposition 5.1. *When \mathcal{O}_z is an open G_v -orbit in X , the G_v -invariant Poisson cohomology $H_{\Pi_v, G_v}^\bullet(\mathcal{O}_z)$ is isomorphic to the de Rham cohomology of X .*

Proof. As in the proof of Theorem 4.1, let $x = g_z B \in X$ be an arbitrary point in \mathcal{O}_z , where $g_z \in \mathcal{Z}$ is in the coset z , and let $\mathcal{V}(\mathfrak{l}_x) = \text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})$. Since \mathcal{O}_z is open, the stabilizer subalgebra of \mathfrak{g}_v at x is $\mathfrak{g}_v \cap \mathcal{V}(\mathfrak{l}_x) = \text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z})$. By [9, Theorem 7.5], the G_v -invariant Poisson cohomology $H_{\Pi_v, G_v}^\bullet(\mathcal{O}_z)$ is isomorphic to the relative Lie algebra cohomology of the Lie algebra $\mathcal{V}(\mathfrak{l}_x) \otimes \mathbb{C}$ relative to the subalgebra

$(\text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z})) \otimes \mathbb{C}$. Thus the G_v -invariant Poisson cohomology is isomorphic to the \mathfrak{h} -invariant part of the Lie algebra cohomology of the direct sum Lie algebra $\mathfrak{n} \oplus \mathfrak{n}$ with coefficients in \mathbb{C} , which by Kostant's theorem [7], is isomorphic to the de Rham cohomology of X . \square

6. REMARKS

We have constructed a Poisson structure Π_v on X for each Vogan diagram v for \mathfrak{g} (which is not necessarily normalized). In particular, each Bruhat cell C_w in X carries the Poisson structure Π_v . It would be interesting to study connections between the Poisson structures for different v . Especially interesting are the properties of Π_v that depend only on the inner class d of the real form \mathfrak{g}_v . We also remark that the Poisson structure Π_v is defined on the whole variety \mathcal{L} of Lagrangian subalgebras of \mathfrak{g} . We have only been looking at the restriction of Π_v to a particular G -orbit, namely the G -orbit through the Lagrangian subalgebra $\mathfrak{t} + \mathfrak{n}$. There are many other interesting G -orbits in \mathcal{L} , such as the G -orbit through a given real form of \mathfrak{g} . It would be interesting to study the properties of the Poisson structure Π_v on these orbits as well as on their closures, with respect to both the classical topology and the Zariski topology.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ARIZONA, TUCSON, ARIZONA 85721-0089
E-mail address: `foth@math.arizona.edu`

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF HONG KONG, POKFULAM, HONG KONG
E-mail address: `jhlu@maths.hku.hk`