POISSON STRUCTURES ON COMPLEX FLAG MANIFOLDS ASSOCIATED WITH REAL FORMS

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Dedicated to Alan Weinstein on the occasion of his 60th birthday

ABSTRACT. For a complex semisimple Lie group \(G\) and a real form \(G_0\) we define a Poisson structure on the variety of Borel subgroups of \(G\) with the property that all \(G_0\)-orbits in \(X\) as well as all Bruhat cells (for a suitable choice of a Borel subgroup of \(G\)) are Poisson submanifolds. In particular, we show that every non-empty intersection of a \(G_0\)-orbit and a Bruhat cell is a regular Poisson manifold, and we compute the dimension of its symplectic leaves.

1. Introduction

Let \(G\) be a connected and simply-connected complex semisimple Lie group with Lie algebra \(\mathfrak{g}\), and let \(X\) be the variety of Borel subalgebras of \(\mathfrak{g}\). In this paper we use a real form \(\mathfrak{g}_0\) of \(\mathfrak{g}\) to define a Poisson structure on \(X\). This Poisson structure depends on a choice of a Borel subalgebra \(\mathfrak{b}\) of \(\mathfrak{g}\) such that \(\mathfrak{g}_0 \cap \mathfrak{b}\) is a maximally compact Cartan subalgebra of \(\mathfrak{g}_0\). Instead of dealing with each real form individually, we fix a Borel subalgebra \(\mathfrak{b}\) of \(\mathfrak{g}\) and a Cartan subalgebra \(\mathfrak{h} \subset \mathfrak{b}\). Then, as is shown in [6], a real form \(\mathfrak{g}_v\) of \(\mathfrak{g}\) can be constructed from each Vogan diagram \(v\) for \(\mathfrak{g}\) such that \(\mathfrak{g}_v \cap \mathfrak{b}\) is a maximally compact Cartan subalgebra of \(\mathfrak{g}_v\). The corresponding Poisson structure on \(X\) is denoted by \(\Pi_v\).

Let \(G_v\) be the real form of \(G\) corresponding to \(\mathfrak{g}_v\), and let \(B\) be the Borel subgroup of \(G\) with Lie algebra \(\mathfrak{b}\). The Poisson structure \(\Pi_v\) has the property that each \(G_v\)-orbit as well as each \(B\)-orbit in \(X\) is a Poisson submanifold. The \(B\)-orbits in \(X\) will be referred to as the Bruhat cells. We compute the rank of \(\Pi_v\). In particular, if a \(G_v\)-orbit \(O\) meets a Bruhat cell \(C\), they intersect transversally, and we find that all the symplectic leaves in \(O \cap C\) have the same dimension, so \(O \cap C\) is a regular Poisson manifold. Moreover, we show that all symplectic leaves in each connected component of \(O \cap C\) are translates of each other by elements of a Cartan subgroup of \(G_v\). We also show that the \(G_v\)-invariant Poisson cohomology for each open \(G_v\)-orbit in \(X\) is isomorphic to the de Rham cohomology of \(X\).

Results similar to those presented here for the full flag manifold \(X = G/B\) are also valid for a partial flag manifold \(G/P\), where \(P\) is a parabolic subgroup of...
$G$ containing $B$. We will treat these more general cases as well as some further properties of $\Pi_2$ in a future paper.

Throughout this paper, if $V$ is a set and $\sigma$ is an involution on $V$, we will use $V^\sigma$ to denote the fixed point set of $\sigma$ in $V$.

2. Real forms of $\mathfrak{g}$ and Vogan diagrams

Let $\mathfrak{g}$ be a complex simple Lie algebra. In this section we recall the classification of real forms of $\mathfrak{g}$ by Vogan diagrams. Details can be found in [3] Chapter 6).

Suppose that $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$ and that $\tau_0$ is the corresponding complex-conjugate linear involution on $\mathfrak{g}$. Let $\theta_0$ be a Cartan involution of $\mathfrak{g}_0$, and let $\mathfrak{h}_0$ be a $\theta_0$-stable maximally compact Cartan subalgebra of $\mathfrak{g}_0$. Set $t_0 = h_0^{\theta_0}$ and $a_0 = h_0^{-\theta_0}$ so that $\mathfrak{h}_0 = t_0 + a_0$. Let $\gamma_0$ be the complexification of $\theta_0$. Then the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0 + i\mathfrak{a}_0$ of $\mathfrak{g}$ is $\gamma_0$-stable. Let $\Delta$ be the root system for $(\mathfrak{g}, \mathfrak{h})$.

Since $\mathfrak{h}_0$ is a maximally compact Cartan subalgebra of $\mathfrak{g}_0$, there exists $x_0 \in i\mathfrak{a}_0$ that is regular for $\Delta$. Define the subset $\Delta^+$ of positive roots in $\Delta$ by $\alpha \in \Delta^+$ if and only if $\alpha(x_0) > 0$. Then $\gamma_0(\Delta^+) = \Delta^+$. Let $\Sigma \subset \Delta^+$ be the set of simple roots in $\Delta^+$. Then $\gamma_0(\Sigma) = \Sigma$, so $\gamma_0$ gives rise to an involutive automorphism of the Dynkin diagram of $\mathfrak{g}$. Let $\mathcal{I}$ be the set of non-compact imaginary simple roots. The Vogan diagram of $\mathfrak{g}_0$ associated to the triple $(\theta_0, \mathfrak{h}_0, \Delta^+)$ is the Dynkin diagram $D(\mathfrak{g})$ of $\mathfrak{g}$, together with an involutive automorphism $\gamma_0$ on $D(\mathfrak{g})$ and the vertices corresponding to the simple roots in $\mathcal{I}$ painted black.

In general, a Vogan diagram for $\mathfrak{g}$ is defined to be a triple $(D(\mathfrak{g}), d, \mathcal{I})$, where $D(\mathfrak{g})$ is the Dynkin diagram of $\mathfrak{g}$, $d$ is an involutive automorphism of $D(\mathfrak{g})$, and $\mathcal{I}$ is a subset of vertices of $D(\mathfrak{g})$ such that $d(\alpha) = \alpha$ for each $\alpha \in \mathcal{I}$. Every Vogan diagram for $\mathfrak{g}$ comes from a real form of $\mathfrak{g}$ (see below), although two different Vogan diagrams can come from isomorphic real forms. A non-redundant list of Vogan diagrams with the corresponding isomorphism class of real forms for all simple Lie algebras is given in [3]. Every Vogan diagram in the list in [3] is normalized in the sense that at most one vertex is painted black.

For the purpose of defining Poisson structures on the variety of Borel subalgebras of $\mathfrak{g}$, we now recall the explicit construction of a real form of $\mathfrak{g}$ from a Vogan diagram [6, Theorem 6.88]. We need to fix the following data for $\mathfrak{g}$.

Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let $\Delta$ be the root system for $(\mathfrak{g}, \mathfrak{h})$. Fix a choice of positive roots $\Delta^+$ and let $\Sigma$ be the basis of simple roots. Let $\langle \, , \rangle$ be the Killing form of $\mathfrak{g}$ and let root vectors $\{E_\alpha : \alpha \in \Delta\}$ be chosen such that $[E_\alpha, E_{-\alpha}] = H_\alpha$ for each $\alpha \in \Delta^+$, where $H_\alpha$ is the unique element of $\mathfrak{h}$ defined by $\langle H, H_\alpha \rangle = \alpha(H)$ for all $H \in \mathfrak{h}$, and such that the numbers $m_{\alpha, \beta}$ given by $[E_\alpha, E_\beta] = m_{\alpha, \beta}E_{\alpha + \beta}$ when $\alpha + \beta \in \Delta$ are real. Define a compact real form $\mathfrak{t}$ of $\mathfrak{g}$ as

$$\mathfrak{t} = \text{span}_\mathbb{R} \{iH_\alpha, X_\alpha := E_\alpha - E_{-\alpha}, Y_\alpha := i(E_\alpha + E_{-\alpha})\},$$

and let $\tilde{\theta}$ be the complex conjugation of $\mathfrak{g}$ defining $\mathfrak{t}$. If $d$ is an involutive automorphism of the Dynkin diagram of $\mathfrak{g}$, define $\gamma_d$ to be the unique automorphism of $\mathfrak{g}$ satisfying $\gamma_d(H_\alpha) = H_{d(\alpha)}$ and $\gamma_d(E_\alpha) = E_{d(\alpha)}$ for each simple root $\alpha$.

Given a Vogan diagram $v$ for $\mathfrak{g}$, not necessarily normalized, with the involutive diagram automorphism $d$, let $t_v$ be the unique element in the adjoint group of $\mathfrak{g}$ such that

$$\text{Ad}_{t_v}(E_\alpha) = \begin{cases} E_\alpha & \text{if $\alpha$ is a blank vertex in $v$,} \\ -E_\alpha & \text{if $\alpha$ is a painted vertex in $v$.} \end{cases}$$
Define a complex conjugate linear involution
\[ \tau_v := \text{Ad}_{i_v} \circ \gamma_d \circ \theta. \]

**Notation 2.1.** We use \( g_v = g^{\tau_v} \) to denote the real form of \( g \) defined by \( \tau_v \). Set \( \theta_v = \theta|_{g_v} \). Then \( \theta_v \) is a Cartan involution of \( g_v \), and \( h^{\tau_v} \) is a \( \theta_v \)-stable maximally compact Cartan subalgebra of \( g_v \), with \( h = h^{\tau_v} + i h^{\tau_v} \). The complexification of \( \tau_v \) is
\[
\gamma_v := \tau_v \theta = \theta \tau_v = \text{Ad}_{i_v} \gamma_d.
\]
Since \( \gamma_v(\Delta^+) = \Delta^+ \), the Vogan diagram of \( g_v \) associated to the triple \((\theta_v, h^{\tau_v}, \Delta^+)\) is \( v \).

One of the advantages of introducing the real form \( g_v \) is as follows. We say that a real subalgebra \( l \) of \( g \) is Lagrangian if its real dimension is equal to the complex dimension of \( g \) and if \( \text{Im}\langle x_1, x_2 \rangle = 0 \) for all \( x_1, x_2 \in l \). A decomposition \( g = l_1 + l_2 \) is called a Lagrangian splitting if both \( l_1 \) and \( l_2 \) are Lagrangian. Let \( n \) be the subalgebra of \( g \) spanned by the set of all positive root vectors for \( \Delta^+ \). The following fact is easy to prove.

**Lemma 2.2.** Let \( l_d := h^{-\tau_v} + n \). Then \( g = g_v + l_d \) is a Lagrangian splitting of \( g \).

Let \( a = \text{span}_\mathbb{R}\{i H_\alpha : \alpha \in \Sigma\} \), and let \( t = ia \). We note that since
\[ h^{-\tau_v} = h^{-\gamma_d \circ \theta} = t^{-\gamma_d} + a \gamma_d, \]
the Lagrangian complement \( l_d \) of \( g_v \) depends only on \( d \), and in the case when \( d = 1 \), we have \( l_d = a + n \). Note that \( h^{\tau_v} = h^{\gamma_d \circ \theta} = t^{\gamma_d} + a^{-\gamma_d} \) also depends only on \( d \).

**Remark 2.3.** Recall [2] Definition 6.10 that two real forms \( \tau_1 \) and \( \tau_2 \) are said to be in the same inner class if there exists \( g \in \text{Int}(g) \), the adjoint group of \( g \), such that \( \tau_1 = \text{Ad}_g \tau_2 \). Inner classes of real forms are in one-to-one correspondence with involutive automorphisms of the Dynkin diagram of \( g \) [2 Proposition 6.12]. Let \( d \) be an involutive automorphism of \( D(g) \). Then as \( v \) runs over the collection of all Vogan diagrams with \( d \) as the diagram automorphism, the real form \( g_v \) runs over all \( \text{Int}(g) \)-conjugacy classes of real forms of \( g \) in the inner class corresponding to \( d \).

### 3. THE POISSON STRUCTURE \( \Pi_v \) ON \( X \)

Let \( g \) be a complex semi-simple Lie algebra, and let \( X \) be the variety of all Borel subalgebras of \( g \). We keep the notation from Section 2. Let \( v \) be a Vogan diagram for \( g \) and let \( g_v = g^{\tau_v} \) be the real form of \( g \) constructed in Section 2. Let \( G \) be the connected and simply-connected Lie group with Lie algebra \( g \). Without any risk of confusion, we shall also denote by \( \tau_v \) the lift of \( \tau_v \) from \( g \) to \( G \), and we set \( G_v = G^{\tau_v} \). It follows from [3] Theorem 8.2, p. 320 that the group \( G_v \) is connected.

In this section, we will start with a Vogan diagram \( v \) for \( g \) and define a Poisson structure \( \Pi_v \) on \( X \) such that every \( G_v \)-orbit in \( X \) is a Poisson submanifold. This Poisson structure comes from an identification of \( X \) with the \( G \)-orbit through \( t + n \) inside the variety \( L \) of Lagrangian subalgebras of \( g \), which was studied in [3]. We now recall the relevant details.

Set \( n = \text{dim} g \) and let \( \text{Gr}_R(n, g) \) be the Grassmannian of real \( n \)-dimensional subspaces of \( g \). The set \( L \) of all Lagrangian subalgebras of \( g \) is naturally a real subvariety of \( \text{Gr}_R(n, g) \). The natural action of \( G \) on \( \text{Gr}_R(n, g) \) gives rise to a Lie algebra anti-homomorphism \( \kappa \) from \( g \) to the Lie algebra of vector fields on \( \text{Gr}_R(n, g) \),
whose extension from $\wedge^2\mathfrak{g}$ to the space of bi-vector fields on $\text{Gr}_R(n, \mathfrak{g})$ will also be denoted by $\kappa$. Given a Lagrangian splitting $\mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$, we define the element $R_{\mathfrak{l}_1, \mathfrak{l}_2} \in \wedge^2 \mathfrak{g}$ by
\begin{equation}
(R_{\mathfrak{l}_1, \mathfrak{l}_2}, (x_1 + \xi_1) \wedge (x_2 + \xi_2)) = \langle \xi_2, x_1 \rangle - \langle \xi_1, x_2 \rangle, \quad x_1, x_2 \in \mathfrak{l}_1, \xi_1, \xi_2 \in \mathfrak{l}_2,
\end{equation}
where $\langle \cdot, \cdot \rangle = \text{Im} \langle \cdot, \cdot \rangle$. Set $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2} = \frac{1}{2} \kappa(R_{\mathfrak{l}_1, \mathfrak{l}_2})$. Clearly, $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ is tangent to every $G$-orbit in $\text{Gr}_R(n, \mathfrak{g})$, so it is tangent to $\mathcal{L}$.

**Theorem 3.1 (\cite{3} Theorems 2.14 and 2.18).** The bi-vector field $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ restricts to a Poisson structure on $\mathcal{L}$. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are the connected subgroups of $G$ with Lie algebras $\mathfrak{l}_1$ and $\mathfrak{l}_2$ respectively, then all the $\mathcal{L}_1$- as well as $\mathcal{L}_2$-orbits in $\mathcal{L}$ are Poisson submanifolds with respect to $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$.

For $l \in \mathcal{L}$, let $\mathfrak{n}(l)$ be the normalizer subalgebra of $l$ in $\mathfrak{l}_1$. Let $\mathfrak{m}(l)$ be the annihilator of $\mathfrak{n}(l)$ in $l$, i.e. $\mathfrak{m}(l) = \{x \in l : \langle x, y \rangle = 0 \ \forall y \in \mathfrak{n}(l)\} \subset l$, and let $\mathcal{V}(l) = \mathfrak{n}(l) + \mathfrak{m}(l)$.

**Proposition 3.2 (\cite{3} Theorem 2.21, \cite{3} Corollary 7.3).** For each $l \in \mathcal{L}$, the space $\mathcal{V}(l)$ is a Lagrangian subalgebra of $\mathfrak{g}$. The co-dimension of the symplectic leaf of $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ through $l$ in the orbit $\mathcal{L}_1 \cdot l$ is equal to $\text{dim}(\mathcal{V}(l) \cap \mathfrak{l}_2)$.

**Notation 3.3.** Let $v$ be a Vogan diagram for $\mathfrak{g}$. We denote by $\Pi_v$, the Poisson structure on $\mathcal{L}$ defined by the Lagrangian splitting $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$ in Lemma 2.2. Let $H$, $N$, and $B$ be respectively the connected subgroups of $G$ with Lie algebras $\mathfrak{h}$, $\mathfrak{n}$, and $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, so $B = HN$. Identify the $G$-orbit through $t + \mathfrak{n} \in \mathcal{L}$ with $G/B \cong X$. The induced Poisson structure on $X$ will also be denoted by $\Pi_v$. Let $H^{-\gamma \circ \theta} = \{h \in H : \gamma \circ \theta(h) = h^{-1}\}$ and let $L_{\mathfrak{d}} = H^{-\gamma \circ \theta}N$. By the Bruhat lemma, orbits of $L_{\mathfrak{d}}$ in $X \cong G/B$, which are the same as the $N$-orbits in $X$, are labeled by the elements in the Weyl group $W$ of $\Delta$. We refer to these $N$-orbits as the Bruhat cells in $X$.

By \cite{3} Theorem 2.18, we have

**Proposition 3.4.** Each $G_v$-orbit in $X$ as well as each Bruhat cell in $X$ is a Poisson submanifold with respect to $\Pi_v$.

When $v$ is the Vogan diagram with $d = 1$ and no vertex painted, we have $\tau_v = \theta$, so $\mathfrak{g}_v = \mathfrak{k}$. The Poisson structure $\Pi_v$ in this case was first introduced in \cite{11} and \cite{13}, and it has the property that its symplectic leaves are precisely the Bruhat cells (hence the name “Bruhat Poisson structure” in \cite{11}). In \cite{3} and \cite{10} this Poisson structure was related to some earlier work of Kostant \cite{7} and of Kostant-Kumar \cite{8} on the Schubert calculus on $X$.

The splitting $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$ naturally defines a Lie bialgebra structure on $\mathfrak{g}_v$ and therefore a Poisson Lie group structure on $G_v$ \cite{11}. All the $G_v$-orbits in $\mathcal{L}$ become $G_v$-Poisson homogeneous spaces \cite{3, 9}. We remark that in \cite{11}, Andruskiewitsch and Jancsa classified non-triangular Lie bialgebra structures on $\mathfrak{g}_v$ using Belavin-Drinfeld triples. The one defined by the splitting $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$ comes from the standard Belavin-Drinfeld triple. We refer to \cite{1} for details.

**Example.** Here we take $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and
\begin{equation}
\mathfrak{g}_v = \mathfrak{su}(1, 1) = \left\{ \begin{pmatrix} ix & y + iz \\ y - iz & -ix \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.
\end{equation}
Then \( d = 1 \) and \( t_d = a + n \) consists of upper triangular matrices in \( \mathfrak{sl}(2, \mathbb{C}) \) with real diagonal entries. Identify \( G/B \) of these three orbits is clearly a Poisson submanifold.

Since \( g \) in terms of the holomorphic coordinate \( z \)

Thus \( \Pi \) vanishes at the “North pole” \((0, 0, 1)\) and at every point on the Equator \( x_3 = 0 \). Under this identification, there are exactly three orbits of \( \text{SU}(1) \) on \( S^2 \): the Northern hemisphere, the Equator, and the Southern hemisphere. Each of these three orbits is clearly a Poisson submanifold.

**4. Symplectic leaves of \( \Pi_v \) in \( X \)**

Suppose that \( O \) is a \( G_v \)-orbit in \( X \) and \( C \) is a Bruhat cell such that \( O \cap C \neq \emptyset \). Since \( g = g_v + t_d \), \( O \) and \( C \) intersect transversally. By Proposition 5.4, \( O \cap C \) is a Poisson submanifold of \( \Pi_v \). In this section we show that \( (O \cap C, \Pi_v) \) is a regular Poisson manifold, and we compute the dimension of its symplectic leaves.

It is well known [13] that there are only finitely many \( G_v \)-orbits in \( X \). We first recall from [12 Section 6] some facts about these orbits.

Let \( N_G(\mathfrak{h}) \) be the normalizer subgroup of \( \mathfrak{h} \) in \( G \). Set

\[
Z = \{ g \in G : g^{-1} \tau_v(g) \in N_G(\mathfrak{h}) \}.
\]

Then \( H \) acts on \( Z \) from the right by right multiplication, and \( G_v \) acts on \( Z \) from the left by left multiplication. Let \( Z \) be the double coset space

\[
Z = G_v \backslash Z/H.
\]

For each \( z \in Z \), choose any \( g_z \in Z \) in the double coset \( z \) and define \( O_z \) to be the \( G_v \)-orbit in \( X \) through \( g_z B \in X \cong G/B \). Clearly, \( O_z \) is independent of the choice of \( g_z \). According to [12 Theorem 6.1.4], the map \( z \mapsto O_z \) is a one-to-one correspondence between the set \( Z \) and the set of \( G_v \)-orbits in \( X \). Let \( W = N_G(\mathfrak{h})/H \) be the Weyl group. Thus we also have the map

\[
\varphi : Z \longrightarrow W : \ z = G_v g_z H \longmapsto g_z^{-1} \tau_v(g_z) H \in W.
\]
According to [12, Theorem 6.4.2], the codimension of the $G_{z}$-orbit $O_{z}$ in $X$ equals $l(\varphi(z))$, where $l$ is the length function on the Weyl group $W$. We also introduce the map

$$\sigma_z = \varphi(z)\tau_v : \mathfrak{h} \longrightarrow \mathfrak{h}.$$  

For any $g_{z}$ in the double coset $z$, we also have $\sigma_z = \text{Ad}_{g_{z}}^{-1} \circ \tau_v \circ \text{Ad}_{g_{z}}$, so $\sigma_z$ is an involution.

Assume now that $z \in \mathbb{Z}$ and $w \in W$ are such that $O_{z} \cap C_{w} \neq \emptyset$, where $C_{w}$ is the Bruhat cell in $X$ corresponding to $w$, i.e., the $N$-orbit through $w \in G/B$. Then $\dim_{\mathbb{R}} C_{w} = 2l(w)$, and since $O_{z}$ and $C_{w}$ intersect transversally, we have

$$\dim(O_{z} \cap C_{w}) = 2l(w) - l(\varphi(z)).$$

Now define

$$\delta_{z,w} = \dim(\mathfrak{h}^{\sigma_z n_{w}^{-1}} \cap \mathfrak{h}^{-\tau_v}).$$

**Theorem 4.1.** Each symplectic leaf in the intersection $O_{z} \cap C_{w}$ has dimension equal to

$$\dim(O_{z} \cap C_{w}) - \delta_{z,w} = 2l(w) - l(\varphi(z)) - \delta_{z,w}.$$

**Proof.** We use Proposition [32] to compute dimensions of the symplectic leaves in $O_{z} \cap C_{w}$. Let $x = g_{z}B \in X$ be a point in $O_{z} \cap C_{w}$, where $g_{z} \in \mathbb{Z}$ lies in the double coset $z$. Let $l_{x} = \text{Ad}_{g_{z}}(t + n) \in L$. Let $n(l_{x}) = \mathfrak{g}_{0} \cap \text{Ad}_{g_{z}}(\mathfrak{h} + n)$ be the normalizer subalgebra of $l_{x}$ in $\mathfrak{g}_{0}$, let $m(l_{x})$ be the annihilator subspace of $n(l_{x})$ in $l_{x}$, and let $\mathcal{V}(l_{x}) = n(l_{x}) + m(l_{x})$. We claim that $\mathcal{V}(l_{x}) = \text{Ad}_{g_{z}}(\mathfrak{h}^{\sigma_z} + n)$. Indeed, it follows from the definition of $\sigma_z$ that

$$\text{Ad}_{g_{z}}(\mathfrak{h}^{\sigma_z}) \subset \mathfrak{g}_{0} \cap \text{Ad}_{g_{z}}(\mathfrak{h} + n) = n(l_{x}).$$

It is also clear that $\text{Ad}_{g_{z}} n \subset m(l_{x})$, so

$$\text{Ad}_{g_{z}}(\mathfrak{h}^{\sigma_z} + n) \subset n(l_{x}) + m(l_{x}) = \mathcal{V}(l_{x}).$$

Since both $\text{Ad}_{g_{z}}(\mathfrak{h}^{\sigma_z} + n)$ and $\mathcal{V}(l_{x})$ have the same dimension, they must coincide.

Now let $S_{x}$ be the symplectic leaf of $\Pi_{x}$ in $X$ through $x$. By Proposition [32] the codimension of $S_{x}$ in $O_{z}$ is equal to $\dim(\mathcal{V}(l_{x}) \cap l_{d})$. Let $w \in N_{G}(\mathfrak{h})$ be a representative of $w$ in $K$. Since $x \in C_{w}$, there exist $n \in N$ and $b \in B$ such that $g_{z} = nbw$. Then we have

$$\mathcal{V}(l_{x}) \cap l_{d} = (\text{Ad}_{n lb}(\mathfrak{h}^{\sigma_z} + n)) \cap (\mathfrak{h}^{-\tau_v} + n) = \text{Ad}_{n}(((\text{Ad}_{w}(\mathfrak{h}^{\sigma_z} + n)) \cap (\mathfrak{h}^{-\tau_v} + n)) = \text{Ad}_{n} \left(\mathfrak{h}^{\sigma_z n_{w}^{-1}} \cap \mathfrak{h}^{-\tau_v} + (\text{Ad}_{w} n) \cap n\right),$$

where in the last line we have the direct sum of vector spaces. Since

$$\dim(\text{Ad}_{w} n) \cap n = \dim_{\mathbb{R}} X - \dim_{\mathbb{R}} C_{w},$$

we have

$$\dim(\mathcal{V}(l_{x}) \cap l_{d}) = \delta_{z,w} + \dim_{\mathbb{R}} X - \dim_{\mathbb{R}} C_{w},$$

and thus

$$\dim S_{x} = \dim O_{x} - \dim(\mathcal{V}(l_{x}) \cap l_{d}) = \dim(O_{x} \cap C_{w}) - \delta_{z,w}.\qedhere$$
Note that the number $\delta_{z,w}$ depends only on $d$ and the two Weyl group elements $\varphi(z)$ and $w$. Define $d : W \to W$ by $d(w) = \gamma_d w \gamma_d$. Following [12], we say that $w \in W$ is a $d$-twisted involution if $d(w) = w^{-1}$. Denote by $\mathcal{I}_d$ the set of all $d$-twisted involutions in $W$. Since for every $g \in G$ we have $\tau_v(g^{-1} \tau_v(g)) = (g^{-1} \tau_v(g))^{-1}$, every $\varphi(z)$ is in $\mathcal{I}_d$. The Weyl group $W$ acts on $\mathcal{I}_d$ by 

$$w_1 \ast w = w_1 \, w_1^{-1} \, (w_1^{-1})^{-1}$$

for $w_1 \in W$ and $w \in \mathcal{I}_d$, and the set $\varphi(Z) \subset \mathcal{I}_d$ is $W$-invariant. In fact, the $W$-action on $G/H$, given by $w \cdot yH = g w^{-1} H$, commutes with the left action of $G_v$ by left multiplication, and thus induces a left action of $W$ on $Z$, which we denote by $w \cdot z$ for $w \in W$ and $z \in Z$. It is easy to see that $\varphi : Z \to W$ is $W$-equivariant, i.e. $\varphi(w \cdot z) = w \ast \varphi(z)$ for all $w \in W$ and $z \in Z$. Similarly, the involution $\tau_v : G \to G$ gives rise to an involution on $Z$ which depends only on $d$. Denote this involution by $z \mapsto d(z)$. Then we also have $\varphi(d(z)) = d \varphi(z) = \varphi(z)^{-1}$. As maps on $\mathfrak{h}$, we see that $w \sigma z w^{-1} = (w \ast \varphi(z)) \tau_v$. Thus we also have

$$\delta_{z,w} = \dim (\mathfrak{h} \cdot (w \ast \varphi(z)) \tau_v \cap \mathfrak{h}^{-\tau_v}).$$

**Corollary 4.2.** 1) When $w \ast \varphi(z) = 1$, symplectic leaves of $\Pi_v$ in $O_z \cap C_w$ are precisely its connected components.

2) Every open orbit $O_z$ has an open symplectic leaf $O_z \cap C_w$, where $w_0$ is the longest element in $W$.

3) If $d = 1$, symplectic leaves in an open orbit $O_z$ are precisely the connected components of intersections of Bruhat cells with $O_z$.

**Proof.** 1) When $w \ast \varphi(z) = 1$, we have $\delta_{z,w} = 0$, so every symplectic leaf in $O_z \cap C_w$ is open in $O_z \cap C_w$.

2) Since $C_{w_0}$ is dense in $X$, it intersects with every open orbit $O_z$. Since an orbit $O_z$ is open if and only if $\varphi(z) = 1$, statement 2) follows from 1) and the fact that $w_0$ commutes with $d$. The fact that $C_{w_0} \cap O_z$ is connected follows from the observation that $O_z$ is a connected open complex submanifold of $X$ and thus $O_z \cap (X \setminus C_{w_0})$ is a divisor in $O_z$.

3) follows directly from 1). \qed

Now consider the group $H^{\tau_v} = H \cap G_v$. Since the centralizer of $\mathfrak{h}^{\tau_v}$ in $G_v$ also centralizes $\mathfrak{h}$, we see that $H^{\tau_v}$ is the Cartan subgroup of $G_v$ corresponding to the Cartan subalgebra $\mathfrak{h}^{\tau_v}$. Then according to [9] Proposition 7.90 the group $H^{\tau_v}$ is connected.

The Poisson structure $\Pi_v$ on $X$ is $H^{\tau_v}$-invariant. Indeed, let $R \in \wedge^2 \mathfrak{g}$ be the element given in [4] for $I_1 = \mathfrak{g}_v$ and $I_2 = \mathfrak{i}_d$. We can also represent $R$ as $R = \sum_i \xi_i \wedge y_i$, where $\{y_i\}$ is a basis of $\mathfrak{g}_v$, and $\{\xi_i\}$ is the dual basis of $\mathfrak{i}_d$ with respect to the pairing between $\mathfrak{g}_v$ and $\mathfrak{i}_d$ given by $(\cdot, \cdot)$, the imaginary part of the Killing form on $\mathfrak{g}$. If $h \in H^{\tau_v}$, then $\{\text{Ad}_h y_i\}$ is a basis of $\mathfrak{g}_v$, and $\{\text{Ad}_h \xi_i\}$ is its dual basis. Thus $\text{Ad}_h R = R$.

Let $z \in Z$ and $w \in W$ be such that $O_z$ and $C_w$ have a non-empty intersection, and let $x \in O_z \cap C_w$. Clearly, $H^{\tau_v}$ leaves $O_z \cap C_w$ invariant. Since the Poisson structure $\Pi_v$ is $H^{\tau_v}$-invariant, if $S_x$ is the symplectic leaf of $\Pi_v$ through $x$, then $h S_x := \{h x_1 : x_1 \in S_x\}$ is the symplectic leaf of $\Pi_v$ through $hx$. Define

$$F_x := \bigcup_{h \in H^{\tau_v}} h S_x.$$

**Proposition 4.3.** For any $x \in X$, the set $F_x$ is a connected component of $O_z \cap C_w$. 

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Proposition 5.1. As in the proof of Theorem 4.1, let
\[ h \rightarrow \mathfrak{h}^\tau_v \]
for \( h \rightarrow \mathfrak{h}^\tau_v \) is the projection with respect to the decomposition \( \mathfrak{h} = \mathfrak{h}^\tau_v + \mathfrak{h}^{\tau_w} \). Assume the claim. Then since the kernel of the map \( p : \mathfrak{h}^\tau_v \rightarrow \mathfrak{h}^{\tau_w} \) has dimension
\[ \dim(\mathfrak{h}^\tau_v) = \dim(\mathfrak{h}^\tau_w) + \delta_{z,w} = \delta_{z,w}. \]
the image of the map
\[ J_x : \mathfrak{h}^\tau_v \rightarrow T_x O_z / T_z S_z : y \mapsto X_y(x) + T_z S_z \]
has dimension equal to \( \dim(\mathfrak{h}^\tau_v) - \dim(\mathfrak{h}^\tau_w) = \delta_{z,w} \). Thus \( J_x \) is onto, and the \( H^\tau_v \)-orbit in \( O_z \cap C_w \) through \( x \) is transversal to the symplectic leaf \( S_x \). It follows that \( F_x \) is open in \( O_z \cap C_w \).

It remains to prove the claim. Also denote by \( p : \mathfrak{g} \rightarrow \mathfrak{g}_w \), the projection with respect to the decomposition \( \mathfrak{g} = \mathfrak{g}_w + \mathfrak{h}^\tau \), and let \( q \) be the projection \( q : \mathfrak{g}_w \rightarrow \mathfrak{g}_w / \mathfrak{g}_w \cap \text{Ad}_{\mathfrak{g}} \mathfrak{b} \cong T_z O_z \). Then by [3] Corollary 7.3, we have \( T_x S_z = (q \circ \pi)(\mathcal{V}(I_x)) \), where, as in the proof of Theorem 4.1, \( \mathcal{V}(I_x) = \text{Ad}_{\mathfrak{g}}(\mathfrak{h}^{\tau_w} + n) \). Let \( y \in \mathfrak{h}^\tau_v \). If \( X_y(x) \in T_z S_z \), then there exist \( y_1 \in \mathfrak{g}_w \) and \( y_2 \in \mathfrak{g}_w \) with \( y_1 + y_2 \in \mathcal{V}(I_x) \) such that \( y - y_2 \in \mathfrak{g}_w \cap \text{Ad}_{\mathfrak{g}} \mathfrak{b} \subset \mathcal{V}(I_x) \). Thus \( y + y_1 = y - y_2 + y_1 + y_2 \in \mathcal{V}(I_x) \). Write \( y_1 = \xi_1 + u_1 \), where \( \xi_1 \in \mathfrak{h}^{\tau_w} \) and \( u_1 \in n \). Then there exist \( \xi_2 \in \mathfrak{h}^{\tau_w} \) and \( u_2 \in n \) such that \( y + \xi_1 + u_1 = \text{Ad}_{\mathfrak{g}}(\xi_2 + u_2) \). Write \( g_z = n\mathfrak{w}b \), where \( n \in N, b \in \mathfrak{b}, \) and \( \mathfrak{w} \) is a representative of \( w \) in \( K \). Write \( \text{Ad}_{n\mathfrak{w}}(y + \xi_1 + u_1) = y + \xi_1 + u_1' \) and \( \text{Ad}_{\mathfrak{w}}(\xi_2 + u_2) = \xi_2 + u_2' \), where \( u_1', u_2' \in n \). Then we have
\[ y + \xi_1 + u_1' = \text{Ad}_{\mathfrak{w}}(\xi_2 + u_2'). \]
Since \( y + \xi_1, \text{Ad}_{\mathfrak{w}}\xi_2 \in \mathfrak{h} \) and \( u_1', u_2' \in n \), we have
\[ y + \xi_1 = \text{Ad}_{\mathfrak{w}}(\xi_2 + u_2'). \]
Thus \( y \in (\mathfrak{h}^{\tau_w} + n) \). Conversely, if \( y \in \mathfrak{h}^{\tau_w} \) is such that \( y + \xi_1 \in (\mathfrak{h}^{\tau_w} + n) \), then \( \text{Ad}_{\mathfrak{w}} \mathfrak{h}_{\xi_2} \) for some \( \xi_2 \in \mathfrak{h}^{\tau_w} \), write \( y + \xi_1 = \text{Ad}_{\mathfrak{w}}\xi_2 \) for \( \xi_2 \in \mathfrak{h}^{\tau_w} \). Let \( \text{Ad}_{n\mathfrak{w}}(\xi_2 + u_2) = \xi_2 + u_2' \), where \( u_2' \in n \). Then we have
\[ \text{Ad}_{n\mathfrak{w}}(y + \xi_1) = \text{Ad}_{n\mathfrak{w}}(\xi_2 + u_2) \in \mathcal{V}(I_x). \]
On the other hand, let \( \text{Ad}_{n\mathfrak{w}}(y + \xi_1) = y + \xi_1 + u_1 \) with \( u_1 \in n \). We see that \( y = (\text{Ad}_{n}(y + \xi_1)) \) so \( X_y(x) \in T_x S_z \).

5..Invariant Poisson cohomology of open orbits

Let \( O_z \) be a \( G_v \)-orbit in \( X \) equipped with the Poisson structure \( \Pi_v \). Then \((O_z, \Pi_v)\) is a Poisson homogeneous space for the Poisson Lie group \( G_v \). The \( G_v \)-invariant Poisson cohomology of \((O_z, \Pi_v)\), denoted by \( H_{\Pi_v, G_v}^\bullet(O_z) \), is defined as the cohomology of the cochain complex \((\chi^\bullet(O_z)^{G_v}, \partial_{\Pi_v})\), where \( \chi^\bullet(O_z)^{G_v} \) is the space of all \( G_v \)-invariant complex multi-vector fields on \( O_z \), \( d_{\Pi_v}(V) = [\Pi_v, V] \), and \([\cdot, \cdot]\) is the Schouten bracket of the multi-vector fields.

Proposition 5.1. When \( O_z \) is an open \( G_v \)-orbit in \( X \), the \( G_v \)-invariant Poisson cohomology \( H_{\Pi_v, G_v}^\bullet(O_z) \) is isomorphic to the de Rham cohomology of \( X \).

Proof. As in the proof of Theorem 4.1, let \( x = g_z B \in X \) be an arbitrary point in \( O_z \), where \( g_z \in Z \) is in the coset \( z \), and let \( V(I_x) = \text{Ad}_{\mathfrak{g}}(\mathfrak{h}^{\tau_w} + n) \). Since \( O_z \) is open, the stabilizer subalgebra of \( g_z \) at \( x \) is \( \mathfrak{g}_w \cap V(I_x) = \text{Ad}_{\mathfrak{g}}(\mathfrak{h}^{\tau_w}) \). By [3] Theorem 7.5, the \( G_v \)-invariant Poisson cohomology \( H_{\Pi_v, G_v}^\bullet(O_z) \) is isomorphic to the relative Lie algebra cohomology of the Lie algebra \( V(I_x) \otimes \mathbb{C} \) relative to the subalgebra
(\text{Ad}_g, (\mathfrak{h}^*)^+) \otimes \mathbb{C}$. Thus the $G_v$-invariant Poisson cohomology is isomorphic to the $\mathfrak{g}$-invariant part of the Lie algebra cohomology of the direct sum Lie algebra $\mathfrak{n} \oplus \mathfrak{n}$ with coefficients in $\mathbb{C}$, which by Kostant’s theorem [7], is isomorphic to the de Rham cohomology of $X$. □

6. Remarks

We have constructed a Poisson structure $\Pi_v$ on $X$ for each Vogan diagram $v$ for $\mathfrak{g}$ (which is not necessarily normalized). In particular, each Bruhat cell $C_w$ in $X$ carries the Poisson structure $\Pi_v$. It would be interesting to study connections between the Poisson structures for different $v$. Especially interesting are the properties of $\Pi_v$ that depend only on the inner class $d$ of the real form $\mathfrak{g}_v$. We also remark that the Poisson structure $\Pi_v$ is defined on the whole variety $\mathcal{L}$ of Lagrangian subalgebras of $\mathfrak{g}$. We have only been looking at the restriction of $\Pi_v$ to a particular $G$-orbit, namely the $G$-orbit through the Lagrangian subalgebra $\mathfrak{t} + \mathfrak{n}$. There are many other interesting $G$-orbits in $\mathcal{L}$, such as the $G$-orbit through a given real form of $\mathfrak{g}$. It would be interesting to study the properties of the Poisson structure $\Pi_v$ on these orbits as well as on their closures, with respect to both the classical topology and the Zariski topology.

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