POISSON STRUCTURES ON COMPLEX FLAG MANIFOLDS
ASSOCIATED WITH REAL FORMS

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Dedicated to Alan Weinstein on the occasion of his 60th birthday

ABSTRACT. For a complex semisimple Lie group $G$ and a real form $G_0$ we define a Poisson structure on the variety of Borel subgroups of $G$ with the property that all $G_0$-orbits in $X$ as well as all Bruhat cells (for a suitable choice of a Borel subgroup of $G$) are Poisson submanifolds. In particular, we show that every non-empty intersection of a $G_0$-orbit and a Bruhat cell is a regular Poisson manifold, and we compute the dimension of its symplectic leaves.

1. Introduction

Let $G$ be a connected and simply-connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$, and let $X$ be the variety of Borel subalgebras of $\mathfrak{g}$. In this paper we use a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ to define a Poisson structure on $X$. This Poisson structure depends on a choice of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{g}_0 \cap \mathfrak{b}$ is a maximally compact Cartan subalgebra of $\mathfrak{g}_0$. Instead of dealing with each real form individually, we fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Then, as is shown in [6], a real form $\mathfrak{g}_v$ of $\mathfrak{g}$ can be constructed from each Vogan diagram $v$ for $\mathfrak{g}$ such that $\mathfrak{g}_v \cap \mathfrak{b}$ is a maximally compact Cartan subalgebra of $\mathfrak{g}_v$. The corresponding Poisson structure on $X$ is denoted by $\Pi_v$.

Let $G_v$ be the real form of $G$ corresponding to $\mathfrak{g}_v$, and let $B$ be the Borel subgroup of $G$ with Lie algebra $\mathfrak{b}$. The Poisson structure $\Pi_v$ has the property that each $G_v$-orbit as well as each $B$-orbit in $X$ is a Poisson submanifold. The $B$-orbits in $X$ will be referred to as the Bruhat cells. We compute the rank of $\Pi_v$. In particular, if a $G_v$-orbit $\mathcal{O}$ meets a Bruhat cell $\mathcal{C}$, they intersect transversally, and we find that all the symplectic leaves in $\mathcal{O} \cap \mathcal{C}$ have the same dimension, so $\mathcal{O} \cap \mathcal{C}$ is a regular Poisson manifold. Moreover, we show that all symplectic leaves in each connected component of $\mathcal{O} \cap \mathcal{C}$ are translates of each other by elements of a Cartan subgroup of $G_v$. We also show that the $G_v$-invariant Poisson cohomology for each open $G_v$-orbit in $X$ is isomorphic to the de Rham cohomology of $X$.

Results similar to those presented here for the full flag manifold $X = G/B$ are also valid for a partial flag manifold $G/P$, where $P$ is a parabolic subgroup of $G$. The corresponding Poisson structure $\Pi_v$ is defined similarly. The results about the symplectic leaves in $\mathcal{O} \cap \mathcal{C}$ also hold in this case.
Let $\mathfrak{g}$ be a complex simple Lie algebra. In this section we recall the classification of real forms of $\mathfrak{g}$ by Vogan diagrams. Details can be found in [6, Chapter 6].

Suppose that $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$ and that $\tau_0$ is the corresponding complex-conjugate linear involution on $\mathfrak{g}$. Let $\theta_0$ be a Cartan involution of $\mathfrak{g}_0$, and let $\mathfrak{h}_0$ be a $\theta_0$-stable maximally compact Cartan subalgebra of $\mathfrak{g}_0$. Set $t_0 = \mathfrak{h}_0^0$ and $a_0 = \mathfrak{h}_0^{-\theta_0}$ so that $\mathfrak{h}_0 = t_0 + a_0$. Let $\gamma_0$ be the complexification of $\theta_0$. Then the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0 + i\mathfrak{h}_0$ of $\mathfrak{g}$ is $\gamma_0$-stable. Let $\Delta$ be the root system for $(\mathfrak{g}, \mathfrak{h})$. Since $\mathfrak{h}_0$ is a maximally compact Cartan subalgebra of $\mathfrak{g}_0$, there exists $x_0 \in t_0$ that is regular for $\Delta$. Define the subset $\Delta^+$ of positive roots in $\Delta$ by $\alpha \in \Delta^+$ if and only if $\alpha(x_0) > 0$. Then $\gamma_0(\Delta^+) = \Delta^+$. Let $\Sigma \subset \Delta^+$ be the set of simple roots in $\Delta^+$. Then $\gamma_0(\Sigma) = \Sigma$, so $\gamma_0$ gives rise to an involutive automorphism of the Dynkin diagram of $\mathfrak{g}$. Let $\mathcal{I}$ be the set of non-compact imaginary simple roots. The Vogan diagram of $\mathfrak{g}_0$ associated to the triple $(\theta_0, \mathfrak{h}_0, \Delta^+)$ is the Dynkin diagram $D(\mathfrak{g})$ of $\mathfrak{g}$, together with an involutive automorphism $\gamma_0$ on $D(\mathfrak{g})$ and the vertices corresponding to the simple roots in $\mathcal{I}$ painted black.

In general, a Vogan diagram for $\mathfrak{g}$ is defined to be a triple $(D(\mathfrak{g}), d, \mathcal{I})$, where $D(\mathfrak{g})$ is the Dynkin diagram of $\mathfrak{g}$, $d$ is an involutive automorphism of $D(\mathfrak{g})$, and $\mathcal{I}$ is a subset of vertices of $D(\mathfrak{g})$ such that $d(\alpha) = \alpha$ for each $\alpha \in \mathcal{I}$. Every Vogan diagram for $\mathfrak{g}$ comes from a real form of $\mathfrak{g}$ (see below), although two different Vogan diagrams can come from isomorphic real forms. A non-redundant list of Vogan diagrams with the corresponding isomorphism class of real forms for all simple Lie algebras is given in [6]. Every Vogan diagram in the list in [6] is normalized in the sense that at most one vertex is painted black.

For the purpose of defining Poisson structures on the variety of Borel subalgebras of $\mathfrak{g}$, we now recall the explicit construction of a real form of $\mathfrak{g}$ from a Vogan diagram [6, Theorem 6.88]. We need to fix the following data for $\mathfrak{g}$.

Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let $\Delta$ be the root system for $(\mathfrak{g}, \mathfrak{h})$. Fix a choice of positive roots $\Delta^+$ and let $\Sigma$ be the basis of simple roots. Let $\langle \langle,\rangle \rangle$ be the Killing form of $\mathfrak{g}$ and let root vectors $\{E_\alpha : \alpha \in \Delta\}$ be chosen such that $\langle E_\alpha, E_-\alpha \rangle = H_\alpha$ for each $\alpha \in \Delta^+$, where $H_\alpha$ is the unique element of $\mathfrak{h}$ defined by $\langle H, H_\alpha \rangle = \alpha(H)$ for all $H \in \mathfrak{h}$, and such that the numbers $m_{\alpha,\beta}$ given by $[E_\alpha, E_\beta] = m_{\alpha,\beta}E_{\alpha+\beta}$ when $\alpha + \beta \in \Delta$ are real. Define a compact real form $\mathfrak{k}$ of $\mathfrak{g}$ as

$$\mathfrak{k} = \text{span}_\mathbb{R}\{iH_\alpha, X_\alpha := E_\alpha - E_-\alpha, Y_\alpha := i(E_\alpha + E_-\alpha)\},$$

and let $\theta$ be the complex conjugation of $\mathfrak{g}$ defining $\mathfrak{k}$. If $d$ is an involutive automorphism of the Dynkin diagram of $\mathfrak{g}$, define $\gamma_d$ to be the unique automorphism of $\mathfrak{g}$ satisfying $\gamma_d(H_\alpha) = H_{d(\alpha)}$ and $\gamma_d(E_\alpha) = E_{d(\alpha)}$ for each simple root $\alpha$.

Given a Vogan diagram $v$ for $\mathfrak{g}$, not necessarily normalized, with the involutive diagram automorphism $d$, let $t_v$ be the unique element in the adjoint group of $\mathfrak{g}$ such that

$$Ad_{t_v}(E_\alpha) = \begin{cases} E_\alpha & \text{if } \alpha \text{ is a blank vertex in } v, \\ -E_\alpha & \text{if } \alpha \text{ is a painted vertex in } v. \end{cases}$$
Define a complex conjugate linear involution
\[ \tau_v := \text{Ad}_{v^\tau} \circ \gamma_d \circ \theta. \]

**Notation 2.1.** We use \( g_v = g^{\tau_v} \) to denote the real form of \( g \) defined by \( \tau_v \). Set \( \theta_v = \theta|_{g_v} \). Then \( \theta_v \) is a Cartan involution of \( g_v \), and \( h_v^{\tau_v} \) is a \( \theta_v \)-stable maximally compact Cartan subalgebra of \( g_v \), with \( h = h_v^{\tau_v} + i h_v^{\tau_v} \). The complexification of \( \tau_v \) is
\[ \gamma_v := \tau_v \theta = \text{Ad}_{v^\tau} \gamma_d. \]

Since \( \gamma_v(\Delta^+) = \Delta^+ \), the Vogan diagram of \( g_v \) associated to the triple \( (\theta_v, h_v^{\tau_v}, \Delta^+) \) is \( v \).

One of the advantages of introducing the real form \( g_v \) is as follows. We say that a real subalgebra \( l \) of \( g \) is Lagrangian if its real dimension is equal to the complex dimension of \( g \) and if \( \text{Im} \langle x_1, x_2 \rangle = 0 \) for all \( x_1, x_2 \in l \). A decomposition \( g = l_1 + l_2 \) is called a Lagrangian splitting if both \( l_1 \) and \( l_2 \) are Lagrangian. Let \( n \) be the subalgebra of \( g \) spanned by the set of all positive root vectors for \( \Delta^+ \). The following fact is easy to prove.

**Lemma 2.2.** Let \( l_d := h^{-\tau_v + n} \). Then \( g = g_v + l_d \) is a Lagrangian splitting of \( g \).

Let \( a = \text{span}_R \{ iH_\alpha : \alpha \in \Sigma \} \), and let \( t = i a \). We note that since
\[ h^{-\tau_v} = h^{-\gamma_d \theta} = t^{-\gamma_d} + a^{-\gamma_d}, \]
the Lagrangian complement \( l_d \) of \( g_v \) depends only on \( d \), and in the case when \( d = 1 \), we have \( l_d = a + n \). Note that \( h^{\tau_v} = h^{\gamma_d \theta} = t^{\gamma_d} + a^{\gamma_d} \) also depends only on \( d \).

**Remark 2.3.** Recall [2] Definition 6.10 that two real forms \( \tau_1 \) and \( \tau_2 \) are said to be in the same inner class if there exists \( g \in \text{Int}(g) \), the adjoint group of \( g \), such that \( \tau_1 = \text{Ad}_g \tau_2 \). Inner classes of real forms are in one-to-one correspondence with involutive automorphisms of the Dynkin diagram of \( g \) [2 Proposition 6.12]. Let \( d \) be an involutive automorphism of \( D(g) \). Then as \( v \) runs over the collection of all Vogan diagrams with \( d \) as the diagram automorphism, the real form \( g_v \) runs over all \( \text{Int}(g) \)-conjugacy classes of real forms of \( g \) in the inner class corresponding to \( d \).

3. The Poisson Structure \( \Pi_v \) on \( X \)

Let \( g \) be a complex semi-simple Lie algebra, and let \( X \) be the variety of all Borel subalgebras of \( g \). We keep the notation from Section 2. Let \( v \) be a Vogan diagram for \( g \) and let \( g_v = g^{\tau_v} \) be the real form of \( g \) constructed in Section 2. Let \( G \) be the connected and simply-connected Lie group with Lie algebra \( g \). Without any risk of confusion, we shall also denote by \( \tau_v \) the lift of \( \tau_v \) from \( g \) to \( G \), and we set \( G_v = G^{\tau_v} \). It follows from [3] Theorem 8.2, p. 320] that the group \( G_v \) is connected.

In this section, we will start with a Vogan diagram \( v \) for \( g \) and define a Poisson structure \( \Pi_v \) on \( X \) such that every \( G_v \)-orbit in \( X \) is a Poisson submanifold. This Poisson structure comes from an identification of \( X \) with the \( G \)-orbit through \( t + n \) inside the variety \( L \) of Lagrangian subalgebras of \( g \), which was studied in [3]. We now recall the relevant details.

Set \( n = \text{dim}_R g \) and let \( \text{Gr}_R(n, g) \) be the Grassmannian of real \( n \)-dimensional subspaces of \( g \). The set \( L \) of all Lagrangian subalgebras of \( g \) is naturally a real subvariety of \( \text{Gr}_R(n, g) \). The natural action of \( G \) on \( \text{Gr}_R(n, g) \) gives rise to a Lie algebra anti-homomorphism \( \kappa \) from \( g \) to the Lie algebra of vector fields on \( \text{Gr}_R(n, g) \),
whose extension from $\wedge^2 g$ to the space of bi-vector fields on $Gr_G(n, g)$ will also be denoted by $\kappa$. Given a Lagrangian splitting $g = l_1 + l_2$, we define the element $R_{l_1, l_2} \in \wedge^2 g$ by
\begin{equation}
(R_{l_1, l_2}, (x_1 + \xi_1) \wedge (x_2 + \xi_2)) = \langle \xi_2, x_1 \rangle - \langle \xi_1, x_2 \rangle, \quad x_1, x_2 \in l_1, \xi_1, \xi_2 \in l_2,
\end{equation}
where $\langle \cdot, \cdot \rangle = \text{Im}(\langle \cdot, \cdot \rangle)$.

Set $\Pi_{l_1, l_2} = \frac{1}{2}\kappa(R_{l_1, l_2})$. Clearly, $\Pi_{l_1, l_2}$ is tangent to every $G$-orbit in $Gr_G(n, g)$, so it is tangent to $L$.

**Theorem 3.1 (\cite{3} Theorems 2.14 and 2.18).** The bi-vector field $\Pi_{l_1, l_2}$ restricts to a Poisson structure on $L$. If $L_1$ and $L_2$ are the connected subgroups of $G$ with Lie algebras $l_1$ and $l_2$ respectively, then all the $L_1$- as well as $L_2$-orbits in $L$ are Poisson submanifolds with respect to $\Pi_{l_1, l_2}$.

For $l \in L$, let $n(l)$ be the normalizer subalgebra of $l$ in $l_1$. Let $m(l)$ be the annihilator of $n(l)$ in $l$, i.e. $m(l) = \{x \in l : \langle x, y \rangle = 0 \ \forall y \in n(l)\} \subseteq l$, and let $\mathcal{V}(l) = n(l) + m(l)$.

**Proposition 3.2 (\cite{3} Theorem 2.21, \cite{3} Corollary 7.3).** For each $l \in L$, the space $\mathcal{V}(l)$ is a Lagrangian subalgebra of $g$. The co-dimension of the symplectic leaf of $\Pi_{l_1, l_2}$ through $l$ in the orbit $L_1 \cdot l$ is equal to $\dim(\mathcal{V}(l) \cap l_2)$.

**Notation 3.3.** Let $v$ be a Vogan diagram for $g$. We denote by $\Pi_v$, the Poisson structure on $L$ defined by the Lagrangian splitting $g = g_v + l_d$ in Lemma 2.2. Let $H$, $N$, and $B$ be respectively the connected subgroups of $G$ with Lie algebras $h$, $n$, and $b = b + n$, so $B = HN$. Identify the $G$-orbit through $t + n \in L$ with $G/B \cong X$. The induced Poisson structure on $X$ will also be denoted by $\Pi_v$. Let $H^{-\gamma_d \theta} = \{h \in H : \gamma_d \circ \theta(h) = h^{-1}\}$ and let $L_d = H^{-\gamma_d \theta}N$. By the Bruhat lemma, orbits of $L_d$ in $X \cong G/B$, which are the same as the $N$-orbits in $X$, are labeled by the elements in the Weyl group $W$ of $\Delta$. We refer to these $N$-orbits as the Bruhat cells in $X$.

By \cite{3} Theorem 2.18, we have

**Proposition 3.4.** Each $G_v$-orbit in $X$ as well as each Bruhat cell in $X$ is a Poisson submanifold with respect to $\Pi_v$.

When $v$ is the Vogan diagram with $d = 1$ and no vertex painted, we have $\tau_v = \theta$, so $g_v = t$. The Poisson structure $\Pi_v$ in this case was first introduced in \cite{11} and \cite{13}, and it has the property that its symplectic leaves are precisely the Bruhat cells (hence the name “Bruhat Poisson structure” in \cite{11}). In \cite{3} and \cite{10} this Poisson structure was related to some earlier work of Kostant \cite{7} and of Kostant-Kumar \cite{8} on the Schubert calculus on $X$.

The splitting $g = g_v + l_d$ naturally defines a Lie bialgebra structure on $g_v$ and therefore a Poisson Lie group structure on $G_v$ \cite{11}. All the $G_v$-orbits in $L$ become $G_v$-Poisson homogeneous spaces \cite{3, 9}. We remark that in \cite{11}, Andruskiewitsch and Juncsa classified non-triangular Lie bialgebra structures on $g_v$ using Belavin-Drinfeld triples. The one defined by the splitting $g = g_v + l_d$ comes from the standard Belavin-Drinfeld triple. We refer to \cite{1} for details.

**Example.** Here we take $g = sl(2, \mathbb{C})$ and

\[ g_v = su(1, 1) = \left\{ \begin{pmatrix} ix & y + iz \\ iz & y - ix \end{pmatrix} : x, y, z \in \mathbb{R} \right\}. \]
Then \( d = 1 \) and \( \mathfrak{l}_g = \mathfrak{a} + \mathfrak{n} \) consists of upper triangular matrices in \( \mathfrak{sl}(2, \mathbb{C}) \) with real diagonal entries. Identify \( G/B \) with \( \mathbb{P}^1 \) via the action

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [w_0 : w_1] = [aw_0 + bw_1 : cw_0 + dw_1]
\]

of \( G \) on \( \mathbb{P}^1 \) and by taking \([1 : 0] \in \mathbb{P}^1 \) as the basepoint. There are two Bruhat cells: the zero-dimensional basepoint \([1 : 0]\), and the other being the rest:

\[
U_1 = \mathbb{P}^1 \setminus \{[1 : 0]\} = \{[w_0 : w_1] : w_1 \neq 0\}.
\]

In terms of the holomorphic coordinate \( z \) on \( U_1 \) given by \( z = w_0/w_1 \), the Poisson structure \( \Pi_v \), up to a scalar multiple, is given by

\[
\Pi_v = i(1 - |z|^2) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}}.
\]

Setting \( u = 1/z \), we see that in the \( u \)-coordinate on the open set

\[
U_0 = \{[w_0 : w_1] \in \mathbb{P}^1, w_0 \neq 0\} = \{[1 : u], u \in \mathbb{C}\},
\]

we have

\[
\Pi_v = i(|u|^2 - 1)|u|^2 \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial \bar{u}}.
\]

Thus \( \Pi_v \) vanishes precisely at the basepoint \([1 : 0]\) and at every point of the form \([z : 1]\) with \(|z| = 1\). If we identify \( \mathbb{P}^1 \) with the unit sphere \( S^2 \) in \( \mathbb{R}^3 \) via

\[
P^1 \rightarrow S^2: [w_0, w_1] \mapsto \left( \frac{2\text{Re}(w_0\bar{w_1})}{|w_0|^2 + |w_1|^2}, \frac{2\text{Im}(w_0\bar{w_1})}{|w_0|^2 + |w_1|^2}, \frac{|w_0|^2 - |w_1|^2}{|w_0|^2 + |w_1|^2} \right),
\]

then we see that \( \Pi_v \) vanishes at the “North pole” \((0, 0, 1)\) and at every point on the Equator \( x_3 = 0 \). Under this identification, there are exactly three orbits of \( \text{SU}(1, 1) \) on \( S^2 \): the Northern hemisphere, the Equator, and the Southern hemisphere. Each of these three orbits is clearly a Poisson submanifold.

4. Symplectic leaves of \( \Pi_v \) in \( X \)

Suppose that \( \mathcal{O} \) is a \( G_v \)-orbit in \( X \) and \( C \) is a Bruhat cell such that \( \mathcal{O} \cap C \neq \emptyset \). Since \( g = g_v + \mathfrak{l}_d \), \( \mathcal{O} \) and \( C \) intersect transversally. By Proposition 3.4 \( \mathcal{O} \cap C \) is a Poisson submanifold of \( \mathcal{O}_v \). In this section we show that \( (\mathcal{O} \cap C, \Pi_v) \) is a regular Poisson manifold, and we compute the dimension of its symplectic leaves.

It is well known [13] that there are only finitely many \( G_v \)-orbits in \( X \). We first recall from [12, Section 6] some facts about these orbits.

Let \( N_G(\mathfrak{h}) \) be the normalizer subgroup of \( \mathfrak{h} \) in \( G \). Set

\[
Z = \{g \in G : g^{-1}v(g) \in N_G(\mathfrak{h})\}.
\]

Then \( H \) acts on \( Z \) from the right by right multiplication, and \( G_v \) acts on \( Z \) from the left by left multiplication. Let \( Z \) be the double coset space

\[
Z = G_v \backslash Z/H.
\]

For each \( z \in Z \), choose any \( g_z \in Z \) in the double coset \( z \) and define \( \mathcal{O}_z \) to be the \( G_v \)-orbit in \( X \) through \( g_zB \in X \cong G/B \). Clearly, \( \mathcal{O}_z \) is independent of the choice of \( g_z \). According to [12, Theorem 6.1.4], the map \( z \mapsto \mathcal{O}_z \) is a one-to-one correspondence between the set \( Z \) and the set of \( G_v \)-orbits in \( X \). Let \( W = N_G(\mathfrak{h})/H \) be the Weyl group. Thus we also have the map

\[
\varphi : Z \rightarrow W : z = G_v g_z H \mapsto g_z^{-1}v(g_z)H \in W.
\]
According to [12, Theorem 6.4.2], the codimension of the $G_v$-orbit $O_z$ in $X$ equals $l(\varphi(z))$, where $l$ is the length function on the Weyl group $W$. We also introduce the map

$$\sigma_z = \varphi(z)\tau_v : \mathfrak{h} \rightarrow \mathfrak{h}.$$ 

For any $g_z$ in the double coset $z$, we also have $\sigma_z = \text{Ad}_{g_z}^{-1} \circ \tau_v \circ \text{Ad}_{g_z}$, so $\sigma_z$ is an involution.

Assume now that $z \in Z$ and $w \in W$ are such that $O_z \cap C_w \neq \emptyset$, where $C_w$ is the Bruhat cell in $X$ corresponding to $w$, i.e. the $N$-orbit through $w \in G/B$. Then $\dim C_w = 2l(w)$, and since $O_z$ and $C_w$ intersect transversally, we have

$$\dim(O_z \cap C_w) = 2l(w) - l(\varphi(z)).$$

Now define

$$\delta_{z,w} = \dim(\mathfrak{h}^{\sigma_z,\mathfrak{w}^{-1}} \cap \mathfrak{h}^{-\tau_v}).$$

**Theorem 4.1.** Each symplectic leaf in the intersection $O_z \cap C_w$ has dimension equal to

$$\dim(O_z \cap C_w) - \delta_{z,w} = 2l(w) - l(\varphi(z)) - \delta_{z,w}.$$

**Proof.** We use Proposition [3] to compute dimensions of the symplectic leaves in $O_z \cap C_w$. Let $x = g_zB \in X$ be a point in $O_z \cap C_w$, where $g_z \in Z$ lies in the double coset $z$. Let $l_x = \text{Ad}_{g_z}(t+n) \in \mathcal{L}$. Let $n(l_x) = g_v \cap \text{Ad}_{g_z}(\mathfrak{h} + \mathfrak{n})$ be the normalizer subalgebra of $l_x$ in $g_v$, let $m(l_x)$ be the annihilator subspace of $n(l_x)$ in $l_x$, and let $\mathcal{V}(l_x) = n(l_x) + m(l_x)$. We claim that $\mathcal{V}(l_x) = \text{Ad}_{g_z}((\mathfrak{h}^{\sigma_z} + \mathfrak{n})$. Indeed, it follows from the definition of $\sigma_z$ that

$$\text{Ad}_{g_z}((\mathfrak{h}^{\sigma_z}) \subset g_v \cap \text{Ad}_{g_z}(\mathfrak{h} + \mathfrak{n}) = n(l_x).$$

It is also clear that $\text{Ad}_{g_z} \subset m(l_x), so$

$$\text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + \mathfrak{w}) \subset m(l_x) + m(l_x) = \mathcal{V}(l_x).$$

Since both $\text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + \mathfrak{w})$ and $\mathcal{V}(l_x)$ have the same dimension, they must coincide.

Now let $S_x$ be the symplectic leaf of $\Pi_v$ in $X$ through $x$. By Proposition [3] the codimension of $S_x$ in $O_z$ is equal to $\dim(V(l_x) \cap \mathfrak{i})$. Let $\hat{w} \in N_G(\mathfrak{h})$ be a representative of $w$ in $K$. Since $x \in C_w$, there exist $n \in N$ and $b \in B$ such that $g_z = nb\hat{w}$. Then we have

$$\mathcal{V}(l_x) \cap \mathfrak{i} = (\text{Ad}_{n\hat{w}b}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})) \cap (\mathfrak{h}^{-\tau_v} + \mathfrak{n})$$

$$= \text{Ad}_n \left( (\text{Ad}_{\hat{w}}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})) \cap (\mathfrak{h}^{-\tau_v} + \mathfrak{n}) \right)$$

$$= \text{Ad}_n \left( \mathfrak{h}^{\sigma_z,\mathfrak{i}^{-1}} \cap \mathfrak{h}^{-\tau_v} + (\text{Ad}_{\hat{w}}(\mathfrak{n}) \cap \mathfrak{n}) \right),$$

where in the last line we have the direct sum of vector spaces. Since

$$\dim(\text{Ad}_{\hat{w}}(\mathfrak{n}) \cap \mathfrak{n} = \dim_X - \dim_C C_w,$$

we have

$$\dim(\mathcal{V}(l_x) \cap \mathfrak{i}) = \delta_{z,w} + \dim_X - \dim C_w,$$

and thus

$$\dim S_x = \dim O_z - \dim(\mathcal{V}(l_x) \cap \mathfrak{i}) = \dim(O_z \cap C_w) - \delta_{z,w}. \quad \square$$
Note that the number $\delta_{z,w}$ depends only on $d$ and the two Weyl group elements $\varphi(z)$ and $w$. Define $d : W \to W$ by $d(w) = \gamma_d w \gamma_d$. Following [12], we say that $w \in W$ is a $d$-twisted involution if $d(w) = w^{-1}$. Denote by $I_d$ the set of all $d$-twisted involutions in $W$. Since for every $g \in G$ we have $\tau_v(g^{-1} \tau_v(g)) = (g^{-1} \tau_v(g))^{-1}$, every $\varphi(z)$ is in $I_d$. The Weyl group $W$ acts on $I_d$ by

$$w_1 * w = w_1 w d(w_1^{-1})$$

for $w_1 \in W$, and $w \in I_d$.

and the set $\varphi(Z) \subset I_d$ is $W$-invariant. In fact, the $W$-action on $G/H$, given by $w \cdot gH = gw^{-1}H$, commutes with the left action of $G_v$ by left multiplication, and thus induces a left action of $W$ on $Z$, which we denote by $w \cdot z$ for $w \in W$ and $z \in Z$. It is also easy to see that $\varphi : Z \to W$ is $W$-equivariant, i.e. $\varphi(w \cdot z) = w \cdot \varphi(z)$ for all $w \in W$ and $z \in Z$. Similarly, the involution $\tau_v : G \to G$ gives rise to an involution on $Z$ which depends only on $d$. Denote this involution by $z \to d(z)$. Then we also have $\varphi(d(z)) = d\varphi(z) = \varphi(z)^{-1}$. As maps on $\mathfrak{h}$, we see that $w \sigma_r w^{-1} = (w \cdot \varphi(z)) \tau_v \sigma_r$. Thus we also have

$$\delta_{z,w} = \dim(\mathfrak{h}^{(w \cdot \varphi(z)) \tau_v \cap \mathfrak{h}^{-\tau_v}}).$$

**Corollary 4.2.** 1) When $w \cdot \varphi(z) = 1$, symplectic leaves of $\Pi_v$ in $O_z \cap C_w$ are precisely its connected components.

2) Every open orbit $O_z$ has an open symplectic leaf $O_z \cap C_{w_0}$, where $w_0$ is the longest element in $W$.

3) If $d = 1$, symplectic leaves in an open orbit $O_z$ are precisely the connected components of intersections of Bruhat cells with $O_z$.

**Proof.** 1) When $w \cdot \varphi(z) = 1$, we have $\delta_{z,w} = 0$, so every symplectic leaf in $O_z \cap C_w$ is open in $O_z \cap C_{w_0}$.

2) Since $C_{w_0}$ is dense in $X$, it intersects with every open orbit $O_z$. Since an orbit $O_z$ is open if and only if $\varphi(z) = 1$, statement 2) follows from 1) and the fact that $w_0$ commutes with $d$. The fact that $C_{w_0} \cap O_z$ is connected follows from the observation that $O_z$ is a connected open complex submanifold of $X$ and thus $O_z \cap (X \setminus C_{w_0})$ is a divisor in $O_z$.

3) follows directly from 1). \qed

Now consider the group $H^{\tau_v} = H \cap G_v$. Since the centralizer of $\mathfrak{h}^{\tau_v}$ in $G_v$ also centralizes $\mathfrak{h}$, we see that $H^{\tau_v}$ is the Cartan subgroup of $G_v$ corresponding to the Cartan subalgebra $\mathfrak{h}^{\tau_v}$. Then according to [8] Proposition 7.90 the group $H^{\tau_v}$ is connected.

The Poisson structure $\Pi_v$ on $X$ is $H^{\tau_v}$-invariant. Indeed, let $R \in \bigwedge^2 \mathfrak{g}$ be the element given in [5] for $l_1 = \mathfrak{g}_a$ and $l_2 = \mathfrak{t}_d$. We can also represent $R$ as $R = \sum_i \xi_i \wedge y_i$, where $\{y_i\}$ is a basis of $\mathfrak{g}_v$, and $\{\xi_i\}$ is the dual basis of $l_d$ with respect to the pairing between $\mathfrak{g}_a$ and $l_d$ given by $\langle \cdot, \cdot \rangle$, the imaginary part of the Killing form on $\mathfrak{g}$. If $h \in H^{\tau_v}$, then $\{\text{Ad}_h y_i\}$ is a basis of $\mathfrak{g}_v$, and $\{\text{Ad}_h \xi_i\}$ is its dual basis. Thus $\text{Ad}_h R = R$.

Let $z \in Z$ and $w \in W$ be such that $O_z$ and $C_w$ have a non-empty intersection, and let $x \in O_z \cap C_w$. Clearly, $H^{\tau_v}$ leaves $O_z \cap C_w$ invariant. Since the Poisson structure $\Pi_v$ is $H^{\tau_v}$-invariant, if $S_z$ is the symplectic leaf of $\Pi_v$ through $x$, then $hS_z := \{hx_1 : x_1 \in S_z\}$ is the symplectic leaf of $\Pi_v$ through $hx$. Define

$$F_x := \bigcup_{h \in H^{\tau_v}} hS_z.$$

**Proposition 4.3.** For any $x \in X$, the set $F_x$ is a connected component of $O_z \cap C_w$. 

When Proposition 5.1.

As in the proof of Theorem 4.1, let $X_y$ be the vector field on $X$ generating the action of $\exp(ty)$ in $H^{\tau_y}$ on $X$. We claim that $X_y(x) \in T_xS_x$ if and only if $y \in p(h^{(w+\varphi(z))\tau_y})$, where $p : h \to h^{\tau_y}$ is the projection with respect to the decomposition $h = h^{\tau_y} + h^{-\tau_y}$. Assume the claim. Then since the kernel of the map $p : h^{(w+\varphi(z))\tau_y} \to h^{\tau_y}$ has dimension equal to $\dim(h^{\tau_y}) = \dim(h^{-\tau_y}) = \delta_z$, the image of the map

$$J_x : \ h^{\tau_y} \to T_xO_z / T_xS_x : \ y \mapsto X_y(x) + T_xS_x$$

has dimension equal to $\dim(h^{\tau_y}) - \dim(h^{(w+\varphi(z))\tau_y}) + \delta_z = \delta_z$. Thus $J_x$ is onto, and the $H^{\tau_y}$-orbit in $O_z \cap C_w$ through $x$ is transversal to the symplectic leaf $S_x$. It follows that $F_x$ is open in $O_z \cap C_w$.

It remains to prove the claim. Also denote by $p : g \to g_v$ the projection with respect to the decomposition $g = g_v + l_d$, and let $q$ be the projection $q : g_v \to g_v / g_v \cap \text{Ad}_{g_v}b = T_xO_z$. Then by [9, Corollary 7.3], we have $T_xS_x = (q \circ p)(V_{(l_z)})$, where, as in the proof of Theorem 4.1, $V_{(l_z)} = \text{Ad}_{g_v}(h^{\tau_z} + n)$. Let $y \in h^{\tau_y}$. If $X_y(x) \in T_xS_x$, then there exist $y_1 \in l_d$ and $y_2 \in g_v$ with $y_1 + y_2 \in V_{(l_z)}$ such that $y - y_2 \in g_v \cap \text{Ad}_{g_v}b \subset V(l_z)$. Thus $y + y_1 = y - y_2 + y_1 + y_2 \in V(l_z)$. Write $y_1 = \xi_1 + u_1$, where $\xi_1 \in h^{-\tau_y}$ and $u_1 \in n$. Then there exist $\xi_2 \in h^{\tau_z}$ and $u_2 \in n$ such that $y + \xi_1 + u_1 = \text{Ad}_{g_v}(\xi_2 + u_2)$. Write $x = \text{nib}(w)$, where $n \in N, b \in B$, and $\hat{w}$ is a representative of $w$ in $K$. Write $\text{Ad}_{n^{-1}}(y + \xi_1 + u_1) = y + \xi_1 + u_1'$ and $\text{Ad}_{n}(\xi_2 + u_2) = \xi_2 + u_2'$, where $u_1', u_2' \in n$. Then we have

$$y + \xi_1 + u_1' = \text{Ad}_w(\xi_2 + u_2').$$

Since $y + \xi_1, \text{Ad}_w\xi_2 \in h$ and $u_1', \text{Ad}_w u_2' \in n + n$, we have $y + \xi_1 = \text{Ad}_w \xi_2 \in h^{(w+\varphi(z))\tau_y}$. Thus $y \in p(h^{(w+\varphi(z))\tau_y})$. Conversely, if $y \in h^{\tau_y}$ is such that $y + \xi_1 \in h^{(w+\varphi(z))\tau_y} = \text{Ad}_w h^{\tau_z}$, write $y + \xi_1 = \text{Ad}_w \xi_2$ for $\xi_2 \in h^{\tau_z}$. Let $\text{Ad}_{n^{-1}} \xi_2 = \xi_2 + u_2$ for some $u_2 \in n$. We then have

$$\text{Ad}_n(y + \xi_1) = \text{Ad}_{nib}(\xi_2 + u_2) \in V_{(l_z)}.$$ 

On the other hand, let $\text{Ad}_n(y + \xi_1) = y + \xi_1 + u_1$ with $u_1 \in n$. We see that $y = p(\text{Ad}_n(y + \xi_1))$ so $X_y(x) \in T_xS_x$.

5. Invariant Poisson cohomology of open orbits

Let $O_z$ be a $G_v$-orbit in $X$ equipped with the Poisson structure $\Pi_v$. Then $(O_z, \Pi_v)$ is a Poisson homogeneous space for the Poisson Lie group $G_v$. The $G_v$-invariant Poisson cohomology of $(O_z, \Pi_v)$, denoted by $H^\bullet_{\Pi_v,G_v}(O_z)$, is defined as the cohomology of the cochain complex $(\chi^\bullet(O_z)^{G_v}, \partial_{\Pi_v})$, where $\chi^\bullet(O_z)^{G_v}$ is the space of all $G_v$-invariant complex multi-vector fields on $O_z$, $d_{\Pi_v}(V) = [\Pi_v, V]$, and $[\cdot, \cdot]$ is the Schouten bracket of the multi-vector fields.

Proposition 5.1. When $O_z$ is an open $G_v$-orbit in $X$, the $G_v$-invariant Poisson cohomology $H^\bullet_{\Pi_v,G_v}(O_z)$ is isomorphic to the de Rham cohomology of $X$.

Proof. As in the proof of Theorem 7.5, the $G_v$-invariant Poisson cohomology $H^\bullet_{\Pi_v,G_v}(O_z)$ is isomorphic to the relative Lie algebra cohomology of the Lie algebra $\mathcal{V}(l_z) \otimes \mathbb{C}$ relative to the subalgebra
Thus the $G_v$-invariant Poisson cohomology is isomorphic to the $\mathfrak{h}$-invariant part of the Lie algebra cohomology of the direct sum Lie algebra $\mathfrak{n} \oplus \mathfrak{n}$ with coefficients in $\mathbb{C}$, which by Kostant’s theorem [7], is isomorphic to the de Rham cohomology of $X$. □

6. Remarks

We have constructed a Poisson structure $\Pi_v$ on $X$ for each Vogan diagram $v$ for $\mathfrak{g}$ (which is not necessarily normalized). In particular, each Bruhat cell $C_w$ in $X$ carries the Poisson structure $\Pi_v$. It would be interesting to study connections between the Poisson structures for different $v$. Especially interesting are the properties of $\Pi_v$ that depend only on the inner class $d$ of the real form $\mathfrak{g}_v$. We also remark that the Poisson structure $\Pi_v$ is defined on the whole variety $L$ of Lagrangian subalgebras of $\mathfrak{g}$. We have only been looking at the restriction of $\Pi_v$ to a particular $G$-orbit, namely the $G$-orbit through the Lagrangian subalgebra $t+n$. There are many other interesting $G$-orbits in $L$, such as the $G$-orbit through a given real form of $\mathfrak{g}$. It would be interesting to study the properties of the Poisson structure $\Pi_v$ on these orbits as well as on their closures, with respect to both the classical topology and the Zariski topology.

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