THE FOURTH POWER MOMENT OF AUTOMORPHIC
L-FUNCTIONS FOR GL(2) OVER A SHORT INTERVAL

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Abstract. In this paper we will prove bounds for the fourth power moment
in the t aspect over a short interval of automorphic L-functions \(L(s, g)\) for
\(GL(2)\) on the central critical line \(Re s = 1/2\). Here \(g\) is a fixed holomorphic
or Maass Hecke eigenform for the modular group \(SL_2(\mathbb{Z})\), or in certain cases,
for the Hecke congruence subgroup \(\Gamma_0(N)\) with \(N > 1\). The short interval
is from a large \(K\) to \(K + K^{103/135+\epsilon}\). The proof is based on an estimate
in the proof of subconvexity bounds for Rankin-Selberg L-function for Maass
forms by Jianya Liu and Yangbo Ye (2002) and Yuk-Kam Lau, Jianya Liu,
and Yangbo Ye (2004), which in turn relies on the Kuznetsov formula (1981)
and bounds for shifted convolution sums of Fourier coefficients of a cusp form

1. Introduction

For the Riemann zeta function and Dirichlet L-functions, estimates for their
power moments on the critical line \(Re s = 1/2\) played central roles in analytic
number theory. Classical results on short intervals

\[
\int_{K}^{K + K^\alpha} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \, dt \ll K^{\alpha + \epsilon}
\]

were proved for \(\alpha = 7/8\) by Heath-Brown [9] and for 2/3 by Iwaniec [11], for any
\(\epsilon > 0\). In this paper, we want to prove a similar result for automorphic L-functions
attached to a certain holomorphic or Maass cusp form \(g\) for \(\Gamma_0(N)\).

To describe our results, we need bounds towards the Ramanujan conjecture for
Maass forms. In terms of representation theory, let \(\pi\) be an automorphic cuspidal
representation of \(GL_2(\mathbb{Q}_A)\) with unitary central character and local Hecke eigen-
values \(\alpha_\pi^{(j)}(p)\) for \(p < \infty\) and \(\mu_\pi^{(j)}(\infty)\) for \(p = \infty, j = 1, 2\). Then bounds toward
the Ramanujan conjecture are

\[
\begin{align*}
|\alpha_\pi^{(j)}(p)| &\leq p^\theta \quad \text{for } p \text{ at which } \pi \text{ is unramified,} \\
|\text{Re}(\mu_\pi^{(j)}(\infty))| &\leq \theta \quad \text{if } \pi \text{ is unramified at } \infty.
\end{align*}
\]

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These bounds were proved for $\theta = 1/4$ by Selberg and Kuznetsov [14], for $\theta = 1/5$ by Shahidi [23] and Luo, Rudnick, and Sarnak [20], for $\theta = 1/9$ by Kim and Shahidi [14], and most recently for $\theta = 7/64$ by Kim and Sarnak [13].

The automorphic $L$-functions we will consider are

$$L(s, g) = \sum_{n \geq 1} \frac{\lambda_g(n)}{n^s} = \prod_{p \nmid N} (1 - \lambda_g(p)p^{-s} + p^{-2s})^{-1} \prod_{p \mid N} (1 - \lambda_g(p)p^{-s})^{-1},$$

where $g$ is a holomorphic or Maass cusp Hecke eigenform for $\Gamma_0(N)$, and its twist by a real primitive character $\chi$ modulo $Q$ with $N|Q$:

$$L(s, g \otimes \chi) = \sum_{n \geq 1} \frac{\lambda_g(n)\chi(n)}{n^s} = \prod_{p \nmid Q} (1 - \chi(p)\lambda_g(p)p^{-s} + p^{-2s})^{-1}$$

for $\Re s > 1$. Following the setting in Conrey and Iwaniec [2], we will assume that $Q$ is odd and square-free, and $\chi$ is the real, primitive character modulo $Q$, i.e., the Jacobi symbol, so that the twisted cusp form $g_\chi$ (see (2.1) and (2.2) below) is a cusp form for $\Gamma_0(N')$. As pointed in [2], p. 1176, $g_\chi$ is primitive even if the Hecke eigenform $g$ itself is not primitive. Our results, nevertheless, are valid in other cases, as long as the twisted $L$-function $L(s, g \otimes \chi)$ has a standard functional equation as in (2.3) (cf. Atkin and Li [1]). In particular, our theorem below is valid for $L(s, g)$ when $N' = 1$. We will assume that $g$ is self-contragredient. If $g$ is holomorphic, we denote its weight by $\ell$. If $g$ is Maass, we denote its Laplace eigenvalue by $1/4 + \ell^2$.

**Theorem 1.1.** Let $g$ be a fixed self-contragredient holomorphic or Maass Hecke eigenform for $\Gamma_0(N)$, and let $\chi$ be a real, primitive character mod $Q$ with $N|Q$. Then

$$\int_{K} |L(\frac{1}{2} + it, g \otimes \chi)|^4 \, dt \ll_{\varepsilon, N, g, Q} (KL)^{1+\varepsilon}$$

for $L = K^{1-1/(4+2\theta)+\varepsilon}$. Here $\theta$ is given by bounds toward the Ramanujan conjecture in (1.1), and we can take $\theta = 7/64$ with $1 - 1/(4 + 2\theta) = 103/135$.

A subconvexity bound for $L(s, g)$ in the $t$ aspect was deduced by Good for holomorphic cusp form $g$ in [6], [7], and [8], and by Meurman for Maass $g$ in [21]:

$$L\left(\frac{1}{2} + it, g\right) \ll_g (1 + |t|)^{1/3} \log^{5/6} (2 + |t|).$$

The goal of the present paper is not an improvement to this subconvexity bound for $L(s, g)$. By a standard argument (cf. Ivic [10], p. 197) though, our Theorem 1.1 implies

$$L\left(\frac{1}{2} + it, g\right) \ll_g (1 + |t|)^{1/2 - 1/(16 + 8\theta) + \varepsilon} = (1 + |t|)^{119/270 + \varepsilon}.$$  

Certainly our (1.3) is not as good as (1.2). Using (1.2), however, one can only get a fourth power moment bound of $(K^{4/3}L)^{1+\varepsilon}$, not as good as our Theorem 1.1.

Subconvexity bounds in the level $N$ aspect and the $\ell$ aspect were studied extensively by Duke, Friedlander, and Iwaniec ([3], [4], [5]), and by Kowalski, Michel, and VanderKam [15].
The proof of Theorem 1.1 is based on an argument in Jianya Liu and Yangbo Ye [18] and Yuk-Kam Lau, Jianya Liu, and Yangbo Ye [17]; see §3 below. In [18] subconvexity bounds for Rankin-Selberg $L$-functions $L(s, f \otimes g)$ were proved as the Laplace eigenvalue of the Maass cusp form $f$ goes to $\infty$, where $g$ is a fixed holomorphic or Maass cusp form. While the exponent $(3 + 2\theta)/4 + \epsilon$ as claimed in [18] does not hold because of a gap in §§4.14 and 4.15, the paper did prove a subconvexity bound

$$L(1/2 + it, f \otimes g) \ll_{N,t,g,\epsilon} k^{(15 + 2\theta)/16 + \epsilon}$$

as pointed out in the first sentence in §4.14 (see Jianya Liu and Yangbo Ye [19]). In [17] (1.4) was improved to a better bound \(O(k^{1-1/(8+4\theta) + \epsilon})\).

What was done in [18] and [17] was to express $L(1/2, f \otimes g)$ in terms of spectral decomposition of $f$ and $g$ by an approximate functional equation. Using the Kuznetsov trace formula ([16]) the spectral sum of $f$ is rewritten in terms of Kloosterman sums. Therefore the central value of $L(s, f \otimes g)$ is essentially expressed as a spectral sum of $g$ with Kloosterman sums as coefficients. An application of bounds for shifted convolution sums of Fourier coefficients of $g$ (Sarnak [22] with an improvement given in [17]) gives a subconvexity bound for $L(s, f \otimes g)$.

In this paper we will proceed to consider the continuous spectrum of the Laplacian in place of the Maass form $f$. This approach is motivated by Conrey and Iwaniec [2].

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2. The approximate functional equation

We know that the twisted $L$-function $L(s, g \otimes \chi)$ is entire, where $\chi$ is a real, primitive character modulo $Q$ with $N|Q$. Note that $L(s, g \otimes \chi)$ is indeed the $L$-function attached to a twisted cusp form $g_\chi$. It is

$$g_\chi(z) = \sum_{n \geq 1} n^{(k-1)/2} \chi(n) \lambda_g(n)e(nz)$$

when $g$ is holomorphic, and

$$g_\chi(z) = y^{1/2} \sum_{n \neq 0} \chi(n) \lambda_g(n) K_1(2\pi|n|y)e(nx)$$

when $g$ is Maass.

In any case, denote by

$$\Lambda(s, g \otimes \chi) = L_\infty(s, g \otimes \chi)L(s, g \otimes \chi)$$

the complete $L$-function, where

$$L_\infty(s, g \otimes \chi) = \prod_{j=1}^{2} \Gamma_\mathbb{R}(s + \mu_{g_\chi}(j))$$

with $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$. Here $\mu_{g_\chi}(1)$ and $\mu_{g_\chi}(2)$ are complex numbers associated to $g_\chi$ at $\infty$. According to Conrey and Iwaniec [2], p. 1188, our twisted cusp form
\( g_\chi \) satisfies the standard functional equation
\[
\Lambda(s, g \otimes \chi) = \varepsilon(s, g_\chi)\Lambda(1 - s, g \otimes \chi),
\]
where \( \varepsilon(s, g_\chi) = \tau(g_\chi)Q_{g_\chi}^{-s} \). Here \( Q_{g_\chi} > 0 \) is the conductor of \( g_\chi \) and \( \tau(g_\chi) \in \mathbb{C}^\times \) satisfies \( \tau(g_\chi)\tau(\bar{g}_\chi) = Q_{g_\chi} \). Since \( g \) is self-contragredient and \( \chi \) is real, we have \( \tau(g_\chi)^2 = Q_{g_\chi} \).

We actually want a functional equation for a product of two such \( L \)-functions:
\[
L(s + ir, g \otimes \chi)L(s - ir, g \otimes \chi) = \gamma(s)L(1 - s - ir, g \otimes \chi)L(1 - s + ir, g \otimes \chi),
\]
where according to (2.3)
\[
\gamma(s) = \tau(g_\chi)^2Q_{g_\chi}^{-2s} \prod_{j=1}^{2} \frac{\Gamma_R(1 - s + i\tau + \mu_{g_\chi}(j))\Gamma_R(1 - s - i\tau + \mu_{g_\chi}(j))}{\Gamma_R(s + i\tau + \mu_{g_\chi}(j))\Gamma_R(s - i\tau + \mu_{g_\chi}(j))} = \left( \frac{\pi^2}{Q_{g_\chi}} \right)^{2s-1} \prod_{j=1}^{2} \frac{\Gamma((1 - s + i\tau + \mu_{g_\chi}(j))/2)\Gamma((1 - s - i\tau + \mu_{g_\chi}(j))/2)}{\Gamma((s + i\tau + \mu_{g_\chi}(j))/2)\Gamma((s - i\tau + \mu_{g_\chi}(j))/2)}.
\]

By Stirling’s formula, we get (see similar computations in [22] and [18])
\[
\gamma(s) = \left( \frac{4\pi^2/Q_{g_\chi}}{(\mu_{g_\chi}(1) + r^2)^{1/2}(\mu_{g_\chi}(2) + r^2)^{1/2}} \right)^{2s-1} (1 + \eta_r(s)),
\]
where the error term \( \eta_r(s) \ll (1 + |s|)^3/(1 + |r|) \). We will consider the case of large \( |r| \) with fixed \( g \); hence \( \gamma(s) \) is asymptotically \((Q_{g_\chi}r^2/(4\pi^2))^{1-2s}\).

Following [22] and [18] again, we can express the central value of
\[
L(s + ir, g \otimes \chi)L(s - ir, g \otimes \chi)
\]
as
\[
(2.4)
\frac{1}{\pi^2} \int_{\text{Re} s = 2} X^s L\left( \frac{1}{2} + s + i\tau, g \otimes \chi \right) L\left( \frac{1}{2} + s - i\tau, g \otimes \chi \right) G(s) \frac{ds}{s}
\]
\[
+ O\left( \left( \int_{\text{Re} s = 2} X^s \eta_r\left( \frac{1}{2} - s \right) L\left( \frac{1}{2} + s + i\tau, g \otimes \chi \right) \right) \times L\left( \frac{1}{2} + s - i\tau, g \otimes \chi \right) G(s) \frac{ds}{s} \right),
\]
where
\[
X = \frac{Q_{g_\chi}}{4\pi^2} (\mu_{g_\chi}(1) + r^2)^{1/2}(\mu_{g_\chi}(2) + r^2)^{1/2}.
\]
Here \( G(s) \) is an analytic function in \(-B \leq \text{Re} s \leq B\) for a fixed \( B > 0 \) satisfying
\[
G(0) = 1, \quad G(s) = G(-s), \quad |G(s)| \ll (1 + |s|)^{-A}
\]
for a fixed large constant \( A \). We note that \( X \) is real (and positive), because of our assumption on \( g \) being self-contragredient.

We may shift the contour in the integral of the big \( O \) term in (2.4) to \text{Re} \( s = 1/2 + \varepsilon \). This way \( X^s \ll r^{1+\varepsilon} \). Recall that \( \eta_r(1/2 - s) \ll (1 + |s|)^3/(1 + |r|) \).

Moreover,
\[
L\left( \frac{1}{2} + s + i\tau, g \otimes \chi \right) L\left( \frac{1}{2} + s - i\tau, g \otimes \chi \right) \ll_{\varepsilon, g} 1
\]
as $r \to \infty$, for Re $s = 1/2 + \varepsilon$, because its Dirichlet series is absolutely convergent.

All these show that the big $O$ term in (2.4) is $\ll r^\varepsilon$.

To compute the main term in (2.4), we expand

$$L(1/2 + s + ir, g \otimes \chi)L(1/2 + s - ir, g \otimes \chi)$$

into its Dirichlet series. For Re $s > 1/2$, we have

(2.5) $$L\left(\frac{1}{2} + s + ir, g \otimes \chi\right)L\left(\frac{1}{2} + s - ir, g \otimes \chi\right)$$

$$= \sum_{m,n \geq 1} \chi(mn)\frac{\lambda_g(m)\lambda_g(n)}{m^{1/2+s+ir}n^{1/2+s-ir}}.$$

As $g$ is a Hecke eigenform, we have

$$\lambda_g(m)\lambda_g(n) = \sum_{d \mid (m,n)} \lambda_g\left(\frac{mn}{d^2}\right).$$

Apply this to the right side of (2.5) and set $m = ad, n = bd$. Then

(2.6) $$L\left(\frac{1}{2} + s + ir, g \otimes \chi\right)L\left(\frac{1}{2} + s - ir, g \otimes \chi\right)$$

$$= \sum_{a,b,d \geq 1} \chi(abd^2)\frac{\lambda_g(ab)}{a^{1/2+2s+ir}b^{1/2+2s-ir}d^{1+2s}}$$

$$= L\left(1 + 2s, \chi^2\right)\sum_{n \geq 1} \frac{\chi(n)\lambda_g(n)}{n^{1/2+s}}d_s(n),$$

where

$$d_s(n) = \sum_{ab = [n]} \left(\frac{a}{b}\right)^s.$$

We remark that in (2.6), the series is actually taken over $n$ which are relatively prime to $Q$.

Consider the Eisenstein series for any fixed cusp $a$ of $\Gamma = \Gamma_0(N)$ defined by

$$E_a(z, s) = \sum_{\gamma \in \Gamma_a \backslash \Gamma} \left(\frac{\text{Im } \sigma_\gamma^{-1} \gamma z}{s}\right)^s$$

for Re $s > 1$ and by analytic continuation for all $s \in \mathbb{C}$. Here $\Gamma_a$ is the stability group of $a$, while $\sigma_a \in SL(2, \mathbb{R})$ is given by $\sigma_a \infty = a$ and $\sigma_a^{-1} \Gamma_a \sigma_a = \Gamma_\infty$. This Eisenstein series is an eigenfunction of the Hecke operators

$$T_n E_a(z, s) = \eta_a(n, s)E_a(z, s),$$

if $(n, N) = 1$. As pointed out in Conrey and Iwaniec [2], for any $n$ relatively prime to $N$, $\eta_a(n, s) = d_{s-1/2}(n)$. Consequently, from (2.6) we get

(2.7) $$L\left(\frac{1}{2} + s + ir, g \otimes \chi\right)L\left(\frac{1}{2} + s - ir, g \otimes \chi\right)$$

$$= L\left(1 + 2s, \chi^2\right)\sum_{n \geq 1} \frac{\chi(n)\lambda_g(n)\eta_a(n, 1/2 + ir)}{n^{1/2+s}}$$

for Re $s > 1/2$. 

Substituting (2.7) into the integral of the main term in (2.4), we get

\[ L \left( \frac{1}{2} + ir, g \otimes \chi \right) L \left( \frac{1}{2} - ir, g \otimes \chi \right) \]

\[ = \frac{1}{\pi i} \int_{\Re = \frac{1}{2}} \left( \sum_{m,n \geq 1} \frac{\chi(nm^2) \lambda_g(n) \eta_a(n, 1/2 + ir)}{(nm^2)^{1/2 + s}} \right) X^s G(s) \frac{ds}{s} + O(r^\varepsilon) \]

\[ = 2 \sum_{m,n \geq 1} \frac{\chi(nm^2) \lambda_g(n) \eta_a(n, 1/2 + ir)}{m \sqrt{n}} \frac{1}{2\pi i} \int_{\Re = \frac{1}{2}} G(s) \left( \frac{nm^2}{X} \right)^{-s} \frac{ds}{s} + O(r^\varepsilon). \]

Denote

\[ V(y) = \frac{1}{2\pi i} \int_{\Re = \frac{1}{2}} G(s) y^{-s} \frac{ds}{s}. \]

Then as in [22] and [18], \[ \lim_{y \to 0} V(y) = 1 \] and \[ V(y) \ll (1 + |y|)^{-B} \] because of our choice of the function \( G(s) \). Therefore

\[ L \left( \frac{1}{2} + ir, g \otimes \chi \right) L \left( \frac{1}{2} - ir, g \otimes \chi \right) \]

\[ = 2 \sum_{m \leq X^{1/2+\varepsilon}} \chi^2(m) \frac{1}{m} \sum_{n \geq 1} \frac{\chi(n) \lambda_g(n) \eta_a(n, 1/2 + ir)}{\sqrt{n}} V \left( \frac{nm^2}{X} \right) \]

\[ + O(r^\varepsilon), \]

because the outer series is negligible if taken over \( m > X^{1/2+\varepsilon} \).

3. Averaging and the Kuznetsov trace formula

According to (2.8), estimation of the central value of our \( L \)-function is reduced to estimation of

\[ S_Y(g, r) = \sum_n \chi(n) \lambda_g(n) \eta_a(n, 1/2 + ir) H \left( \frac{n}{\sqrt{Y}} \right) \]

for fixed \( g \), where \( H \) is a fixed smooth function of compact support contained in \( (1, 2) \).

To prove our results on short intervals, let \( L \) be a number which satisfies \( \sqrt{K} \leq L \leq K/4 \) for large \( K \). Let \( h(t) \) be an even analytic function in \( |\Im t| \leq 1/2 \) satisfying \( h^{(n)}(t) \ll (1 + |t|)^{-N} \) for any \( N > 0 \) in this region. Thus \( h \) is a Schwartz function on \( \mathbb{R} \). We also assume that \( h(t) \geq 0 \) for real \( t \). For example, we may simply take \( h(t) = 1/\cosh(t) \). Denote

\[ \zeta_N(s) = \prod_{p \not\in N} (1 - p^{-s})^{-1}. \]
We want to estimate

\[ I_{K,L} = \int_{\mathbb{R}} \left( h \left( \frac{K-r}{L} \right) + h \left( \frac{K+r}{L} \right) \right) |S_Y(g, r)|^2 \frac{\zeta_N(1+2ir)^2}{\zeta(1+2ir)^2} \, dr \]

\[ = \sum_{m,n} \chi(n) \overline{\chi}(m) \lambda_g(n) \widehat{\lambda_g}(m) \mathcal{H} \left( \frac{n}{Y} \right) \widehat{\mathcal{H}} \left( \frac{m}{Y} \right) \]

\[ \times \int_{\mathbb{R}} \left( h \left( \frac{K-r}{L} \right) + h \left( \frac{K+r}{L} \right) \right) \, dr \]

\[ \times \eta_a(n, 1/2 + ir) \overline{\eta}_a(m, 1/2 + ir) \frac{\zeta_N(1+2ir)^2}{\zeta(1+2ir)^2} \, dr. \]

As in Liu and Ye [18], we apply the Kuznetsov trace formula to the integral on the right side of (3.1):

\[ \pi \sum_{m,n} \chi(n) \overline{\chi}(m) \lambda_g(n) \widehat{\lambda_g}(m) \mathcal{H} \left( \frac{n}{Y} \right) \widehat{\mathcal{H}} \left( \frac{m}{Y} \right) \]

\[ \times \mathcal{F}_{j} \left( h \left( \frac{K-r}{L} \right) + h \left( \frac{K+r}{L} \right) \right) \lambda_{f_j}(n) \overline{\lambda_{f_j}}(m) \]

\[ \times \int_{\mathbb{R}} \left( h \left( \frac{K-r}{L} \right) + h \left( \frac{K+r}{L} \right) \right) \, dr \]

\[ \times \eta_a(n, 1/2 + ir) \overline{\eta}_a(m, 1/2 + ir) \frac{\zeta_N(1+2ir)^2}{\zeta(1+2ir)^2} \, dr. \]

\[ \sum_{m,n} \chi(n) \overline{\chi}(m) \lambda_g(n) \widehat{\lambda_g}(m) \mathcal{H} \left( \frac{n}{Y} \right) \widehat{\mathcal{H}} \left( \frac{m}{Y} \right) \]

\[ \times \frac{\delta_{n,m}}{\pi} \int \tanh(\pi r) \left( h \left( \frac{K-r}{L} \right) + h \left( \frac{K+r}{L} \right) \right) r \, dr. \]

\[ + 2\pi \sum_{m,n} \chi(n) \overline{\chi}(m) \lambda_g(n) \widehat{\lambda_g}(m) \mathcal{H} \left( \frac{n}{Y} \right) \widehat{\mathcal{H}} \left( \frac{m}{Y} \right) \]

\[ \times \mathcal{F}_{j} \left( h \left( \frac{K-r}{L} \right) + h \left( \frac{K+r}{L} \right) \right) \lambda_{f_j}(n) \overline{\lambda_{f_j}}(m) \]

\[ \times \int_{\mathbb{R}} \left( h \left( \frac{K-r}{L} \right) + h \left( \frac{K+r}{L} \right) \right) \, dr \]

\[ \times \eta_a(n, 1/2 + ir) \overline{\eta}_a(m, 1/2 + ir) \frac{\zeta_N(1+2ir)^2}{\zeta(1+2ir)^2} \, dr. \]

Here in (3.2) \( f_j \) are Hecke eigenforms, with Laplace eigenvalues \( 1/4 + k_j^2 \) and Fourier coefficients \( \lambda_{f_j}(n) \), which form an orthonormal basis of the space of Maass cusp forms for \( \Gamma_0(N) \), while in (3.5) \( S(n, m; c) \) is the classical Kloosterman sum.

Recall that \( \chi(n) \lambda_g(n) \) is the \( n \)th Fourier coefficient of the twisted cusp form \( g_{\chi} \) as in (2.1) or (2.2). We want to apply the main estimation in Liu and Ye [18] (§4.1–§4.13) and Lau, Liu, and Ye [17], (2.2), to our (3.4) and (3.5) above. Note that these estimations are based on bounds for shifted convolution sums of Fourier
coefficients of cusp forms proved by Sarnak [22], Appendix, and by Lau, Liu, and Ye [17].

More precisely, (3.4) + (3.5) \( \ll LKY^{1+\varepsilon} \) for \( L = K^{1-1/(4+2\theta)+\varepsilon} \). Since (3.2) and (3.3) are both positive, this implies that (3.3), i.e., \( I_{K,L} \), is bounded by \( O(LKY^{1+\varepsilon}) \) for the same \( L \). By \( \zeta(1+2ir) \ll \log(1+|r|) \), this estimate of \( I_{K,L} \) implies that

\[
(3.6) \quad \int_{K}^{K+L} |S_{Y}(g,r)|^{2} \, dr \ll LKY^{1+\varepsilon}
\]

for the above \( L \).

Now we can go back to the fourth power moment of \( L(1/2+ir, g \otimes \chi) \). Since \( g \) is self-contragredient and \( \chi \) is real, we have from (2.8) that

\[
\int_{K}^{K+L} \left| L\left(\frac{1}{2} + ir, g \otimes \chi \right) \right|^{4} \, dr = \int_{K}^{K+L} \left| L\left(\frac{1}{2} + ir, g \otimes \chi \right) \right|^{2} \, dr \\
\ll \int_{K}^{K+L} \sum_{1 \leq m \leq X^{1/2+\varepsilon}} \chi^{2}(m) \left| \sum_{n \geq 1} \frac{\lambda(n)\lambda_{g}(n)\eta_{a}(n,1/2+ir)}{\sqrt{n}} V\left(\frac{nm^{2}}{X}\right) \right|^{4} \, dr.
\]

Here we can take \( X = K^{2} \) and get

\[
\int_{K}^{K+L} \left| L\left(\frac{1}{2} + ir, g \otimes \chi \right) \right|^{4} \, dr \\
\ll \frac{1}{K^{2}} \int_{K}^{K+L} \left| \sum_{n \geq 1} \chi(n)\lambda_{g}(n)\eta_{a}(n,1/2+ir) \sum_{1 \leq m \leq K^{1+\varepsilon}} \chi^{2}(m) V\left(\frac{nm^{2}/K^{2}}{\sqrt{nm^{2}/K^{2}}}\right) \right|^{2} \, dr.
\]

Now we apply a smooth dyadic subdivision to

\[
\sum_{1 \leq m \leq K^{1+\varepsilon}} \chi^{2}(m) V\left(\frac{nm^{2}/K^{2}}{\sqrt{nm^{2}/K^{2}}}\right),
\]

by dividing the interval \([1, K^{1+\varepsilon}]\) into subintervals of the form \([\alpha, 1.8\alpha]\) and covering, with overlapping, each subinterval by a smooth, nonnegative function of compact support. The total number of subintervals is \( O(\log K) \). This way, we can find a smooth function \( H \) of compact support in \((1,2)\) so that

\[
\int_{K}^{K+L} \left| L\left(\frac{1}{2} + ir, g \otimes \chi \right) \right|^{4} \, dr \\
\ll \frac{\log K}{K^{2}} \int_{K}^{K+L} \max_{1 \leq B \leq K^{2+\varepsilon}} \left| \sum_{n \geq 1} \chi(n)\lambda_{g}(n)\eta_{a}(n,1/2+ir) H \left( \frac{n}{K^{2}/B} \right) \right|^{2} \, dr.
\]
The sum inside the absolute value signs is indeed $S_{K^2/B}(g,r)$. By (3.6), the maximum contribution is from $B = 1$:

$$\int K \left| L\left(\frac{1}{2} + ir, g \otimes \chi \right) \right|^4 dr \ll \frac{\log K}{K^2} \int K \left| S_{K^2}(g,r) \right|^2 dr$$

$$\ll \frac{\log K}{K^2} LK(K^2)^{1+\varepsilon} \ll (KL)^{1+\varepsilon}$$

for $L = K^{1-1/(4+2\theta)+\varepsilon}$. This completes the proof of Theorem 1.1.

**Added in proof**

Recently, Lau, Liu, and Ye further improved the subconvexity bound (1.4) to $k^{3/4+\varepsilon}$. Using this new result, our Theorem 1.1 can be stated for $L = K^{1/2+\varepsilon}$.

**References**


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