THE FOURTH POWER MOMENT OF AUTOMORPHIC L-FUNCTIONS FOR GL(2) OVER A SHORT INTERVAL

YANGBO YE

ABSTRACT. In this paper we will prove bounds for the fourth power moment in the $t$ aspect over a short interval of automorphic $L$-functions $L(s, g)$ for $GL(2)$ on the central critical line Re $s = 1/2$. Here $g$ is a fixed holomorphic or Maass Hecke eigenform for the modular group $SL_2(\mathbb{Z})$, or in certain cases, for the Hecke congruence subgroup $\Gamma_0(N)$ with $N > 1$. The short interval is from a large $K$ to $K + K^{103/135 + \varepsilon}$. The proof is based on an estimate in the proof of subconvexity bounds for Rankin-Selberg $L$-function for Maass forms by Jianya Liu and Yangbo Ye (2002) and Yuk-Kam Lau, Jianya Liu, and Yangbo Ye (2004), which in turn relies on the Kuznetsov formula (1981) and bounds for shifted convolution sums of Fourier coefficients of a cusp form proved by Sarnak (2001) and by Lau, Liu, and Ye (2004).

1. Introduction

For the Riemann zeta function and Dirichlet $L$-functions, estimates for their power moments on the critical line Re $s = 1/2$ played central roles in analytic number theory. Classical results on short intervals

$$\int_{K}^{K+K^{\alpha}} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \, dt \ll K^{\alpha + \varepsilon}$$

were proved for $\alpha = 7/8$ by Heath-Brown [9] and for $2/3$ by Iwaniec [11], for any $\varepsilon > 0$. In this paper, we want to prove a similar result for automorphic $L$-functions attached to a certain holomorphic or Maass cusp form $g$ for $\Gamma_0(N)$.

To describe our results, we need bounds towards the Ramanujan conjecture for Maass forms. In terms of representation theory, let $\pi$ be an automorphic cuspidal representation of $GL_2(\mathbb{Q}_A)$ with unitary central character and local Hecke eigenvalues $\alpha_\pi^{(j)}(p)$ for $p < \infty$ and $\mu_\pi^{(j)}(\infty)$ for $p = \infty$, $j = 1, 2$. Then bounds toward the Ramanujan conjecture are

$$|\alpha_\pi^{(j)}(p)| \leq p^{\theta} \quad \text{for } p \text{ at which } \pi \text{ is unramified},$$

$$|\text{Re}(\mu_\pi^{(j)}(\infty))| \leq \theta \quad \text{if } \pi \text{ is unramified at } \infty.$$
These bounds were proved for \( \theta = 1/4 \) by Selberg and Kuznetsov \[10\], for \( \theta = 1/5 \) by Shahidi \[23\] and Luo, Rudnick, and Sarnak \[20\], for \( \theta = 1/9 \) by Kim and Shahidi \[14\], and most recently for \( \theta = 7/64 \) by Kim and Sarnak \[13\].

The automorphic \( L \)-functions we will consider are

\[
L(s, g) = \sum_{n \geq 1} \frac{\lambda_g(n)}{n^s} = \prod_{p \mid \mathcal{Q}} (1 - \lambda_g(p)p^{-s} + p^{-2s})^{-1},
\]

where \( g \) is a holomorphic or Maass cusp Hecke eigenform for \( \Gamma_0(\mathcal{N}) \), and its twist by a real primitive character \( \chi \) modulo \( \mathcal{Q} \) with \( \mathcal{N} | \mathcal{Q} \):

\[
L(s, g \otimes \chi) = \sum_{n \geq 1} \frac{\lambda_g(n)\chi(n)}{n^s} = \prod_{p \mid \mathcal{Q}} (1 - \chi(p)\lambda_g(p)p^{-s} + p^{-2s})^{-1}
\]

for \( \Re s > 1 \). Following the setting in Conrey and Iwaniec \[2\], we will assume that \( \mathcal{Q} \) is odd and square-free, and \( \chi \) is the real, primitive character modulo \( \mathcal{Q} \), i.e., the Jacobi symbol, so that the twisted cusp form \( g_\chi \) (see (2.1) and (2.2) below) is a cusp form for \( \Gamma_0(\mathcal{N}^2) \). As pointed in \[2\], p. 1176, \( g_\chi \) is primitive even if the Hecke eigenform \( g \) itself is not primitive. Our results, nevertheless, are valid in other cases, as long as the twisted \( L \)-function \( L(s, g \otimes \chi) \) has a standard functional equation as in (2.3) (cf. Atkin and Li \[1\]). In particular, our theorem below is valid for \( L(s, g) \) when \( \mathcal{N} = 1 \). We will assume that \( g \) is self-contragredient. If \( g \) is holomorphic, we denote its weight by \( \ell \). If \( g \) is Maass, we denote its Laplace eigenvalue by \( 1/4 + \ell^2 \).

**Theorem 1.1.** Let \( g \) be a fixed self-contragredient holomorphic or Maass Hecke eigenform for \( \Gamma_0(\mathcal{N}) \), and let \( \chi \) be a real, primitive character mod \( \mathcal{Q} \) with \( \mathcal{N} | \mathcal{Q} \). Then

\[
\int_{K} \left| L \left( \frac{1}{2} + it, g \otimes \chi \right) \right|^4 dt \ll_{\varepsilon, \mathcal{N}, \mathcal{Q}, K} (KL)^{1+\varepsilon}
\]

for \( L = K^{1-1/(4+2\theta)+\varepsilon} \). Here \( \theta \) is given by bounds toward the Ramanujan conjecture in (1.1), and we can take \( \theta = 7/64 \) with \( 1 - 1/(4 + 2\theta) = 103/135 \).

A subconvexity bound for \( L(s, g) \) in the \( t \) aspect was deduced by Good for holomorphic cusp form \( g \) in \[6\], \[7\], and \[8\], and by Meurman for Maass \( g \) in \[21\]:

\[
L \left( \frac{1}{2} + it, g \right) \ll (1 + |t|)^{1/3} \log^{5/6}(2 + |t|).
\]

The goal of the present paper is not an improvement to this subconvexity bound for \( L(s, g) \). By a standard argument (cf. Ivic \[10\], p. 197) though, our Theorem 1.1 implies

\[
L \left( \frac{1}{2} + it, g \right) \ll (1 + |t|)^{1/2 - 1/(16 + 8\theta) + \varepsilon} = (1 + |t|)^{119/270^+\varepsilon}.
\]

Certainly our (1.3) is not as good as (1.2). Using (1.2), however, one can only get a fourth power moment bound of \( (K^{4/3}L)^{1+\varepsilon} \), not as good as our Theorem 1.1.

Subconvexity bounds in the level \( \mathcal{N} \) aspect and the \( \ell \) aspect were studied extensively by Duke, Friedlander, and Iwaniec \([3\), \[4\], \[5\]), and by Kowalski, Michel, and VanderKam \[15\].
The proof of Theorem 1.1 is based on an argument in Jianya Liu and Yangbo Ye [18] and Yuk-Kam Lau, Jianya Liu, and Yangbo Ye [17]; see §3 below. In [18] subconvexity bounds for Rankin-Selberg \( L(s, f \otimes g) \) were proved as the Laplace eigenvalue of the Maass cusp form \( f \) goes to \( \infty \), where \( g \) is a fixed holomorphic or Maass cusp form. While the exponent \( (3 + 2\theta)/4 + \epsilon \) as claimed in [18] does not hold because of a gap in §§4.14 and 4.15, the paper did prove a subconvexity bound

\[
L(1/2 + it, f \otimes g) \ll_{N,t,g,\epsilon} k^{(15+2\theta)/16+\epsilon}
\]

as pointed out in the first sentence in §4.14 (see Jianya Liu and Yangbo Ye [19]). In [17] (1.4) was improved to a better bound

\[
O(k^{1-1/(8+4\theta)+\epsilon}).
\]

What was done in [18] and [17] was to express \( L(s, f \otimes g) \) in terms of spectral decomposition of \( f \) and \( g \) by an approximate functional equation. Using the Kuznetsov trace formula ([16]) the spectral sum of \( f \) is rewritten in terms of Kloosterman sums. Therefore the central value of \( L(s, f \otimes g) \) is essentially expressed as a spectral sum of \( g \) with Kloosterman sums as coefficients. An application of bounds for shifted convolution sums of Fourier coefficients of \( g \) (Sarnak [22] with an improvement given in [17]) gives a subconvexity bound for \( L(s, f \otimes g) \).

In this paper we will proceed to consider the continuous spectrum of the Laplacian in place of the Maass form \( f \). This approach is motivated by Conrey and Iwaniec [2].

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2. The approximate functional equation

We know that the twisted \( L \)-function \( L(s, g \otimes \chi) \) is entire, where \( \chi \) is a real, primitive character modulo \( \mathbb{Q} \) with \( \mathcal{N} \mid \mathbb{Q} \). Note that \( L(s, g \otimes \chi) \) is indeed the \( L \)-function attached to a twisted cusp form \( g_{\chi} \). It is

\[
g_{\chi}(z) = \sum_{n \geq 1} n^{(k-1)/2} \chi(n) \lambda_g(n) e(nz)
\]

when \( g \) is holomorphic, and

\[
g_{\chi}(z) = y^{1/2} \sum_{n \neq 0} \chi(n) \lambda_g(n) K_{id}(2\pi |n| y) e(nx)
\]

when \( g \) is Maass.

In any case, denote by

\[
\Lambda(s, g \otimes \chi) = L_\infty(s, g \otimes \chi)L(s, g \otimes \chi)
\]

the complete \( L \)-function, where

\[
L_\infty(s, g \otimes \chi) = \prod_{j=1}^{2} \Gamma_{\mathbb{R}}(s + \mu_{g_\chi}(j))
\]

with \( \Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2) \). Here \( \mu_{g_\chi}(1) \) and \( \mu_{g_\chi}(2) \) are complex numbers associated to \( g_\chi \) at \( \infty \). According to Conrey and Iwaniec [2], p. 1188, our twisted cusp form
$g_\chi$ satisfies the standard functional equation
\begin{equation}
\Lambda(s, g \otimes \chi) = \varepsilon(s, g_\chi) \Lambda(1-s, g \otimes \chi),
\end{equation}
where $\varepsilon(s, g_\chi) = \tau(g_\chi) Q_{g_\chi}^{-s}$. Here $Q_{g_\chi} > 0$ is the conductor of $g_\chi$ and $\tau(g_\chi) \in \mathbb{C}^\times$ satisfies $\tau(g_\chi) \tau(\bar{g}_\chi) = Q_{g_\chi}$. Since $g$ is self-conjugate and $\chi$ is real, we have $\tau(g_\chi)^2 = Q_{g_\chi}$.

We actually want a functional equation for a product of two such $L$-functions:
\begin{equation}
L(s + ir, g \otimes \chi)L(s - ir, g \otimes \chi) = \gamma(s) L(1-s - ir, g \otimes \chi)L(1-s + ir, g \otimes \chi),
\end{equation}
where according to (2.3)
\begin{equation}
\gamma(s) = \tau(g_\chi)^2 Q_{g_\chi}^{-2s} \prod_{j=1}^{2} \frac{\Gamma(1-s+ir + \mu_{g_\chi}(j))\Gamma(1-s - ir + \mu_{g_\chi}(j))}{\Gamma(s+ir + \mu_{g_\chi}(j))\Gamma(s - ir + \mu_{g_\chi}(j))} = \left(\frac{\pi^2}{Q_{g_\chi}}\right)^{2s-1} \prod_{j=1}^{2} \frac{\Gamma((1-s+ir + \mu_{g_\chi}(j))/2)\Gamma((1-s - ir + \mu_{g_\chi}(j))/2)}{\Gamma((s+ir + \mu_{g_\chi}(j))/2)\Gamma((s - ir + \mu_{g_\chi}(j))/2)}.
\end{equation}
By Stirling’s formula, we get (see similar computations in [22] and [18])
\begin{equation}
\gamma(s) = \left(\frac{4\pi^2/Q_{g_\chi}}{(\mu_{g_\chi}(1)+\varepsilon)^{1/2}(\mu_{g_\chi}(2)+\varepsilon)^{1/2}}\right)^{2s-1} (1 + \eta_r(s)),
\end{equation}
where the error term $\eta_r(s) \ll (1 + |s|)^3/(1 + |r|)$. We will consider the case of large $|r|$ with fixed $g$; hence $\gamma(s)$ is asymptotically $(Q_{g_\chi} r^2/(4\pi^2))^{1-2s}$.

Following [22] and [18] again, we can express the central value of
\begin{equation}
L(s + ir, g \otimes \chi)L(s - ir, g \otimes \chi)
\end{equation}
as
\begin{equation}
L\left(\frac{1}{2} + ir, g \otimes \chi\right)L\left(\frac{1}{2} - ir, g \otimes \chi\right) = \frac{1}{\pi i} \int_{\text{Re } s = 2} X^* L\left(\frac{1}{2} + s + ir, g \otimes \chi\right)L\left(\frac{1}{2} + s - ir, g \otimes \chi\right) G(s) \frac{ds}{s} + O\left(\int_{\text{Re } s = 2} X^* \eta_r\left(\frac{1}{2} - s\right)L\left(\frac{1}{2} + s + ir, g \otimes \chi\right) \times L\left(\frac{1}{2} + s - ir, g \otimes \chi\right) G(s) \frac{ds}{s}\right),
\end{equation}
where
\begin{equation}
X = \frac{Q_{g_\chi}}{4\pi^2} \left(\mu_{g_\chi}(1)+\varepsilon\right)^{1/2}\left(\mu_{g_\chi}(2)+\varepsilon\right)^{1/2}.
\end{equation}
Here $G(s)$ is an analytic function in $-B \leq \text{Re } s \leq B$ for a fixed $B > 0$ satisfying
\begin{equation}
G(0) = 1, \quad G(s) = G(-s), \quad |G(s)| \ll (1 + |s|)^{-A}
\end{equation}
for a fixed large constant $A$. We note that $X$ is real (and positive), because of our assumption on $g$ being self-conjugate.

We may shift the contour in the integral of the big $O$ term in (2.4) to Re $s = 1/2 + \varepsilon$. This way $X^* \ll r^{1+\varepsilon}$. Recall that $\eta_r(1/2 - s) \ll (1 + |s|)^3/(1 + |r|)$.

Moreover,
\begin{equation}
L\left(\frac{1}{2} + s + ir, g \otimes \chi\right)L\left(\frac{1}{2} + s - ir, g \otimes \chi\right) \ll_{\varepsilon, g} 1
\end{equation}
as \( r \to \infty \), for \( \text{Re} \ s = 1/2 + \varepsilon \), because its Dirichlet series is absolutely convergent. All these show that the big \( O \) term in (2.4) is \( \ll r^\varepsilon \).

To compute the main term in (2.4), we expand

\[
L(1/2 + s + ir, g \otimes \chi)L(1/2 + s - ir, g \otimes \chi)
\]

into its Dirichlet series. For \( \text{Re} \ s > 1/2 \), we have

\[
(2.5) \quad L \left( \frac{1}{2} + s + ir, g \otimes \chi \right)L \left( \frac{1}{2} + s - ir, g \otimes \chi \right) = \sum_{m,n \geq 1} \chi(mn) \frac{\lambda_g(m)\lambda_g(n)}{m^{1/2+s+ir}n^{1/2+s-ir}}.
\]

As \( g \) is a Hecke eigenform, we have

\[
\lambda_g(m)\lambda_g(n) = \sum_{d \mid (m,n)} \lambda_g \left( \frac{mn}{d^2} \right).
\]

Apply this to the right side of (2.5) and set \( m = ad, n = bd \). Then

\[
(2.6) \quad L \left( \frac{1}{2} + s + ir, g \otimes \chi \right)L \left( \frac{1}{2} + s - ir, g \otimes \chi \right) = \sum_{a,b,d \geq 1} \chi(abd^2) \frac{\lambda_g(ab)}{a^{1/2+s+ir}b^{1/2+s-ir}d^{1+2s}}
\]

\[
= L(1 + 2s, \chi^2) \sum_{n \geq 1} \frac{\chi(n)\lambda_g(n)\eta_a(n, 1/2 + ir)}{n^{1/2+s}},
\]

where

\[
d_a(n) = \sum_{ab = [n]} \left( \frac{a}{b} \right)^s.
\]

We remark that in (2.6), the series is actually taken over \( n \) which are relatively prime to \( \mathbb{Q} \).

Consider the Eisenstein series for any fixed cusp \( a \) of \( \Gamma = \Gamma_0(N) \) defined by

\[
E_a(z,s) = \sum_{\gamma \in \Gamma \setminus \Gamma_0(N)} (\text{Im} \ \sigma_a^{-1} \gamma z)^s
\]

for \( \text{Re} \ s > 1 \) and by analytic continuation for all \( s \in \mathbb{C} \). Here \( \Gamma_a \) is the stability group of \( a \), while \( \sigma_a \in SL(2, \mathbb{R}) \) is given by \( \sigma_a \infty = a \) and \( \sigma_a^{-1} \Gamma \sigma_a = \Gamma_\infty \). This Eisenstein series is an eigenfunction of the Hecke operators

\[
T_a E_a(z,s) = \eta_a(n,s) E_a(z,s),
\]

if \( (n,N) = 1 \). As pointed out in Conrey and Iwaniec [2], for any \( n \) relatively prime to \( N \), \( \eta_a(n,s) = d_{a-1/2}(n) \). Consequently, from (2.6) we get

\[
(2.7) \quad L \left( \frac{1}{2} + s + ir, g \otimes \chi \right)L \left( \frac{1}{2} + s - ir, g \otimes \chi \right) = L(1 + 2s, \chi^2) \sum_{n \geq 1} \frac{\chi(n)\lambda_g(n)\eta_a(n, 1/2 + ir)}{n^{1/2+s}}
\]

for \( \text{Re} \ s > 1/2 \).
Substituting (2.7) into the integral of the main term in (2.4), we get

\[
L \left( \frac{1}{2} + ir, g \otimes \chi \right) L \left( \frac{1}{2} - ir, g \otimes \chi \right)
= \frac{1}{\pi i} \int_{\text{Re } s = 2} \left( \sum_{m,n \geq 1} \frac{\chi(nm^2) \lambda_g(n) \eta_a(n, 1/2 + ir)}{(nm^2)^{1/2 + s}} \right) X^s G(s) \frac{ds}{s} + O(r^\varepsilon)
= 2 \sum_{m,n \geq 1} \frac{\chi(nm^2) \lambda_g(n) \eta_a(n, 1/2 + ir)}{m \sqrt{n}} \left( \frac{1}{2\pi i} \int_{\text{Re } s = 2} G(s) \left( \frac{nm^2}{X} \right)^{-s} ds \right) + O(r^\varepsilon).
\]

Denote

\[
V(y) = \frac{1}{2\pi i} \int_{\text{Re } s = 2} G(s) y^{-s} \frac{ds}{s}.
\]

Then as in [22] and [18], \( \lim_{y \to 0} V(y) = 1 \) and \( V(y) \ll_B (1 + |y|)^{-B} \) because of our choice of the function \( G(s) \). Therefore

\[
(2.8) \quad L \left( \frac{1}{2} + ir, g \otimes \chi \right) L \left( \frac{1}{2} - ir, g \otimes \chi \right)
= 2 \sum_{1 \leq m \leq X^{1/2 + \varepsilon}} \frac{\chi^2(m)}{m} \sum_{n \geq 1} \frac{\chi(n) \lambda_g(n) \eta_a(n, 1/2 + ir)}{\sqrt{n}} V \left( \frac{nm^2}{X} \right) + O(r^\varepsilon),
\]

because the outer series is negligible if taken over \( m > X^{1/2 + \varepsilon} \).

3. Averaging and the Kuznetsov trace formula

According to (2.8), estimation of the central value of our \( L \)-function is reduced to estimation of

\[
S_Y(g, r) = \sum_n \chi(n) \lambda_g(n) \eta_a(n, 1/2 + ir) H \left( \frac{n}{Y} \right)
\]

for fixed \( g \), where \( H \) is a fixed smooth function of compact support contained in \( (1, 2) \).

To prove our results on short intervals, let \( L \) be a number which satisfies \( \sqrt{K} \leq L \leq K/4 \) for large \( K \). Let \( h(t) \) be an even analytic function in \( |\text{Im } t| \leq 1/2 \) satisfying \( h^{(n)}(t) \ll (1 + |t|)^{-N} \) for any \( N > 0 \) in this region. Thus \( h \) is a Schwartz function on \( \mathbb{R} \). We also assume that \( h(t) \geq 0 \) for real \( t \). For example, we may simply take \( h(t) = 1/\cosh(t) \). Denote

\[
\zeta_N(s) = \prod_{p \nmid N} (1 - p^{-s})^{-1}.
\]
We want to estimate

\[
I_{K,L} = \int_{\mathbb{R}} \left( h\left(\frac{K-r}{L}\right) + h\left(\frac{K+r}{L}\right) \right) \left| S_{\gamma}(g,r) \right|^2 \frac{|\zeta_N(1+2ir)|^2}{|\zeta(1+2ir)|^2} \, dr
\]

\[
= \sum_{m,n} \chi(n)\bar{\chi}(m)\lambda_g(n)\bar{\lambda}_g(m)H\left(\frac{n}{Y}\right)\bar{H}\left(\frac{m}{Y}\right)
\]

\[
\times \int_{\mathbb{R}} \left( h\left(\frac{K-r}{L}\right) + h\left(\frac{K+r}{L}\right) \right) \, dr
\]

\[
\times \eta_a(n,1/2+ir)\bar{\eta}_a(m,1/2+ir) \frac{|\zeta_N(1+2ir)|^2}{|\zeta(1+2ir)|^2} \, dr.
\]

As in Liu and Ye [18], we apply the Kuznetsov trace formula to the integral on the right side of (3.1):

\[
\pi \sum_{m,n} \chi(n)\bar{\chi}(m)\lambda_g(n)\bar{\lambda}_g(m)H\left(\frac{n}{Y}\right)\bar{H}\left(\frac{m}{Y}\right)
\]

\[
\times \sum_{f_j} \left( h\left(\frac{K-k_j}{L}\right) + h\left(\frac{K+k_j}{L}\right) \right) \lambda_{f_j}(n)\bar{\lambda}_{f_j}(m)
\]

\[
+ \sum_{m,n} \chi(n)\bar{\chi}(m)\lambda_g(n)\bar{\lambda}_g(m)H\left(\frac{n}{Y}\right)\bar{H}\left(\frac{m}{Y}\right)
\]

\[
\times \int_{\mathbb{R}} \left( h\left(\frac{K-r}{L}\right) + h\left(\frac{K+r}{L}\right) \right) \, dr
\]

\[
\times \eta_a(n,1/2+ir)\bar{\eta}_a(m,1/2+ir) \frac{|\zeta_N(1+2ir)|^2}{|\zeta(1+2ir)|^2} \, dr.
\]

(3.4)

Here in (3.2) \( f_j \) are Hecke eigenforms, with Laplace eigenvalues \( 1/4+k_j^2 \) and Fourier coefficients \( \lambda_{f_j}(n) \), which form an orthonormal basis of the space of Maass cusp forms for \( \Gamma_0(N) \), while in (3.5) \( S(n,m;c) \) is the classical Kloosterman sum.

Recall that \( \chi(n)\lambda_g(n) \) is the \( n \)th Fourier coefficient of the twisted cusp form \( g_\chi \) as in (2.1) or (2.2). We want to apply the main estimation in Liu and Ye [18] (§4.1–§4.13) and Lau, Liu, and Ye [17], (2.2), to our (3.4) and (3.5) above. Note that these estimations are based on bounds for shifted convolution sums of Fourier
coefficients of cusp forms proved by Sarnak [22], Appendix, and by Lau, Liu, and Ye [17].

More precisely, (3.4) + (3.5) \( \ll LKY^{1+\varepsilon} \) for \( L = K^{1-1/(4+2\theta)+\varepsilon} \). Since (3.2) and (3.3) are both positive, this implies that (3.3), i.e., \( I_{K,L} \), is bounded by \( O(LKY^{1+\varepsilon}) \) for the same \( L \). By \( \zeta(1 + 2ir) \ll \log(1 + |r|) \), this estimate of \( I_{K,L} \) implies that

\[
(3.6) \quad \int_{K}^{K+L} |S_Y(g, r)|^2 \, dr \ll LKY^{1+\varepsilon}
\]

for the above \( L \).

Now we can go back to the fourth power moment of \( L(1/2 + ir, g \otimes \chi) \). Since \( g \) is self-contragredient and \( \chi \) is real, we have from (2.8) that

\[
\int_{K}^{K+L} \left| L\left(\frac{1}{2} + ir, g \otimes \chi \right) \right|^4 \, dr = \int_{K}^{K+L} \left| L\left(\frac{1}{2} + ir, g \otimes \chi \right) L\left(\frac{1}{2} + ir, g \otimes \chi \right) \right|^2 \, dr
\]

\[
\ll \int_{K}^{K+L} \sum_{1 \leq m \leq K^{1/2+\varepsilon}} \chi^2(m) \sum_{n \geq 1} \lambda(g(n) \eta_a(n,1/2+ir)) \frac{V(nm^2)}{\sqrt{n}} \left| \sum_{1 \leq \mu \leq K^{1+\varepsilon}} \chi(n) \lambda_g(n) \eta_a(n,1/2+ir) \right|^2 \, dr.
\]

Here we can take \( X = K^2 \) and get

\[
\int_{K}^{K+L} \left| L\left(\frac{1}{2} + ir, g \otimes \chi \right) \right|^4 \, dr
\]

\[
\ll \frac{1}{K^2} \int_{K}^{K+L} \left| \sum_{n \geq 1} \chi(n) \lambda_g(n) \eta_a(n,1/2+ir) \sum_{1 \leq m \leq K^{1+\varepsilon}} \chi^2(m) \frac{V(nm^2/K^2)}{\sqrt{nm^2/K^2}} \right|^2 \, dr.
\]

Now we apply a smooth dyadic subdivision to

\[
\sum_{1 \leq m \leq K^{1+\varepsilon}} \chi^2(m) \frac{V(nm^2/K^2)}{\sqrt{nm^2/K^2}},
\]

by dividing the interval \([1, K^{1+\varepsilon}]\) into subintervals of the form \([a, 1.8a]\) and covering, with overlapping, each subinterval by a smooth, nonnegative function of compact support. The total number of subintervals is \( O(\log K) \). This way, we can find a smooth function \( H \) of compact support in \((1, 2)\) so that

\[
\int_{K}^{K+L} \left| L\left(\frac{1}{2} + ir, g \otimes \chi \right) \right|^4 \, dr
\]

\[
\ll \frac{\log K}{K^2} \int_{K}^{K+L} \max_{1 \leq B \leq K^{1+\varepsilon}} \left| \sum_{n \geq 1} \chi(n) \lambda_g(n) \eta_a(n,1/2+ir) H\left(\frac{n}{K^2/B}\right) \right|^2 \, dr.
\]
The sum inside the absolute value signs is indeed $S_{K^2/B}(g, r)$. By (3.6), the maximum contribution is from $B = 1$:

$$
\int_{K} \left| L \left( \frac{1}{2} + iz, g \otimes \chi \right) \right|^4 \, dz \ll \frac{\log K}{K^2} \int_{K} \left| S_{K^2}(g, r) \right|^2 \, dr
$$

$$
\ll \frac{\log K}{K^2} L K (K^2)^{1+\epsilon} \ll (KL)^{1+\epsilon}
$$

for $L = K^{1-1/(4+2\theta)+\epsilon}$. This completes the proof of Theorem 1.1.

**Added in proof**

Recently, Lau, Liu, and Ye further improved the subconvexity bound (1.4) to $k^{3/4+\epsilon}$. Using this new result, our Theorem 1.1 can be stated for $L = K^{1/2+\epsilon}$.

**References**


Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242-1419  
E-mail address: yey@math.uiowa.edu