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UNIFORM ASYMPTOTICS FOR JACOBI POLYNOMIALS WITH VARYING LARGE NEGATIVE PARAMETERS— A RIEMANN-HILBERT APPROACH

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Abstract. An asymptotic expansion is derived for the Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(z)$ with varying parameters $\alpha_n = -nA + a$ and $\beta_n = -nB + b$, where $A > 1, B > 1$ and a, b are constants. Our expansion is uniformly valid in the upper half-plane $\overline{\mathbb{C}}^+ = \{z : \operatorname{Im} z \geq 0\}$. A corresponding expansion is also given for the lower half-plane $\overline{\mathbb{C}}^- = \{z : \operatorname{Im} z \leq 0\}$. Our approach is based on the steepest-descent method for Riemann-Hilbert problems introduced by Deift and Zhou (1993). The two asymptotic expansions hold, in particular, in regions containing the curve L, which is the support of the equilibrium measure associated with these polynomials. Furthermore, it is shown that the zeros of these polynomials all lie on one side of L, and tend to L as $n \to \infty$.

1. Introduction

The Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ have the explicit expression

$$P_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{z-1}{2}\right)^k\left(\frac{z+1}{2}\right)^{n-k}. \tag{1.1}$$

When $\alpha, \beta > -1$, these polynomials are orthogonal with respect to the weight function $w(z) = (1-z)^\alpha(1+z)^\beta$ on the interval $(-1, 1)$. In view of its importance in application, there is a tremendous amount of literature on the asymptotic behavior of these polynomials as n grows to infinity. For fixed α and β, classical results on this subject can be found in the definitive book by Szegö [13]; for more recent work, we refer to [8], [16] and the references given there. Some asymptotic results are now also available, when α and β depend on n and tend to infinity with n; see, e.g., [1], [2] and [9].

Formula (1.1) simply gives a polynomial in z, without any restriction on the parameters α and β. When $\alpha, \beta < -1$, these polynomials are of course no longer orthogonal with respect to the weight function $w(z)$ on the interval $(-1, 1)$. Interest seems to be growing in recent years to study the zeros of the polynomials in (1.1), when α and β are negative and tend to $-\infty$ as $n \to \infty$; see [6], [11] and [14]. In

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fact, Temme [14, p. 461] has expressed that new research is needed in this area of asymptotics. Motivated by the work in [9] and [11], let us restrict our attention to the case

$$\alpha = \alpha_n = -An + a \qquad \text{and} \qquad \beta = \beta_n = -Bn + b, \tag{1.2}$$

where A, B and a, b are constants. Furthermore, we shall assume that $A > 1$ and $B > 1$ so that $\alpha_n < -n, \beta_n < -n$ and $2n+\alpha+\beta < -1$ for large n. By the reflection formula $P_n^{(\alpha,\beta)}(z) = (-1)^n P_n^{(\beta,\alpha)}(-z)$, we may also assume that $B \geq A$.

The purpose of this paper is to investigate the behavior of the polynomials $P_n^{(\alpha,\beta)}(z)$ in (1.1) and their zeros, when α and β are given by (1.2). Our approach is to use the steepest descent method introduced by Deift and Zhou [5] for Riemann-Hilbert problems, and further developed in [3] and [4]. The basic idea of this method is to first use a theorem of Fokas, Its and Kitaev [7], in which the orthogonal polynomial in concern appears as an entry in the matrix solution to a 2-dimensional Riemann-Hilbert problem (RHP). Then one applies a sequence of transformations which ultimately leads to a RHP which can be solved asymptotically in various parts of the complex plane. Thus, the final result usually consists of a set of asymptotic expansions, each valid in a different region; cf. [3] and [10]. In this paper, we shall present a modification of this method, and construct an asymptotic expansion for $P_n^{(\alpha_n,\beta_n)}(z)$, which holds uniformly in the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \text{ Im } z \geq 0\}$, where α_n and β_n are given in (1.2). Since the Jacobi polynomials have real coefficients, a uniform asymptotic expansion in the lower half-plane $\mathbb{C}^- = \{z \in \mathbb{C} : \text{ Im } z \leq 0\}$ can be obtained by taking complex conjugates. Our analysis consists of two parts. The first part is essentially to follow the method of Deift and Zhou, and to come up with a formal (heuristic) derivation of an asymptotic approximation. The second part is to prove that this asymptotic approximation is indeed the leading term of a globally uniform asymptotic expansion.

2. Orthogonality of Jacobi polynomials

Two main ingredients in the Riemann-Hilbert approach are: (i) orthogonality of the polynomials with respect to an appropriate weight function on a suitable curve; (ii) asymptotic distribution of the zeros of these polynomials. Both of them have been established recently for the polynomials under consideration; see [9] and [11]. However, for convenience of the reader, we will include in this paper brief sketches of the arguments used. Since some of the well-known identities (e.g., 3-term recurrence relation) no longer hold for the polynomials $P_n^{(\alpha_n,\beta_n)}(z)$, it is more efficient to introduce the polynomials $\{\boldsymbol{P}_k(z)\}_{k=0}^n$ defined by

$$\boldsymbol{P}_k(z) = P_k^{(\alpha_n,\beta_n)}(z), \qquad k = 0, 1, 2, ..., n. \tag{2.1}$$

Clearly, $\boldsymbol{P}_n(z)$ is exactly the Jacobi polynomial of degree n.

Define the weight function

$$\omega(z; \alpha, \beta) = (z-1)^\alpha (z+1)^\beta = e^{\alpha \log(z-1) + \beta \log(z+1)}, \tag{2.2}$$

where the logarithmic functions are defined in the cut plane $\mathbb{C}\backslash(-\infty, -1] \cup [1, \infty)$ with $\arg(0-1) = \pi$ and $\arg(0+1) = 0$, i.e., $\omega(0; \alpha, \beta) = e^{\pi\alpha i}$. Using the results for the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$, the following identities can be easily verified.

(A) Rodrigues' formula

$$\boldsymbol{P}_k(z) = \frac{1}{2^k k!}\frac{\omega^{(k)}(z; k+\alpha_n, k+\beta_n)}{\omega(z; \alpha_n, \beta_n)}, \qquad k = 1, 2, ..., n, \tag{2.3}$$

where $w^{(k)}$ denotes the kth derivative of ω.

(B) Differential equation

$$\begin{aligned}(z^2-1)\boldsymbol{P}_k''(z) + [\alpha_n - \beta_n + (\alpha_n+\beta_n+2)z]\boldsymbol{P}_k'(z) \\ - k(k+\alpha_n+\beta_n+1)\boldsymbol{P}_k(z) = 0, \qquad k = 1, 2, ..., n.\end{aligned} \tag{2.4}$$

(C) Three-term recurrence relation

$$\boldsymbol{P}_{k+1}(z) - (a_k z + b_k)\boldsymbol{P}_k(z) + c_k \boldsymbol{P}_{k-1}(z) = 0, \qquad k = 1, 2, ..., n-1, \tag{2.5}$$

where

$$\begin{aligned} a_k &= \frac{(2k+\alpha_n+\beta_n+1)(2k+\alpha_n+\beta_n+2)}{(2k+2)(k+\alpha_n+\beta_n+1)}, \\ b_k &= \frac{(2k+\alpha_n+\beta_n+1)(\alpha_n^2-\beta_n^2)}{(2k+2)(k+\alpha_n+\beta_n+1)(2k+\alpha_n+\beta_n)}, \\ c_k &= \frac{2(k+\alpha_n)(k+\beta_n)(2k+\alpha_n+\beta_n+2)}{(2k+2)(k+\alpha_n+\beta_n+1)(2k+\alpha_n+\beta_n)}. \end{aligned} \tag{2.6}$$

These formulas will play an important role in our later discussion.

Let Γ be a generic smooth curve in the complex z-plane that crosses the real axis between -1 and 1, and nowhere else; see Figure 1.

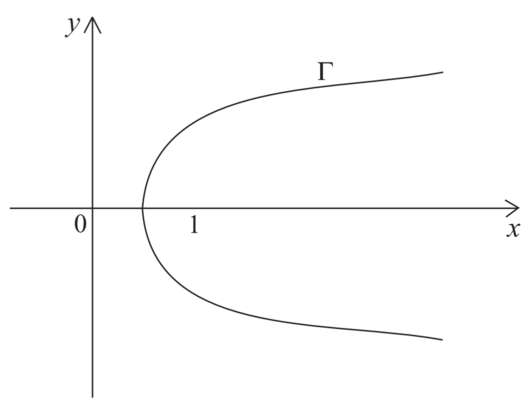

FIGURE 1. Path Γ.

Proposition 1. *If $\alpha_n < -n, \beta_n < -n$ and $2n+\alpha_n+\beta_n < -1$, then*

$$\int_\Gamma z^k \boldsymbol{P}_m(z) w(z; \alpha_n, \beta_n) dz = C'_{m,n}\delta_{k,m} \tag{2.7}$$

for $k = 0, 1, \cdots, m$ and $m = 1, \cdots, n$, where

$$C'_{m,n} = \frac{-2^{m+\alpha_n+\beta_n+2}\pi i e^{i\alpha_n\pi}}{B(-m-\alpha_n, -m-\beta_n)(2m+\alpha_n+\beta_n)} \neq 0, \tag{2.8}$$

$B(x,y)$ being the beta function, i.e., $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$.

Proof. Let $I(m,k)$ denote the integral in (2.7). By Rodrigues' formula, we have

$$I(m,k)=\frac{1}{2^m m!}\int_\Gamma z^k\frac{d^m}{dz^m}[\omega(z;m+\alpha_n,m+\beta_n)]dz.$$

To the last integral, we apply integration by parts m times. This gives

$$\begin{aligned} I(m,k) &= \frac{1}{2^m m!}\sum_{j=0}^{m-1}(-1)^j\frac{d^j}{dz^j}(z^k)\frac{d^{m-j-1}}{dz^{m-j-1}}[(z-1)^{m+\alpha_n}(z+1)^{m+\beta_n}]\Big|_\Gamma \\ &\quad +\frac{(-1)^m}{2^m m!}\int_\Gamma\frac{d^m}{dz^m}(z^k)[(z-1)^{m+\alpha_n}(z+1)^{m+\beta_n}]dz. \end{aligned}$$

As $z\to\infty$, the jth term in the sum behaves like a constant multiple of $z^{k+m+\alpha_n+\beta_n+1}$. Since $2n+\alpha_n+\beta_n<-1$, all terms under the summation vanish for $k=0,1,\cdots,m$. Also, $d^m(z^k)/dz^m=0$ for $k<m$. Hence, $I(m,k)=0$ if $k<m$. For $k=m$, we have by partial integration j times

$$\begin{aligned} I(m,m) &= \frac{(-1)^{m+j}}{2^m}\frac{\Gamma(m+\beta_n+1)\Gamma(m+\alpha_n+1)}{\Gamma(m+\beta_n-j+1)\Gamma(m+\alpha_n+j+1)} \\ &\quad \times\int_\Gamma(z-1)^{m+\alpha_n+j}(z+1)^{m+\beta_n-j}dz. \end{aligned}$$

Choose j so that $-1<m+\alpha_n+j\le 0$. The contour Γ can now be deformed into two straight lines from 1 to ∞, one on the upper edge of the cut along the real axis from 1 to ∞ and the other on the lower edge of the cut. The two resulting integrals can be evaluated in terms of gamma functions. This completes the proof of the proposition. □

Corollary 1. *Under the condition of Proposition* 1*, we have*

$$\int_\Gamma \boldsymbol{P}_m(z)\boldsymbol{P}_k(z)\omega(z;\alpha_n,\beta_n)dz=C_{m,n}\delta_{k,m}, \tag{2.9}$$

for $k,m=0,1,,\cdots,n$*, where*

$$C_{m,n}=\frac{1}{2^m}\binom{2m+\alpha_n+\beta_n}{m}C'_{m,n}. \tag{2.10}$$

This result follows immediately from Proposition 1, by noticing that the leading coefficient in the Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$ is

$$\frac{1}{2^n}\sum_{k=0}^n\binom{n+\alpha}{n-k}\binom{n+\beta}{k}=\frac{1}{2^n}\binom{2n+\alpha+\beta}{n}; \tag{2.11}$$

cf. (1.1).

Remark. Corollary 1 shows that the polynomials $\boldsymbol{P}_k(z),k=0,1,\cdots,n$, are orthogonal on the curve Γ with respect to the weight function $\omega(z;\alpha_n,\beta_n)$. However, in general,

$$\int_\Gamma P_m^{(\alpha_m,\beta_m)}(z)P_k^{(\alpha_k,\beta_k)}(z)\omega(z;\alpha_n,\beta_n)dz\neq 0;$$

i.e., the Jacobi polynomials $P_k^{(\alpha_k,\beta_k)}(z),k=0,1,2,\cdots,n$, are *not* orthogonal on Γ with respect to $\omega(z;\alpha_n,\beta_n)$.

3. Limit distribution of the zeroes

To study the zero distribution of the polynomials $\boldsymbol{P}_n(z) = P_n^{(\alpha_n,\beta_n)}(z)$, we first note that by using the three-term recurrence relation (2.5) and the Gershgorin theorem it can be shown that there exists a constant R_0 such that all zeros of $\boldsymbol{P}_k(z), k = 0, 1, \cdots, n$, lie inside the disk $|z| \leq R_0$; in particular, the zeros of $P_n^{(\alpha_n,\beta_n)}(z)$ are uniformly bounded. Let μ_n denote the zero-counting measure of $P_n^{(\alpha_n,\beta_n)}(z)$, i.e., for any compact subset $K \subset \mathbb{C}$,

$$\int_K d\mu_n = \frac{1}{n}\{\text{the number of zeros of } P_n^{(\alpha_n,\beta_n)}(z) \text{ in } K\}, \tag{3.1}$$

where the zeros are counted with their multiplicity. In term of the Dirac delta function, $d\mu_n$ can be expressed as $d\mu_n = (1/n)\sum_{k=1}^n \delta(z - z_k)$. Furthermore, let μ be the limit distribution of the zeros of $P_n^{(\alpha_n,\beta_n)}(z)$ as $n \to \infty$. The uniform boundedness of the zeros of $P_n^{(\alpha_n,\beta_n)}(z)$ implies that all μ_n have uniformly bounded support; consequently,

$$\mu_n \xrightarrow{*} \mu \qquad \text{as} \quad n \to \infty. \tag{3.2}$$

The following result gives a precise description of the measure μ.

Proposition 2. *The support of μ is an oriented smooth curve L which is symmetric with respect to the real axis, starts from the point s and ends at the point $\bar{s}$, where*

$$s = \frac{B^2 - A^2 + 4i\sqrt{(A-1)(B-1)(A+B-1)}}{(A+B-2)^2}; \tag{3.3}$$

see Figure 2. Furthermore,

$$d\mu(z) = \frac{(A+B-2)R_+(z)}{2\pi i(z^2-1)}, \qquad z \in L, \tag{3.4}$$

in which $R(z) = \sqrt{(z-s)(z-\bar{s})}$ behaves like z as $z \to \infty$ and the complex z-plane is cut along the curve L.

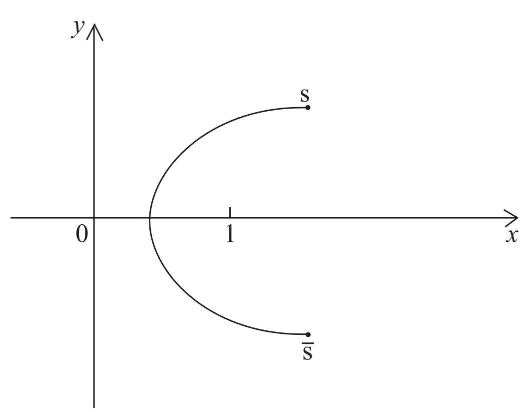

Figure 2. The support of μ.

The results in Proposition 2 are essentially given in [11, Theorem 1]. However, since not many details of proof are provided there, and since some of the formulas to be used in the following "sketch of proof" will become useful later in our discussion, we decide to include an outline of the argument.

Sketch of the proof. Let $z_1, ..., z_n$ be all the zeros of $P_n^{(\alpha_n,\beta_n)}(z)$ and consider the functions

$$h_n(z) = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{z-z_k} = \int\frac{d\mu_n(\xi)}{z-\zeta}, \qquad n=1,2,3,\cdots. \tag{3.5}$$

Clearly,

$$h_n'(z) = \frac{1}{n}\sum_{k=1}^{n}\frac{-1}{(z-z_k)^2} = -\int\frac{d\mu_n(\zeta)}{(z-\zeta)^2}, \qquad n=1,2,3,\cdots. \tag{3.6}$$

Since the zeros of $\boldsymbol{P}_n(z)$ are uniformly bounded, there exists a compact set $\Omega\subset\mathbb{C}$ containing $\{z : |z| \geq R_0+1\}$, on which both $h_n(z)$ and $h_n'(z)$ are analytic and uniformly bounded for all $n=1,2,3,\cdots$. In fact, we have $|h_n(z)|\leq 1$, and $|h_n'(z)|\leq 1$, since $|z-z_k|\geq 1, k=1,\cdots,n$, for all $z\in\Omega$. Hence, by Weierstrass's theorem, there is a subsequence $\{n_k\}$ such that $h_{n_k}(z)$ and $h_{n_k}'(z)$ converge uniformly to an analytic function $h(z)$ and its derivative $h'(z)$, respectively, on the compact set Ω. From (3.5) and the weak $*$ convergence in (3.2), we expect that

$$h(z) = \int\frac{d\mu(\zeta)}{z-\zeta}, \tag{3.7}$$

and $d\mu(z)$ can be determined from $h(z)$.

To find $h(z)$, we first note that $h_n(z) = \boldsymbol{P}_n'(z)/n\boldsymbol{P}_n(z)$. Hence, the differential equation (2.4) can be written as

$$\begin{aligned}(z^2-1)\left[h_n^2(z)+\frac{1}{n}h_n'(z)\right] =& \frac{1}{n}\left[\beta_n-\alpha_n-(\alpha_n+\beta_n+2)z\right]h_n(z)\\ &+\frac{n+\alpha_n+\beta_n+1}{n}.\end{aligned} \tag{3.8}$$

Taking subsequence $\{n_k\}$ in (3.8), and letting $k\to\infty$, we obtain

$$(z^2-1)h^2(z) - [A(z+1)+B(z-1)]\,h(z) + (A+B-1) = 0$$

for $z\in\Omega$, which in turn gives

$$h(z) = \frac{A(z+1)+B(z-1)-(A+B-2)R(z)}{2(z^2-1)}, \qquad z\in\Omega, \tag{3.9}$$

where $R(z)$ is as given in the statement of the proposition. Since $R(z)\sim z$ as $z\to\infty$, it follows that $zh(z)\to 1$ as $z\to\infty$, which is consistent with the fact that $zh_n(z)\to 1$ as $z\to\infty$ for all n. We also note that

$$R(1) = \sqrt{(1-s)(1-\bar{s})} = |1-s| = \frac{2A}{A+B-2},$$

$$R(-1) = \sqrt{(-1-s)(-1-\bar{s})} = -|1+s| = \frac{-2B}{A+B-2},$$

which imply that $h(z)$ has removable singularities at $z=\pm 1$. Since $R(z)$ is analytic everywhere except on the cut along L, $h(z)$ in (3.9) can be analytically continued to $\mathbb{C}\setminus L$ with the values at $z=\pm1$ given by

$$h(\pm 1) = \pm\frac{A+B}{4}. \tag{3.10}$$

To determine $d\mu(\zeta)$, we apply Cauchy's integral formula to $h(z)$; that is, we write

$$h(z) = \frac{1}{2\pi i}\int_\gamma \frac{h(\zeta)}{\zeta - z}d\zeta, \tag{3.11}$$

where γ is a clockwise oriented closed curve enclosing L but not z and ± 1. Taking γ sufficiently close to L, we get

$$h(z) = \frac{1}{2\pi i}\int_L \frac{h_+(\zeta) - h_-(\zeta)}{\zeta - z}d\zeta. \tag{3.12}$$

With $h(z)$ given in (3.9) for all $z \in \mathbb{C}\backslash L$, it follows that

$$h(z) = \frac{1}{2\pi i}\int_L \frac{-(A+B-2)R_+(\zeta)}{(\zeta - z)(\zeta^2 - 1)}d\zeta, \tag{3.13}$$

where $R_+(\zeta)$ denotes the limiting value of $R(\zeta)$ on the left-hand side of L. Comparing (3.13) with (3.7), it is evident that we should take

$$d\mu(\zeta) = \frac{(A+B-2)R_+(\zeta)}{2\pi i(\zeta^2 - 1)}, \qquad \zeta \in L. \tag{3.14}$$

If $s = \bar{s}$, then $A = 1, s = \frac{B+1}{B-1}$ and $R_+(\zeta) = \zeta - s$. Hence

$$d\mu(\zeta) = \frac{B(\zeta - 1) - (\zeta + 1)}{2\pi i(\zeta^2 - 1)}, \qquad \zeta \in L. \tag{3.15}$$

Equations (3.14) and (3.15) determine the measure μ, thus completing the proof. $\square$

Since μ is to be a measure on L, we want $\int_{\bar{s}}^z d\mu(\zeta)$ to be real or equivalently, the function

$$r(z) = \int_{\bar{s}}^z \frac{R_+(\zeta)}{\zeta^2 - 1}d\zeta \tag{3.16}$$

to be purely imaginary for $z \in L$. We define L to be the curve satisfying the equation

$$|\exp\{r(z)\}|^2 = 1. \tag{3.17}$$

In general, equation (3.17) may consist of more than one curve. We use L to denote the one which begins at $\bar{s}$, crosses the real axis between -1 and 1, and ends at s.

Proposition 3. *The measure $d\mu(z)$ is a probability measure on L.*

Proof. We first show that

$$\int_L d\mu(\zeta) = 1. \tag{3.18}$$

If $s = \bar{s}$, then either $A = 1$ or $B = 1$. We consider only the case $A = 1$; the other case is entirely similar. Since $s = \bar{s}, L$ is a closed curve encircling 1, but not -1. Also, we have

$$s = \frac{B+1}{B-1} > 1, \qquad d\mu(\zeta) = \frac{(B-1)(\zeta - s)}{2\pi i(\zeta^2 - 1)}d\zeta.$$

By Cauchy's residue theorem,

$$\int_L d\mu(\zeta) = \frac{B-1}{2\pi i}\int_L \frac{\zeta - s}{\zeta^2 - 1}d\zeta = 1.$$

If $s \neq \bar{s}$, then $A, B \neq -1$. Following the argument used in (3.11)–(3.13), we have from (3.14)

$$\int_L d\mu(\zeta) = \frac{A+B-2}{2\pi i} \int_L \frac{R_+(\zeta)}{\zeta^2-1} d\zeta$$
$$= \frac{A+B-2}{2} \cdot \frac{1}{2\pi i} \int_\gamma \frac{R(\zeta)}{\zeta^2-1} d\zeta,$$

where γ is the closed curve described in (3.11). It encloses L in the clockwise direction, and not the points $+1$ and -1. Again by the residue theorem,

$$\int_L d\mu(\zeta) = \frac{A+B-2}{2} \left[\frac{R(1)}{2} + \frac{R(-1)}{-2} - \lim_{z\to\infty} z \frac{R(z)}{z^2-1} \right].$$

The values of $R(1)$ and $R(-1)$ have already been given before, and the limit in the last equation is equal to 1 since $R(z) \sim z$ as $z \to \infty$; see the equations following (3.9). A simple calculation shows that (3.18) again holds.

Next, we demonstrate that as z moves from $\bar{s}$ to s along L, the value of the integral $I(z) = \int_{\bar{s}}^z d\mu(\zeta)$ increases from 0 to 1. Let the parametric equation of the arc length of L be denoted by

$$z = z(t), \qquad t \in [0,1], \qquad z(0) = \bar{s}, \qquad z(1) = s.$$

For any $z = z(x) \in L \backslash \{s, \bar{s}\}, x \in [0,1]$, we have

$$I(z) = \int_{\bar{s}}^z d\mu(\zeta) = \frac{A+B-2}{2\pi i} \int_0^x \frac{R(z(t))}{z^2(t)-1} z'(t) dt := J(x).$$

Clearly, $J(x) \in \mathbb{R}, J(0) = 0, J(1) = 1$ and

$$J'(x) = \frac{A+B-2}{2\pi i} \frac{R(z(x))}{z^2(x)-1} z'(x) \neq 0, \qquad x \in (0,1).$$

Hence, $J(x)$ is monotonically increasing on $[0,1]$; i.e., $I(z)$ is monotonically increasing as z moves from $\bar{s}$ to s along L, thus proving that $d\mu(\zeta)$ is a probability measure. □

So far we have not proved that L is the limit distribution curve of the zeros of $P_n^{(\alpha_n,\beta_n)}(z)$. We shall establish this fact later in Section 9 below. Here, let us first consider some special cases in which the curve L given in (3.17) can be described more precisely.

When $A = 1$ and $B > 1$, we have $s = (B+1)/(B-1) > 1$ and $R(z) = z - s$. The equation for L can be simplified to

$$\left| \left(\frac{z-1}{s-1} \right)^{s-1} \left(\frac{s+1}{z+1} \right)^{s+1} \right| = 1$$

or, equivalently,

$$\left| \frac{z+1}{s+1} \right|^B = \left| \frac{z-1}{s-1} \right|. \tag{3.19}$$

In this case, L is a simple closed curve enclosing $z = 1$ but not $z = -1$, passing through $z = s$ and a point $\delta_B \in (0,1)$. As $B \to 1^+$, we have $s \to +\infty$ and $\delta_B \to 0^+$. As $B \to +\infty$, we have $s \to 1^+$ and $\delta_B \to 1^-$. The curve L when $A = 1$ and $B = 2$ and the corresponding zeros of $P_n^{(\alpha_n,\beta_n)}(z)$ when $a = 0.5$ and $b = 1$ are shown in Figures 3 and 4 for $n = 60$ and $n = 100$, respectively.

When $A = B > 1$, $s = i\frac{\sqrt{2A-1}}{A-1}$ is purely imaginary, and L is the line segment on the y-axis, starting from $\bar{s}$ and ending at s.

When $B > A > 2$, we have

$$0 < \mathrm{Re}\, s = \frac{B^2 - A^2}{(A+B-2)^2} < 1,$$

and L is a smooth curve starting from $\bar{s}$, ending at s, and passing through a point $\delta_{A,B} \in (0,1)$.

In the case when $B > A$ and $1 < A < 2$, the above equality still holds. A routine computation also shows that Re $s < 1$ if $B < \frac{(A-1)^2+1}{2-A}$, Re $s = 1$ if $B = \frac{(A-1)^2+1}{2-A}$, and Re $s > 1$ if $B > \frac{(A-1)^2+1}{2-A}$. Furthermore, as $B \to +\infty$, we have Re $s \to 1^+$.

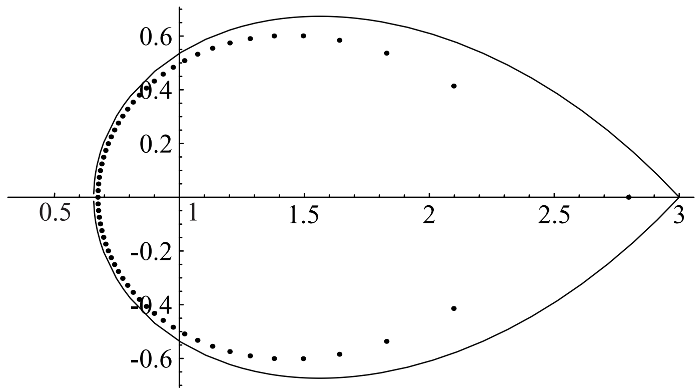

FIGURE 3. Curve L when $A = 1, B = 2$ and zeros of $P_n^{(\alpha_n,\beta_n)}(z)$ when $a = 0.5, b = 1$ and $n = 60$.

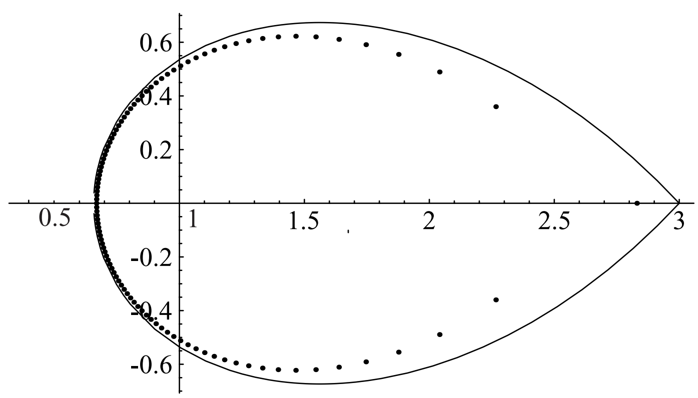

FIGURE 4. Curve L when $A = 1, B = 2$ and zeros of $P_n^{(\alpha_n,\beta_n)}(z)$ when $a = 0.5, b = 1$ and $n = 100$.

4. Auxiliary functions and the right path

A key step in using the RHP approach to analyze the asymptotic behavior of orthogonal polynomials is to choose a right path. In Corollary 1, we have already seen that the polynomials $\boldsymbol{P}_k, z = 0, 1, \cdots, n$, are orthogonal with respect to the weight function $\omega(z; \alpha_n, \beta_n)$ on a quite general class of curves Γ (Figure 1). Now we will have to choose a path which is more suitable for our later use; the asymptotic zero distribution curve L (Figure 2) will form part of our desired path Γ. We recall from Proposition 3 that the integral $I(z) = \int_{\bar{s}}^{z} d\mu(\zeta)$ is real and increases from 0 to 1 as z moves from $\bar{s}$ to s along L.

Choose a curve Σ_1, which starts from $\bar{s}$, goes to infinity in the direction of the positive real-axis, and stays below the real line. Furthermore, on Σ_1 we require the function

$$K_1(z) = \frac{A+B-2}{2\pi i} \int_{\bar{s}}^{z} \frac{R(\zeta)}{\zeta^2 - 1} d\zeta, \qquad z \in \mathbb{C} \backslash L, \tag{4.1}$$

to be purely imaginary and $iK_1(z)$ to be positive. This is possible, since $\frac{R(\zeta)}{\zeta^2-1} \sim \frac{1}{\zeta}$ for $\zeta \in \mathbb{R}$ and $\zeta \to +\infty$ and hence $K_1(z) \sim \frac{A+B-2}{2\pi i} \log z$ for $z \in \mathbb{R}$ and $z \to +\infty$. The last quantity is purely imaginary, and $K_1(\bar{s}) = 0$. Similarly, we can choose the curve Σ_2 on which

$$K_2(z) = \frac{A+B-2}{2\pi i} \int_{s}^{z} \frac{R(\zeta)}{\zeta^2 - 1} d\zeta, \qquad z \in \mathbb{C} \backslash L, \tag{4.2}$$

is purely imaginary and $iK_2(z)$ is positive. Σ_2 lies in the upper half-plane, and is symmetric to Σ_1 with respect to the real-axis. The desired path is $\Gamma = \Sigma_1 \cup L \cup \Sigma_2$, and we orient it in the counterclockwise direction. This path divides the complex plane $\mathbb{C}$ into two regions Ω_+ and Ω_-, as shown in Figure 5.

When $\bar{s} = s \in \mathbb{R}, \Sigma_1$ and Σ_2 are, respectively, the lower and upper edge of the infinite interval $[s, +\infty)$. Also we point out that the path Γ and the real line $\mathbb{R}$ divide the complex z-plane into four disjoint regions: $D_1 = \mathbb{C}^+ \cap \Omega_+, D_2 = \mathbb{C}^+ \cap \Omega_-, D_3 = \mathbb{C}^- \cap \Omega_+$ and $D_4 = \mathbb{C}^- \cap \Omega_-$; see Figure 6.

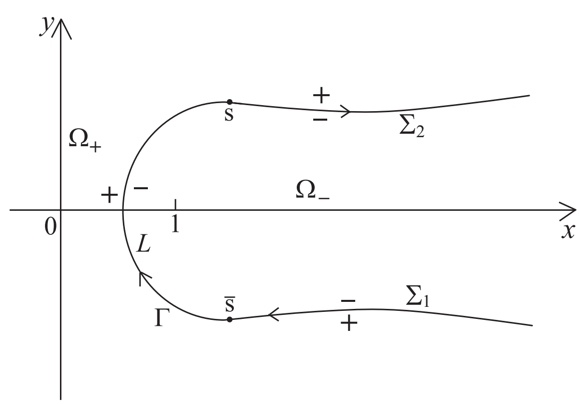

FIGURE 5. Path $\Gamma = \Sigma_1 \cup L \cup \Sigma_2$ and regions $\Omega_\pm$.

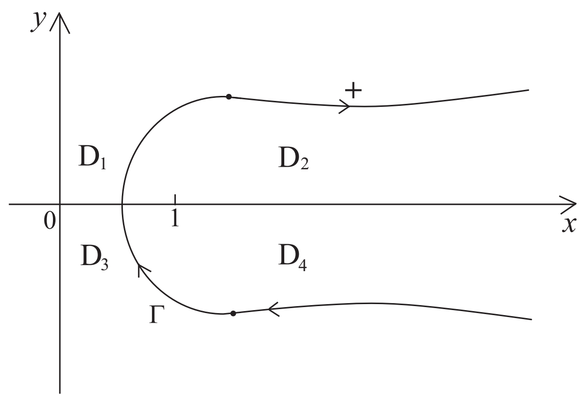

FIGURE 6. Regions D_1, D_2, D_3 and D_4.

To obtain the asymptotic behavior of $\boldsymbol{P}_n(z) = P_n^{(\alpha_n,\beta_n)}(z)$, it is equivalent to consider the monic polynomial

$$\pi_n(z) = \prod_{k=1}^{n}(z - z_k) = \gamma_n^{-1} P_n^{(\alpha_n,\beta_n)}(z),$$

where $z_1, \cdots, z_n$ are the zeros of $P_n^{(\alpha_n,\beta_n)}(z)$, and γ_n is the leading coefficient in $P_n^{(\alpha_n,\beta_n)}(z)$; cf. (2.11). Note that

$$\pi_n(z) = \exp\left[\sum_{k=1}^{n} \log(z - z_k)\right] = \exp\left[n \int \log(z - \zeta) d\mu_n(\zeta)\right].$$

This leads us to the following definition.

Definition 1. The so-called *g-function* is the complex logarithmic potential of μ defined by

$$g(z) = \int_L \log(z - \zeta) d\mu(\zeta), \qquad z \in \mathbb{C}\backslash(L \cup \Sigma_1), \tag{4.3}$$

where for each ζ we view $\log(z - \zeta)$ as an analytic function of z with branch cut starting from $z = \zeta$ to $z = \infty$ along the curve $L \cup \Sigma_1$ (see Figure 7).

The ϕ-function is defined by

$$\phi(z) = \frac{A + B - 2}{2} \int_s^z \frac{R(\zeta)}{\zeta^2 - 1} d\zeta, \qquad z \in \mathbb{C}\backslash L \cup \Sigma_1 \cup (-\infty, -1] \cup [1, \infty), \tag{4.4}$$

where the path of integration from s to z lies wholly inside the region $\mathbb{C}\backslash L \cup \Sigma_1 \cup (-\infty, 1] \cup [1, \infty)$; see Figure 7.

Similarly, we define

$$\tilde{g}(z) = \int_L \log(z - \zeta) d\mu(\zeta), \qquad z \in \mathbb{C}\backslash L \cup \Sigma_2, \tag{4.5}$$

and

$$\widetilde{\phi}(z) = \frac{A + B - 2}{2} \int_{\bar{s}}^z \frac{R(\zeta)}{\zeta^2 - 1} d\zeta, \qquad z \in \mathbb{C}\backslash L \cup \Sigma_2 \cup (-\infty, -1] \cup [1, \infty), \tag{4.6}$$

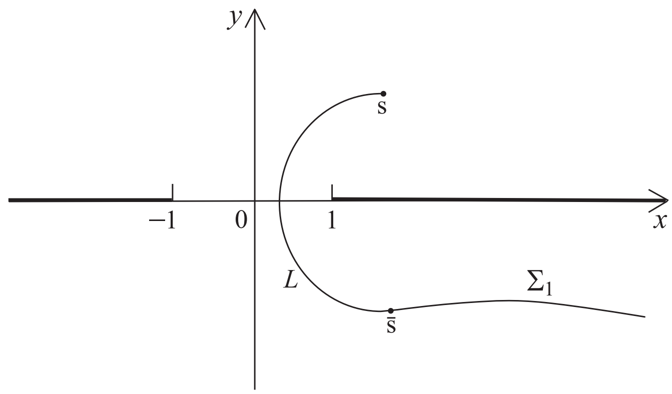

FIGURE 7. The cut of $\log(z-\zeta)$ in the g-function and the integration path of $\phi(z)$.

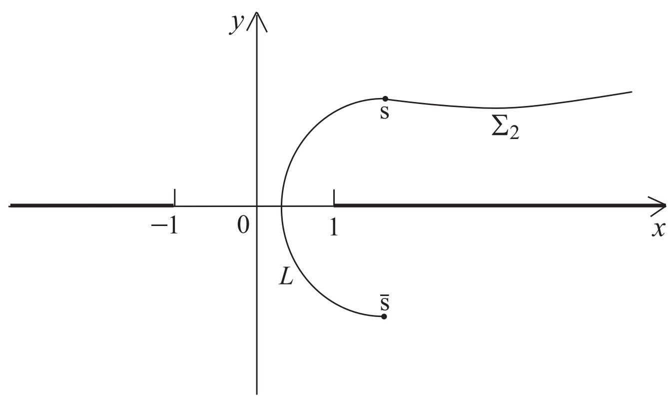

FIGURE 8. The cut of $\log(z-\zeta)$ in the $\tilde{g}$-function and the integration path of $\widetilde{\phi}(z)$.

where for each ζ, $\log(z-\zeta)$ is an analytic function of z with a branch cut from ζ to ∞ along the curve $L \cup \Sigma_2$. The path of integration in (4.6) from $\bar{s}$ to z lies wholly inside the region $\mathbb{C}\backslash L \cup \Sigma_2 \cup (-\infty, -1] \cup [1, \infty)$; see Figure 8.

From (4.3) and (4.5), it is easily seen that $g(z) \sim \log z$ and $\tilde{g}(z) \sim \log z$, as $z \to \infty$. Also, we have formally $\pi_n(z) \sim e^{ng(z)}$ and $\pi_n(z) \sim e^{n\tilde{g}(z)}$, as $z \to \infty$, where the $\log z$ in $g(z)$ and $\tilde{g}(z)$ may differ by $2\pi i$ for z in certain regions of the complex plane in view of the different cuts that have been chosen for them. Furthermore, since $\int_{\bar{s}}^{z} = \int_{\bar{s}}^{s} + \int_{s}^{z}$, the functions ϕ and $\tilde{\phi}$ are related by

$$\tilde{\phi}(z) - \phi(z) = \pm \pi i, \qquad z \in \Omega_{\pm}. \tag{4.7}$$

From the definitions of ϕ and $\tilde{\phi}$, we also have the important relationship

$$\tilde{\phi}(z) = \overline{\phi(\overline{z})}, \qquad z \in \mathbb{C}\backslash\Gamma \cup (-\infty, -1] \cup [1, \infty). \tag{4.8}$$

The auxiliary functions $g(z)$, $\tilde{g}(z)$, $\phi(z)$ and $\tilde{\phi}(z)$ play an important role in the asymptotic representation of the Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(z)$. In the remaining part of this section, we shall discuss some of their properties. For convenience, let us fix the orientation of the real axis so that it always traverses from $-\infty$ to $+\infty$; that is, the $+$ side of $\mathbb{R}$ lies in the upper half-plane $\mathbb{C}^+$, and the $-$ side lies in the lower half-plane $\mathbb{C}^-$. The point where the curve L intersects the real line $\mathbb{R}$ will be denoted by L_x.

Proposition 4. *Define the constants*

$$l := 2g(s) - [A\log(s-1) + B\log(s+1)], \tag{4.9a}$$

$$\tilde{l} := 2\tilde{g}(\bar{s}) - [A\log(\bar{s}-1) + B\log(\bar{s}+1)]. \tag{4.9b}$$

The g-function (respectively, *$\tilde{g}$*-function) *and the ϕ-function (respectively, $\tilde{\phi}$-function) are related through the equations*

$$g(z) + \phi(z) = \frac{1}{2}\left[A\log(z-1) + B\log(z+1) + l\right], \tag{4.10a}$$

$$\tilde{g}(z) + \tilde{\phi}(z) = \frac{1}{2}\left[A\log(z-1) + B\log(z+1) + \tilde{l}\right], \tag{4.10b}$$

$$\tilde{g}(z) = \begin{cases} g(z), & z \in \Omega_+, \\ g(z) + 2\pi i, & z \in \Omega_-, \end{cases} \tag{4.11}$$

and

$$\tilde{l} = l + 2\pi i, \tag{4.12}$$

where $\log(z-1)$ *and* $\log(z+1)$ *are defined in the complex plane with cuts on* $[1,\infty)$ *and* $(-\infty,-1]$*, respectively.*

Proof. On account of (4.3), (3.13) and (3.14), we have

$$g'(z) = \int_L \frac{d\mu(\zeta)}{z-\zeta} = h(z), \qquad z \in \mathbb{C}\backslash L \cup \Sigma_1.$$

From (3.9) it follows that for $z \in \mathbb{C}\backslash L \cup \Sigma_1 \cup (-\infty,-1] \cup [1,\infty)$,

$$\begin{aligned} g(z) &= g(s) + \int_s^z h(\zeta)d\zeta \\ &= g(s) + \int_s^z \frac{A(\zeta+1) + B(\zeta-1) - (A+B-2)R(\zeta)}{2(\zeta^2-1)}d\zeta \\ &= \frac{1}{2}[A\log(z-1) + B\log(z+1) + l] - \phi(z). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \tilde{g}(z) &= \tilde{g}(s) + \int_{\bar{s}}^z h(\zeta)d\zeta \\ &= \frac{1}{2}[A\log(z-1) + B\log(z+1) + \tilde{l}] - \tilde{\phi}(z) \end{aligned}$$

for $z \in \mathbb{C}\backslash L \cup \Sigma_2 \cup (-\infty,-1] \cup [1,\infty)$, thus proving (4.10a) and (4.10b). To demonstrate (4.11), we note that $\tilde{g}'(z) - g'(z) = 0$ for $z \in \Omega_\pm$. Hence, $g(z) = \tilde{g}(z)+$ constant. The fact that the value of the constant differs for z in Ω_+ and for z in Ω_- is because different cuts have been taken for the logarithmic function in $g(z)$

and $\tilde{g}(z)$. The final result (4.21) follows directly from (4.7), (4.10a), (4.10b) and (4.11). $\square$

By using (4.7), (4.10a) and (4.10b), we also have

Corollary 2. *The g-function and the $\tilde{g}$-function have the following jump properties:*

$$g_+(z) - g_-(z) = \begin{cases} 2\pi i, & z \in \Sigma_1, \\ 2\phi_-(z) = -2\phi_+(z), & z \in L, \end{cases} \tag{4.13}$$

$$\tilde{g}_+(z) - \tilde{g}_-(z) = \begin{cases} 2\pi i, & z \in \Sigma_2, \\ 2\tilde{\phi}_-(z) = -2\tilde{\phi}_+(z), & z \in L, \end{cases} \tag{4.14}$$

$$g_+(z) + g_-(z) = \begin{cases} A\log(z-1) + B\log(z+1) + l, & z \in L, \\ A\log(z-1) + B\log(z+1) + l - 2\tilde{\phi}(z), & z \in \Sigma_1, \\ A\log(z-1) + B\log(z+1) + l - 2\phi(z), & z \in \Sigma_2. \end{cases} \tag{4.15}$$

Proposition 5. *Let L_x denote the point where the curve L intersects the real-axis. The values of the function $\phi(z)$ on $L \cup \Sigma_2 \cup (-\infty, -1] \cup [1, \infty)$ are given by*

$$\phi(z) > 0, \qquad \arg\phi(z) = 0, \qquad z \in \Sigma_2; \tag{4.16}$$

$$\phi(s) = 0, \qquad \phi_\pm(\bar{s}) = \mp\pi i, \qquad \phi_\pm(L_x) = \mp\frac{\pi}{2}i; \tag{4.17}$$

$$\operatorname{Re}\phi_\pm(z) = 0, \qquad \arg\phi_\pm(z) = \pm\frac{3\pi}{2}, \qquad z \in L\backslash\{s, \bar{s}\}; \tag{4.18}$$

(4.19)

$$\operatorname{Re}\phi(z) < 0, \qquad \operatorname{Im}\phi(z) = -\frac{\pi}{2}, \qquad \pi < \arg\phi(z) < \frac{3\pi}{2}, \qquad z \in (-1, L_x);$$

$$\begin{aligned} &-\infty < \operatorname{Re}\phi(z) < \infty, && \operatorname{Im}\phi_+(z) = \frac{B-1}{2}\pi, \\ &0 < \arg\phi_+(z) < \pi, && z \in (-\infty, -1]; \end{aligned} \tag{4.20}$$

(4.21)

$$\operatorname{Re}\phi(z) < 0, \qquad \operatorname{Im}\phi(z) = \frac{\pi}{2}, \qquad -\frac{3\pi}{2} < \arg\phi(z) < -\pi, \qquad z \in (L_x, 1);$$

$$\begin{aligned} &-\infty < \operatorname{Re}\phi(z) < \infty, && \operatorname{Im}\phi_+(z) = \frac{1-A}{2}\pi, \\ &-\pi < \arg\phi_+(z) < 0, && z \in [1, \infty). \end{aligned} \tag{4.22}$$

Proof. By the way Σ_2 was chosen, it is obvious that $\phi(z)$ is positive on Σ_2 and (4.16) holds. Let $U_\delta = \{z \in \mathbb{C} : |z - s| < \delta\}$ be a circular neighborhood of s. Clearly, $\phi(z)$ can be expressed as

$$\begin{aligned} \phi(z) &= \frac{A+B-2}{2}\int_s^z \frac{(\zeta - s)^{1/2}(\zeta - \bar{s})^{1/2}}{\zeta^2 - 1} d\zeta \\ &= \frac{A+B-2}{2}(z-s)^{3/2}\sum_{k=0}^{\infty} c_k (z-s)^k, \qquad c_0 = \frac{2(2i\operatorname{Im} s)^{\frac{1}{2}}}{3(s^2-1)} \neq 0, \end{aligned} \tag{4.23}$$

where $(z-s)^{3/2}$ is analytic in $\mathbb{C}$ cut along the curve $L \cup \Sigma_1$. Since

$$\phi_\pm(z) = \pm\pi i \int_s^z d\mu(\zeta), \qquad z \in L,$$

$\phi_\pm(z)$ is purely imaginary. In view of (4.23), $\phi_\pm(z)$ also have, respectively, the arguments $\pm\frac{3\pi}{2}$ on L. The results in (4.17) and (4.18) now follow immediately.

To prove (4.19) and (4.20), we first note that for any $x_1 \in (-1, L_x)$, we have

$$\phi_+(L_x) - \phi_+(x_1) = \frac{A+B-2}{2} \int_{x_1}^{L_x} \frac{R(x)}{x^2-1} dx.$$

The denominator under the integral sign is negative. Since

$$\arg R(x) = \frac{1}{2}[\arg(x-s) + \arg(x-\bar{s})] = \pi,$$

the numerator $R(x)$ is also negative. Hence, the right-hand side of the above equation is real and positive. In view of (4.18), Re $\phi(x_1)$ is negative and Im $\phi(x_1) = -\frac{\pi}{2}$. Furthermore, for any $x_1, x_2 \in (-\infty, -1]$, we have

$$\phi_+(x_2) - \phi_-(x_-) = \frac{A+B-2}{2} \int_{x_1}^{x_2} \frac{R(x)}{x^2-1} dx. \tag{4.24}$$

By a similar argument as above, the right-hand side is real and negative; hence, Im $\phi_+(x) =$ constant. To find the constant, we let C_ε denote the half circle

$$C_\varepsilon : z + 1 = \varepsilon e^{i\theta}, \qquad 0 \le \theta \le \pi.$$

From (4.24), it follows that

$$\phi_+(-1-\varepsilon) = \frac{A+B-2}{2} \int_{C_\varepsilon} \frac{R(\zeta)}{\zeta^2-1} d\zeta + \phi_+(-1+\varepsilon).$$

Thus, for any $x \in (-\infty, -1)$, we have

$$\begin{aligned} \text{Im } \phi_+(x) &= \text{Im } \phi_+(-1-\varepsilon) \\ &= \text{Im } \left[\lim_{\varepsilon\to 0} \frac{A+B-2}{2} \int_{C_\varepsilon} \frac{R(\zeta)}{\zeta^2-1} d\zeta\right] + \lim_{\varepsilon\to 0} \text{Im } \phi_+(-1+\varepsilon) \\ &= -\frac{\pi}{2} + \text{Im } \left[\frac{A+B-2}{2}\pi i \frac{R(-1)}{-2}\right] = \frac{B-1}{2}\pi; \end{aligned}$$

cf. the equation preceding (3.10). The proofs of (4.21) and (4.22) are similar, and hence will not be given. □

A geometrical interpretation of the above result is stated in the following corollary.

Corollary 3. *Let D_1 be the shaded region depicted in Figure 9, and let D_2 denote the region $\mathbb{C}^+\backslash D_1$; cf. Figure 6. The function ϕ maps D_1 and D_2 onto the regions*

$$\left\{z : 0 < \text{Im } z < \frac{B-1}{2}\pi,\ \text{Re } z \ge 0\right\} \cup \left\{z : -\frac{\pi}{2} < \text{ Im } z < \frac{B-1}{2}\pi, \text{Re } z < 0\right\}, \tag{4.25}$$

and

$$\left\{z : \frac{1-A}{2} < \text{Im } z < 0, \text{Re } z > 0\right\} \cup \left\{z : \frac{1-A}{2} < \text{Im } z < \frac{\pi}{2}, \text{Re } z < 0\right\}, \tag{4.26}$$

respectively. Therefore,

$$\begin{aligned} 0 < \arg\phi(z) < \frac{3}{2}\pi, \qquad & z \in D_1, \\ -\frac{3}{2}\pi < \arg\phi(z) < 0, \qquad & z \in D_2. \end{aligned}$$

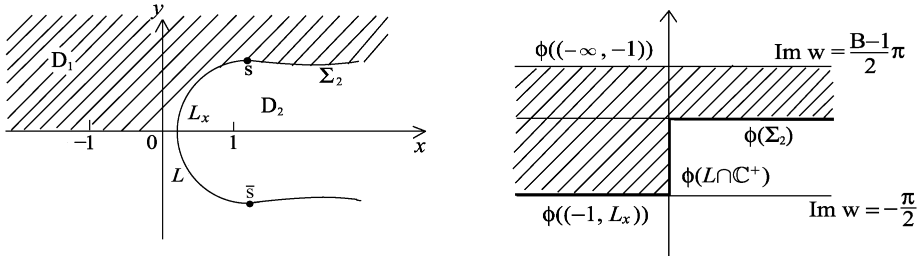

FIGURE 9. The image of D_1 under $\phi(z)$.

In view of the conjugate symmetry of $\tilde{\phi}(z)$ and $\phi(z)$, we also have properties for $\tilde{\phi}(z)$ which correspond to those stated in Proposition 5 and Corollary 3. Moreover, a combination of (4.7), (4.8), (4.20) and (4.22) gives the following result.

Corollary 4. *The function $\phi(z)$ and $\tilde{\phi}(z)$ satisfy the following relations on $\mathbb{R}$:*

$$\tilde{\phi}(z)-\phi(z)=\pi i, \qquad z\in(-1,L_x); \qquad \tilde{\phi}(z)-\phi(z)=-\pi i, \qquad z\in(L_x,1);$$
$$\phi_+(z)-\tilde{\phi}_-(z)=(B-1)\pi i,\ z\in(-\infty,-1);$$
$$\tilde{\phi}_-(z)-\phi_+(z)=(A-1)\pi i,\ z\in(1,\infty).$$

5. FIRST RIEMANN-HILBERT PROBLEM

With the above preparatory material, we are now ready to construct a 2-dimensional Riemann-Hilbert problem whose matrix solution has the Jacobi polynomial $P_n^{(\alpha_n,\beta_n)}(z)$ in one of its entries. Let the integer n be large but not fixed. For convenience, we shall consider only a special case of (1.2); namely, $\alpha_n=-nA$ and $\beta_n=-nB$, with $B>A>1$ so that $\alpha_n<-n$, $\beta_n<-n$ and $2n+\alpha_n+\beta_n<-1$. The general case can be treated through a limiting process. Let $\pi_k(z)=z^k+\cdots$ be the monic polynomials of $\boldsymbol{P}_k(z), k=0,1,\cdots,n$; they are orthogonal with respect to the weight function $w(z)=w(z;\alpha_n,\beta_n)=(z-1)^{-nA}(z+1)^{-nB}$ on the curve $\Gamma=L\cup\Sigma_1\cup\Sigma_2$; see Figure 5. Our first Riemann-Hilbert problem (RHP) is to find a 2×2 matrix-valued function $Y(z):\mathbb{C}\backslash\Gamma\to\mathbb{C}^{2\times 2}$ satisfying

(Y_a) $Y(z)$ is analytic in $\mathbb{C}\backslash\Gamma$,

(Y_b) for $z\in\Gamma$,

$$Y_+(z)=Y_-(z)\begin{pmatrix}1 & \omega(z)\\ 0 & 1\end{pmatrix}, \tag{5.1}$$

(Y_c) as $z\to\infty$,

$$Y(z)=\left(I+O\left(\frac{1}{z}\right)\right)\begin{pmatrix}z^n & 0\\ 0 & z^{-n}\end{pmatrix}, \qquad z\in\mathbb{C}\backslash\Gamma.$$

Theorem 1. *The unique solution to the above RHP for Y is given by*

$$Y(z)=\begin{pmatrix}\pi_n(z), & C\left[\pi_n\omega\right](z)\\ d_{n-1}\pi_{n-1}(z), & d_{n-1}C\left[\pi_{n-1}\omega\right](z)\end{pmatrix}, \qquad z\in\mathbb{C}\backslash\Gamma, \tag{5.2}$$

where

$$d_{n-1}=-\left[\frac{1}{2\pi i}\int_\Gamma \pi_{n-1}^2(\zeta)\omega(\zeta;\alpha_n,\beta_n)d\zeta\right]^{-1}$$

is a nonzero constant, and

$$C[f](z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta, \qquad z \in \mathbb{C}\backslash\Gamma,$$

is the Cauchy transform of f.

Proof. The proof is rather routine, and we give only a brief version of it. For details, see [9] or [10].

$Y(z)$ obviously satisfies the condition in (Y_a). The jump condition (5.1) is verified by using the Plemelj formula. Let $y_{ij}(z), i, j = 1, 2$, denote the entries in $Y(z)$. Clearly, $y_{11}(z)$ and $y_{21}(z)$ satisfy the condition in (Y_c). To show that the other two entries also satisfy this condition, we expend the kernel $\frac{1}{\zeta - z}$ in the Cauchy transform into a geometric series with remainder, and apply the orthogonality property established in Proposition 1. Thus,

$$\begin{aligned} y_{12}(z) &= \frac{1}{2\pi i}\int_\Gamma \frac{\pi_n(\zeta)\omega(\zeta;\alpha_n,\beta_n)}{\zeta - z}d\zeta \\ &= \frac{1}{2\pi i}\frac{1}{z^n}\int_\Gamma \pi_n(\zeta)\omega(\zeta;\alpha_n,\beta_n)\frac{\zeta^n}{\zeta - z}d\zeta \\ &= O\left(\frac{1}{z^{n+1}}\right), \qquad \text{as } z \to \infty \text{ in } \mathbb{C}\backslash\Gamma, \end{aligned}$$

and

$$\begin{aligned} y_{22}(z) &= \frac{d_{n-1}}{2\pi i}\int_\Gamma \frac{\pi_{n-1}(\zeta)\omega(\zeta;\alpha_n,\beta_n)}{\zeta - z}d\zeta \\ &= -\frac{d_{n-1}}{2\pi i}\frac{1}{z^n}\int_\Gamma \pi_{n-1}(\zeta)\zeta^{n-1}\omega(\zeta;\alpha_n,\beta_n)d\zeta + O\left(\frac{1}{z^{n+1}}\right) \\ &= \frac{1}{z^n} + O\left(\frac{1}{z^{n+1}}\right), \qquad \text{as } z \to \infty \text{ in } \mathbb{C}\backslash\Gamma. \end{aligned}$$

To prove that $Y(z)$ given in (5.2) is the unique solution, we consider the scalar function $\det Y(z)$. Clearly, it is analytic in $\mathbb{C}\backslash\Gamma$. Since the determinant of the jump matrix in (5.1) is equal to 1, we have $(\det Y)_+(z) = (\det Y)_-(z)$ for $z \in \Gamma$. Hence, $\det\, Y(z)$ is an entire function. Condition (Y_c) implies that $\det\, Y(z)$ is bounded and $\det\, Y(z) \to 1$ as $z \to \infty$. By Liouville theorem, $\det\, Y(z) \equiv 1$; i.e., the matrix $Y(z)$ is invertible. Suppose that the RHP for Y has another solution, say $\widetilde{Y}(z)$. Then, the matrix $G(z) \equiv Y^{-1}(z)\widetilde{Y}(z)$ satisfies the following conditions:

(G_a) $G(z)$ is analytic in $\mathbb{C}\backslash\Gamma$,

(G_b) $G_+(z) = G_-(z)$ for $z \in \Gamma$,

(G_c) $G(z) = I + O\left(\frac{1}{z}\right)$ as $z \to \infty$ in $\mathbb{C}\backslash\Gamma$.

These three conditions together imply that all entries of $G(z)$ are entire functions, and $G(z) \to I$ as $z \to \infty$. Thus, $G(z) \equiv I$ or, equivalently, $\widetilde{Y}(z) \equiv Y(z)$. □

6. Normalization and contour deformation

One of the first steps in the Riemann-Hilbert method is to transform condition (Y_c) into a standard form. To do this, we introduce the matrix

(6.1) $$U(z) = e^{-\frac{1}{2}nl\sigma_3}Y(z)e^{-n(g(z)-\frac{1}{2}l)\sigma_3},$$

where σ_3 is the Pauli matrix $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $g(z)$ is the g-function defined in (4.3) and l is the constant given in (4.9a). Since $g(z) \sim \log z$ as $z \to \infty$, it is readily verified that $U(z)$ is the unique solution to the following RHP:

(U_a) $U(z)$ is analytic in $\mathbb{C}\backslash\Gamma$,

(U_b) for $z \in \Gamma$,

$$U_+(z) = U_-(z)\begin{pmatrix} e^{-n(g_+(z)-g_-(z))} & w(z)e^{n(g_+(z)+g_-(z)-l)} \\ 0 & e^{n(g_+(z)-g_-(z))} \end{pmatrix}, \tag{6.2}$$

(U_c) as $z \to \infty$,

$$U(z) = I + O\left(\frac{1}{z}\right) \qquad \text{for } z \in \mathbb{C}\backslash\Gamma.$$

On account of (4.11) and (4.13)–(4.15), the jump matrix in (6.2) has different representations on L, Σ_1 and Σ_2. Indeed, we have

$$U_+(z) = U_-(z)\begin{pmatrix} e^{2n\phi_+(z)} & 1 \\ 0 & e^{2n\phi_-(z)} \end{pmatrix}, \qquad z \in L, \tag{6.3}$$

$$U_+(z) = U_-(z)\begin{pmatrix} 1 & e^{-2n\tilde{\phi}(z)} \\ 0 & 1 \end{pmatrix}, \qquad z \in \Sigma_1, \tag{6.4}$$

and

$$U_+(z) = U_-(z)\begin{pmatrix} 1 & e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix}, \qquad z \in \Sigma_2, \tag{6.5}$$

where ϕ and $\tilde{\phi}$ are the auxiliary functions defined in (4.4) and (4.6).

Since $\phi_\pm(z)$ are purely imaginary on L (see (4.18)), the diagonal entries of the jump matrix in (6.3) are rapidly oscillatory as $n \to \infty$. Furthermore, since $\phi(z)$ is positive on Σ_2 and $\tilde{\phi}(z)$ is positive in Σ_1 (cf. (4.16)), the jump matrices in (6.4) and (6.5) are exponentially decaying to the identity matrix as $n \to \infty$. To the jump matrix in (6.3), we do the factorization

$$\begin{pmatrix} e^{2n\phi_+(z)} & 1 \\ 0 & e^{2n\phi_-(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{2n\phi_-(z)} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ e^{2n\phi_+(z)} & 1 \end{pmatrix}, \tag{6.6}$$

where use has been made of (4.13). The first and third matrices on the right-hand side have the analytic continuation $\begin{pmatrix} 1 & 0 \\ e^{2n\phi(z)} & 1 \end{pmatrix}$ on two sides of L.

To find the behavior of $U(z)$ as $n \to \infty$, we recall the properties of ϕ in Proposition 5 and Corollary 3 in Section 4. As z moves from the $+$ side of Σ_2 to the $+$ side of $L, \arg\phi(z)$ increases from 0 to $3\pi/2$. Hence, near the left side of L, we have $\pi/2 < \arg\phi(z) < 3\pi/2$, i.e., Re $\phi(z) < 0$; see (4.26). Similarly, as z moves from the $-$ side of Σ_2 to the $-$ side of $L, \arg\phi(z)$ decreases from 0 to $-3\pi/2$. So near the right side of L, we have $-3\pi/2 < \arg\phi(z) < -\pi/2$, which again means Re $\phi(z) < 0$; see (4.27) and Figure 9. Therefore, we can choose two curves Σ_3 and Σ_4 joining s and $\bar{s}$, contained in Ω_+ and Ω_-, respectively, such that

$$\text{Re }\phi(z) < 0, \qquad z \in \Sigma_3 \cup \Sigma_4\backslash\{s, \bar{s}\}; \tag{6.7}$$

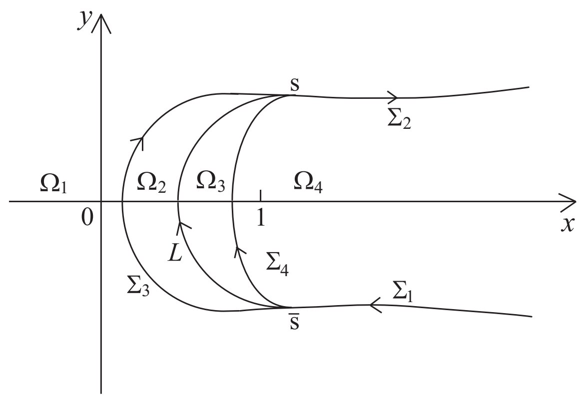

FIGURE 10. Curves Σ_1 and Σ_2 and regions $\Omega_i, i = 1, \cdots, 4$.

see Figure 10. By conjugate symmetry, we also have

$$\mathrm{Re}\, \tilde{\phi}(z) < 0, \qquad z \in \Sigma_3 \cup \Sigma_4 \backslash \{s, \bar{s}\}.$$

Let $\Sigma = \Gamma \cup \Sigma_3 \cup \Sigma_4 = L \cup \Sigma_1 \cup \cdots \cup \Sigma_4$. This curve divides the complex plane $\mathbb{C}$ into four disjoint regions $\Omega_i, i = 1, \cdots, 4$; see Figure 10.

Furthermore, we have

$$\mathrm{Re}\, \phi(z) < 0, \qquad \mathrm{Re}\, \tilde{\phi}(z) < 0, \qquad z \in \Omega_2 \cup \Omega_3. \tag{6.8}$$

Now we define the second transformation $U \to T$ by

$$T(z) = \begin{cases} U(z), & z \in \Omega_1 \cup \Omega_4, \\ U(z) \begin{pmatrix} 1 & 0 \\ -e^{2n\phi(z)} & 1 \end{pmatrix}, & z \in \Omega_2, \\ U(z) \begin{pmatrix} 1 & 0 \\ e^{2n\phi(z)} & 1 \end{pmatrix}, & z \in \Omega_3. \end{cases} \tag{6.9}$$

From the RHP for $U(z)$ and the factorization in (6.6), one can show by a straightforward calculation that $T(z)$ is the unique solution of the following RHP:

(T_a) $T(z)$ is analytic in $\mathbb{C} \backslash \Sigma$,

(T_b) for $z \in \Sigma$, we have

$$T_+(z) = T_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in L \backslash \{s, \bar{s}\}, \tag{6.10}$$

$$T_+(z) = T_-(z) \begin{pmatrix} 1 & 0 \\ e^{2n\phi(z)} & 1 \end{pmatrix}, \qquad z \in \Sigma_3 \cup \Sigma_4, \tag{6.11}$$

$$T_+(z) = T_-(z) \begin{pmatrix} 1 & e^{-2n\tilde{\phi}(z)} \\ 0 & 1 \end{pmatrix}, \qquad z \in \Sigma_1, \tag{6.12}$$

$$T_+(z) = T_-(z) \begin{pmatrix} 1 & e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix}, \qquad z \in \Sigma_2, \tag{6.13}$$

(T_c) for $z \notin \Sigma \cup (-\infty, -1] \cup [1, \infty)$,

$$T(z) = I + O\left(\frac{1}{z}\right) \qquad \text{as } z \to \infty.$$

The advantage of $T(z)$ over $U(z)$ is that for z in $\Sigma_1 \cup \cdots \cup \Sigma_4$ but $z \notin \{s, \bar{s}\}$, the jump matrices all decay exponentially to the identity matrix. This suggests that the leading-order term in the asymptotic expansion of $T(z)$ may be the solution $N(z)$ to the RHP:

(N_a) $N(z)$ is analytic in $\mathbb{C} \backslash L$,

(N_b) for $z \in L \backslash \{s, \bar{s}\}$,

$$N_+(z) = N_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{6.14}$$

(N_c) as $z \to \infty$,

$$N(z) = I + O\left(\frac{1}{z}\right).$$

This is a standard problem, and can be solved explicitly; see [3, p. 1520].

Its solution is given by

$$N(z) = \frac{1}{2} \begin{pmatrix} a(z) + a(z)^{-1} & i(a(z)^{-1} - a(z)) \\ i(a(z) - a(z)^{-1}) & a(z) + a(z)^{-1} \end{pmatrix}, \tag{6.15}$$

where

$$a(z) = \frac{(z-s)^{1/4}}{(z-\bar{s})^{1/4}} \tag{6.16}$$

with a branch cut along the curve L and $a(z) \to 1$ as $z \to \infty$. The matrix $N(z)$ can also be expressed in terms of the function $R(z) = \sqrt{(z-s)(z-\bar{s})}$ mentioned in Proposition 2. Indeed, since

$$R'(z) = \frac{1}{2}\left[a(z) + a(z)^{-1}\right]^2 - 1 = \frac{1}{2}\left[a(z) - a(z)^{-1}\right]^2 + 1,$$

we have

$$N(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + R'(z)} & -\sqrt{1 - R'(z)} \\ \sqrt{1 - R'(z)} & \sqrt{1 + R'(z)} \end{pmatrix}. \tag{6.17}$$

Note that

$$\det N(z) = \frac{1}{4}\left[a(z) + a(z)^{-1}\right]^2 - \frac{1}{4}\left[a(z) - a(z)^{-1}\right]^2 = 1.$$

Hence it follows

$$N(z)^{-1} = \frac{1}{2}\begin{pmatrix} a(z) + a(z)^{-1} & i(a(z) - a(z)^{-1}) \\ i(a(z)^{-1} - a(z)) & a(z) + a(z)^{-1} \end{pmatrix}. \tag{6.18}$$

It is also worth-while to point out that $N(z)$ and $N(z)^{-1}$ have the following factorizations:

$$\begin{aligned} N(z) &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} a(z)^{-\sigma_3} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} a(z)^{\sigma_3} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}, \end{aligned} \tag{6.19}$$

$$\begin{aligned} N(z)^{-1} &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} a(z)^{\sigma_3} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} a(z)^{-\sigma_3} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}, \end{aligned} \tag{6.20}$$

where σ_3 is the Pauli matrix used in (6.1).

7. Construction of parametrix

In §6, we have indicated that a reasonable asymptotic approximation to $T(z)$ is given by the matrix $N(z)$. From this, one can work backwards to get (heuristically) the asymptotic formula

$$Y(z) \sim e^{\frac{1}{2}nl\sigma_3} N(z) e^{n(g(z) - \frac{1}{2}l)\sigma_3},$$

which by (6.19) can be written as

$$\begin{aligned} Y(z) \sim & e^{\frac{1}{2}nl\sigma_3} \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} a(z)^{-\sigma_3} \\ & \times \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} e^{-n\phi(z)\sigma_3} e^{n\left(g(z) + \phi(z) - \frac{1}{2}l\right)\sigma_3} \end{aligned}$$

or, equivalently,

$$\begin{aligned} Y(z) \sim & \frac{1}{2} e^{\frac{1}{2}nl\sigma_3} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} [a(z)]^{-\sigma_3} \\ & \times \begin{pmatrix} e^{-n\phi(z)} & ie^{n\phi(z)} \\ -e^{-n\phi(z)} & ie^{n\phi(z)} \end{pmatrix} e^{n\left(g(z) + \phi(z) - \frac{1}{2}l\right)\sigma_3}. \end{aligned} \tag{7.1}$$

To find a parametrix for the RHP of Y, we first look for a matrix which is asymptotic to

$$\begin{pmatrix} e^{-n\phi(z)} & ie^{n\phi(z)} \\ -e^{-n\phi(z)} & ie^{n\phi(z)} \end{pmatrix}. \tag{7.2}$$

The entries in this matrix suggest that we should try a matrix whose elements are the Airy function $\mathrm{Ai}(\xi)$ and its derivative $\mathrm{Ai}'(\xi)$, where $\frac{2}{3}\xi^{3/2} = n\phi(z)$. Indeed, from (4.26) and (4.27), we have

$$-\frac{3}{2}\pi < \arg\phi(z) < \frac{3}{2}\pi.$$

Since $\arg\phi(z) = 0$ for $z \in \Sigma_2$ and $\arg\phi_\pm(z) = \mp\frac{3}{2}\pi$ for $z \in L\backslash\{s, \bar{s}\}$, the function

$$\xi = f(z) = \left[\frac{3}{2}n\phi(z)\right]^{\frac{2}{3}} \tag{7.3}$$

is well-defined and analytic in $\mathbb{C}^+$ with a branch cut along the curve L. In (7.3), we take the branch so that $f(z) > 0$ for $z \in \Sigma_2$. Furthermore, Corollary 3 gives

$$0 < \arg f(z) < \pi \qquad\qquad \text{for } z \in D_1.$$

As long as $z \neq s$, we have $\xi = f(z) \longrightarrow \infty$ as $n \to \infty$; cf. (7.3). From the well-known asymptotic behavior of the Airy function [12, p. 392], we have

$$\mathrm{Ai}(\xi) \sim \frac{1}{2\sqrt{\pi}}(f(z))^{-\frac{1}{4}}e^{-n\phi(z)},$$
$$\mathrm{Ai}'(\xi) \sim -\frac{1}{2\sqrt{\pi}}(f(z))^{\frac{1}{4}}e^{-n\phi(z)}.$$

Let $\omega = e^{2\pi i/3}$. Then, we also have

$$\mathrm{Ai}(\omega^2\xi) = \mathrm{Ai}(\omega^{-1}\xi) \sim \frac{e^{i\pi/6}}{2\sqrt{\pi}}(f(z))^{-\frac{1}{4}}e^{n\phi(z)},$$
$$\mathrm{Ai}'(\omega^2\xi) = \mathrm{Ai}'(\omega^{-1}\xi) \sim -\frac{e^{-i\pi/6}}{2\sqrt{\pi}}(f(z))^{\frac{1}{4}}e^{n\phi(z)}.$$

The last four asymptotic formulas show that the matrix in (7.2) is the leading-order term in the asymptotic expansion of the matrix

$$2\sqrt{\pi}[(f(z)]^{\frac{1}{4}\sigma_3}\begin{pmatrix} \mathrm{Ai}(\xi) & -\omega^2\mathrm{Ai}(\omega^2\xi) \\ \mathrm{Ai}'(\xi) & -\omega\mathrm{Ai}'(\omega^2\xi) \end{pmatrix} \tag{7.4}$$

for $z \in D_1$. Similarly, Corollary 3 implies

$$-\pi < \arg\phi(z) < 0 \qquad\qquad \text{for } z \in D_2.$$

So, for $z \neq \bar{s}$, the matrix in (7.2) is the leading-order term in the asymptotic expansion of the matrix

$$2\sqrt{\pi}[(f(z)]^{\frac{1}{4}\sigma_3}\begin{pmatrix} \mathrm{Ai}(\xi) & \omega\mathrm{Ai}(\omega\xi) \\ \mathrm{Ai}'(\xi) & \omega^2\mathrm{Ai}'(\omega^2\xi) \end{pmatrix} \tag{7.5}$$

for $z \in D_2$, where ξ is again given by (7.3).

In the lower half-plane $\mathbb{C}^-$, there is a result corresponding to (7.1); it involves $\tilde{g}, \tilde{\phi}$ and $\tilde{l}$. Indeed, we have

$$\begin{aligned} Y(z) \sim &\frac{1}{2}e^{\frac{1}{2}n\tilde{l}\sigma_3}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}[a(z)]^{\sigma_3} \\ &\times \begin{pmatrix} e^{-n\tilde{\phi}(z)} & -ie^{n\tilde{\phi}(z)} \\ -e^{-n\tilde{\phi}(z)} & -ie^{n\tilde{\phi}(z)} \end{pmatrix} e^{n\left(\tilde{g}(z)+\tilde{\phi}(z)-\frac{1}{2}\tilde{l}\right)\sigma_3} \end{aligned} \tag{7.6}$$

for $z \in \mathbb{C}^-$. Let

$$\tilde{\xi} = \tilde{f}(z) = \left[\frac{3}{2} n \tilde{\phi}(z)\right]^{\frac{2}{3}}, \qquad z \in \mathbb{C}^-, \tag{7.7}$$

where we again take the branch cut along the curve L and choose $\tilde{f}(z) > 0$ for $z \in \Sigma_1$. As before, it can be shown that the matrix in (7.6) is the leading term in the asymptotic expansion of the matrices

$$2\sqrt{\pi}[(\tilde{f}(z)]^{\frac{1}{4}\sigma_3} \begin{pmatrix} \mathrm{Ai}(\tilde{\xi}) & -\omega \mathrm{Ai}(\omega\tilde{\xi}) \\ \mathrm{Ai}'(\tilde{\xi}) & -\omega^2 \mathrm{Ai}'(\omega\xi) \end{pmatrix} \tag{7.8}$$

for $z \in D_3$, and

$$2\sqrt{\pi}[(\tilde{f}(z)]^{\frac{1}{4}\sigma_3} \begin{pmatrix} \mathrm{Ai}(\tilde{\xi}) & \omega^2 \mathrm{Ai}(\omega^2\tilde{\xi}) \\ \mathrm{Ai}'(\tilde{\xi}) & \omega \mathrm{Ai}'(\omega^2\tilde{\xi}) \end{pmatrix} \tag{7.9}$$

for $z \in D_4$, where D_3 and D_4 are complex conjugate regions of D_1 and D_2, respectively. Define the matrix

$$P(z) = \begin{cases} \text{the matrix in (7.4)}, & z \in D_1, \\ \text{the matrix in (7.5)}, & z \in D_2, \\ \text{the matrix in (7.8)}, & z \in D_3, \\ \text{the matrix in (7.9)}, & z \in D_4. \end{cases} \tag{7.10}$$

It is easily verified that $P(z)$ is conjugate symmetric in $\mathbb{C}\backslash\mathbb{R}$. Using the well-known formulas

$$\mathrm{Ai}(\xi) + \omega \mathrm{Ai}(\omega\xi) + \omega^2 \mathrm{Ai}(\omega^2\xi) = 0$$

and

$$\mathrm{Ai}'(\xi) + \omega^2 \mathrm{Ai}'(\omega\xi) + \omega \mathrm{Ai}'(\omega^2\xi) = 0,$$

it can also be shown that $P(z)$ satisfies the jump condition

$$P_+(z) = P_-(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad z \in \Gamma. \tag{7.11}$$

The heuristic argument given thus far suggests that the matrix $Y_*(z)$ defined by

$$\begin{aligned} Y_*(z) = &\sqrt{\pi} e^{\frac{1}{2}nl\sigma_3} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \left[\frac{f(z)^{\frac{1}{4}}}{a(z)}\right]^{\sigma_3} \\ &\cdot P(z) e^{n(g(z)+\phi(z)-\frac{1}{2}l)\sigma_3}, \qquad z \in \mathbb{C}^+\backslash\Gamma, \end{aligned} \tag{7.12}$$

and

$$\begin{aligned} Y_*(z) = &\sqrt{\pi} e^{\frac{1}{2}n\tilde{l}\sigma_3} \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} \left[a(z)\tilde{f}(z)^{\frac{1}{4}}\right]^{\sigma_3} \\ &\cdot P(z) e^{n(\tilde{g}(z)+\tilde{\phi}(z)-\frac{1}{2}\tilde{l})\sigma_3}, \qquad z \in \mathbb{C}^-\backslash\Gamma, \end{aligned} \tag{7.13}$$

is indeed a reasonable parametrix for RHP of $Y(z)$.

To establish the above claim rigorously, we need to prove that (i) $Y_*(z)$ satisfies the jump condition (Y_b) in (5.1), and (ii) $Y_*(z)$ has the same behavior as $Y(z)$ given in (Y_c) as $z \to \infty$. Note that $Y_*(z)$ is not analytic on $\mathbb{R}$. Thus, we also have to show that the jump matrix

$$J(x) = Y_{*_-}(x)[Y_{*_+}(x)]^{-1}, \qquad x \in \mathbb{R}, \tag{7.14}$$

is asymptotically equal to the identity matrix I. To this end, we introduce the notation

$$\begin{aligned} E(z) :=& \sqrt{\pi} e^{\frac{1}{2} n l \sigma_3} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \left[\frac{f(z)^{\frac{1}{4}}}{a(z)} \right]^{\sigma_3}, \qquad & z \in \mathbb{C}^+ \backslash \Gamma, \\ Q(z) :=& P(z) e^{n(g(z)+\phi(z)-\frac{1}{2}l)\sigma_3}, & z \in \mathbb{C}^+ \backslash \Gamma, \end{aligned}$$

$$\begin{aligned} \widetilde{E}(z) :=& \sqrt{\pi} e^{\frac{1}{2} n \tilde{l} \sigma_3} \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} [a(z) \tilde{f}(z)^{\frac{1}{4}}]^{\sigma_3}, \qquad & z \in \mathbb{C}^+ \backslash \Gamma, \\ \tilde{Q}(z) :=& P(z) e^{n(\tilde{g}(z)+\tilde{\phi}(z)-\frac{1}{2}\tilde{l})\sigma_3}, & z \in \mathbb{C}^+ \backslash \Gamma, \end{aligned}$$

so that

$$Y_*(z) = \begin{cases} E(z)Q(z), & z \in \mathbb{C}^+ \backslash \Gamma, \\ \widetilde{E}(z)\widetilde{Q}(z), & z \in \mathbb{C}^- \backslash \Gamma. \end{cases} \tag{7.15}$$

From (4.10) and (2.2), one easily sees that $e^{n(g(z)+\phi(z)-\frac{1}{2}l)} = e^{n(\tilde{g}(z)+\tilde{\phi}(z)-\frac{1}{2}\tilde{l})} = w(z;\alpha_n,\beta_n)^{-\frac{1}{2}} = w(z)^{-1/2}$. Hence, from (7.11), it follows that

$$Q_+(z) = Q_-(z) \begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{C}^+ \cap \Gamma, \tag{7.16}$$

$$\widetilde{Q}_+(z) = \widetilde{Q}_-(z) \begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{C}^- \cap \Gamma. \tag{7.17}$$

On the other hand, we know from (4.23), (6.16) and (7.1) that $f(z)^{\frac{1}{4}}/a(z)$ is analytic on $\Gamma \cap \mathbb{C}^+$. By the same argument, $\tilde{f}(z)^{\frac{1}{4}} a(z)$ is analytic on $\Gamma \cap \mathbb{C}^-$. A combination of (7.15), (7.16) and (7.17) gives

$$Y_{*_+}(z) = Y_{*_-}(z) \begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix}, \qquad z \in \Gamma \backslash \{L_x\}. \tag{7.18}$$

To show that $Y_*(z)$ has the same behavior as $Y(z)$ as $z \to \infty$, we consider only the case $z \in D_1$. The other cases can be dealt with in a similar manner. From (7.12) and (7.10), we have

$$\begin{aligned} Y_*(z) =& \sqrt{\pi} e^{\frac{1}{2} n l \sigma_3} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \left[\frac{f(z)^{\frac{1}{4}}}{a(z)} \right]^{\sigma_3} \\ & \times \begin{pmatrix} \mathrm{Ai}\,(\xi) & -\omega^2\,\mathrm{Ai}(\omega^2 \xi) \\ \mathrm{Ai}'(\xi) & -\omega \mathrm{Ai}'(\omega^2 \xi) \end{pmatrix} e^{n(g(z)+\phi(z)-\frac{1}{2}l)\sigma_3}. \end{aligned}$$

Recall that the matrix in (7.2) is the leading term in the asymptotic expansion of the above matrix; see (7.4). Hence

$$\begin{aligned} Y_*(z) \sim & e^{\frac{1}{2}nl\sigma_3}\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}[a(z)]^{-\sigma_3} \\ & \times \begin{pmatrix} e^{-n\phi(z)} & ie^{n\phi(z)} \\ -e^{-n\phi(z)} & ie^{n\phi(z)} \end{pmatrix} e^{n(g(z)+\phi(z)-\frac{1}{2}l)\sigma_3}. \end{aligned}$$

Upon simplification, we further obtain from (6.19),

$$\begin{aligned} Y_*(z) \sim & e^{\frac{1}{2}nl\sigma_3}\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} a(z)^{-\sigma_3}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} e^{n(g(z)-\frac{1}{2}l)\sigma_3} \\ = & e^{\frac{1}{2}nl\sigma_3}N(z)e^{n(g(z)-\frac{1}{2}l)\sigma_3}. \end{aligned}$$

Since $g(z) \sim \log z$ as $z \to \infty$, the behavior of $N(z)$ implies

$$Y_*(z) = \left[I + O\left(\frac{1}{z}\right)\right] e^{ng(z)\sigma_3} = \left[I + O\left(\frac{1}{z}\right)\right]\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}; \tag{7.19}$$

see (N_c).

Finally, we come to the estimation of the jump matrix $J(x)$ in (7.14). We first note that from (7.10) and (7.4), we have

$$\det P(z) = \omega(\omega-1)\mathrm{Ai}(\xi)\mathrm{Ai}'(\omega^2\xi).$$

Since $\mathrm{Ai}(\xi)$ and $-\omega^2\mathrm{Ai}(\omega^2\xi)$ are two linearly independent solutions of the Airy equation, the determinant of the matrix in (7.4) is essentially the Wronskian of these two solutions. From a formula of the Wronskian, it follows that

$$\det P(z) = \omega(\omega-1)\mathrm{Ai}(0)\mathrm{Ai}'(0) = \frac{i}{2\pi}, \qquad z \in \mathbb{C}^+;$$

cf. [12, p. 142 and p. 392]. Therefore,

$$P^{-1}(z) = -2\pi i\begin{pmatrix} -\omega\mathrm{Ai}'(\omega^2\xi) & \omega^2\mathrm{Ai}(\omega^2\xi) \\ -\mathrm{Ai}'(\xi) & \mathrm{Ai}(\xi) \end{pmatrix}, \qquad z \in D_1, \tag{7.20}$$

and

$$P^{-1}(z) = -2\pi i\begin{pmatrix} \omega^2\mathrm{Ai}'(\omega^2\xi) & -\omega\mathrm{Ai}(\omega\xi) \\ -\mathrm{Ai}'(\xi) & \mathrm{Ai}(\xi) \end{pmatrix}, \qquad z \in D_2. \tag{7.21}$$

For $x \in (L_x, \infty)$, we have from (7.14), (7.13) and (7.12)

$$\begin{aligned} J(x) = & e^{\frac{1}{2}n\tilde{l}\sigma_3}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}[a(x)\tilde{f}_-(x)^{\frac{1}{4}}]^{\sigma_3}P_-(x)e^{n(\tilde{g}_-(x)+\tilde{\phi}_-(x)-\frac{1}{2}\tilde{l})\sigma_3} \\ & \cdot e^{-n(g_+(x)+\phi_+(x)-\frac{1}{2}l)\sigma_3}P_+(x)^{-1}\left[\frac{f_-(x)^{\frac{1}{4}}}{a(x)}\right]^{-\sigma_3}\frac{1}{2}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} e^{-\frac{1}{2}nl\sigma_3}. \end{aligned} \tag{7.22}$$

(Note: $g(z)$ actually has no cut on $\mathbb{R}^+$, and the cut of f is along the curve Γ.) In (7.21), we replace the Airy functions in the matrices $P_-(x)$ and $P_+(x)^{-1}$ by the leading terms in their asymptotic expansions; see the connection between (7.6) and (7.9). On account of (6.19) and (6.20), we obtain

$$J(x) \sim e^{\frac{1}{2}n\tilde{l}\sigma_3} N(x) e^{n(\tilde{g}_-(x)-\frac{1}{2}\tilde{l})\sigma_3} e^{-n(g_+(x)-\frac{1}{2}l)\sigma_3} N(x)^{-1} e^{-\frac{1}{2}nl\sigma_3}.$$

The right-hand side is exactly the identity matrix, thus proving that for $x \in (L_x, \infty)$, $J(x) \sim I$ as $n \to \infty$.

The case for $x \in (-\infty, L_x)$ can be handled in a similar manner.

8. Uniform asymptotic expansions

Since $Y(z)$ and $Y_*(z)$ have the same jump matrix on Γ, the matrix

$$S(z) = Y(z)Y_*(z)^{-1}, \qquad z \in \mathbb{C}\backslash\mathbb{R} \cup \Gamma, \tag{8.1}$$

satisfies the relation

$$S_+(z) = S_-(z), \qquad z \in \Gamma;$$

that is, $S(z)$ is analytic in $\mathbb{C}^+$. Furthermore, it is easily verified that $S(z)$ is a solution of the RHP:

(S_a) $S(z)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$,

(S_b) for $x \in \mathbb{R}$,

$$S_+(x) = S_-(x)J(x),$$

where $J(x)$ is the jump matrix defined in (7.14),

(S_c) as $z \to \infty$,

$$S(z) = I + O\left(\frac{1}{z}\right), \qquad z \notin \mathbb{R}.$$

We shall show that $S(z)$ has an asymptotic expansion of the form

$$S(z) \sim I + \sum_{k=1}^{\infty} \frac{S_k(z)}{n^k}, \qquad z \in \mathbb{C}^+, \tag{8.2}$$

as $n \to \infty$, where the coefficients $S_k(z)$ are analytic functions in $\mathbb{C}\backslash\mathbb{R}$. The desired expansion for the monic Jacobi polynomial $\pi_n(z)$ can then be obtained from (8.1), since it is in the (1,1) entry of the matrix $Y(z)$; see (5.2). To achieve this, we first need to derive an expansion for the jump matrix $J(x)$.

For $x \in (L_x, \infty)$, we have from the well-known results of the Airy function

$$\mathrm{Ai}(f_+(x)) \sim \frac{1}{2\sqrt{\pi}}(f_+(x))^{-\frac{1}{4}} e^{-n\phi_+(x)} \sum_{k=0}^{\infty}(-1)^k s_k (n\phi_+(x))^{-k}, \tag{8.3a}$$

$$\mathrm{Ai}'(f_+(x)) \sim -\frac{1}{2\sqrt{\pi}}(f_+(x))^{\frac{1}{4}} e^{-n\phi_+(x)} \sum_{k=0}^{\infty}(-1)^k t_k \left(n\phi_+(x)\right)^{-k}, \tag{8.3b}$$

$$\mathrm{Ai}(\omega f_+(x)) \sim \frac{e^{-i\pi/6}}{2\sqrt{\pi}}(f_+(x))^{-\frac{1}{4}} e^{n\phi_+(x)} \sum_{k=0}^{\infty} s_k \left(n\phi_+(x)\right)^{-k}, \tag{8.3c}$$

$$\mathrm{Ai}'(\omega f_+(x)) \sim -\frac{e^{i\pi/6}}{2\sqrt{\pi}}(f_+(x))^{\frac{1}{4}} e^{n\phi_+(x)} \sum_{k=0}^{\infty} t_k \left(n\phi_+(x)\right)^{-k}, \tag{8.3d}$$

where $\omega = e^{2\pi i/3}$, $f_+(x)$ and $\phi_+(x)$ are related by (7.3), and the coefficients s_k and t_k are explicitly given with $s_0 = t_0 = 1$. Corresponding results can be given for $\mathrm{Ai}(\tilde{f}_-(x))$, $\mathrm{Ai}'(\tilde{f}_-(x))$, $\mathrm{Ai}(\omega^2\tilde{f}_-(x))$ and $\mathrm{Ai}'(\omega^2\tilde{f}_-(x))$.

The asymptotic expansions of the matrices $P_-(x)$ and $P_+(x)^{-1}$ can then be obtained by replacing the Airy functions by their asymptotic expansions in the matrices in (7.10) and (7.20). Insert the resulting expansions in (7.21), and simplify the expression. This leads to the result

$$J(x) \sim I + \sum_{p=1}^{\infty} \frac{J_p(x)}{n^p}, \tag{8.4}$$

where

$$J_1(x) = O\left(\frac{1}{\tilde{\phi}_-(x)\phi_+(x)}\right) = O\left(\frac{1}{(\log x)^2}\right), \qquad x \to \infty, \tag{8.5}$$

and

$$J_p(x) = O\left(\frac{1}{(\log x)^p}\right), \qquad p = 2, 3, \cdots, \qquad x \to \infty. \tag{8.6}$$

For convenience, we put $J^*(x) := J(x) - I$ so that

$$J^*(x) \sim \sum_{k=1}^{\infty} \frac{J_k(x)}{n^k}. \tag{8.7}$$

Coupling (S_b) and (8.4), we have

$$S_+(x) = S_-(x)[I + J^*(x)] = S_-(x)\left[I + O\left(\frac{1}{n}\right)\right], \qquad x \in (L_x, +\infty). \tag{8.8}$$

In a similar manner, we can show that (8.8) also holds for $x \in (-\infty, L_x)$. Thus, we have established the equation

$$(S_+(x) - I) - (S_-(x) - I) = S_-(x)J^*(x), \qquad x \in \mathbb{R}, \tag{8.9}$$

where the coefficients $J_k(x)$ in the expansion (8.7) are $O(1/(\log|x|)^2)$ as $|x| \to \infty$.

By $(S_c), S_\pm(x) = I + O(1/|x|)$ as $|x| \to \infty$. From (8.9) and the Plemelj formula, it follows that

$$S(z) - I = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{S_-(x)J^*(x)}{x - z}dx, \qquad z \in \mathbb{C}\backslash\mathbb{R}. \tag{8.10}$$

Inserting (8.7) into (8.10) and carrying out integration termwise, we obtain (8.2), i.e.,

$$S(z) \sim I + \sum_{k=1}^{\infty} \frac{S_k(z)}{n^k} = \sum_{k=0}^{\infty} \frac{S_k(z)}{n^k}, \tag{8.11}$$

where we have put $S_0(z) := I$. The coefficients $S_k(z)$ can be determined recursively. Indeed, from (8.11), we have

$$S_-(x) \sim \sum_{k=1}^{\infty} (S_k)_-(x)\frac{1}{n^k}, \qquad x \in \mathbb{R}. \tag{8.12}$$

Substituting (8.7) and (8.12) in (8.10) gives

$$S(z) = I + \frac{1}{2\pi i}\sum_{k=1}^{\infty} \frac{1}{n^k}\int_{-\infty}^{\infty} \left[\sum_{j=1}^{k} (S_{k-j})_-(x)J_j(x)\right]\frac{dx}{x - z}. \tag{8.13}$$

A comparison of the expansions in (8.11) and (8.13) yields the recursion formula

$$S_k(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[\sum_{j=1}^{k}(S_{k-j})_-(x)J_j(x)\right]\frac{dx}{x-z}, \qquad k = 1, 2, \cdots. \tag{8.14}$$

In particular, we have $S_0(z) = I$,

$$S_1(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{J_1(x)}{x-z}dx,$$
$$S_2(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[(S_1)_-(x)J_1(x) + J_2(x)\right]\frac{dx}{x-z}.$$

Theorem 2. *Let the element in the ith row and jth column of the matrix $S_k(z)$ be denoted by $s_{ij}^{(k)}(z)$. Then, under the assumptions imposed on the parameters α_n and β_n at the beginning of Section* 5, *we have*

$$\begin{aligned}\pi_n(z) \sim & \sqrt{\pi}\exp\left\{-\frac{1}{2}[\alpha_n\log(z-1) + \beta_n\log(z+1) - nl]\right\}\\ & \cdot\left\{\frac{f(z)^{1/4}}{a(z)}\mathrm{Ai}(f(z))\Big[1 + \sum_{k=1}^{\infty}\frac{s_{11}^{(k)}(z) - ie^{-nl}s_{12}^{(k)}(z)}{n^k}\Big]\right.\\ & \left. - \frac{a(z)}{f(z)^{1/4}}\mathrm{Ai}'(f(z))\Big[1 + \sum_{k=1}^{\infty}\frac{s_{11}^{(k)}(z) + ie^{-nl}s_{12}^{(k)}(z)}{n^k}\Big]\right\}\end{aligned} \tag{8.15}$$

as $n \to \infty$. This expansion holds uniformly for z in the closed upper half-plane $\overline{\mathbb{C}^+}$; in particular, for $z \in \Gamma \cap \overline{\mathbb{C}^+}$.

With l and $f(z)$ replaced by $\tilde{l}$ and $\tilde{f}(z)$, expansion (8.15) *also holds uniformly in the lower half-plane $\overline{\mathbb{C}^-}$.*

Proof. We consider only the case when z lies in the upper half-plane. The proof in the other case is very similar.

Let $y_{ij}(z)$ and $y_{*_{ij}}(z)$ denote, respectively, the elements in $Y(z)$ and $Y_*(z)$. Since $Y(z) = S(z)Y_*(z)$, it follows from (5.2) that

$$\pi_n(z) = y_{11}(z) = s_{11}(z)y_{*_{11}}(z) + s_{12}(z)y_{*_{21}}(z). \tag{8.16}$$

From (8.11), we also have

$$s_{11}(z) \sim 1 + \sum_{k=1}^{\infty}\frac{s_{11}^{(k)}(z)}{n^k},$$
$$s_{12}(z) \sim \sum_{k=1}^{\infty}\frac{s_{12}^{(k)}(z)}{n^k}.$$

To find $y_{*_{11}}(z)$ and $y_{*_{21}}(z)$, we return to the matrix $Y_*(z)$ given in (7.12). By the definition in (7.10), the matrix $P(z)$ in (7.12) is given by either the one in (7.4) or the one in (7.5) according as $z \in D_1$ or $z \in D_2$. In both cases, the entries in the first column of $P(z)$ are the same. If $p_{ij}(z)$ denotes the entry in the (i,j)-position of $P(z)$, then we have $p_{11}(z) = \mathrm{Ai}(f(z))$ and $p_{21}(z) = \mathrm{Ai}'(f(z))$. A straightforward calculation now gives

$$y_{11}(z) = \sqrt{\pi}\left[\frac{f(z)^{1/4}}{a(z)}\mathrm{Ai}(f(z)) - \frac{a(z)}{f(z)^{1/4}}\mathrm{Ai}'(f(z))\right]e^{n(g(z)+\phi(z))},$$

$$y_{21}(z) = -i\sqrt{\pi}\left[\frac{f(z)^{1/4}}{a(z)}\mathrm{Ai}(f(z)) + \frac{a(z)}{f(z)^{1/4}}\mathrm{Ai}'(f(z))\right]e^{n(g(z)+\phi(z)-l)}.$$

The desired result follows, upon inserting the last four equations into (8.16). □

9. Asymptotics for the zeros

In Proposition 2, we have proved that L is the support of μ, which is the weak-$*$ limit of the zero-counting measure μ_n of $P_n^{(\alpha_n,\beta_n)}(z)$. This is a rather abstract statement; it does not clearly say that all zeros of $P_n^{(\alpha_n,\beta_n)}(z)$ tend to L as $n \to \infty$. (See a remark following the proof of Proposition 3.) The latter more concrete statement is the content of our next theorem.

Theorem 3. *Assume that the conditions on the parameters α_n and β_n imposed at the beginning of Section 5 again hold.*

(i) *All zeros of $P_n^{(\alpha_n,\beta_n)}(z)$ tend to L as $n \to \infty$; that is, for any small neighborhood $U(L)$ of L, there exists an integer n_0 such that for $n > n_0$, all zeros of $P_n^{(\alpha_n,\beta_n)}(z)$ are in $U(L)$.*

(ii) *All zeros of $P_n^{(\alpha_n,\beta_n)}(z)$ are on the $-$ side of L; i.e., there exists an integer n_0 such that for all $n > n_0$, no zeros of $P_n^{(\alpha_n,\beta_n)}(z)$ are on the $+$ side of L away from the endpoints s and $\bar{s}$.*

Proof. (i) From Theorem 2, we have

$$\begin{aligned}\pi_n(z) =& \sqrt{\pi}\exp\left\{\frac{n}{2}\left[A\log(z-1) + B\log(z+1) + l\right]\right\}\\ &\times\left\{\frac{f(z)^{\frac14}}{a(z)}\mathrm{Ai}(f(z))\left[1+O\left(\frac{1}{n}\right)\right]\right.\\ &\left.\quad - \frac{a(z)}{f(z)^{1/4}}\mathrm{Ai}'(f(z))\left[1+O\left(\frac{1}{n}\right)\right]\right\}.\end{aligned} \tag{9.1}$$

On account of (4.10a), the quantity inside the exponential equation is equal to $g(z)+\phi(z)$. For $z \in D_1\backslash U(L)$, the asymptotic behavior of $\mathrm{Ai}(f(z))$ and $\mathrm{Ai}'(f(z))$ are given by the two asymptotic formulas following equation (7.3). Inserting these two formulas into (9.1), we obtain upon simplification

$$\pi_n(z) = e^{ng(z)}\frac{a(z)+a(z)^{-1}}{2}\left[1+O\left(\frac{1}{n}\right)\right].$$

For large n, the right-hand side of this equation clearly does not vanish, thus proving that $P_n^{(\alpha_n,\beta_n)}(z)$ has no zero in $D_1\backslash U(L)$. A similar argument shows that $P_n^{(\alpha_n,\beta_n)}(z)$ also has no zero in $D_i\backslash U(L), i = 2,3,4$. Therefore, all zeros of $P_n^{(\alpha_n,\beta_n)}(z)$ lie in $U(L)$.

(ii) For z near L and on the $+$ side of L, but away from the end-points s and $\bar{s}$, we have

$$\frac{2\pi}{3} < \arg f(z) = \arg\left(\frac{3}{2}n\phi(z)\right)^{\frac23} < \pi;$$

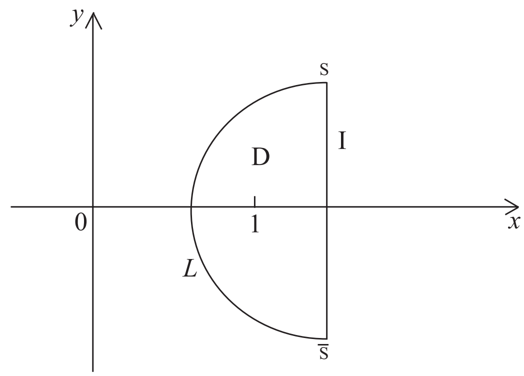

FIGURE 11. Region D.

cf. Propositions 5 and 6. For $f(z)$ in this sector, the well-known asymptotic results of the Airy function give

$$\mathrm{Ai}(f(z)) = \frac{1}{\sqrt{\pi}} f(z)^{-1/4} \left[\frac{e^{-n\phi(z)} + ie^{n\phi(z)}}{2}\right] \left[1 + O\left(\frac{1}{n}\right)\right],$$
$$\mathrm{Ai}'(f(z)) = \frac{1}{\sqrt{\pi}} f(z)^{1/4} \left[-\frac{e^{-n\phi(z)} + ie^{n\phi(z)}}{2}\right] \left[1 + O\left(\frac{1}{n}\right)\right];$$

see Olver [12, p. 118] or Wong [15, p. 94]. Substituting these into (9.1), we get

$$\begin{aligned}\pi_n(z) = e^{ng(z)} \Bigg\{ &\frac{a(z)^{-1} + a(z)}{2}\left[1 + O\left(\frac{1}{n}\right)\right] \\ &+ \frac{a(z) - a(z)^{-1}}{2i} e^{2n\phi(z)} \left[1 + O\left(\frac{1}{n}\right)\right] + O\left(\frac{1}{n}\right)\Bigg\}.\end{aligned} \tag{9.2}$$

Since z is away from s and $\bar{s}$, the function $a(z)$ is non-vanishing and bounded. Hence, equation (9.2) can be simplified to

$$\pi_n(z) = e^{ng(z)} \left[\frac{a(z)^{-1} + a(z)}{2} + \frac{a(z) - a(z)^{-1}}{2i} e^{2n\phi(z)} + O\left(\frac{1}{n}\right)\right].$$

In terms of the function $R(z)$, the last equation can be written as

$$\pi_n(z) = e^{ng(z)} \sqrt{\frac{1 + R'(z)}{2}} \left[1 - \sqrt{\frac{1 - R'(z)}{1 + R'(z)}} e^{2n\phi(z)} + O\left(\frac{1}{n}\right)\right]; \tag{9.3}$$

see (6.15) and (6.17).

We now prove that

$$\left|\frac{1 - R'(z)}{1 + R'(z)}\right| < 1 \qquad \text{for all } z \notin \overline{D}, \tag{9.4}$$

where D is the region bounded by L and the line segment I joining the end-points of L; see Figure 11. Note that $R'(z) \to 1$ as $z \to \infty$. Hence, (9.4) holds for all sufficiently large z. Since the quotient in (9.4) is analytic away from L, it is continuous in $\mathbb{C}\backslash L$. Thus, it suffices to prove that

$$\left|\frac{1 - R'(z)}{1 + R'(z)}\right| = 1 \qquad \text{if and only if } z \in I. \tag{9.5}$$

Simple calculation shows that equality in (9.5) holds if and only if Re $R'(z) = 0$, which is equivalent to $R'(z)^2 < 0$. Also, we have

$$R'(z)^2 = \frac{(z - \text{Re } s)^2}{(z - \text{Re } s)^2 + (\text{Im } s)^2}. \tag{9.6}$$

If $z \in I$, then it is obvious from (9.6) that $R'(z)^2 < 0$. On the other hand, if $R'(z)^2 < 0$, then one has from (9.6)

$$(z - \text{Re } s)^2 < 0 \qquad \text{and} \qquad (z - \text{Re } s)^2 + (\text{Im } z)^2 > 0,$$

which in turn implies

$$z - \text{Re } s = iy, \qquad -\text{Im } s < y < \text{Im } s,$$

i.e., $z \in I$. This establishes (9.5), and hence (9.6).

Now, if z lies on the $+$ side of L and is away from s and $\bar{s}$, then $z \notin \bar{D}$ and the inequality in (9.4) holds. Furthermore, since Re $\phi(z) < 0$ by (6.8), we in fact have

$$\left| \frac{1 - R'(z)}{1 + R'(z)} e^{2n\phi(z)} \right| < 1.$$

From (9.4), it follows that $\pi_n(z) \neq 0$, thus completing the proof. □

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