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ENTIRE MAJORANTS VIA EULER–MACLAURIN SUMMATION

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ABSTRACT. It is the aim of this article to give extremal majorants of type $2\pi\delta$ for the class of functions $f_n(x) = \operatorname{sgn}(x)x^n$, where $n \in \mathbb{N}$. As applications we obtain positive definite extensions to $\mathbb{R}$ of $\pm(it)^{-m}$ defined on $\mathbb{R}\backslash[-1,1]$, where $m \in \mathbb{N}$, optimal bounds in Hilbert-type inequalities for the class of functions $(it)^{-m}$, and majorants of type 2π for functions whose graphs are trapezoids.

1. INTRODUCTION AND NOTATION

An entire function $F(z)$ is said to be of type $2\pi\delta$ if

$$|F(z)| \le A_\varepsilon \exp\big(|z|(2\pi\delta + \varepsilon)\big) \tag{1}$$

for every $\varepsilon > 0$ and some constant $A_\varepsilon > 0$ depending on ε (in the notation of [2] this a function of order 1 and type $2\pi\delta$). The set of all functions of type $2\pi\delta$ that are real in $\mathbb{R}$ will be denoted by $E(2\pi\delta)$.

By the Paley-Wiener Theorem (cf. [2]), functions in $E(2\pi\delta) \cap L^2(\mathbb{R})$ have a Fourier transform with support in $[-\delta, \delta]$, where the Fourier transform of $f \in L^2(\mathbb{R})$ is given by

$$\mathcal{F}f(t) := \lim_{N\to\infty} \int_{-N}^{N} f(x)e(-tx)dx.$$

Here the notation $e(y) = \exp(2\pi i y)$ is used.

In the 1930's A. Beurling studied the entire function

$$B(z) := \frac{\sin^2 \pi z}{\pi^2}\Big(\sum_{n=0}^{\infty}(z-n)^{-2} - \sum_{n=-\infty}^{-1}(z-n)^{-2} + 2z^{-1}\Big). \tag{2}$$

He found that $B(z)$ satisfies the following extremal property: $B(z)$ is of type 2π, $B(x) \ge \operatorname{sgn}(x)$ for all $x \in \mathbb{R}$, $\int_{\mathbb{R}}(B - \operatorname{sgn}) = 1$, and any $F \in E(2\pi)$ with $F \ge \operatorname{sgn}$ on the real line and $F \ne B$ satisfies $\int_{\mathbb{R}}(F - \operatorname{sgn}) > 1$.

This motivates

Definition 1. Let $f : \mathbb{R} \to \mathbb{R}$. For $F \in E(2\pi\delta)$ consider the conditions

(i) $f(x) \le F(x)$ for all $x \in \mathbb{R}$,

(ii) $\displaystyle\int_{\mathbb{R}}(F - f) = \min_{\substack{G\in E(2\pi\delta)\\ G\ge f}} \int_{\mathbb{R}}(G - f)$.

A function $F \in E(2\pi\delta)$ satisfying (i) and (ii) is called an extremal majorant of type $2\pi\delta$ of f. Extremal minorants are defined with the obvious modifications.

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A. Selberg discovered $B(z)$ independently, and he used it to obtain a sharp form of the large sieve inequality ([11], chapter 20; see also [8]).

A general method to construct candidates for extremal majorants when $f \in L^2(\mathbb{R})$ is given by S. W. Graham and J. D. Vaaler in [3]. Their applications include a finite form of the Wiener-Ikehara Tauberian theorem (see also [5], chapter 5), a proof of the large sieve inequality, and inequalities for character sums.

Although Beurling never published his results, an account can be found in the survey [12] by Vaaler.

The function $B(z)$ can be used to give a short and elegant proof for a general form of Hilbert's inequality (cf. [11], chapter 20, and [12], Theorem 16; for the first proof cf. [9]). This result will be generalized in Corollary 2.

It is the purpose of this note to give extremal minorants and majorants for the class of functions

$$f_n(x) := \operatorname{sgn}(x)x^n,$$

where $n \in \mathbb{N}_0$. The way extremal minorants and majorants are obtained is similar to the method of [12], except that the Euler-Maclaurin summation formula is employed rather than the arithmetic-geometric mean inequality.

As usual, $\operatorname{sgn}(x)$ denotes the symmetric signum function, i.e. $\operatorname{sgn}(x) = -1$ for $x < 0$, $\operatorname{sgn}(x) = 1$ for $x > 0$, and $\operatorname{sgn}(0) = 0$. Also, $\operatorname{sgn}_+(x)$ denotes the right-continuous signum function, i.e. $\operatorname{sgn}_+(x) = \operatorname{sgn}(x)$ for $x \neq 0$ and $\operatorname{sgn}_+(0) = 1$. The expression $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

2. Main results

Given $\alpha \in \mathbb{R}$, let

$$F_\alpha(z) := \pi^{-2} \sin^2 \pi(z - \alpha) \text{ for } z \in \mathbb{C}.$$

The following definition provides us with the candidates for extremal minorants and majorants of $f(x) = \operatorname{sgn}(x)x^n$.

Definition 2. Define for $0 \leq \alpha \leq 1$, $z \in \mathbb{C}$, and $n \in \mathbb{N}_0$

$$H_n(z;\alpha) := F_\alpha(z)\Big(z^n \sum_{k=-\infty}^{\infty} \frac{\operatorname{sgn}_+(k)}{(x-k-\alpha)^2} + 2\sum_{k=1}^{n} B_{k-1}(\alpha) z^{n-k} + 2\frac{B_n(\alpha)}{z - \{\alpha\}} \Big),$$

where $\{\alpha\}$ denotes the fractional part of α, and $B_n(\alpha)$ is the n-th Bernoulli polynomial (cf. Section 4). For $n = 0$ the second sum is assigned the value zero.

The equality $B(z) = H_0(z;0)$ holds, where $B(z)$ is Beurling's function defined in (2).

Note that $H_n(z;\alpha)$ is real entire, because the zeros of F_α cancel the poles of the first and the last terms in the parenthesis, and the second term is a polynomial.

Next it will be shown that $H_n(\delta z;\alpha)$ is of type $2\pi\delta$. The expressions obtained by multiplying $F_\alpha(z)$ with the second and the third terms in the parenthesis of Definition 2 are of type 2π. It remains to estimate the first term in Definition 2. The series $\sum_\ell |z - \ell - \alpha|^{-2} F_\alpha(z - \ell)$ is bounded uniformly for all z satisfying $|z - k - \alpha| < 1/4$ with some $k \in \mathbb{Z}$. Moreover, for all z and k satisfying $|z - k - \alpha| \geq 1/4$, the sum $\sum_\ell |z - \ell - \alpha|^{-2}$ is bounded uniformly in z. Since $F_\alpha(z)$ is of type 2π, it follows that $H_n(z;\alpha)$ is of type 2π as well, and one obtains from (1) that $H_n(\delta z;\alpha)$ is of type $2\pi\delta$.

It will be shown in (37) and (38) that the function $H_n(x;\alpha)$ is an extremal function for $\operatorname{sgn}(x)x^n$ precisely when the 1–periodic function

$$B_{n+1}(\alpha) - B_{n+1}(t+\alpha-[t+\alpha])$$

has no changes of sign for all $t \in \mathbb{R}$ (here $[x]$ denotes the greatest integer less than or equal to x). This motivates the following choices for the values of α.

Let $n \in \mathbb{N}$. The function $B_{2n}(t)$ $(n \geq 1)$ has exactly one zero in the interval $(0, 1/2)$. Denote this zero by z_{2n}, and let $z_0 = 0$. By a result of D. H. Lehmer [6] the inequality $1/4 - \pi^{-1}2^{-2n-1} < z_{2n} < 1/4$ holds for $n \in \mathbb{N}$. The odd Bernoulli polynomials $B_{2n+1}(t)$ have zeros at $t = 0$ and $t = 1/2$, but no zeros in the interval $(0, 1/2)$ (cf. Section 4).

Define two sequences $\{\alpha_n\}_{n\in\mathbb{N}_0}$ and $\{\beta_n\}_{n\in\mathbb{N}_0}$ by

$$\begin{array}{llllll} \alpha_{4k} & := & 1 - z_{4k}, & \beta_{4k} & := & z_{4k}, \\ \alpha_{4k+1} & := & 0, & \beta_{4k+1} & := & \frac{1}{2}, \\ \alpha_{4k+2} & := & z_{4k+2}, & \beta_{4k+2} & := & 1 - z_{4k+2}, \\ \alpha_{4k+3} & := & \frac{1}{2}, & \beta_{4k+3} & := & 0, \end{array} \tag{3}$$

where $k \in \mathbb{N}_0$. Note that $B_{n+1}(t)$ assumes a maximum in $[0, 1]$ at $t = \alpha_n$, and $B_{n+1}(t)$ assumes a minimum in $[0, 1]$ at $t = \beta_n$ (cf. Lemma 5).

With these definitions $H_n(z;\alpha_n)$ and $H_n(z;\beta_n)$ turn out to be the extremal minorant and the extremal majorant of $\operatorname{sgn}(x)x^n$, respectively:

Theorem 1. *Let $n \in \mathbb{N}_0$. The inequality*

$$\delta^{-n}H_n(\delta x;\alpha_n) \leq \operatorname{sgn}(x)x^n \leq \delta^{-n}H_n(\delta x;\beta_n) \tag{4}$$

holds for all $x \in \mathbb{R}$. Moreover,

(i) *for every function $F \in E(2\pi\delta)$ satisfying $F(x) \geq \operatorname{sgn}(x)x^n$*

$$\int_{\infty}^{\infty} \big(F(x) - \operatorname{sgn}(x)x^n\big)dx \geq -2\frac{B_{n+1}(\beta_n)}{(n+1)\delta^{n+1}} \tag{5}$$

with equality exactly for $F(x) = \delta^{-n}H_n(\delta x;\beta_n)$, and

(ii) *for every function $F \in E(2\pi\delta)$ satisfying $G(x) \leq \operatorname{sgn}(x)x^n$*

$$\int_{\infty}^{\infty} \big(\operatorname{sgn}(x)x^n - G(x)\big)dx \geq 2\frac{B_{n+1}(\alpha_n)}{(n+1)\delta^{n+1}} \tag{6}$$

with equality exactly for $G(x) = \delta^{-n}H_n(\delta x;\alpha_n)$.

Let S be $\mathbb{R}$ or $\mathbb{Z}$. A function $f : S \to \mathbb{C}$ is called *positive definite* if for every $N \in \mathbb{N}$, any $a_1, ..., a_n \in \mathbb{C}$, and any $x_1, ..., x_n \in S$, the inequality

$$\sum_{\nu,\mu=1}^{N} a_\nu \overline{a}_\mu f(x_\nu - x_\mu) \geq 0 \tag{7}$$

holds.

Let $m \in \mathbb{N}$. As a first corollary of Theorem 1, positive definite extensions to $\mathbb{R}$ of the functions $\pm m!(2\pi i t)^{-m}$ restricted to $\mathbb{R}\backslash[-1, 1]$ are obtained. Define

$$s_{m,\alpha}(t) := -2\sum_{k=0}^{\infty} \frac{B_{k+m}(\alpha)}{(k+1)!}\Big(\frac{k+1}{k+m} - |t|\Big)(-2\pi i t)^k, \tag{8}$$

where $0 \leq \alpha \leq 1$, $m \in \mathbb{N}$, and $|t| < 1$.

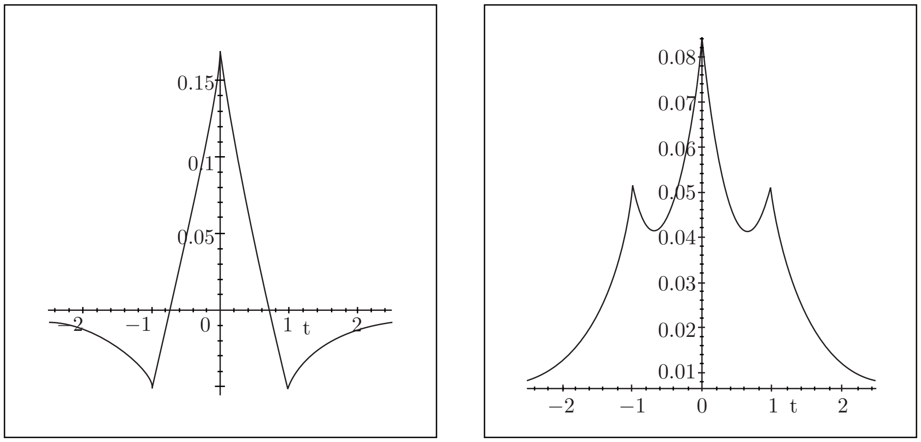

FIGURE 1. Plots of $f_2(t)$ and $g_2(t)$ (cf. Corollary 1)

Corollary 1. *Let $m \in \mathbb{N}$. The following functions are positive definite on $\mathbb{R}$:*

$$f_m(t) = \begin{cases} m!(2\pi i t)^{-m} & \textit{if } |t| \geq 1, \\ s_{m,\alpha_{m-1}}(t) & \textit{else}, \end{cases}$$

$$g_m(t) = \begin{cases} -m!(2\pi i t)^{-m} & \textit{if } |t| \geq 1, \\ -s_{m,\beta_{m-1}}(t) & \textit{else}. \end{cases}$$

The following functions are positive definite on $\mathbb{Z}$:

$$p_m(k) = \begin{cases} m!(2\pi i k)^{-m} & \textit{if } k \neq 0, \\ B_m(\alpha_{m-1}) & \textit{if } k = 0, \end{cases}$$

$$q_m(k) = \begin{cases} -m!(2\pi i k)^{-m} & \textit{if } k \neq 0, \\ -B_m(\beta_{m-1}) & \textit{if } k = 0. \end{cases}$$

Moreover, $f_m(0) = B_m(\alpha_{m-1})$, $g_m(0) = -B_m(\beta_{m-1})$, and the values $f_m(0)$, $g_m(0)$, $p_m(0)$, $q_m(0)$ are all minimal in the sense that none of $\pm m!(2\pi i t)^{-m}$ (resp. $\pm(2\pi i k)^{-m}$) restricted to $\mathbb{R}\backslash[-1,1]$ (resp. $\mathbb{Z}\backslash\{0\}$) can have a positive extension to $\mathbb{R}$ (resp. $\mathbb{Z}$) having a smaller value at the origin.

The proof of Corollary 1 will be given in Section 6. A consequence of this corollary are sharp bounds in certain Hilbert-type inequalities. Let $(a_\nu)_{\nu=1}^N$ be a finite sequence of complex numbers, and let $\{\lambda_\nu\}_{\nu=1}^N$ be a set of real numbers which are well-spaced in the sense that $|\lambda_\nu - \lambda_\mu| \geq \delta > 0$ for all $\nu \neq \mu$, and let $h(t)$ $(t \in \mathbb{R})$ be a hermitian function, i.e. $h(-t) = \overline{h(t)}$. We are interested in optimal bounds $L_\delta(h)$ and $U_\delta(h)$ such that

$$-L_\delta(h) \sum_{\nu=1}^N |a_\nu|^2 \leq \sum_{\substack{\mu,\nu=1 \\ \mu\neq\nu}}^N a_\nu \overline{a}_\mu h(\lambda_\nu - \lambda_\mu) \leq U_\delta(h) \sum_{\nu=1}^N |a_\nu|^2 \tag{9}$$

holds independently of $N \in \mathbb{N}$, and independently of the sequences $\{a_\nu\}_{\nu=1}^N$ and $\{\lambda_\nu\}_{\nu=1}^N$.

For $h_1(t) = (it)^{-1}$ the problem of finding the best possible values for $L(h_1)$ and $U(h_1)$ was solved by Montgomery and Vaughan [9]. As mentioned in the Introduction, Beurling's majorant $B(z)$ can be used to give a proof of Montgomery and Vaughan's result (cf. [12], Theorem 16, [11], chapter 20). Their result will be extended to the functions

$$h_m(t) = (it)^{-m}, \text{ where } m \in \mathbb{N}. \tag{10}$$

Corollary 2. *Let $m \in \mathbb{N}$, let $\delta > 0$, and let L_δ, U_δ be as in (9). In this case the optimal bounds are*

$$\begin{aligned} L_\delta\big((it)^{-m}\big) &= (2\pi)^m \frac{B_m(\alpha_{m-1})}{m!\delta^m}, \\ U_\delta\big((it)^{-m}\big) &= -(2\pi)^m \frac{B_m(\beta_{m-1})}{m!\delta^m}. \end{aligned}$$

For example, since $-2\pi^2 B_2(1/2) = \pi^2 B_2(0) = \zeta(2)$ we obtain for $m = 2$ and $\delta = 1$ that

$$-\zeta(2)\sum_{\nu=1}^{N} |a_\nu|^2 \le \sum_{\substack{\mu,\nu=1 \\ \mu\neq\nu}}^{N} \frac{a_\nu \overline{a}_\mu}{(\lambda_\nu - \lambda_\mu)^2} \le 2\zeta(2)\sum_{\nu=1}^{N} |a_\nu|^2 \tag{11}$$

for all $N \in \mathbb{N}$ and all sequences (a_ν), $\{\lambda_\nu\}$ as above.

For $m = 2$ and $\delta = 1$ we have the following extremal configurations. An extremal configuration for the upper bound is given by $\lambda_\nu := \nu$, $a_\nu := 1$, and $N \to \infty$, since

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{\nu,\mu=1 \\ n\neq m}}^{N} \frac{1}{(\nu-\mu)^2} = \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{k=1-N \\ k\neq 0}}^{N-1} \frac{N-|k|}{k^2} = 2\zeta(2).$$

An extremal configuration for the lower bound is given by $\lambda_\nu := \nu$, $a_\nu := (-1)^\nu$ and $N \to \infty$, since

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{\nu,\mu=1 \\ n\neq m}}^{N} \frac{(-1)^{\nu-\mu}}{(\nu-\mu)^2} = \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{k=1-N \\ k\neq 0}}^{N-1} (-1)^k \frac{N-|k|}{k^2} = -\zeta(2).$$

Note that $L_\delta((it)^{-m}) = U_\delta((it)^{-m})$ for odd-valued, but not for even-valued $m \in \mathbb{N}$.

The proof of Corollary 2 will be given in Section 6.

Another application is the following result originally obtained by J. J. Holt (cf. [4], Theorem 1 and Corollary 1). Let $\alpha > 0$, and define

$$R_\alpha(x) = \alpha^{-1}(|x+\alpha| - |x|) \text{ for all } x \in \mathbb{R}. \tag{12}$$

Holt obtained extremal majorants and minorants of type 2π for $R_\alpha(x)$ in the case that $\alpha \in A := (0, 1/2] \cup \{k + 1/2 : k \in \mathbb{N}\}$, and he obtained non-extremal minorants and majorants of type 2π for all other $\alpha > 0$. We will obtain Holt's result for $\alpha \in A$, and we will give slightly better (also non-extremal) majorants and minorants for all positive $\alpha \notin A$. Define

$$M_\alpha(x) = \begin{cases} H_0(x;0) & \text{if } 0 < \alpha \leq 1/2, \\ \alpha^{-1}(H_1(x+\alpha;1/2) - H_1(x;0)) & \text{if } \alpha > 1/2, \end{cases} \tag{13}$$

$$m_\alpha(x) = \begin{cases} H_0(x+\alpha;1) & \text{if } 0 < \alpha \leq 1/2, \\ \alpha^{-1}(H_1(x+\alpha;0) - H_1(x;1/2)) & \text{if } \alpha > 1/2. \end{cases} \tag{14}$$

For $0 < \alpha \leq 1/2$ the inequality $H_0(x+\alpha;1) \leq R_\alpha(x) \leq H_0(x;0)$ holds (cf. [4], Corollary 1). For any $\alpha > 0$ we have by Theorem 1 that $H_1(x+\alpha;0) \leq |x+\alpha| \leq H_1(x+\alpha;1/2)$ and $-H_1(x;1/2) \leq -|x| \leq -H_1(x;0)$. So for all $x \in \mathbb{R}$

$$m_\alpha(x) \leq R_\alpha(x) \leq M_\alpha(x). \tag{15}$$

Moreover for $0 < \alpha \leq 1/2$, $\int (H_0(x;0) - R_\alpha(x))dx = \int (R_\alpha(x) - H_0(x+\alpha;1))dx = 1-\alpha$. Since $-B_2(1/2) + B_2(0) = 1/12 + 1/6 = 1/4$, Theorem 1 implies for $\alpha > 1/2$

$$\int_{\mathbb{R}} (M_\alpha - R_\alpha) = \int_{\mathbb{R}} (R_\alpha - m_\alpha) = (4\alpha)^{-1}. \tag{16}$$

Define

$$d(\alpha) = \begin{cases} 1-\alpha & \text{if } 0 < \alpha \leq 1/2, \\ (4\alpha)^{-1} & \text{if } \alpha > 1/2. \end{cases} \tag{17}$$

We have shown

Corollary 3. *The functions M_α and m_α are of type 2π, and they majorize and minorize R_α, respectively, on the real line. Moreover,*

$$\int_{\mathbb{R}} (M_\alpha - R_\alpha) = \int_{\mathbb{R}} (R_\alpha - m_\alpha) = d(\alpha).$$

We use Corollary 3 to obtain majorants and minorants of type 2π for trapezoids. Define $f_{\alpha,\beta,\gamma}(x) = \frac{1}{2}(R_\alpha(x) + R_\gamma(\beta - x))$. The graph of $f_{\alpha,\beta,\gamma}(x)$ is a trapezoid with base-length $\alpha+\beta+\gamma$, top-length β, height 1, and left point at $x = -\alpha$. Define

$$M_{\alpha,\beta,\gamma}(x) = \frac{1}{2}(M_\alpha(x) + M_\gamma(\beta - x)), \tag{18}$$

$$m_{\alpha,\beta,\gamma}(x) = \frac{1}{2}(m_\alpha(x) + m_\gamma(\beta - x)). \tag{19}$$

From Corollary 3 we obtain

Corollary 4. *$M_{\alpha,\beta,\gamma}$ and $m_{\alpha,\beta,\gamma}$ are functions of type 2π, they satisfy*

$$m_{\alpha,\beta,\gamma}(x) \leq f_{\alpha,\beta,\gamma}(x) \leq M_{\alpha,\beta,\gamma}(x)$$

for all real x, and

$$\int_{\mathbb{R}} (M_{\alpha,\beta,\gamma} - f_{\alpha,\beta,\gamma}) = \int_{\mathbb{R}} (f_{\alpha,\beta,\gamma} - m_{\alpha,\beta,\gamma}) = \frac{1}{2}(d(\alpha) + d(\gamma)).$$

3. Outline of the proofs

We will first deal with the case $\delta = 1$. At the end of this section we will indicate how to obtain majorants for any $\delta > 0$. Let $\delta = 1$. Since most of the following statements are concerned with the difference of $H_n(x;\alpha)$ and $\operatorname{sgn}(x)x^n$, we define

$$\psi_{n,\alpha}(x) := H_n(x;\alpha) - \operatorname{sgn}(x)x^n. \tag{20}$$

The proof of Theorem 1 is divided into a series of lemmata whose proofs are given in Section 5.

Lemma 1. *Let $0 \le \alpha \le 1$ and $n \in \mathbb{N}_0$. The function $\psi_{n,\alpha}(x)$ $(x \in \mathbb{R})$ is absolutely integrable. Moreover, if $\{\alpha_n\}_{n\in\mathbb{N}_0}$ and $\{\beta_n\}_{n\in\mathbb{N}_0}$ are defined by* (3), *then*

$$H_n(x;\alpha_n) \le \operatorname{sgn}(x)x^n \le H_n(x;\beta_n).$$

Since $\psi_{n,\alpha}(x)$ is integrable, its Fourier transform exists. Its value is given by

Lemma 2. *Let $0 \le \alpha \le 1$ and $n \in \mathbb{N}_0$. We have*

$$\begin{aligned}\mathcal{F}\psi_{n,\alpha}(t) = -2\sum_{k=0}^{\infty}\frac{B_{k+n+1}(\alpha)}{(k+1)!}\Big(\frac{k+1}{k+n+1} - |t|\Big)(-2\pi i t)^k \qquad&\\ + \frac{B_n(\alpha)}{\pi i}\operatorname{sgn}(t)\big(e(-\{\alpha\}t) - 1\big) \text{ for } |t| < 1,&\end{aligned} \tag{21}$$

$$\mathcal{F}\psi_{n,\alpha}(t) = -\frac{2\cdot n!}{(2\pi i t)^{n+1}} \text{ for } |t| \ge 1. \tag{22}$$

By taking the value of $\mathcal{F}\psi_{n,\alpha}(t)$ at $t = 0$ in Lemma 2, we obtain the equalities in (5) for $F(x) = H_n(x;\beta_n)$ and in (6) for $G(x) = H_n(x;\alpha_n)$.

The next lemma establishes the extremality properties of $H_n(x;\alpha)$.

Lemma 3. *Let $n \in \mathbb{N}_0$, and let $F_n, G_n \in E(2\pi)$ be real entire functions such that*

$$G_n(x) \le \operatorname{sgn}(x)x^n \le F_n(x)$$

for all $x \in \mathbb{R}$. Then

$$\int_{-\infty}^{\infty}(F_n(x) - \operatorname{sgn}(x)x^n)dx \ge -\frac{2}{n+1}\min_{0\le t\le 1} B_{n+1}(t), \tag{23}$$

$$\int_{-\infty}^{\infty}(\operatorname{sgn}(x)x^n - G_n(x))dx \ge \frac{2}{n+1}\max_{0\le t\le 1} B_{n+1}(t). \tag{24}$$

Moreover, in (23) *and* (24) *equality can hold only for the minorants and majorants defined in Lemma* 1.

The proof of Theorem 1 is completed by considering the case of arbitrary $\delta > 0$. Defining $f_\delta(x) := f(\delta x)$ for any $f \in L^2(\mathbb{R})$ we note that $\widehat{f_\delta}(t) = \delta^{-1}f(t/\delta)$. Moreover,

$$\delta^{-n}\psi_{n,\alpha}(\delta x) = \delta^{-n}H_n(\delta x;\alpha) - \operatorname{sgn}(x)x^n.$$

This implies $\delta^{-n}\widehat{\psi}_{n,\alpha}(0) = \delta^{-n-1}\widehat{\psi}_{n,\alpha}(0)$, which completes the proof of Theorem 1.

4. Bernoulli functions and Euler-Maclaurin summation

In this section we give a brief review of some facts about Bernoulli polynomials that we will need in our proofs. Most of these facts are taken from [1], [7], and [10].

The Bernoulli polynomials $B_n(x)$ can be defined by the power series expansion

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}t^n, \tag{25}$$

where $|t|<2\pi$, the Bernoulli numbers B_n by

$$B_n=B_n(0), \tag{26}$$

and the Bernoulli periodic functions $\mathcal{B}_n(t)$ by

$$\mathcal{B}_n(t)=B_n(t-[t]). \tag{27}$$

The Bernoulli polynomials satisfy $B_n'(t)=nB_{n-1}(t)$ and

$$\int_0^1 B_n(t)dt=0.$$

This implies that for $0\le\alpha\le 1$ the Bernoulli periodic functions have the antiderivatives

$$\int_0^x \mathcal{B}_n(t+\alpha)dt=\frac{1}{n+1}\big(\mathcal{B}_{n+1}(x+\alpha)-B_{n+1}(\alpha)\big). \tag{28}$$

For $n\ge 1$ the Bernoulli periodic functions have the Fourier series expansion

$$\mathcal{B}_n(t)=-\frac{n!}{(2\pi i)^n}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{1}{k^n}e(kt), \tag{29}$$

which is valid for $t\in\mathbb{R}\backslash\mathbb{Z}$ with symmetric summation if $n=1$, and it is valid for $t\in\mathbb{R}$ if $n\ge 2$.

We will need the Euler-Maclaurin summation formula in the following form:

Lemma 4. *For* $0\le\alpha\le 1$, $x>0$ *and any* $\mu\in\mathbb{N}$

$$\sum_{n=1}^{\infty}\frac{1}{(x+n-\alpha)^2}=\sum_{n=1}^{\mu}\frac{B_{n-1}(\alpha)}{x^n}+(\mu+1)\int_0^{\infty}\frac{B_\mu(\alpha)-\mathcal{B}_\mu(t+\alpha)}{(x+t)^{\mu+2}}dt. \tag{30}$$

Proof. Induction on μ. For $0\le\alpha<1$ we obtain with integration by parts

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(x+n-\alpha)^2}&=\int_{0+}^{\infty}\frac{1}{(x+t)^2}d[t+\alpha]=\int_0^{\infty}\frac{1}{(x+t)^2}dt+\int_{0+}^{\infty}\frac{d[t+\alpha]-dt}{(x+t)^2}\\&=\frac{B_0(\alpha)}{x}+2\int_0^{\infty}\frac{B_1(\alpha)-\mathcal{B}_1(t+\alpha)}{(x+t)^3}dt,\end{aligned}$$

and for $\alpha=1$ we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(x+n-1)^2}&=\frac{1}{x^2}+\sum_{n=1}^{\infty}\frac{1}{(x+n)^2}=\frac{1}{x}+2\int_0^{\infty}\frac{1+B_1(0)-\mathcal{B}_1(t)}{(x+t)^3}dt\\&=\frac{B_0(1)}{x}+2\int_0^{\infty}\frac{B_1(1)-\mathcal{B}_1(t+1)}{(x+t)^3}dt,\end{aligned}$$

since $\mathcal{B}_1(t)$ is 1-periodic. This establishes (30) for $\mu=1$.

The remaining part of the induction follows with repeated applications of integrations by parts using (28). ☐

We will need the extrema of the Bernoulli polynomials in the interval $[0,1]$. The locations of these extrema are collected in the following lemma. These facts come from [10], chapter 2.

Lemma 5. *Let* $0 \le x \le 1$ *and* $n \ge 1$.

- (i) $B_{4n}(x)$ *assumes its maximum value at* $x = 1/2$ *and its minimum value at* $x = 0$, $x = 1$.
- (ii) $B_{4n+1}(x)$ *assumes its minimum value at a unique* $\alpha \in (0, 1/2)$ *and its maximum value at* $1 - \alpha \in (1/2, 1)$.
- (iii) $B_{4n-2}(x)$ *assumes its maximum value at* $x = 0$, $x = 1$ *and its minimum value at* $x = 1/2$.
- (iv) $B_{4n-1}(x)$ *assumes its maximum value at a unique* $\alpha \in (0, 1/2)$ *and its minimum value at* $1 - \alpha \in (1/2, 1)$.

Finally, $B_0(x) = 1$ and $B_1(x) = x - 1/2$. As was pointed out in Section 2, Lehmer showed in [6] that the zeros z_{2n} of the even Bernoulli polynomial in $(0, 1/2)$ (or, what amounts to the same thing, the extrema of the odd Bernoulli polynomials in $(0, 1/2)$) satisfy

$$\frac{1}{4} - \frac{1}{\pi 2^{2n+1}} < z_{2n} < \frac{1}{4}.$$

Decimal approximations for the first four z_{2n} are $z_2 = 0.2113$, $z_4 = 0.2403$, $z_6 = 0.2475$, $z_8 = 0.2494$.

5. Proof of the lemmata

Proof of Lemma 1. Let $x \in \mathbb{R}$ and $0 \le \alpha \le 1$. Recall

$$\psi_{n,\alpha}(x) = H_n(x;\alpha) - \operatorname{sgn}(x) x^n. \tag{31}$$

We will consider the cases $x > 0$ and $x < 0$ separately. Let $x > 0$. We have by Lemma 4 with $\mu = n + 1$ that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \frac{\operatorname{sgn}_+(k+\alpha)}{(x-k-\alpha)^2} + 2\sum_{\ell=1}^{n} \frac{B_{\ell-1}(\alpha)}{x^\ell} + \frac{2B_n(\alpha)}{x^n(x-\{\alpha\})} - \sum_{k=-\infty}^{\infty} \frac{1}{(x-k-\alpha)^2}\\
&\quad = -2\sum_{k=1}^{\infty} \frac{1}{(x+k-\alpha)^2} + 2\sum_{\ell=1}^{n+1} \frac{B_{\ell-1}(\alpha)}{x^\ell} + \frac{B_n(\alpha)}{x^n}\Big(\frac{2}{x-\{\alpha\}} - \frac{2}{x}\Big)\\
&\quad = -2(n+2)\int_0^\infty \frac{B_{n+1}(\alpha) - \mathcal{B}_{n+1}(t+\alpha)}{(x+t)^{n+3}}\, dt + O\left(x^{-n-2}\right) \qquad (32)\\
&\quad \ll x^{-n-2},
\end{aligned}$$

because $B_{n+1}(\alpha) - \mathcal{B}_{n+1}(t+\alpha)$ is bounded. Since for $x > 0$

$$x^n = x^n \Big(\frac{\sin \pi(x-\alpha)}{\pi}\Big)^2 \sum_{k=-\infty}^{\infty} \frac{1}{(x-k-\alpha)^2},$$

we obtain

$$\psi_{n,\alpha}(x) = H_n(x;\alpha) - x^n = O(x^{-2}) \tag{33}$$

for $x > 0$.

Now let $x < 0$. Putting $y = -x > 0$ and using $B_\ell(\alpha) = (-1)^\ell B_\ell(1-\alpha)$ we obtain with a similar computation that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \frac{\operatorname{sgn}_+(k+\alpha)}{(x-k-\alpha)^2} + 2\sum_{\ell=1}^{n} \frac{B_{\ell-1}(\alpha)}{x^\ell} + \frac{2B_n(\alpha)}{x^n(x-\{\alpha\})} + \sum_{k=-\infty}^{\infty} \frac{1}{(x-k-\alpha)^2} \\
&\quad = 2(n+2)\int_0^\infty \frac{B_{n+1}(1-\alpha) - \mathcal{B}_{n+1}(t-\alpha)}{(y+t)^{n+3}} dt + O(y^{-n-2}) \qquad (34) \\
&\quad \ll y^{-n-2}.
\end{aligned}$$

We obtain for $x < 0$ that

$$\psi_{n,\alpha}(x) = H_n(x;\alpha) + x^n = O(x^{-2}). \qquad (35)$$

Equalities (33) and (35) prove the first statement of Lemma 1.

For the second statement we use the representation for $\psi_{n,\alpha}(x)$ derived in (32) and (34). If

$$\frac{B_n(\alpha)}{x^n}\Big(\frac{2}{x-\{\alpha\}} - \frac{2}{x}\Big) = 0, \qquad (36)$$

then (32) implies for $x > 0$

$$\psi_{n,\alpha}(x) = -2(n+2)F(x-\alpha)x^n \int_0^\infty \frac{B_{n+1}(\alpha) - \mathcal{B}_{n+1}(t+\alpha)}{(x+t)^{n+3}}\, dt, \qquad (37)$$

and (34) implies for $x < 0$

$$\psi_{n,\alpha}(x) = 2(n+2)F(x-\alpha)(-x)^n \int_0^\infty \frac{B_{n+1}(1-\alpha) - \mathcal{B}_{n+1}(t+1-\alpha)}{(-x+t)^{n+3}}\, dt. \qquad (38)$$

If $B_{n+1}(t)$ restricted to $[0,1]$ has a maximum at $t = \alpha$, then it has a minimum at $t = 1-\alpha$ if n is even, and a maximum if n is odd, since $B_\ell(\alpha) = (-1)^\ell B_\ell(\alpha)$. This implies that for such α the expressions $B_{n+1}(\alpha) - \mathcal{B}_{n+1}(t+\alpha)$ and $B_{n+1}(1-\alpha) - \mathcal{B}_{n+1}(t+1-\alpha)$ do not change their signs for $t \in [0,\infty)$, and since $-x^n = (-x)^n(-1)^{n+1}$ we obtain that for such α the expressions in (37) and (38) are either both positive or both negative for all x in the respective ranges. Moreover, $\psi_{n,\alpha} \geq 0$ if $B_{n+1}(t)$ assumes its minimum on $[0,1]$ at $t = \alpha$, and $\psi_{n,\alpha} \leq 0$ if $B_{n+1}(t)$ assumes its maximum at $t = \alpha$.

Since by Lemma 5 the function $B_{n+1}(t)$ assumes its minimum on $[0,1]$ at $t = \beta_n$ and its maximum at $t = \alpha_n$, we have

$$H_n(x;\alpha_n) \leq \operatorname{sgn}(x)x^n \leq H_n(x;\beta_n),$$

and this finishes the proof of Lemma 1. □

(36) is satisfied for $\alpha = 0$. This provides us with the majorant. Since $\operatorname{sgn}(x)$ is odd, any majorant $M(x)$ of $\operatorname{sgn}(x)$ gives rise to a minorant $-M(-x)$ of $\operatorname{sgn}(x)$, and $-H_0(-x;0) = H_0(x;1)$.

Proof of Lemma 2. Recall $\operatorname{sgn}_+(x) = \operatorname{sgn}(x+)$, and let

$$F(z) = \pi^{-2}\sin^2 \pi z \text{ for } z \in \mathbb{C}.$$

Performing the index shift $k+n+1 \mapsto k$ in the series representing $\mathcal{F}\psi_{n,\alpha}(t)$ for $|t|<1$ leads to (21) in the form in which we will prove it:

$$\begin{aligned}\mathcal{F}\psi_{n,\alpha}(t) = -2 &\sum_{k=n+1}^{\infty} \frac{B_k(\alpha)}{(k-n)!}\Big(\frac{k-n}{k} - |t|\Big)(-2\pi i t)^{k-n-1} \\ &+ \frac{B_n(\alpha)}{\pi i}\operatorname{sgn}(t)\big(e(-\{\alpha\}t)-1\big) \text{ for } |t|<1. \end{aligned} \tag{39}$$

The first part of the proof will be similar to the proof of Theorem 6 in [12]. Define

$$H_{0,K}(x,\alpha) := F(x-\alpha)\Big(\sum_{k=-K}^{K-1} \frac{\operatorname{sgn}_+(k+\alpha)}{(x-k-\alpha)^2} + \frac{2}{x-\{\alpha\}}\Big).$$

With the Fourier expansions

$$\frac{F(x)}{x^2} = \int_{-1}^{1} (1-|t|)e(xt)dt, \tag{40}$$

$$\frac{F(x)}{x} = \frac{1}{2\pi i}\int_{-1}^{1} \operatorname{sgn}(t)e(xt)dt, \tag{41}$$

we obtain

$$\begin{aligned}H_{0,K}(x,\alpha) = \int_{-1}^{1} (1-|t|)\Big[\sum_{k=0}^{K-1} e(-(k+\alpha)t) - \sum_{k=-K}^{-1} e(-(k+\alpha)t)\Big]e(xt)dt \\ + \frac{1}{\pi i}\int_{-1}^{1} \operatorname{sgn}(t)e(-\{\alpha\}t)e(xt)dt.\end{aligned}$$

We have for $t \neq 0$

$$\sum_{k=0}^{K-1} e(-(k+\alpha)t) - \sum_{k=-K}^{-1} e(-(k+\alpha)t) = 2\frac{e(-\alpha t)}{1-e(-t)}(1-\cos 2\pi K t),$$

and since the last expression is bounded in a neighborhood of $t=0$ we obtain

$$\begin{aligned}H_{0,K}(x,\alpha) = \int_{-1}^{1} (1-|t|)\Big[\frac{2e(-\alpha t)}{1-e(-t)} - e(-\alpha t)\frac{2\cos 2\pi K t}{1-e(-t)}\Big]e(xt)dt \\ + \frac{1}{\pi i}\int_{-1}^{1} \operatorname{sgn}(t)e(-\{\alpha\}t)e(xt)dt.\end{aligned}$$

In order to apply the lemma of Riemann-Lebesgue we have to remove the poles in the fractions of the first integral. We do this by differentiating both sides with respect to x and dividing the resulting expression by 2. We obtain

$$\begin{aligned}\frac{1}{2}H'_{0,K}(x,\alpha) = \int_{-1}^{1} (1-|t|)\Big[\frac{2\pi i t\, e(-\alpha t)}{1-e(-t)} - e(-\alpha t)\frac{2\pi i t \cos 2\pi K t}{1-e(-t)}\Big]e(xt)dt \\ + \int_{-1}^{1} |t|e(-\alpha t)e(xt)dt.\end{aligned}$$

By the lemma of Riemann-Lebesgue we have

$$\lim_{K\to\infty}\int_{-1}^{1} \frac{2\pi i t \cos 2\pi K t}{1-e(-t)}e(xt)dt = 0.$$

Since $\{H_{0,K}(x,\alpha)\}_{K\in\mathbb{N}}$ is a sequence of entire functions that converges to $H_0(x,\alpha)$ uniformly on any compact subset of $\mathbb{C}$, the sequence of derivatives $\{H'_{0,K}(x,\alpha)\}_{K\in\mathbb{N}}$ converges to $H'_0(x,\alpha)$ uniformly on any compact subset of $\mathbb{C}$. Thus

$$\frac{1}{2}H'_0(x,\alpha) = \int_{-1}^{1}\Big[(1-|t|)\frac{2\pi it\, e(-\alpha t)}{1-e(-t)} + |t|e(-\{\alpha\}t)\Big]e(xt)dt,$$

and using (25) we obtain

$$\begin{aligned}\mathcal{F}\Big[\frac{1}{2}H'_0(x,\alpha)\Big](t) &= (1-|t|)\sum_{k=0}^{\infty}\frac{B_k(\alpha)}{k!}(-2\pi it)^k + |t|e(-\{\alpha\}t)\\ &= 1 + (1-|t|)\sum_{k=1}^{\infty}\frac{B_k(\alpha)}{k!}(-2\pi it)^k + |t|\big(e(-\{\alpha\}t)-1\big)\end{aligned} \tag{42}$$

for $|t|<1$, and $\mathcal{F}\big[\frac{1}{2}H'_0(x,\alpha)\big](t)=0$ for $|t|\geq 1$.

Now we can prove (22) and (39) by induction on n. The difference $\psi_{0,\alpha}(x) = H_0(x,\alpha) - \operatorname{sgn}(x)$ is absolutely integrable by Lemma 1, so its Fourier transform exists.

From

$$\frac{1}{2}\int_{-\infty}^{\infty} e(-xt)d\psi_{0,\alpha}(x) = \mathcal{F}\Big[\frac{1}{2}H'_0(x,\alpha)\Big](t) - 1$$

we obtain with (42) and $2\pi it\mathcal{F}f(t) = \mathcal{F}[f'](t)$ that for $|t|<1$

$$\begin{aligned}\mathcal{F}\psi_{0,\alpha}(t) &= \frac{1}{\pi it}\Big(\mathcal{F}\Big[\frac{1}{2}H'_0(x,\alpha)\Big]-1\Big)\\ &= \frac{1}{\pi it}\Big((1-|t|)\sum_{k=1}^{\infty}\frac{B_k(\alpha)}{k!}(-2\pi it)^k + |t|\big(e(-\{\alpha\}t)-1\big)\Big)\\ &= -2(1-|t|)\sum_{k=1}^{\infty}\frac{B_k(\alpha)}{k!}(-2\pi it)^{k-1} + \frac{\operatorname{sgn}(t)}{\pi i}\big(e(-\{\alpha\}t)-1\big),\end{aligned}$$

and this is (39) for $n=0$. Moreover, for $|t|\geq 1$

$$\mathcal{F}\psi_{0,\alpha}(t) = \frac{1}{\pi it}\Big(\mathcal{F}\Big[\frac{1}{2}H'_0(x,\alpha)\Big]-1\Big) = -\frac{1}{\pi it},$$

and this is (22) for $n=0$.

Induction step. Assume that (22) and (39) are true for some $n\in\mathbb{N}_0$. From Definition 2 with n and $n+1$ we obtain

$$H_{n+1}(z;\alpha) = zH_n(z;\alpha) + 2F(z-\alpha)\,\frac{B_{n+1}(\alpha)-\{\alpha\}B_n(\alpha)}{z-\{\alpha\}} \tag{43}$$

for any $z\in\mathbb{C}$. Since by equation (39) the Fourier transforms of $\psi_{n,\alpha}$ and $\psi_{n+1,\alpha}$ exist, we obtain with (43) and (41) for $|t|<1$

$$\mathcal{F}\psi_{n+1,\alpha}(t) = -\frac{1}{2\pi i}\frac{d}{dt}\mathcal{F}\psi_{n,\alpha}(t) + \frac{1}{\pi i}(B_{n+1}(\alpha)-\{\alpha\}B_n(\alpha))\operatorname{sgn}(t)e(-\{\alpha\}t). \tag{44}$$

By the induction hypothesis, (39) holds for n, i.e. for $|t|<1$

$$\mathcal{F}\psi_{n,\alpha}(t) = -2\sum_{k=n+1}^{\infty}\frac{B_k(\alpha)}{(k-n)!}\Big(\frac{k-n}{k}-|t|\Big)(-2\pi i t)^{k-n-1} + \frac{B_n(\alpha)}{\pi i}\operatorname{sgn}(t)\big(e(-\{\alpha\}t)-1\big). \tag{45}$$

For $k \geq n+2$

$$\frac{d}{dt}\Big(\frac{k-n}{k}-|t|\Big)t^{k-n-1} = (k-n)\Big(\frac{k-n-1}{k}-|t|\Big)t^{k-n-2}. \tag{46}$$

Inserting (45) in (44), splitting off the first term of the series, and applying (46) to the remaining part of the series, we obtain for $|t|<1$

$$\begin{aligned}\mathcal{F}\psi_{n+1,\alpha}(t) &= -2\sum_{k=n+2}^{\infty}\frac{B_k(\alpha)}{(k-n-1)!}\Big(\frac{k-n-1}{k}-|t|\Big)(-2\pi i t)^{k-n-2}\\ &\quad - \frac{B_{n+1}(\alpha)\operatorname{sgn}(t)}{\pi i} + \frac{\operatorname{sgn}(t)B_n(\alpha)}{(-2\pi i)\pi i}(-2\pi i\{\alpha\})e(-\{\alpha\}t)\\ &\quad + \frac{1}{\pi i}(B_{n+1}(\alpha)-\{\alpha\}B_n(\alpha))\operatorname{sgn}(t)e(-\{\alpha\}t)\\ &= -2\sum_{k=n+2}^{\infty}\frac{B_k(\alpha)}{(k-n-1)!}\Big(\frac{k-n-1}{k}-|t|\Big)(-2\pi i t)^{k-n-2}\\ &\quad + \frac{B_{n+1}(\alpha)}{\pi i}\operatorname{sgn}(t)(e(-\{\alpha\}t)-1),\end{aligned}$$

and this is (39) for $n+1$.

Since the Fourier transform of $(x-\{\alpha\})^{-1}\sin^2\pi(x-\alpha)$ equals zero outside the interval $[-1,1]$, we have with (43) for $|t|\geq 1$

$$\mathcal{F}\psi_{n+1,\alpha}(t) = -\frac{1}{2\pi i}\frac{d}{dt}\mathcal{F}\psi_{n,\alpha}(t) = -\frac{2(n+1)!}{(2\pi i t)^{n+2}},$$

and this is (22) for $n+1$. □

Proof of Lemma 3. Let $0\leq\alpha\leq 1$, and let $F_n\in E(2\pi)$ be a majorant for $\operatorname{sgn}(x)x^n$. Assume that

$$\int_{-\infty}^{\infty}(F_n(x)-\operatorname{sgn}(x)x^n)dx<\infty.$$

Let $\psi_n(x)=F_n(x)-\operatorname{sgn}(x)x^n$, and recall that $\psi_{n,\alpha}(x)=H_n(x;\alpha)-\operatorname{sgn}(x)x^n$. Since $F_n(x)-H_n(x;\alpha)$ is an absolutely integrable function in $E(2\pi)$, we know by the Paley-Wiener Theorem that the support of its Fourier transform is a subset of $[-1,1]$, i.e.

$$\mathcal{F}[F_n(x)-H_n(x,\alpha)](t)=0 \text{ for } |t|\geq 1.$$

It follows from Lemma 2 that

$$\mathcal{F}\psi_n(t)=\mathcal{F}\psi_{n,\alpha}(t)=-\frac{2n!}{(2\pi i t)^{n+1}} \text{ for } |t|\geq 1. \tag{47}$$

Now use (47), the Poisson summation formula and (29) to obtain that

$$0 \leq \sum_{\ell=-\infty}^{\infty} \psi_n(\ell + t) = \mathcal{F}\psi_n(0) - \frac{2}{n+1} \sum_{k \neq 0} \frac{(n+1)!}{(2\pi i k)^{n+1}} e(kt) \tag{48}$$
$$= \mathcal{F}\psi_n(0) + \frac{2}{n+1} \mathcal{B}_{n+1}(t),$$

and since this has to hold for all $t \in [0,1]$,

$$\mathcal{F}\psi_n(0) \geq -\frac{2}{n+1} \min_{0 \leq t \leq 1} B_{n+1}(t). \tag{49}$$

Similarly, with $\phi_n(x) = \operatorname{sgn}(x) x^n - G_n(x)$

$$\mathcal{F}\phi_n(0) \geq \frac{2}{n+1} \max_{0 \leq t \leq 1} B_{n+1}(t). \tag{50}$$

Vaaler showed in Theorem 9 of [12] that any integrable function in $E(2\pi)$ is already uniquely determined by its values and the values of its first derivative at the integers, and he used this result to prove the case $n = 0$ of Lemma 3. We will use his argument.

Let $0 \leq \alpha \leq 1$ be such that $B_{n+1}(t)$ has its minimum on $[0,1]$ at $t = \alpha$. If $F_n \in E(2\pi)$ is chosen such that F_n is a majorant of $\operatorname{sgn}(x) x^n$ with

$$\mathcal{F}\psi_n(0) = -\frac{2}{n+1} B_{n+1}(\alpha),$$

then we have equality in (48) for $t = \alpha$. This means that

$$F(\alpha + k) = \operatorname{sgn}_+(\alpha + k)(\alpha + k)^n \text{ for all } k \in \mathbb{Z}.$$

The same is true for $H_n(x;\alpha)$ by construction. If $\alpha = 0$ or 1, let $n \geq 2$. Since both $F_n(x)$ and $H_n(x;\alpha)$ are majorants of $\operatorname{sgn}(x) x^n$, they must have the same derivatives at the numbers $\alpha + k$, namely $n \cdot \operatorname{sgn}(\alpha + k)(\alpha + k)^{n-1}$. From Theorem 9 of [12] we obtain

$$F_n(z) - H_n(z;\alpha) = 0$$

for all $z \in \mathbb{C}$. The computation for $G_n(z)$ follows along the same lines.

If $n = 0, 1$ and $\alpha = 0, 1$, then we cannot immediately conclude that $F_n(x)$ and $H_n(x;\alpha)$ have equal derivatives at $x = 0$. However, as in the proof of Theorem 8 in [12]

$$F_n(z) - H_n(z;\alpha) = (F_n'(0) - H_n'(0;\alpha)) \pi^{-2} x^{-1} \sin^2 \pi z,$$

and since $x^{-1} \sin^2 \pi x$ is not integrable on the real line, we must have $F_n'(0) = H_n'(0;\alpha)$. Thus, $F_n(z) = H_n(z;\alpha)$ holds in this case as well. □

6. Proofs of Corollaries 1 and 2

Proof of Corollary 1. We will prove the statements about f_m and p_m of Corollary 1. Let $n \in \mathbb{N}_0$. By Theorem 1

$$\phi_{n,\alpha_n}(x) = \operatorname{sgn}(x) x^n - H_n(x;\alpha_n) \geq 0,$$

by Lemma 1 the function is integrable on $\mathbb{R}$, and by Lemma 2

$$\mathcal{F}\phi_{n,\alpha_n}(t) = \frac{2 \cdot n!}{(2\pi i t)^{n+1}}$$

for $|t| \geq 1$. By the easy implication of Bochner's theorem, $\mathcal{F}\phi_{n,\alpha_n}$ is positive definite. Equation (21) of Lemma 2 yields the explicit representation of $\mathcal{F}\phi_{n,\alpha_n}(t)$

for $|t| < 1$. Note that the last term in (21) is equal to zero, since by definition one of the equations $B_n(\alpha_n) = 0$, $\alpha_n = 0$, or $\alpha_n = 1$ holds. Performing the substitution $m = n + 1$ yields the statements about f_m.

To verify the claims about p_m consider the function $p_{m,c} : \mathbb{Z} \to \mathbb{C}$ ($c \in \mathbb{R}$) defined by

$$p_{m,c}(k) = \begin{cases} m!\,(2\pi ik)^{-m} & \text{if } k \neq 0 \\ c & \text{if } k = 0 \end{cases} \quad (k \in \mathbb{Z}).$$

By (29)

$$\sum_{k\in\mathbb{Z}} p_{m,c}(k)e(kt) = c - \mathcal{B}_m(t),$$

and this is non-negative if, and only if,

$$c \geq \mathcal{B}_m(t) \text{ for all } t \in [0,1].$$

We obtain, using Bochner's theorem, that $p_{m,c}$ is a positive definite function on $\mathbb{Z}$ if, and only if, $c \geq \max \mathcal{B}_m(t) = B_m(\alpha_{m-1})$, which shows that $p_m(0) = B_m(\alpha_{m-1})$ is the minimal value that gives rise to a positive extension of $p_m(k) = (2\pi ik)^{-m}$ ($k \neq 0$) to $\mathbb{Z}$. Moreover, if $c < B_m(\alpha_{m-1})$, then there exist $N \in \mathbb{N}$, numbers $a_\nu \in \mathbb{C}$, and distinct numbers $\lambda_\nu \in \mathbb{Z}$ such that

$$(51) \qquad \sum_{\nu,\mu=1}^{N} a_\nu \overline{a}_\mu f(\lambda_\nu - \lambda_\mu) = \sum_{\nu,\mu=1}^{N} a_\nu \overline{a}_\mu p_{m,c}(\lambda_\nu - \lambda_\mu) < 0,$$

which shows that the value $f_m(0) = B_m(\alpha_{m-1})$ of Corollary 1 is optimal as well.

The statements about g_m and q_m follow similarly by considering

$$\psi_{n,\beta_n}(x) = H_n(x;\beta_n) - \operatorname{sgn}(x)x^n$$

instead of ϕ_{n,α_n}. □

Proof of Corollary 2. We will prove the corollary for $\delta = 1$. The general case follows by noting that $|\lambda_\nu - \lambda_\mu| \geq \delta > 0$ implies $|\lambda_\nu/\delta - \lambda_\mu/\delta| \geq 1$.

From Corollary 1 (i) we obtain that for any $N \in \mathbb{N}$, $a_\nu \in \mathbb{C}$, and $\lambda_\nu \in \mathbb{R}$,

$$\sum_{\nu,\mu=1}^{N} a_\nu \overline{a}_\mu f_m(\lambda_\nu - \lambda_\mu) \geq 0.$$

If we require additionally that $|\lambda_\nu - \lambda_\mu| \geq 1$ for all $\nu \neq \mu$, then after a multiplication by $m!^{-1}(2\pi)^m$ we obtain

$$\sum_{\substack{\nu,\mu=1 \\ \nu\neq\mu}}^{N} a_\nu \overline{a}_\mu (i(\lambda_\nu - \lambda_\mu))^{-m} \geq -f_m(0)\frac{(2\pi)^m}{m!}\sum_{\nu=1}^{N} |a_\nu|^2.$$

This shows that the function $(it)^{-m}$ satisfies (9) with $L_1((it)^{-m})$ as in Corollary 2. The optimality of $L_1((it)^{-m})$ follows from (51). (Note that the set of integers $\{\lambda_\nu\}$ used in (51) obviously satisfies $|\lambda_\nu - \lambda_\mu| \geq 1$ for all $\nu \neq \mu$.)

The validity of $U_1((it)^{-m})$ is verified in the same way using g_m and q_m. □

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