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UNITAL BIMODULES OVER THE SIMPLE JORDAN SUPERALGEBRA $D(t)$

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Abstract. We classify indecomposable finite dimensional bimodules over Jordan superalgebras $D(t)$, $t \neq -1, 0, 1$.

1. Introduction

Throughout this paper all algebras are considered over a ground field k of characteristic zero. A (linear) Jordan algebra is a vector space J with a binary bilinear operation $(x, y) \to xy$ satisfying the following identities:

$$xy = yx,$$
$$(x^2y)x = x^2(yx).$$

Respectively a Jordan superalgebra is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra $J = J_{\bar{0}} + J_{\bar{1}}$ satisfying the graded identities

$$xy = (-1)^{|x||y|}yx,$$
$$((xy)z)t + (-1)^{|y||z|+|y||t|+|z||t|}((xt)z)y + (-1)^{|x||y|+|x||z|+|x||t|+|z||t|}((yt)z)x$$
$$= (xy)(zt) + (-1)^{|y||z|}(xz)(yt) + (-1)^{|t|(|y|+|z|)}(xt)(yz).$$

In [K2] (see also I. L. Kantor [Ka1, Ka2]), V. Kac classified simple finite dimensional Jordan superalgebras over an algebraically closed field of zero characteristic. This classification included the 1-parametric family of 4-dimensional superalgebras D_t, which corresponds to the 1-parametric family of 17-dimensional Lie superalgebras via the Tits-Kantor-Koecher construction,

$$D_t = (ke_1 + ke_2) + (kx + ky),$$

with the products

$$e_i^2 = e_i, \quad e_1e_2 = 0, \quad e_ix = \frac{1}{2}x, \quad e_iy = \frac{1}{2}y, \quad xy = e_1 + te_2, \qquad i = 1, 2.$$

The superalgebra D_t is simple if $t \neq 0$. For $t = 0$ the superalgebra D_0 is a unital hull of the 3-dimensional nonunital Kaplansky superalgebra K_3, $D_0 = K_3 + k1$. In the case $t = -1$, the superalgebra D_{-1} is isomorphic to the Jordan superalgebra $M_{1,1}(k)^+$ of 2×2 matrices.

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A Jordan bimodule V over a Jordan (super)algebra J is a vector space with operations $V \times J \to V$, $J \times V \to V$ such that the split null extension $V + J$ is a Jordan (super)algebra (see [J1]).

We denote a Jordan triple product by $\{x, y, z\} = (xy)z + x(yz) - (-1)^{|x||y|}y(xz)$.

Let e be the identity of J and let $V = \{e, V, e\} + \{1 - e, V, e\} + \{1 - e, V, 1 - e\}$ be the Peirce decomposition. Then $\{e, V, e\}$ is a unital bimodule over J, that is, e is an identity of $\{e, V, e\} + J$. The component $\{1 - e, V, e\}$ is a one-sided module, that is, $\{J, \{1 - e, V, e\}, J\} = (0)$.

Finally, $\{1 - e, V, 1 - e\}$ is a bimodule with zero multiplication.

One-sided finite dimensional bimodules over D_t were classified in [MZ]. In this paper we classify finite dimensional unital bimodules, thus finishing the classification of finite dimensional bimodules over D_t. Bimodules over semisimple finite dimensional Jordan algebras have been completely classified by N. Jacobson (see [J1], [J3]).

2. Irreducible modules

For a set X let $V(X)$ denote the free D_t-bimodule on the set of free generators X (see [J1]). Consider the linear operator $R(a) : V(X) \to V(X)$, $v \to v \cdot a$, $v \in V(X)$, $a \in J$.

The algebra $M(J)$ generated by all operators $R(a)$, $a \in J$, is called the universal multiplicative enveloping algebra of J (see [J1]).

Assume that $t \neq -1$ and consider the following elements of $M(D_t)$:

$$E = \frac{2}{t+1}R(x)^2,\ F = \frac{-2}{t+1}R(y)^2,\ H = \frac{-2}{t+1}(R(x)R(y) + R(y)R(x)).$$

It is easy to see that $[F, H] = 2F$, $[E, H] = -2E$, $[E, F] = H$, $kE + kF + kH \simeq sl_2(k)$.

Note that $xH = -x$, $yH = y$.

Definition 2.1. For $\sigma \in \{\bar{0}, \bar{1}\}$, $i \in \{0, 1, \frac{1}{2}\}$, $\lambda \in k$, a *Verma module* $V(\sigma, i, \lambda)$ is defined as a unital D_t-bimodule presented by one generator v of parity σ and the relations $vR(e_1) = iv$, $vR(y) = 0$, $vH = \lambda v$.

Remark. $V(\sigma, i, \lambda)^{op} = V(1 - \sigma, i, \lambda)$.

Lemma 2.1. *For an arbitrary* $\lambda \in k$, $V(\sigma, \frac{1}{2}, \lambda) \neq (0)$.

Proof. Let $(k, +)$ be the additive group of the field k. We will denote elements of $(k, +)$ as u^μ, $\mu \in k$, $u^\mu u^\nu = u^{\mu+\nu}$. Consider the group algebra $\Lambda = k(k, +) = \{\sum_i \mu_i u^{\nu_i}; \mu_i, \nu_i \in k\}$ and the derivation $D : u^\mu \to \mu u^{\mu-1}$ of Λ.

Let $R = \langle \Lambda, D \rangle = \sum_{i \geq 0} \Lambda D^i$, $aD - Da = D(a)$, $a \in \Lambda$, be the Weyl algebra.

Consider the following 2×2 matrices over R:

$$x = \begin{pmatrix} 0 & -(t+1)D + (1-t)u^{-1} \\ -(t+1)D & 0 \end{pmatrix},\ y = \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that $\frac{1}{2}(xy - yx) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$, so e_1, e_2, x, y span a superalgebra which is isomorphic to D_t.

Consider the odd element $v = \begin{pmatrix} 0 & u^\lambda \\ u^\lambda & 0 \end{pmatrix}$. Then $v \cdot e_i = \frac{1}{2}v$ and $vy - yv = 0$. It is straightforward to verify that $vH = \lambda v$. This implies that $V(\bar{1}, \frac{1}{2}, \lambda) \neq (0)$. Consequently, the module $V(\bar{0}, \frac{1}{2}, \lambda) \neq (0)$. Lemma 2.1 is proved.

Lemma 2.2. *$V(\sigma, 1, \lambda) = (0)$ unless $\lambda = \frac{-2}{t+1}$.*

Proof. Let $U(x,y) = R(x)R(y) - R(y)R(x) - R([x,y])$. Clearly, $vU(x,y) = -\{x, v, y\} \in \{e_2, V, e_2\}$. Hence, $vU(x,y)R(e_1) = 0$. We have

$$\begin{aligned} vU(x,y)R(e_1) &= v(R(x)R(y) - R(y)R(x) - R(e_1 + te_2))R(e_1) \\ &= v(R(x)R(y) + R(y)R(x) - R(e_1 + te_2))R(e_1) \\ &= v(-\frac{t+1}{2}H - R(e_1 + te_2))R(e_1) \\ &= -\frac{t+1}{2}\lambda v - v = -(\lambda\frac{t+1}{2} + 1)v = 0, \end{aligned}$$

which implies $v = 0$ unless $\lambda = -\frac{2}{t+1}$. Lemma 2.2 is proved.

Lemma 2.3. *$V(\sigma, 0, \lambda) = 0$ unless $\lambda = \frac{-2t}{t+1}$.*

Proof. Arguing as above we get

$$\begin{aligned} vU(x,y)R(e_2) &= v(R(x)R(y) - R(y)R(x) - R(e_1 + te_2))R(e_2) \\ &= v(-\frac{t+1}{2}H - R(e_1 + te_2))R(e_2) = (-\frac{t+1}{2}\lambda - t)v = 0, \end{aligned}$$

which implies $v = 0$ unless $\lambda = -\frac{2t}{t+1}$. Lemma 2.3 is proved.

To establish that $V(\sigma, 1, -\frac{2}{t+1}) \neq (0)$, $V(\sigma, 0, -\frac{2t}{t+1}) \neq (0)$ we need to recall some facts about one-sided bimodules (see [MZ]).

A Jordan bimodule V over a Jordan (super)algebra J is said to be *one-sided* if $\{J, V, J\} = (0)$. If J is special with a universal associative enveloping algebra $U(J)$ (see [J1]), then a one-sided bimodule V can be viewed as a right $U(J)$-module. For $a \in J$, $v \in V$ we have $v \cdot a = \frac{1}{2}va$, where the left-hand side is the bimodule action, whereas the right-hand side is the right module action by $a \in U(J)$. Similarly, we can make V a left module over $U(J)$ via $\frac{1}{2}av = a \cdot v$.

The tensor product $V \otimes V$ is an associative bimodule over $U(J) : a(v \otimes w)b = av \otimes wb;\ a, b \in J;\ v, w \in V$.

Consider the elements $e = \frac{1}{4(1+t)}x^2$, $f = -\frac{1}{1+t}y^2$, $h = -\frac{1}{2(1+t)}(xy + yx)$ in $U(D_t)$. We have $[e, f] = h$, $[f, h] = 2f$, $[e, h] = -2e$.

In [MZ] the following one-sided Verma bimodules over D_t were introduced.[1]

The right module $V_1(t)$ over $U(D_t)$ is presented by one even generator v and the relations $ve_1 = v$, $vy^2 = 0$, $vh = (\frac{-1-2t}{1+t})v$.

The right module $V_2(t)$ over $U(D_t)$ is presented by one even generator v and the relations $ve_1 = v$, $vy = 0$, $vh = -\frac{1}{1+t}v$.

The tensor square $V_2(t) \otimes V_2(t)$ is a bimodule over $U(D_t)$, hence a Jordan bimodule over D_t via

$$(v' \otimes v'') \cdot a = \frac{1}{2}(v' \otimes v''a + (-1)^{|a|(|v'|+|v''|)}av' \otimes v'').$$

[1]Our notation differs from [MZ]. For example, we assume that $\frac{1}{2}(xy - yx) = e_1 + te_2$ in $U(D_t)$.

We have

$$(v \otimes v) \cdot e_1 = v \otimes v,\ (v \otimes v) \cdot y = 0,\ (v \otimes v)H = -\frac{2}{t+1}(v \otimes v).$$

We proved

Lemma 2.4. $V(\sigma, 1, -\frac{2}{t+1}) \neq (0)$.

Now consider the tensor square $V_1(t) \otimes V_1(t)$ and the element $vy \otimes vy$. We have $(vy \otimes vy) \cdot e_1 = 0$, $(vy \otimes vy) \cdot e_2 = vy \otimes vy$, $(vy \otimes vy) \cdot y = 0$, $(vy \otimes vy)H = -\frac{2t}{t+1} vy \otimes vy$.

We proved

Lemma 2.5. $V(\sigma, 0, -\frac{2t}{1+t}) \neq (0)$.

Lemma 2.6. *For an arbitrary $\lambda \in k$:*

(a) *The elements $vR(x)^j$, $vR(x)R(e_1)R(x)^j$, $j \geq 0$, form a basis of $V(\sigma, \frac{1}{2}, \lambda)$.*

(b) *The elements $vR(x)^j$, $j \geq 0$, form a basis of $V(\sigma, i, \lambda)$, $i = 0$ (and $\lambda = \frac{-2t}{1+t}$) or 1 (and $\lambda = \frac{-2}{1+t}$).*

Proof. (a) Let us show that the elements $vR(x)^j$, $vR(x)R(e_1)R(x)^j$, $j \geq 0$, span $V(\sigma, \frac{1}{2}, \lambda)$.

Let v be a highest weight vector in a Verma module $V(\sigma, \frac{1}{2}, \lambda)$. Consider an operator $W = R(a_1) \cdots R(a_s)$, where $a_i = e_1$ or x or y, $1 \leq i \leq s$.

We will use an induction on s and assume that operators of length $< s$ map v into a linear combination of suggested elements.

If $a_1 = e_1$, then $vW = \frac{1}{2} vR(a_2) \cdots R(a_s)$. If $a_1 = y$, then $vW = 0$. Let $a_1 = x$. If $a_2 = y$, then

$$vW = v(R(x)R(y) + R(y)R(x))R(a_2) \cdots R(a_s) = -\frac{1+t}{2} \lambda v R(a_2) \cdots R(a_s).$$

If $a_j = y$, where $j \geq 3$, then we use the Jordan identity to move y to the left modulo shorter operators. Thus we can assume that $a_j = e_1$ or x, $1 \leq j \leq s$.

Let j be the minimal index such that $a_j = e_1$. We see that $j \geq 2$. If $j \geq 3$, then we can again use the Jordan identity to move e_1 to the left. Hence, either e_1 does not occur in W or $j = 2$. Let $W = R(x)R(e_1)R(a_3) \cdots R(a_s)$. Let $j' \geq 3$ be the minimal index such that $a_{j'} = e_1$. If $j' = 3$, then $vR(x)R(e_1)R(e_1) = vR(x)R(e_1)$.

If $j' = 4$, then $vR(x)R(e_1)R(x)R(e_1) = \frac{1}{2} vR(x)R(e_1)R(x)$. If $j' \geq 5$, then we again use the Jordan identity to decrease j'. We proved that $V(\sigma, \frac{1}{2}, \lambda)$ is spanned by $vR(x)^j$, $vR(x)R(e_1)R(x)^{j-1}$. Now, it remains to prove that these elements are linearly independent.

We have

$$vR(x)^{2i}R(y) = \frac{1+t}{2} i vR(x)^{2i-1},\ i \geq 1;$$

$$vR(x)^{2i+1}R(y) = -\frac{1+t}{2}(\lambda - i) vR(x)^{2i};\ vR(x)^{2i}R(e_1) = \frac{1}{2} vR(x)^{2i};$$

$$vR(x)^{2i+1}R(e_1) = vR(x)R(e_1)R(x)^{2i};$$

$$vR(x)R(e_1)R(x)^{2i}R(y) = -\frac{(t+1)(\lambda-1)+2}{4} vR(x)^{2i} + \frac{1+t}{2} i vR(x)R(e_1)R(x)^{2i-1},\quad i \geq 1;$$

$$vR(x)R(e_1)R(y) = -\frac{(t+1)(\lambda-1)+2}{4} v;$$

$$vR(x)R(e_1)R(x)^{2i+1}R(y) = \frac{(t+1)(\lambda-1)+2}{4}vR(x)^{2i+1}$$
$$-\frac{1+t}{2}(\lambda-i-1)vR(x)R(e_1)R(x)^{2i};$$
$$vR(x)R(e_1)R(x)^{2i}R(e_1) = vR(x)R(e_1)R(x)^{2i};$$
$$vR(x)R(e_1)R(x)^{2i+1}R(e_1) = \frac{1}{2}vR(x)R(e_1)R(x)^{2i+1}.$$

Let us first check linear independence under the assumption that $\lambda \notin \mathbb{Z}_{\geq 0}$, $\lambda \neq \frac{t-1}{t+1}$, $\lambda \neq \frac{1-t}{t+1}$. We need to verify that $vR(x)R(e_1)R(x)^i \neq 0$, $vR(x)R(e_2)R(x)^i \neq 0$ for all $i \geq 0$.

From

$$vR(x)R(e_1)R(y) = -\frac{(t+1)(m-1)+2}{4}v \neq 0,$$
$$vR(x)R(e_2)R(y) = -\frac{(t+1)(m+1)-2}{4}v \neq 0$$

it follows that $vR(x)R(e_1) \neq 0$, $vR(x)R(e_2) \neq 0$. Now suppose that

$$vR(x)R(e_1)R(x)^i \neq 0, vR(x)R(e_2)R(x)^i \neq 0 \quad \text{for } i \leq k.$$

Let k be even. Then

$$\begin{aligned} vR(x)R(e_1)R(x)^{k+1}R(y) &= \frac{(t+1)(\lambda-1)+2}{4}vR(x)^{k+1} \\ &\quad - \frac{1+t}{2}(\lambda - \frac{k+2}{2})vR(x)R(e_1)R(x)^k \\ &= (\frac{(t+1)(\lambda-1)+2}{4} - \frac{1+t}{2}(\lambda - \frac{k+2}{2}))vR(x)R(e_1)R(x)^k \\ &\quad + \frac{(t+1)(\lambda-1)+2}{4}vR(x)R(e_2)R(x)^k \neq 0. \end{aligned}$$

The case when k is odd can be treated similarly, and similarly

$$vR(x)R(e_2)R(x)^{k+1}R(y) \neq 0.$$

We have proved assertion (a) for $\lambda \notin \mathbb{Z}_{\geq 0}$, $\lambda \neq \frac{t-1}{t+1}$, $\lambda \neq \frac{1-t}{t+1}$.

Now consider a vector space $\tilde{V}$ with a basis $v_i,\ i \in \mathbb{Z}_{\geq 0}$; $v'_j,\ j \in \mathbb{Z}_{>0}$, and define a D_t-bimodule structure via:

$$v_iR(x) = v_{i+1}; v_0R(y) = 0; v_{2i}R(y) = \frac{1+t}{2}iv_{2i-1}, i \geq 1;$$
$$v_{2i+1}R(y) = -\frac{1+t}{2}(\lambda - i)v_{2i}; v_{2i}R(e_1) = \frac{1}{2}v_i, v_{2i+1}R(e_1) = v'_{2i+1};$$
$$v'_jR(x) = v'_{j+1}; v'_{2i+1}R(y) = -\frac{(t+1)(\lambda-1)+2}{4}v_{2i} + \frac{1+t}{2}iv'_{2i}, i \geq 1;$$
$$\begin{aligned} v'_1R(y) &= -\frac{(t+1)(\lambda-1)+2}{4}v_0; v'_{2i}R(y) \\ &= -\frac{1+t}{2}(\lambda - i)v'_{2i-1} + \frac{(t+1)(\lambda-1)+2}{4}v_{2i-1}, i \geq 1; \end{aligned}$$
$$v'_{2i}R(e_1) = \frac{1}{2}v'_{2i}; v'_{2i+1}R(e_1) = v'_{2i+1}.$$

For a fixed i the equalities

$$w(R(a)R(b)R(c) + (-1)^{|a||b|+|a||c|+|b||c|}R(c)R(b)R(a) + (-1)^{|b||c|}R((ac)b) - R(ab)R(c) - (-1)^{|b||c|}R(ac)R(b) - (-1)^{|a|(|b|+|c|)}R(bc)R(a)) = 0,$$

where $w = v_i$ or v_i'; $a, b, c = x$ or y or e_1 amount to a bunch of at most quadratic equalities involving λ. Since all these equalities hold for all $\lambda \notin \mathbb{Z}_{\geq 0}$, $\lambda \neq \frac{t-1}{t+1}$, $\lambda \neq \frac{1-t}{t+1}$, it follows that these equalities hold for all λ. Hence for all λ, $\tilde{V}$ is a Jordan bimodule over D_t with a highest weight element v_0 and the highest weight λ. This implies assertion (a) of the lemma.

Now consider the bimodules $V(\sigma, i, \lambda)$, $i = 0$ or 1. Arguing as above we can prove that $V(\sigma, i, \lambda)$ is spanned by $vR(x)^i$, $i \geq 0$. To show that the elements $vR(x)^i$ are all nonzero, we can use the embedding of $V(\sigma, i, \lambda)$ into the tensor product of one-sided Verma modules as in the proofs of Lemmas 2.4 and 2.5. Lemma 2.6 is proved.

Corollary 2.1. *Every nonzero Verma bimodule $V(\sigma, i, \lambda)$ contains a largest proper sub-bimodule $M(\sigma, i, \lambda)$. Hence there exists a unique irreducible D_t-bimodule $Irr(\sigma, i, \lambda) = V(\sigma, i, \lambda)/M(\sigma, i, \lambda)$ generated by an element of the highest weight λ.*

Lemma 2.7. *Every finite dimensional irreducible D_t-bimodule is isomorphic to $Irr(\sigma, i, \lambda)$ for some σ, i, λ.*

Proof. Let V be a finite dimensional irreducible D_t-bimodule. Then V is a module over the Lie algebra $sl_2(k) = kE + kF + kH$. From the representation theory of $sl_2(k)$ (see [J2]) it follows that the action of H on V is diagonalizable, $V = \sum_\gamma V_\gamma$ is the sum of eigenspaces. Choose an eigenvalue λ such that $V_\lambda \neq (0)$, $V_{\lambda+1} = (0)$.

Let $0 \neq v \in V_{\lambda,\sigma}$, $\sigma = \bar{0}$ or $\bar{1}$. Consider a Peirce decomposition, $v = v_0 + v_1 + v_{\frac{1}{2}}$. Clearly $v_i \in V_\lambda$, $i = 0$ or 1 or $\frac{1}{2}$, and therefore $v_i y = 0$. If $v_i \neq 0$, then v_i generates the bimodule V, which implies $V \simeq Irr(\sigma, i, \lambda)$. Lemma 2.7 is proved.

Lemma 2.8. *Suppose that $V(\sigma, i, \lambda) \neq (0)$. If dim $Irr(\sigma, i, \lambda) < \infty$, then $\lambda \in \mathbb{Z}_{\geq 0}$. If $i = 0$ or 1 and $\lambda \in \mathbb{Z}_{\geq 0}$, then $dimIrr(\sigma, i, \lambda) < \infty$. For $t \neq \pm 1$ the bimodule $V(\sigma, \frac{1}{2}, 0)$ is infinite dimensional and irreducible.*

Proof. From the representation theory of $sl_2(k)$ it follows that $\dim Irr(\sigma, i, \lambda) < \infty$ implies $\lambda \in \mathbb{Z}_{\geq 0}$.

Let $m \in \mathbb{Z}_{\geq 0}$, $i = 0$ or 1, or $m \in \mathbb{Z}_{>0}$, $i = \frac{1}{2}$. Let us show that $vR(x)^{2m+1}$ generates a proper sub-bimodule V' of $V(\sigma, i, m)$. We have

$$vR(x)^{2m+1}R(y) = vR(x)^{2m}(R(x)R(y) + R(y)R(x)) - vR(x)^{2m}R(y)R(x);$$

$$vR(x)^{2m}(R(x)R(y) + R(y)R(x)) = vR(x)^{2m}(-\frac{1+t}{2}H)$$

$$= -\frac{1+t}{2}(m - 2m)vR(x)^{2m} = m\frac{1+t}{2}vR(x)^{2m};$$

$$[R(x)^2, R(y)] = \frac{1+t}{2}R(x);$$

$$vR(x)^{2m}R(y) = \sum_{j=0}^{m-1} vR(x)^{2(m-j-1)}[R(x)^2, R(y)]R(x)^{2j} = m\frac{1+t}{2}vR(x)^{2m-1}.$$

This proves that $vR(x)^{2m+1}R(y) = 0$.

If $i = 0$ or 1, then $vR(x)^{2m+1}$ belongs to the $\frac{1}{2}$-Peirce component. By Lemma 2.6 the sub-bimodule V' is spanned by

$$vR(x)^{2m+1}R(x)^j,\ vR(x)^{2m+1}R(x)R(e_1)R(x)^j, \quad j \geq 0.$$

Since all these elements belong to eigenvalues $\leq -(m+1)$ with respect to H, we conclude that $v \notin V'$.

Let $i = \frac{1}{2}$. The element $vR(x)^{2m+1}R(e_1)R(y)$ belongs to the $\frac{1}{2}$-Peirce component and $vR(x)^{2m+1}R(e_1)R(y)R(y) = 0$. By Lemma 2.6, the sub-bimodule V_1' generated by $vR(x)^{2m+1}R(e_1)R(y)$ is spanned by $vR(x)^{2m+1}R(e_1)R(y)R(x)^j$, $vR(x)^{2m+1}R(e_1)R(y)R(x)R(e_1)R(x)^j$, $j \geq 0$.

Similarly, the sub-bimodule V_2' generated by $vR(x)^{2m+1}R(e_2)R(y)$ is spanned by $vR(x)^{2m+1}R(e_2)R(y)R(x)^j$, $vR(x)^{2m+1}R(e_2)R(y)R(x)R(e_1)R(x)^j$, $j \geq 0$.

The element $vR(x)^{2m+1}R(e_1)$ lies in the 1-Peirce component and

$$vR(x)^{2m+1}R(e_1)R(y) \equiv 0 \bmod V_1'.$$

By Lemma 2.6 the sub-bimodule generated by $vR(x)^{2m+1}R(e_1)$ is spanned by

$$vR(x)^{2m+1}R(e_1)R(x)^j, j \geq 0, \bmod V_1'.$$

The sub-bimodule generated by $vR(x)^{2m+1}R(e_2)$ is spanned by

$$vR(x)^{2m+1}R(e_2)R(x)^j, j \geq 0, \bmod V_2'.$$

Finally, we conclude that V' is spanned by

$$\begin{gathered} vR(x)^{2m+1}R(e_k)R(y)R(x)^j, \\ vR(x)^{2m+1}R(e_k)R(x)R(e_1)R(x)^j, \\ vR(x)^{2m+1}R(e_k)R(x)^j, \quad j \geq 0,\ k = 1 \text{ or } 2. \end{gathered}$$

If $m \geq 1$, then all the elements above have weights $< m$. Hence V' is proper.

It is easy to see that the bimodule $W(\sigma, i, m) = V(\sigma, i, m)/V'$ is finite dimensional. It remains to show that the Verma bimodule $V(\sigma, \frac{1}{2}, 0)$ is infinite dimensional and irreducible.

We have

$$\begin{aligned} vR(x)R(y) &= v(R(x)R(y) + R(y)R(x)) = -\frac{t+1}{2}vH = 0; \\ vR(x)R(e_1)R(y) &= v(-R([x,y]\cdot e_1) + R(x \cdot e_1)R(y) - R(y \cdot e_1)R(x) \\ &\quad +R([x,y])R(e_1)) = \frac{t-1}{4}v; \\ vR(x)R(e_2)R(y) &= \frac{1-t}{4}v. \end{aligned}$$

The Verma module over the Lie algebra $sl_2(k)$ with maximal eigenvalue -1 is irreducible and infinite dimensional (see [J2]).

If $t \neq 1$, then $vR(x) \neq 0$ and $vR(x)R(e_1) \neq 0$. Similarly, $vR(x)R(e_2) \neq 0$.

Both elements belong to the eigenvalue -1 with respect to H and

$$vR(x)R(y)^2 = vR(x)R(e_1)R(y)^2 = 0.$$

Hence $\sum_{j=0}^{\infty} kvR(x)^{2j+1}$, $\sum_{j=0}^{\infty} kvR(x)R(e_1)R(x)^{2j}$, $\sum_{j=0}^{\infty} kvR(x)R(e_2)R(x)^{2j}$ are infinite dimensional irreducible $sl_2(k)$-modules. In particular, the module V is infinite dimensional.

Let V' be a proper nonzero sub-bimodule of $V = V(\sigma, \frac{1}{2}, 0)$. Then $\alpha vR(x)^h + \beta vR(x)R(e_1)R(x)^{h-1} \in V'$ for some $h \geq 1$; $\alpha, \beta \in k$; $(\alpha, \beta) \neq (0,0)$.

Applying $R(x)$ if necessary, we will assume that h is odd. Then

$$vR(x)R(e_1)R(x)^{h-1}R(e_2) = 0$$

and therefore $\alpha vR(x)R(e_2)R(x)^{h-1} \in V'$.

Suppose that $\alpha \neq 0$. Then $\sum_{j=0}^{\infty} kvR(x)R(e_2)R(x)^{2j} \subset V'$, $vR(x)R(e_2) \in V'$ and, finally, $vR(x)R(e_2)R(y) = \frac{1-t}{4}v \in V'$, a contradiction.

Hence $\alpha = 0$, hence $vR(x)R(e_1)R(x)^{h-1} \in V'$. Arguing as above we get $\sum_{j=0}^{\infty} kvR(x)R(e_1)R(x)^{2j} \subseteq V'$, $vR(x)R(e_1) \in V'$, $vR(x)R(e_1)R(y) = \frac{t-1}{4}v$. Lemma 2.8 is proved.

Remark. In the same way we can prove that if $V(\sigma, i, \lambda) \neq (0)$ and $Irr(\sigma, i, \lambda)$ is infinite dimensional, then $V(\sigma, i, \lambda)$ is irreducible.

Remark. For $t = 1$, $\sum_{j=0}^{\infty} kvR(x)^j + \sum_{j=0}^{\infty} kvR(x)R(e_1)R(x)^j$ is a proper sub-bimodule of $V(\sigma, \frac{1}{2}, 0)$. Hence, dim $Irr(\sigma, \frac{1}{2}, 0) = 1$.

Theorem 2.1. *If $t \neq -1$ is not of the type $-\frac{m}{m+2}$, $m \geq 0$; $-\frac{m+2}{m}$, $m \geq 1$; or 1, then the only unital finite dimensional irreducible D_t-bimodules are*

$$(*) \qquad Irr(\sigma, \frac{1}{2}, m),\ m \geq 1.$$

If $t = 1$, then add the one-dimensional bimodules $Irr(\sigma, \frac{1}{2}, 0)$, $\sigma = \bar{0}, \bar{1}$ to the series $(*)$.

If $t = -\frac{m+2}{m}$, $m \geq 1$, then add the bimodules $V(\sigma, 1, m)$, $\sigma = \bar{0}, \bar{1}$ to $(*)$.

If $t = -\frac{m}{m+2}$, $m \geq 0$, then add the bimodules $V(\sigma, 0, m)$, $\sigma = \bar{0}, \bar{1}$ to $(*)$.

Let $m \in \mathbb{Z}_{>0}$. As in the proof of Lemma 2.8, let V' denote the sub-bimodule of $V(\sigma, i, m)$ generated by $vR(x)^{2m+1}$. We proved that the quotient module $W(\sigma, i, m) = V(\sigma, i, m)/V'$ is finite dimensional

Lemma 2.9. *$W(\sigma, i, m)$ is the largest finite dimensional homomorphic image of $V(\sigma, i, m)$.*

Proof. Let $\tilde{V}$ be a sub-bimodule of $V(\sigma, i, m)$ such that $\dim V(\sigma, i, m)/\tilde{V} < \infty$. From the representation theory of $sl_2(k)$ it follows that $vR(x)^{2(m+1)} \in \tilde{V}$.

Now $vR(x)^{2(m+1)}R(y) = (m+1)\frac{1+t}{2}vR(x)^{2m+1}$. Hence $vR(x)^{2m+1} \in \tilde{V}$ and therefore $V' \subseteq \tilde{V}$. Lemma 2.9 is proved.

Lemma 2.10. *Let V be a unital D_t-bimodule, $t \neq 0, 1$.*

(a) *If $V_{\bar{0}} = (0)$, then $V = (0)$.*

(b) *Let R be the subalgebra of $End(V)$ generated by all multiplications $R(a)$, $a \in D_t$. Clearly, $R = R_{\bar{0}} + R_{\bar{1}}$, $V_{\bar{i}}R_{\bar{j}} \subseteq V_{\overline{i+j}}$. If $V_{\bar{0}}$ is an irreducible module over $R_{\bar{0}}$, then V is an irreducible D_t-bimodule.*

Proof. (a) If $V_{\bar{0}} = (0)$, then $V = V_{\bar{1}}$ and $Vx = 0 = Vy$.

Since $t \neq 0$, then e_1 and e_2 play a symmetric role and we can assume that $V = \{e_1, V, e_2\}$ or $V = \{e_1, V, e_1\}$.

If $V = \{e_1, V, e_2\}$, then for an arbitrary $v \in V$ we have

$$v(R(x)R(e_1)R(y) - R(y)R(e_1)R(x) + R([x,y]e_1) - R(xe_1)R(y)$$
$$+R(ye_1)R(x) - R([x,y])R(e_1)) = 0.$$

Hence $ve_1 - (v(e_1 + te_2))e_1 = 0$, that is, $\frac{1}{2}v - \frac{1}{4}(1+t)v = 0$. This implies that $\frac{1-t}{4}v = 0$ and then $v = 0$ since $t \neq 1$.

If $V = \{e_1, V, e_1\}$ and $v \in V$, then

$$v(R(x)R(y)R(e_1) - R(e_2)R(y)R(x) + R([xe_2, y]) - R(xe_2)R(y) \\ +R(ye_2)R(x) - R([x,y])R(e_2)) = 0.$$

Then $v(\frac{1}{2}(e_1 + te_2)) - (v(e_1 + te_2))e_2 = \frac{1}{2}v = 0$, that is, $v = 0$.

(b) Let V' be a nonzero D_t-sub-bimodule of V. By (a) $V'_{\bar{0}} \neq (0)$. Since $V'_{\bar{0}}$ is a module over $R_{\bar{0}}$, it follows that $V'_{\bar{0}} = V_{\bar{0}}$. Now $(V/V')_{\bar{0}} = (0)$. By (a) $V = V'$. Lemma 2.10 is proved.

Let $m \geq 1$, $W = W(1, \frac{1}{2}, m)$. We have

$$vR(x)R(e_1)R(y) = -\frac{(t+1)(m-1)+2}{4}v,$$
$$vR(x)R(e_2)R(y) = -\frac{(t+1)(m+1)-2}{4}v.$$

If $t \neq -\frac{m+1}{m-1}$, $t \neq -\frac{m-1}{m+1}$, then $vR(x)R(e_1) \neq 0$, $vR(x)R(e_2) \neq 0$ in W. In this case, the even part $W_{\bar{0}}$ is a direct sum of two $sl_2(k)$-modules generated by $vR(x)R(e_1)$, $vR(x)R(e_2)$. Both elements belong to the eigenvalue $m-1$ with respect to H, hence $\dim W_{\bar{0}} = 2m$.

Let $\xi = -\frac{(1+t)(m-1)+2}{2m(1+t)}$. Then $(\xi vR(x)^2 + vR(x)R(e_1)R(x))R(y)^2 = 0$.

In this case $W_{\bar{1}}$ is a direct sum of two irreducible $sl_2(k)$-modules generated by the elements v and $\xi vR(x)^2 + vR(x)R(e_1)R(x)$ respectively. Hence $dimW_{\bar{1}} = (m+1) + (m-1) = 2m$.

It is easy to see that in this case the module W is irreducible,

$$W = Irr(1, \frac{1}{2}, m).$$

If $t = -\frac{m+1}{m-1}$, then $vR(x)R(e_1)R(y) = 0$. We remark though that the element $vR(x)R(e_1)$ is not equal to zero in W. Indeed, it follows from Lemma 2.6(a) that $vR(x)R(e_1) \neq 0$ in $V(\sigma, \frac{1}{2}, m)$. Now it remains to notice that all eigenvalues of the operator H that occur in V' are smaller than $m-1$.

Hence the element $vR(x)R(e_1)$ generates a proper sub-bimodule W' of W. The even part $W'_{\bar{0}}$ is the irreducible $sl_2(k)$-module, hence W' is irreducible, $W' \simeq Irr(0, 1, m-1)$.

The even part of the quotient W/W' is an irreducible $sl_2(k)$-module of dimension m. Hence W/W' is irreducible, $W/W' \simeq Irr(1, \frac{1}{2}, m)$. The odd part $(W/W')_{\bar{1}}$ is also an irreducible $sl_2(k)$-module generated by v, hence $dim(W/W') = m+1$.

If $t = -\frac{m-1}{m+1}$, then

$$0 \longrightarrow Irr(0,0,m-1) \longrightarrow W \longrightarrow Irr(1, \frac{1}{2}, m) \longrightarrow 0$$

is an exact sequence and as above $dimIrr(1, \frac{1}{2}, m) = 2m+1$.

For $t = -\frac{m+2}{m}$, $m \geq 1$, we have $W(1,1,m) \simeq Irr(1,1,m)$; both the even and the odd parts are irreducible $sl_2(k)$-modules of dimensions m and $m+1$ respectively.

For $t = -\frac{m}{m+2}$, $m \geq 0$, $W(1,0,m) \simeq Irr(1,0,m)$ and $Irr(1,0,m)_{\bar{0}}$, $Irr(1,0,m)_{\bar{1}}$ are again irreducible $sl_2(k)$-modules of dimensions m and $m+1$ respectively.

Corollary 2.2. *The only finite dimensional irreducible bimodules of the (nonunital) Kaplansky superalgebra K_3 are $Irr(\sigma, \frac{1}{2}, m)$, $m \geq 1$, and $Irr(\sigma, 0, 0)$. We have $dimIrr(\sigma, \frac{1}{2}, m) = 4m$ if $m \geq 2$, $dimIrr(\sigma, \frac{1}{2}, 1) = 3$, $dimIrr(\sigma, 0, 0) = 1$.*

Proof. The unital hull of K_3 is D_t, where $t = 0$. Every bimodule over K_3 has a structure of a unital bimodule over the unital hull of K_3. Now it remains to apply Theorem 2.1.

3. Indecomposable modules

Lemma 3.1. *Let $W \subseteq \{e_i, V_{\bar{0}}, e_i\}$, $i = 1$ or 2, be a module over $sl_2(k) = kE + kF + kH$. Then the D_t-sub-bimodule of V generated by W is $\tilde{W} = W + WU(x,y) + WR(x) + WR(y)$.*

Proof. It is straightforward that W, $WU(x,y)$, $WR(x) + WR(y)$ belong to 1, 0, $\frac{1}{2}$-Peirce components respectively. Hence we need only to verify that $\tilde{W}R(x)$, $\tilde{W}R(y)$ lie in $\tilde{W}$.

We have

$$R(x)R(y) = \frac{1}{2}(U(x,y) - \frac{1+t}{2}H - R(e_1 + te_2));$$
$$R(y)R(x) = \frac{1}{2}(-U(x,y) - \frac{1+t}{2}H + R(e_1 + te_2));$$
$$\begin{aligned} U(x,y)R(x) &= (R(x)R(y) - R(y)R(x) - R(e_1+te_2))R(x) = R(x)R(y)R(x) \\ &\quad -R(y)R(x)^2 - R(e_1+te_2)R(x) = (R(x)R(y) + R(y)R(x))R(x) \\ &\quad -2R(x)^2R(y) - 2R(yR(x)^2) - R(e_1+te_2)R(x) \\ &= -\frac{t+1}{2}HR(x) - (t+1)ER(y) + (t+1)R(x) - R(e_1+te_2)R(x), \end{aligned}$$

which implies that $WU(x,y)R(x) \subseteq \tilde{W}$.

Similarly, $WU(x,y)R(y) \subseteq \tilde{W}$. Lemma 3.1 is proved.

The operator $U(x,y)$ commutes with E, F, H. Let $W \subseteq \{e_1, V_{\bar{0}}, e_1\}$ be an irreducible $sl_2(k)$-module. Then the restriction of $U(x,y)$ to W is an isomorphism $W \to WU(x,y)$, $WU(x,y) \subseteq \{e_2, V_{\bar{0}}, e_2\}$, or a zero mapping. By Schur's Lemma $U(x,y)^2_{|W} = \alpha Id_W$, $\alpha \in k$.

Let v be a highest weight vector of W, $vR(y)^2 = 0$, $vH = mv$, $m \in \mathbb{Z}_{\geq 0}$.

Lemma 3.2. *If $WU(x,y)^2 \neq (0)$, then $\tilde{W}$ is an irreducible D_t-bimodule.*

Proof. We showed above that $W + WU(x,y)$ is a direct sum of two isomorphic irreducible $sl_2(k)$-modules. Let W' be a nonzero $R_{\bar{0}}$-submodule of $W + WU(x,y)$, $w_1 + w_2U(x,y) \in W'$, $w_1, w_2 \in W$. Clearly, $w_1 = (w_1 + w_2U(x,y))R(e_1) \in W'$, $w_2U(x,y) = (w_1 + w_2U(x,y))R(e_2) \in W'$. If $w_1 \neq 0$, then $W \subseteq W'$ and $WU(x,y) \subseteq W'$, hence $W' = W + WU(x,y) \neq 0$. If $w_2U(x,y) \neq 0$, then $0 \neq w_2U(x,y)^2 \in W'$ and we argue as above.

We proved that $W + WU(x,y)$ is an irreducible $R_{\bar{0}}$-module. By Lemma 2.10(b) the bimodule $\tilde{W}$ is irreducible. Lemma 3.2 is proved.

Similarly, if $W \subseteq \{e_2, V_{\bar{0}}, e_2\}$ is an irreducible $sl_2(k)$-module and $WU(x,y)^2 \neq (0)$, then $\tilde{W}$ is an irreducible D_t-bimodule.

Lemma 3.3. *If $W \subseteq \{e_1, V_{\bar{0}}, e_1\}$ is an irreducible $sl_2(k)$-module of highest weight m and $WU(x,y)^2 = (0)$, then $t = -\frac{m}{m+2}$ or $-\frac{m+2}{m}$, $m \in \mathbb{Z}_{>0}$.*

Proof. Let $w \in W$ be a vector of maximal weight, $wF = 0$, $wH = mw$. Hence $w(R(x) \cdot R(y)) = -\alpha m w$, with $\alpha = \frac{1+t}{2}$.

Then

$$\begin{aligned} wU(x,y)^2 &= w(2R(x)R(y) - R(x)\cdot R(y) - R(e_1+te_2))(R(x).R(y) \\ &\qquad - 2R(y)R(x) - R(e_1+te_2)) \\ &= w(2R(x)R(y) + \alpha m - 1)(-\alpha m - 2R(xy)R(x) - t) \\ &= -4\alpha wR(y)R(x) - 2(\alpha m - 1)wR(y)R(x) \\ &\qquad - 2(\alpha m + t)wR(x)R(y) - (\alpha m + t)(\alpha m - 1)w \\ &= (\alpha m + t)(\alpha m + 1)w. \end{aligned}$$

So $wU(x,y)^2 = 0$, implies that $\alpha m + t = 0$ or $\alpha m + 1 = 0$, that is, $t = -\frac{m}{m+2}$ or $t = -\frac{m+2}{m}$.

Definition 3.1. An element v of a unital D_t-bimodule is said to be a highest weight element if $vR(y) = 0$, $vH = \lambda v$ for some $\lambda \in k$ and v lies in some Peirce component with respect to e_1, e_2.

Lemma 3.4. *An arbitrary finite dimensional unital D_t-bimodule, $t \neq 0, 1$, is generated by its highest weight elements.*

Proof. Let V be a nonzero unital D_t-bimodule, $t \neq 0, 1$. Let $\tilde{V}$ be a sub-bimodule generated by all highest weight elements of V. Let $W \subseteq \{e_1, V_{\bar{0}}, e_1\}$ be an irreducible $sl_2(k)$-submodule with a highest weight element v, that is, $vH = mv$, $m \in \mathbb{Z}_{\geq 0}$, $vR(y)^2 = 0$, v generates W.

If $WU(x,y)^2 \neq (0)$, then by Lemma 3.2 the bimodule $\tilde{W} = W + WU(x,y) + WR(x) + WR(y)$ is irreducible, hence $\tilde{W} \subseteq \tilde{V}$.

Suppose that $WU(x,y)^2 = (0)$ and therefore $t = -\frac{m}{m+2}$ or $t = -\frac{m+2}{m}$.

Let $v' = vU(x,y)$. Then $v'H = mv'$, $v'R(y)^2 = 0$ and $v'U(x,y) = 0$. We have $v'R(y) \in \tilde{V}$. Now,

$$v'R(y)R(x) = \frac{1}{2}v'(-U(x,y) - \frac{1+t}{2}H + R(e_1+te_2)) = \frac{1}{2}(-\frac{1+t}{2}m + t)v'.$$

The element v' lies in $\tilde{V}$ unless $-\frac{1+t}{2}m + t = 0$, which is equivalent to $t = -\frac{m}{m-2}$. The latter contradicts our assumption that $t = -(\frac{m+2}{m})^{\pm 1}$. We proved that $v' \in \tilde{V}$.

The element $vR(y)$ also lies in $\tilde{V}$. We have

$$vR(y)R(x) = \frac{1}{2}v(-U(x,y) - \frac{1+t}{2}H + R(e_1+te_2)) = \frac{1}{2}(-\frac{1+t}{2}m + 1)v$$

mod $\tilde{V}$.

Hence $v \in \tilde{V}$ unless $-\frac{1+t}{2}m + 1 = 0$, which is equivalent to $t = -\frac{m-2}{m} \neq -(\frac{m+2}{m})^{\pm 1}$. Hence $v \in \tilde{V}$.

We proved that $\{e_1, V_{\bar{0}}, e_1\} \subseteq \tilde{V}$. Similarly, $\{e_2, V_{\bar{0}}, e_2\} \subseteq \tilde{V}$. The even part of the bimodule $\{e_1, V_{\bar{0}}, e_1\} + \{e_2, V_{\bar{0}}, e_2\} + \{e_1, V_{\bar{1}}, e_2\}$ lies in $\tilde{V}$. By Lemma 2.10(a) $\{e_1, V_{\bar{1}}, e_2\} \subseteq \tilde{V}$. Passing to opposites, we get

$$\{e_1, V_{\bar{1}}, e_1\} + \{e_2, V_{\bar{1}}, e_2\} + \{e_1, V_{\bar{0}}, e_2\} \subseteq \tilde{V}.$$

Hence $\tilde{V} = V$. Lemma 3.4 is proved.

Theorem 3.1. *Suppose that t is not of the type $-\frac{m}{m+2},\ -\frac{m+2}{m},\ 0,\ 1,\ m \in \mathbb{Z}_{>0}$. Then every finite dimensional unital bimodule V over D_t is completely reducible.*

Proof. An arbitrary finite dimensional highest weight bimodule over D_t is a homomorphic image of some bimodule $W(\sigma, \frac{1}{2}, m)$, which was shown to be irreducible. Hence V is a sum of irreducible sub-bimodules. Theorem 3.1 is proved.

Theorem 3.2. *If $t = -\frac{m+1}{m-1}$ or $t = -\frac{m-1}{m+1}$, $m \geq 2$, then $W(\sigma, \frac{1}{2}, m)$, $\sigma = \bar{0}$ or $\bar{1}$, are the only finite dimensional indecomposible D_t-bimodules, which are not irreducible.*

Proof. Let $t = -\frac{m+1}{m-1}$, $m \geq 2$. We have proved that

$$(0) \longrightarrow Irr(0,1,m-1) \longrightarrow W(1,\frac{1}{2},m) \longrightarrow Irr(1,\frac{1}{2},m) \longrightarrow (0)$$

is an exact sequence. Let us show that $W(1, \frac{1}{2}, m)$ is not isomorphic to $Irr(0, 1, m-1) \oplus Irr(1, \frac{1}{2}, m)$. Indeed, in both bimodules the eigenspaces that correspond to the eigenvalue m are one-dimensional. However, in $W(1, \frac{1}{2}, m)$ this eigenspace is not killed by $R(x)R(e_1)$, whereas in $Irr(0, 1, m-1) \oplus Irr(1, \frac{1}{2}, m)$ it is killed by $R(x)R(e_1)$. Hence $W(1, \frac{1}{2}, m)$ is indecomposable. Similarly, $W(0, \frac{1}{2}, m)$ is indecomposable.

Now let V be an indecomposable D_t-bimodule. By Lemma 3.4 V is a sum of highest weight bimodules, $V = \sum_{i=1}^{s} V_i$. We showed above that all these bimodules V_i are either irreducible or isomorphic to $W(\sigma, \frac{1}{2}, m)$.

If at least one bimodule, say V_s, is irreducible, then either $V = (\sum_{i=0}^{s-1})V_i \oplus V_s$, which contradicts indecomposability of V or $V = \sum_{i=1}^{s-1} V_i$.

Suppose therefore that all summands are of the types $W(0, \frac{1}{2}, m)$, $W(1, \frac{1}{2}, m)$. Let $V_i \simeq W(0, \frac{1}{2}, m)$, $1 \leq i \leq k$; $V_i \simeq W(1, \frac{1}{2}, m)$, $k+1 \leq i \leq s$.

The sub-bimodule $\sum_{i=1}^{k} V_i$ contains only irreducible sub-bimodules of type $Irr(1, 1, m-1)$, whereas the sub-bimodule $\sum_{i=k+1}^{s} V_i$ contains only irreducible sub-bimodules of type $Irr(0, 1, m-1)$. The bimodules $Irr(1, 1, m-1)$, $Irr(0, 1, m-1)$ are not isomorphic.

Hence $V = (\sum_{i=1}^{k} V_i) \oplus (\sum_{i=k+1}^{s} V_i)$ is a direct sum, a contradiction.

Now suppose that all summands V_i are of the type $W(1, \frac{1}{2}, m)$. Let $v_i \in V_{i\bar{1}}$ be a highest weight element of the bimodule V_i. If $V_s \cap \sum_{i=1}^{s-1} V_i \neq (0)$, then $v_s R(x)R(e_1) \in \sum_{i=1}^{s-1} V_i$, $v_s R(x)R(e_1) = \sum_{i=1}^{s-1} \alpha_i v_i R(x)R(e_1)$, $\alpha_i \in k$. We have $(v_s - \sum_{i=1}^{s-1} \alpha_i v_i)R(x)R(e_1) = 0$.

Hence either $v_s - \sum_{i=1}^{s-1} \alpha_i v_i = 0$ or the sub-bimodule V_s' generated by $v_s - \sum_{i=1}^{s-1} \alpha_i v_i$ is isomorphic to $Irr(1, \frac{1}{2}, m)$. Hence either $V = \sum_{i=1}^{s-1} V_i$ or $V = (\sum_{i=1}^{s-1} V_i) \oplus V_s'$.

We proved that $V \simeq W(1, \frac{1}{2}, m)$. The case of $t = -\frac{m-1}{m+1}$, $m \geq 2$ is treated similarly. Theorem 3.2 is proved.

References

[J1] N. Jacobson, *Structure and Representation of Jordan algebras*, Amer. Math. Soc., Providence, R.I., (1969). MR0251099 (40:4330)

[J2] N. Jacobson, *Lie Algebras*, Dover Publications, Inc., New York, (1979); Republication of the 1962 original. MR0559927 (80k:17001)

[J3] N. Jacobson, General representation theory of Jordan algebras, *Trans. Amer. Math. Soc.* **70**, 509-530, (1951). MR0041118 (12:797d)

[K1] V. G. Kac, Lie Superalgebras, *Advances in Mathematics* **26**, 8-96, (1977). MR0486011 (58:5803)

[K2] V. G. Kac, Classification of simple Z-graded Lie superalgebras and simple Jordan superalgebras, *Comm. in Algebra* **5** (13), 1375-1400, (1977). MR0498755 (58:16806)

[Ka1] I. L. Kantor, Connection between Poisson brackets and Jordan and Lie superalgebras, *Lie theory, differential equations and representation theory*, 213-225, Montreal (1989). MR1121965 (92f:17040)

[Ka2] I. L. Kantor, Jordan and Lie superalgebras defined by a Poisson algebra, *Algebra and Analysis*, 55-80. *Amer. Math. Soc. Transl. Ser. (2)*, **151**, (1992). MR1191172 (93j:17004)

[Kp1] I. Kaplansky, Superalgebras, *Pacific J. of Math.* **86**, 93-98, (1980). MR0586871 (81j:17006)

[Kp2] I. Kaplansky, Graded Jordan Algebras I, *Preprint*.

[MZ] C. Martínez and E. Zelmanov, Specializations of Jordan Superalgebras, *Canad. Math. Bull.* **45** (4), 653-671, (2002). MR1941232 (2003k:17040)

[ZSSS] K. A. Zhevlakov, A. M. Slinko, I. P. Shestakov and A. I. Shirshov, *Rings that are nearly associative*, Academic Press, New York, 1982. MR0668355 (83i:17001)

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