UNITAL BIMODULES OVER THE SIMPLE JORDAN SUPeralGebra \( D(t) \)

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Abstract. We classify indecomposable finite dimensional bimodules over Jordan superalgebras \( D(t) \), \( t \neq -1, 0, 1 \).

1. Introduction

Throughout this paper all algebras are considered over a ground field \( k \) of characteristic zero. A (linear) Jordan algebra is a vector space \( J \) with a binary bilinear operation \( (x, y) \to xy \) satisfying the following identities:

\[
xy = yx,
\]

\[
(x^2 y)x = x^2 (yx).
\]

Respectively a Jordan superalgebra is a \( \mathbb{Z}/2\mathbb{Z} \)-graded algebra \( J = J_0 + J_1 \) satisfying the graded identities

\[
xy = (-1)^{|x||y|} yx,
\]

\[
((xy)z)t + (-1)^{|y||z|+|z||t|+|x||t|} ((xt)z)y + (-1)^{|x||y|+|x||z|} (xy)(zt)z = (xy)(zt) + (-1)^{|y||z|} (xz)(yt) + (-1)^{|y||z|} (yt) (xz) (zt).
\]

In [K2] (see also I. L. Kantor [Ka1, Ka2], V. Kac classified simple finite dimensional Jordan superalgebras over an algebraically closed field of zero characteristic. This classification included the 1-parametric family of 4-dimensional superalgebras \( D_t \), which corresponds to the 1-parametric family of 17-dimensional Lie superalgebras via the Tits-Kantor-Koecher construction,

\[
D_t = (ke_1 + ke_2) + (kx + ky),
\]

with the products

\[
e_i^2 = e_i, \; e_1 e_2 = 0, \; e_i x = \frac{1}{2} x, \; e_i y = \frac{1}{2} y, \; xy = e_1 + te_2, \; i = 1, 2.
\]

The superalgebra \( D_t \) is simple if \( t \neq 0 \). For \( t = 0 \) the superalgebra \( D_0 \) is a unital hull of the 3-dimensional nonunital Kaplansky superalgebra \( K_3 \), \( D_0 = K_3 + k1 \). In the case \( t = -1 \), the superalgebra \( D_{-1} \) is isomorphic to the Jordan superalgebra \( M_{1,1}(k)^\dagger \) of \( 2 \times 2 \) matrices.

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A Jordan bimodule $V$ over a Jordan (super)algebra $J$ is a vector space with operations $V \times J \to V$, $J \times V \to V$ such that the split null extension $V + J$ is a Jordan (super)algebra (see [J1]).

We denote a Jordan triple product by $\{x, y, z\} = (xy)z + x(yz) - (-1)^{|x||y|}y(xz)$.

Let $e$ be the identity of $J$ and let $V = \{e, V, e\} + \{1 - e, V, e\} + \{1 - e, V, 1 - e\}$ be the Peirce decomposition. Then $\{e, V, e\}$ is a unital bimodule over $J$, that is, $e$ is an identity of $\{e, V, e\} + J$. The component $\{1 - e, V, e\}$ is a one-sided module, that is, $\{J, \{1 - e, V, e\}, J\} = (0)$.

Finally, $\{1 - e, V, 1 - e\}$ is a bimodule with zero multiplication.

One-sided finite dimensional bimodules over $D_t$ were classified in [MZ]. In this paper we classify finite dimensional unital bimodules, thus finishing the classification of finite dimensional bimodules over $D_t$. Bimodules over semisimple finite dimensional Jordan algebras have been completely classified by N. Jacobson (see [J1], [J3]).

2. Irreducible modules

For a set $X$ let $V(X)$ denote the free $D_t$-bimodule on the set of free generators $X$ (see [J1]). Consider the linear operator $R(a) : V(X) \to V(X)$, $v \mapsto v \cdot a$, $v \in V(X)$, $a \in J$.

The algebra $M(J)$ generated by all operators $R(a)$, $a \in J$, is called the universal multiplicative enveloping algebra of $J$ (see [J1]).

Assume that $t \neq -1$ and consider the following elements of $M(D_t)$:

\[ E = \frac{2}{t+1} R(x)^2, \quad F = \frac{2}{t+1} R(y)^2, \quad H = \frac{2}{t+1} (R(x)R(y) + R(y)R(x)). \]

It is easy to see that $[F, H] = 2F$, $[E, H] = -2E$, $[E, F] = H$, $kE + kF + kH \simeq sl_2(k)$.

Note that $xH = -x$, $yH = y$.

**Definition 2.1.** For $\sigma \in \{0, 1\}$, $i \in \{0, 1, \frac{1}{2}\}$, $\lambda \in k$, a Verma module $V(\sigma, i, \lambda)$ is defined as a unital $D_t$-bimodule presented by one generator $v$ of parity $\sigma$ and the relations $vR(e_1) = iv$, $vR(y) = 0$, $vH = \lambda v$.

**Remark.** $V(\sigma, i, \lambda)^{op} = V(1 - \sigma, i, \lambda)$.

**Lemma 2.1.** For an arbitrary $\lambda \in k$, $V(\sigma, \frac{1}{2}, \lambda) \neq (0)$.

**Proof.** Let $(k, +)$ be the additive group of the field $k$. We will denote elements of $(k, +)$ as $u^\mu$, $\mu \in k$, $u^\mu u^\nu = u^{\mu + \nu}$. Consider the group algebra $\Lambda = k(k, +) = \{\sum_\mu u^\mu; \mu, \nu \in k\}$ and the derivation $D : u^\mu \to \mu u^{\mu - 1}$ of $\Lambda$.

Let $R = (\Lambda, D) = \sum_{i \geq 0} \Lambda D^i$, $aD - Da = D(a)$, $a \in \Lambda$, be the Weyl algebra.

Consider the following $2 \times 2$ matrices over $R$:

\[
\begin{pmatrix}
0 & (t+1)D - (1-t)u^{-1}
\end{pmatrix},
\begin{pmatrix}
0 & u
\end{pmatrix},
\begin{pmatrix}
1 & 0
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0
0 & 1
\end{pmatrix}.
\]

It is easy to see that $\frac{1}{2} (xy - yx) = \begin{pmatrix}
1 & 0
0 & t
\end{pmatrix}$, so $e_1, e_2, x, y$ span a superalgebra which is isomorphic to $D_t$. 

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Consider the odd element \( v = \begin{pmatrix} 0 & v^\lambda \\ v^\lambda & 0 \end{pmatrix} \). Then \( v \cdot e_i = \frac{1}{2}v \) and \( vy - yv = 0 \). It is straightforward to verify that \( vH = \lambda v \). This implies that \( V(\frac{1}{2}, \frac{1}{2}, \lambda) \neq 0 \). Consequently, the module \( V(0, \frac{1}{2}, \lambda) \neq 0 \). Lemma 2.1 is proved.

**Lemma 2.2.** \( V(\sigma, 1, \lambda) = 0 \) unless \( \lambda = \frac{-2}{t+1} \).

**Proof.** Let \( U(x, y) = R(x)R(y) - R(y)R(x) - R([x, y]) \). Clearly, \( vU(x, y) = -\{x, v, y\} \in \{v_2, V, e_2\} \). Hence, \( vU(x, y)R(e_1) = 0 \). We have

\[
vU(x, y)R(e_1) = v(R(x)R(y) - R(y)R(x) - R(e_1 + te_2))R(e_1)
\]

\[
= v(R(x)R(y) + R(y)R(x) - R(e_1 + te_2))R(e_1)
\]

\[
= v(-\frac{t+1}{2}H - R(e_1 + te_2))R(e_1)
\]

\[
= -\frac{t+1}{2}\lambda v - v = -(\lambda\frac{t+1}{2} + 1)v = 0,
\]

which implies \( v = 0 \) unless \( \lambda = \frac{-2}{t+1} \). Lemma 2.2 is proved.

**Lemma 2.3.** \( V(\sigma, 0, \lambda) = 0 \) unless \( \lambda = \frac{-2t}{t+1} \).

**Proof.** Arguing as above we get

\[
vU(x, y)R(e_2) = v(R(x)R(y) - R(y)R(x) - R(e_1 + te_2))R(e_2)
\]

\[
= v(-\frac{t+1}{2}H - R(e_1 + te_2))R(e_2) = -(\lambda\frac{t+1}{2} - t)v = 0,
\]

which implies \( v = 0 \) unless \( \lambda = \frac{-2t}{t+1} \). Lemma 2.3 is proved.

To establish that \( V(\sigma, 1, \frac{-2}{t+1}) \neq 0 \), \( V(\sigma, 0, \frac{-2t}{t+1}) \neq 0 \) we need to recall some facts about one-sided bimodules (see [MZ]).

A Jordan bimodule \( V \) over a Jordan (super)algebra \( J \) is said to be one-sided if \( \{J, V, J\} = 0 \). If \( J \) is special with a universal associative enveloping algebra \( U(J) \) (see [M]), then a one-sided bimodule \( V \) can be viewed as a right \( U(J) \)-module. For \( a \in J, v \in V \) we have \( v \cdot a = \frac{1}{2}va \), where the left-hand side is the bimodule action, whereas the right-hand side is the right module action by \( a \in U(J) \). Similarly, we can make \( V \) a left module over \( U(J) \) via \( \frac{1}{2}av = a \cdot v \).

The tensor product \( V \otimes V \) is an associative bimodule over \( U(J) : a(v \otimes w)b = av \otimes wb; a, b \in J; v, w \in V \).

Consider the elements \( e = \frac{1}{4(t+1)}x^2, f = -\frac{1}{t+1}y^2, h = -\frac{1}{(t+1)^2}(xy + yx) \) in \( U(D_t) \). We have \([e, f] = h, [f, h] = 2f, [e, h] = -2e\).

In [MZ] the following one-sided Verma bimodules over \( D_t \) were introduced[1]

The right module \( V_1(t) \) over \( U(D_t) \) is presented by one even generator \( v \) and the relations \( ve_1 = v, vy^2 = 0, vh = (\frac{-t-2}{t+1})v \).

The right module \( V_2(t) \) over \( U(D_t) \) is presented by one even generator \( v \) and the relations \( ve_1 = v, vy = 0, vh = (\frac{-t}{t+1})v \).

The tensor square \( V_1(t) \otimes V_2(t) \) is a bimodule over \( U(D_t) \), hence a Jordan bimodule over \( D_t \) via

\[
(v' \otimes v'') \cdot a = \frac{1}{2}(v' \otimes v''a + (-1)^{|a|(|v'|+|v''|)}av' \otimes v'').
\]

[1] Our notation differs from [MZ]. For example, we assume that \( \frac{1}{2}(xy - yx) = e_1 + te_2 \) in \( U(D_t) \).
We have
\[(v \otimes v) \cdot e_1 = v \otimes v, \quad (v \otimes v) \cdot y = 0, \quad (v \otimes v)H = -\frac{2}{t+1}(v \otimes v).\]

We proved

**Lemma 2.4.** \(V(\sigma, 1, -\frac{2}{t+1}) \neq (0).\)

Now consider the tensor square \(V_1(t) \otimes V_1(t)\) and the element \(v y \otimes v y\). We have
\[(v y \otimes v y) \cdot e_1 = 0, \quad (v y \otimes v y) \cdot e_2 = v y \otimes v y, \quad (v y \otimes v y) \cdot y = 0, \quad (v y \otimes v y)H = -\frac{2t}{t+1}vy \otimes vy.\]

We proved

**Lemma 2.5.** \(V(\sigma, 0, -\frac{2t}{t+1}) \neq (0).\)

**Lemma 2.6.** For an arbitrary \(\lambda \in k:\)
(a) The elements \(v R(x)^j, v R(x) R(e_1)R(x)^j, \ j \geq 0, \) form a basis of \(V(\sigma, \frac{1}{t}, \lambda).\)
(b) The elements \(v R(x)^j, \ j \geq 0, \) form a basis of \(V(\sigma, i, \lambda), \ i = 0 \) (and \(\lambda = \frac{2i}{t+1}\)) or 1 (and \(\lambda = \frac{2i-2}{t+1}\)).

**Proof.** (a) Let us show that the elements \(v R(x)^j, v R(x) R(e_1)R(x)^j, \ j \geq 0, \) span \(V(\sigma, \frac{1}{t}, \lambda).\)

Let \(v\) be a highest weight vector in a Verma module \(V(\sigma, \frac{1}{t}, \lambda).\) Consider an operator \(W = R(a_1) \cdots R(a_s), \) where \(a_i = e_1 \) or \(x\) or \(y, \ 1 \leq i \leq s.\)

We will use an induction on \(s\) and assume that operators of length \(< s\) map \(v\) into a linear combination of suggested elements.

If \(a_1 = e_1, \) then \(vW = \frac{1}{t}v R(a_2) \cdots R(a_s).\) If \(a_1 = y, \) then \(vW = 0.\) Let \(a_1 = x.\)

If \(a_2 = y, \) then
\[vW = v(R(x)R(y) + R(y)R(x))R(a_2) \cdots R(a_s) = -\frac{1}{2} \lambda v R(a_2) \cdots R(a_s).\]

If \(a_j = y, \) where \(j \geq 3, \) then we use the Jordan identity to move \(y\) to the left modulo shorter operators. Thus we can assume that \(a_j = e_1\) or \(x, \ 1 \leq j \leq s.\)

Let \(j\) be the minimal index such that \(a_j = e_1.\) We see that \(j \geq 2.\) If \(j \geq 3, \) then we can again use the Jordan identity to move \(e_1\) to the left. Hence, either \(e_1\) does not occur in \(W\) or \(j = 2.\)

Let \(W = R(x) R(e_1)R(a_3) \cdots R(a_s).\) Let \(j' \geq 3\) be the minimal index such that \(a_{j'} = e_1.\) If \(j' = 3, \) then \(vR(x) R(e_1) R(e_1) = vR(x) R(e_1) R(e_1).\)

If \(j' = 4, \) then \(vR(x) R(e_1) R(e_1) R(e_1) = \frac{1}{t} vR(x) R(e_1) R(x).\) If \(j' \geq 5, \) then we again use the Jordan identity to decrease \(j'.\) We proved that \(V(\sigma, \frac{1}{t}, \lambda)\) is spanned by \(v R(x)^j, v R(x) R(e_1) R(x)^j - 1.\) Now, it remains to prove that these elements are linearly independent.

We have
\[v R(x)^{2i} R(y) = \frac{1 + t}{2} iv R(x)^{2i-1}, \ i \geq 1;\]
\[v R(x)^{2i+1} R(y) = -\frac{1 + t}{2} (\lambda - i) v R(x)^{2i}; \ v R(x)^{2i} R(e_1) = \frac{1}{2} v R(x)^{2i};\]
\[v R(x)^{2i+1} R(e_1) = v R(x) R(e_1) R(x)^{2i};\]
\[v R(x) R(e_1) R(x)^{2i} R(y) = -\frac{(t + 1)(\lambda - 1) + 2}{4} v R(x)^{2i} \]
\[+ \frac{1 + t}{2} iv R(x) R(e_1) R(x)^{2i-1}, \ i \geq 1;\]
\[v R(x) R(e_1) R(y) = -\frac{(t + 1)(\lambda - 1) + 2}{4} v;\]
Let us first check linear independence under the assumption that \( \lambda \notin \mathbb{Z}_{\geq 0}, \lambda \neq \frac{t-1}{t+1}, \). We need to verify that \( vR(x)vR(e_1)x^i \neq 0, vR(x)vR(e_2)x^i \neq 0 \) for all \( i \geq 0 \).

From
\[
vR(x)vR(e_1)y = -\frac{(t+1)(m-1)+2}{4}v \neq 0,
\]
\[
vR(x)vR(e_2)y = -\frac{(t+1)(m+1)-2}{4}v \neq 0
\]

it follows that \( vR(x)vR(e_1) \neq 0, vR(x)vR(e_2) \neq 0 \). Now suppose that \( vR(x)vR(e_1)x^i \neq 0, vR(x)vR(e_2)x^i \neq 0 \) for \( i \leq k \).

Let \( k \) be even. Then
\[
vR(x)vR(e_1)x^{k+1}y = \frac{(t+1)(\lambda - 1) + 2}{4}vR(x)^{k+1}
\]
\[
- \frac{1+t}{2}(\lambda - \frac{k+2}{2})vR(x)vR(e_1)x^k
\]
\[
= \frac{(t+1)(\lambda - 1) + 2}{4} - \frac{1+t}{2}(\lambda - \frac{k+2}{2})vR(x)vR(e_1)x^k
\]
\[
+ \frac{(t+1)(\lambda - 1) + 2}{4}vR(x)vR(e_2)x^k \neq 0.
\]

The case when \( k \) is odd can be treated similarly, and similarly
\[
vR(x)vR(e_2)x^{k+1}y \neq 0.
\]

We have proved assertion (a) for \( \lambda \notin \mathbb{Z}_{\geq 0}, \lambda \neq \frac{t-1}{t+1}, \lambda \neq \frac{t-1}{t+1}. \)

Now consider a vector space \( \hat{V} \) with a basis \( v_i, i \in \mathbb{Z}_{\geq 0}; v'_j, j \in \mathbb{Z}_{>0} \), and define a \( D_t \)-bimodule structure via:
\[
v_iR(x) = v_{i+1}; v_0R(y) = 0; v_{2i}R(y) = \frac{1+t}{2}v_{2i-1}, i \geq 1;
\]
\[
v_{2i+1}R(y) = -\frac{1+t}{2}(\lambda - i)v_{2i}; v_{2i}R(e_1) = \frac{1}{2}v_i, v_{2i+1}R(e_1) = v'_{2i+1};
\]
\[
v'_jR(x) = v'_{j+1}; v'_{2i+1}R(y) = -\frac{(t+1)(\lambda - 1) + 2}{4}v_{2i} + \frac{1+t}{2}v'_{2i}, i \geq 1;
\]
\[
v'_iR(y) = -\frac{(t+1)(\lambda - 1) + 2}{4}v_0; v'_iR(y)
\]
\[
= -\frac{1+t}{2}(\lambda - i)v'_{2i-1} + \frac{(t+1)(\lambda - 1) + 2}{4}v_{2i-1}, i \geq 1;
\]
\[
v'_{2i}R(e_1) = \frac{1}{2}v'_{2i}; v'_{2i+1}R(e_1) = v'_{2i+1}.
\]
For a fixed $i$ the equalities
\[ w(R(a)R(b)R(c) + (-1)^{|a||b|+|a||c|+|b||c|}R(c)R(b)R(a) + (-1)^{|b||c|}R(ac)b) \]
\[ - R(ab)R(c) - (-1)^{|b||c|}R(ac)b - (-1)^{|b||c|}R(bc)R(a)) = 0, \]
where $w = v_i$ or $v_i'$; $a, b, c = x$ or $y$ or $e_1$ amount to a bunch of at most quadratic equalities involving $\lambda$. Since all these equalities hold for all $\lambda \notin \mathbb{Z}_{\geq 0}$, $\lambda \neq \frac{1}{2}$, $\lambda \neq \frac{1}{2t+1}$, it follows that these equalities hold for all $\lambda$. Hence for all $\lambda$, $\tilde{V}$ is a Jordan bimodule over $D_t$ with a highest weight element $v_0$ and the highest weight $\lambda$.

This implies assertion (a) of the lemma.

Now consider the bimodules $V(\sigma, i, \lambda)$, $i = 0$ or 1. Arguing as above we can prove that $V(\sigma, i, \lambda)$ is spanned by $vR(x)^i$, $i \geq 0$. To show that the elements $vR(x)^i$ are all nonzero, we can use the embedding of $V(\sigma, i, \lambda)$ into the tensor product of one-sided Verma modules as in the proofs of Lemmas 2.4 and 2.5. Lemma 2.6 is proved.

**Corollary 2.1.** Every nonzero Verma bimodule $V(\sigma, i, \lambda)$ contains a largest proper sub-bimodule $M(\sigma, i, \lambda)$. Hence there exists a unique irreducible $D_t$-bimodule $\text{Irr}(\sigma, i, \lambda) = V(\sigma, i, \lambda)/M(\sigma, i, \lambda)$ generated by an element of the highest weight $\lambda$.

**Lemma 2.7.** Every finite dimensional irreducible $D_t$-bimodule is isomorphic to $\text{Irr}(\sigma, i, \lambda)$ for some $\sigma, i, \lambda$.

**Proof.** Let $V$ be a finite dimensional irreducible $D_t$-bimodule. Then $V$ is a module over the Lie algebra $\mathfrak{sl}_2(k) = kE + kF + kH$. From the representation theory of $\mathfrak{sl}_2(k)$ (see [L2]) it follows that the action of $H$ on $V$ is diagonalizable, $V = \bigoplus V_\lambda$ is the sum of eigenspaces. Choose an eigenvalue $\lambda$ such that $V_\lambda \neq (0), V_{\lambda + 1} = (0)$.

Let $0 \neq v \in V_{\lambda, \sigma}$, $\sigma = 0$ or 1. Consider a Peirce decomposition, $v = v_0 + v_1 + v_2$. Clearly $v_1 \in V_\lambda$, $i = 0$ or 1 or $\frac{1}{2}$, and therefore $v_1 y = 0$. If $v_1 \neq 0$, then $v_1$ generates the bimodule $V$, which implies $V \simeq \text{Irr}(\sigma, i, \lambda)$. Lemma 2.7 is proved.

**Lemma 2.8.** Suppose that $V(\sigma, i, \lambda) \neq (0)$. If $\dim \text{Irr}(\sigma, i, \lambda) < \infty$, then $\lambda \in \mathbb{Z}_{\geq 0}$.

If $i = 0$ or 1 and $\lambda \in \mathbb{Z}_{\geq 0}$, then $\dim \text{Irr}(\sigma, i, \lambda) < \infty$. For $t \neq \pm 1$ the bimodule $V(\sigma, \lambda, 0)$ is infinite dimensional and irreducible.

**Proof.** From the representation theory of $\mathfrak{sl}_2(k)$ it follows that $\dim \text{Irr}(\sigma, i, \lambda) < \infty$ implies $\lambda \in \mathbb{Z}_{\geq 0}$.

Let $m \in \mathbb{Z}_{\geq 0}$, $i = 0$ or 1, or $m \in \mathbb{Z}_{\geq 0}$, $i = \frac{1}{2}$. Let us show that $vR(x)^{2m+1}$ generates a proper sub-bimodule $V'$ of $V(\sigma, i, m)$. We have
\[
 vR(x)^{2m+1} R(y) = vR(x)^{2m}(R(x)R(y) + R(y)R(x)) - vR(x)^{2m} R(y)R(x); \\
 vR(x)^{2m}(R(x)R(y) + R(y)R(x)) = vR(x)^{2m}(\frac{1 + t}{2} H) \\
 = -\frac{1 + t}{2} (m - 2m) vR(x)^{2m} = \frac{1 + t}{2} vR(x)^{2m}; \\
 [R(x)^2, R(y)] = \frac{1 + t}{2} R(x); \\
 vR(x)^{2m} R(y) = \sum_{j=0}^{m-1} vR(x)^{(2m-j-1)} [R(x)^2, R(y)] R(x)^2 j = \frac{1 + t}{2} vR(x)^{2m-1}. 
\]
This proves that $vR(x)^{2m+1} R(y) = 0$. 

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If \( i = 0 \) or \( 1 \), then \( vR(x)^{2m+1} \) belongs to the \( \frac{1}{2} \)-Peirce component. By Lemma 2.6 the sub-bimodule \( V' \) is spanned by 
\[
vR(x)^{2m+1} R(x)^j, \quad vR(x)^{2m+1} R(x) R(e_1) R(x)^j, \quad j \geq 0.
\]
Since all these elements belong to eigenvalues \( \leq -(m+1) \) with respect to \( H \), we conclude that \( v \notin V' \).

Let \( i = \frac{1}{2} \). The element \( vR(x)^{2m+1} R(e_1) R(y) \) belongs to the \( \frac{1}{2} \)-Peirce component and \( vR(x)^{2m+1} R(e_1) R(y) R(y) = 0 \). By Lemma 2.6, the sub-bimodule \( V_1' \) generated by \( vR(x)^{2m+1} R(e_1) R(y) \) is spanned by \( vR(x)^{2m+1} R(e_1) R(y) R(x)^j, \quad vR(x)^{2m+1} R(e_1) R(y) R(x) R(e_1) R(x)^j, \quad j \geq 0. \)

Similarly, the sub-bimodule \( V_2' \) generated by \( vR(x)^{2m+1} R(e_2) R(y) \) is spanned by \( vR(x)^{2m+1} R(e_2) R(y) R(x)^j, \quad vR(x)^{2m+1} R(e_2) R(y) R(x) R(e_1) R(x)^j, \quad j \geq 0. \)

The element \( vR(x)^{2m+1} R(e_1) \) lies in the 1-Peirce component and 
\[
vR(x)^{2m+1} R(e_1) R(y) \equiv 0 \mod V_1'.
\]
By Lemma 2.6 the sub-bimodule generated by \( vR(x)^{2m+1} R(e_1) \) is spanned by 
\[
vR(x)^{2m+1} R(e_1) R(x)^j, \quad j \geq 0, \mod V_1'.
\]
The sub-bimodule generated by \( vR(x)^{2m+1} R(e_2) \) is spanned by 
\[
vR(x)^{2m+1} R(e_2) R(x)^j, \quad j \geq 0, \mod V_2'.
\]
Finally, we conclude that \( V' \) is spanned by
\[
vR(x)^{2m+1} R(e_k) R(y) R(x)^j, \quad vR(x)^{2m+1} R(e_k) R(x) R(e_1) R(x)^j, \quad vR(x)^{2m+1} R(e_k) R(x)^j, \quad j \geq 0, \quad k = 1 \) or \( 2 \).

If \( m \geq 1 \), then all the elements above have weights \( < m \). Hence \( V' \) is proper.

It is easy to see that the bimodule \( W(\sigma, i, m) = V(\sigma, i, m)/V' \) is finite dimensional. It remains to show that the Verma bimodule \( V(\sigma, \frac{1}{2}, 0) \) is infinite dimensional and irreducible.

We have
\[
vR(x) R(y) = vR(x) R(y) + R(y) R(x) = -\frac{t+1}{2} vH = 0;
\]
\[
vR(x) R(e_1) R(y) = v(-R([x, y] \cdot e_1) + R(x \cdot e_1) R(y) - R(y \cdot e_1) R(x)
+ R([x, y]) R(e_1)) = \frac{t-1}{4} v;
\]
\[
vR(x) R(e_2) R(y) = \frac{1-t}{4} v.
\]

The Verma module over the Lie algebra \( sl_2(k) \) with maximal eigenvalue \(-1\) is irreducible and infinite dimensional (see [12]).

If \( t \neq 1 \), then \( vR(x) \neq 0 \) and \( vR(x) R(e_1) \neq 0 \). Similarly, \( vR(x) R(e_2) \neq 0 \).

Both elements belong to the eigenvalue \(-1\) with respect to \( H \) and 
\[
vR(x) R(y)^2 = vR(x) R(e_1) R(y)^2 = 0.
\]

Hence \( \sum_{j=0}^{\infty} kvR(x)^{2j+1}, \sum_{j=0}^{\infty} kvR(x) R(e_1) R(x)^{2j}, \sum_{j=0}^{\infty} kvR(x) R(e_2) R(x)^{2j} \) are infinite dimensional irreducible \( sl_2(k) \)-modules. In particular, the module \( V \) is infinite dimensional.

Let \( V' \) be a proper nonzero sub-bimodule of \( V = V(\sigma, \frac{1}{2}, 0) \). Then \( \alpha vR(x)^h + \beta vR(x) R(e_1) R(x)^{h-1} \in V' \) for some \( h \geq 1 \); \( \alpha, \beta \in k \); \( (\alpha, \beta) \neq (0, 0) \).
Applying $R(x)$ if necessary, we will assume that $h$ is odd. Then

$$vR(x)R(e_1)R(x)^{h-1}R(e_2) = 0$$

and therefore $\alpha vR(x)R(e_2)R(x)^{h-1} \in V'$.

Suppose that $\alpha \neq 0$. Then $\sum_{j=0}^{\infty} kvR(x)R(e_2)R(x)^{2j} \subset V'$, $vR(x)R(e_2) \in V'$ and, finally, $vR(x)R(e_2)R(y) = \frac{1-t}{4}v \in V'$, a contradiction.

Hence $\alpha = 0$, hence $vR(x)R(e_1)R(x)^{h-1} \in V'$. Arguing as above we get $\sum_{j=0}^{\infty} kvR(x)R(e_1)R(x)^{2j} \subseteq V'$, $vR(x)R(e_1) \in V'$, $vR(x)R(e_1)R(y) = \frac{1-t}{4}v$. Lemma 2.8 is proved.

Remark. In the same way we can prove that if $V(\sigma, i, \lambda) \neq (0)$ and $\text{Irr}(\sigma, i, \lambda)$ is infinite dimensional, then $V(\sigma, i, \lambda)$ is irreducible.

Remark. For $t = 1$, $\sum_{j=0}^{\infty} kvR(x)^{j} + \sum_{j=0}^{\infty} kvR(x)R(1)R(x)^{j}$ is a proper sub-bimodule of $\text{Irr}(\sigma, \frac{1}{2}, 0)$. Hence, $\dim \text{Irr}(\sigma, \frac{1}{2}, 0) = 1$.

**Theorem 2.1.** If $t \neq -1$ is not of the type $\frac{m}{m+2}$, $m \geq 0$; $\frac{m+2}{m}$, $m \geq 1$; or $1$, then the only unital finite dimensional irreducible $D_1$-bimodules are

$$(*) \quad \text{Irr}(\sigma, \frac{1}{2}, m), \ m \geq 1.$$

If $t = 1$, then add the one-dimensional bimodules $\text{Irr}(\sigma, \frac{1}{2}, 0)$, $\sigma = 0, 1$ to the series $(*)$.

If $t = -\frac{m+2}{m}$, $m \geq 1$, then add the bimodules $V(\sigma, 1, m)$, $\sigma = 0, 1$ to $(*)$.

If $t = -\frac{m}{m+2}$, $m \geq 0$, then add the bimodules $V(\sigma, 0, m)$, $\sigma = 0, 1$ to $(*)$.

Let $m \in \mathbb{Z}_{>0}$. As in the proof of Lemma 2.8, let $V'$ denote the sub-bimodule of $V(\sigma, i, m)$ generated by $vR(x)^{2m+1}$. We proved that the quotient module $W(\sigma, i, m) = V(\sigma, i, m)/V'$ is finite dimensional.

**Lemma 2.9.** $W(\sigma, i, m)$ is the largest finite dimensional homomorphic image of $V(\sigma, i, m)$.

**Proof.** Let $\tilde{V}$ be a sub-bimodule of $V(\sigma, i, m)$ such that $\dim V(\sigma, i, m)/\tilde{V} < \infty$. From the representation theory of $sl_2(k)$ it follows that $vR(x)^{2(m+1)} \in \tilde{V}$.

Now $vR(x)^{2(m+1)}R(y) = (m+1)\frac{1-t}{4}vR(x)^{2m+1}$. Hence $vR(x)^{2m+1} \in \tilde{V}$ and therefore $V' \subseteq \tilde{V}$. Lemma 2.9 is proved.

**Lemma 2.10.** Let $V$ be a unital $D_1$-bimodule, $t \neq 0, 1$.

(a) If $V_0 = (0)$, then $V = (0)$.

(b) Let $R$ be the subalgebra of $\text{End}(V)$ generated by all multiplications $R(a)$, $a \in D_1$. Clearly, $R = R_0 + R_1$, $V_1R_j \subseteq V_{1+j}$. If $V_0$ is an irreducible module over $R_0$, then $V$ is an irreducible $D_1$-bimodule.

**Proof.** (a) If $V_0 = (0)$, then $V = V_1$ and $Vx = 0 = Vy$.

Since $t \neq 0$, then $e_1$ and $e_2$ play a symmetric role and we can assume that $V = \{e_1, V, e_2\}$ or $V = \{e_1, V, e_1\}$.

If $V = \{e_1, V, e_2\}$, then for an arbitrary $v \in V$ we have

$$vR(x)R(e_1)R(y) - R(y)R(e_1)R(x) + R([x, y]e_1) - R(xe_1)R(y) + R(ye_1)R(x) - R([x, y]e_1) = 0.$$ 

Hence $v - (v(e_1 + te_2))e_1 = 0$, that is, $\frac{1}{2}v - \frac{1}{4}(1 + t)v = 0$. This implies that $\frac{1-t}{4}v = 0$ and then $v = 0$ since $t \neq 1$. 

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If $V = \{e_1, V, e_1\}$ and $v \in V$, then

$$v(R(x)R(y)R(e_1) - R(e_2)R(y)R(x) + R([xe_2, y]) - R(xe_2)R(y) + R(ye_2)R(x) - R([x,y])R(e_2)) = 0.$$ 

Then $v(\frac{1}{2}(e_1 + te_2)) - (v(e_1 + te_2))e_2 = \frac{1}{2}v = 0$, that is, $v = 0$.

(b) Let $V'$ be a nonzero $D_i$-sub-bimodule of $V$. By (a) $V'_0 \neq (0)$. Since $V'_0$ is an irreducible $vR$ module over $R$, it follows that $V'_0 = V'_0$. Now $(V/V'_0) = (0)$. By (a) $V = V'$.

Lemma 2.10 is proved.

Let $m \geq 1$, $W = W(1, \frac{1}{2}, m)$. We have

$$vR(x)R(e_1)R(y) = - \frac{(t + 1)(m - 1) + 2}{4}v,$$

$$vR(x)R(e_2)R(y) = - \frac{(t + 1)(m + 1) - 2}{4}v.$$ 

If $t \neq \frac{m+1}{m-1}$, then $vR(x)R(e_1)R(y) \neq 0$. We remark though that the element $vR(x)R(e_1)$ is not equal to zero in $W$. Indeed, it follows from Lemma 2.6(a) that $vR(x)R(e_1) \neq 0$ in $V(\sigma, \frac{1}{2}, m)$. Now it remains to notice that all eigenvalues of the operator $H$ that occur in $V'$ are smaller than $m - 1$.

Hence the element $vR(x)R(e_1)$ generates a proper sub-bimodule $W'$ of $W$. The even part $W'_0$ is the irreducible $sl_2(k)$-module, hence $W'$ is irreducible, $W' \simeq \text{Irr}(0, 1, m - 1)$.

The even part of the quotient $W/W'$ is an irreducible $sl_2(k)$-module of dimension $m$. Hence $W/W'$ is irreducible, $W/W' \simeq \text{Irr}(1, \frac{1}{2}, m)$. The odd part $(W/W')_1$ is also an irreducible $sl_2(k)$-module generated by $v$, hence $\dim(W/W') = m + 1$.

If $t = -\frac{m+1}{m-1}$, then

$$0 \longrightarrow \text{Irr}(0, 0, m-1) \longrightarrow W \longrightarrow \text{Irr}(1, \frac{1}{2}, m) \longrightarrow 0$$

is an exact sequence and as above $\dim\text{Irr}(1, \frac{1}{2}, m) = 2m + 1$.

For $t = -\frac{m+2}{m}$, $m \geq 1$, we have $W(1, 1, m) \simeq \text{Irr}(1, 1, m)$; both the even and the odd parts are irreducible $sl_2(k)$-modules of dimension $m$ and $m + 1$ respectively.

For $t = -\frac{m}{m+2}$, $m \geq 0$, $W(1, 0, m) \simeq \text{Irr}(1, 0, m)$ and $\text{Irr}(1, 0, m)_0$, $\text{Irr}(1, 0, m)_1$ are again irreducible $sl_2(k)$-modules of dimensions $m$ and $m + 1$ respectively.

Corollary 2.2. The only finite dimensional irreducible bimodules of the (nonunital) Kaplansky superalgebra $K_3$ are $\text{Irr}(\sigma, \frac{1}{2}, m)$, $m \geq 1$, and $\text{Irr}(\sigma, 0, 0)$. We have $\dim\text{Irr}(\sigma, \frac{1}{2}, m) = 4m$ if $m \geq 2$, $\dim\text{Irr}(\sigma, \frac{1}{2}, 1) = 3$, $\dim\text{Irr}(\sigma, 0, 0) = 1$. 
Proof: The unital hull of $K_3$ is $D_t$, where $t = 0$. Every bimodule over $K_3$ has a structure of a unital bimodule over the unital hull of $K_3$. Now it remains to apply Theorem 2.1.

3. INDECOMPOSABLE MODULES

Lemma 3.1. Let $W \subseteq \{ e_i, V_0, e_i \}$, $i = 1$ or 2, be a module over $sl_2(k) = kE + kF + kH$. Then the $D_t$-sub-bimodule of $V$ generated by $W$ is $\tilde{W} = W + WU(x, y) + WR(x) + WR(y)$.

Proof. It is straightforward that $W$, $WU(x, y)$, $WR(x) + WR(y)$ belong to $1$, $0$, $\frac{1}{2}$-Peirce components respectively. Hence we need only to verify that $WR(x)$, $WR(y)$ lie in $\tilde{W}$.

We have
\[
R(x)R(y) = \frac{1}{2}(U(x, y) - \frac{1+t}{2}H - R(e_1 + te_2));
\]
\[
R(y)R(x) = \frac{1}{2}(-U(x, y) - \frac{1+t}{2}H + R(e_1 + te_2));
\]
\[
U(x, y)R(x) = (R(x)R(y) - R(y)R(x) - R(e_1 + te_2))R(x) = R(x)R(y)R(x)
\]
\[-R(y)R(x)^2 - R(e_1 + te_2)R(x) = (R(x)R(y) + R(y)R(x))R(x)
\]
\[-2R(x)^2R(y) - 2R(yR(x)^2) - R(e_1 + te_2)R(x)
\]
\[= -\frac{t+1}{2}HR(x) - (t+1)ER(y) + (t+1)R(x) - R(e_1 + te_2)R(x),
\]
which implies that $WU(x, y)R(x) \subseteq \tilde{W}$.

Similarly, $WU(x, y)R(y) \subseteq \tilde{W}$. Lemma 3.1 is proved.

The operator $U(x, y)$ commutes with $E, F, H$. Let $W \subseteq \{ e_i, V_0, e_i \}$ be an irreducible $sl_2(k)$-module. Then the restriction of $U(x, y)$ to $W$ is an isomorphism $W \rightarrow WU(x, y)$, $WU(x, y) \subseteq k[e_2, V_0, e_2]$, or a zero mapping. By Schur’s Lemma $U(x, y)^2|_W = \alpha Id_W$, $\alpha \in k$.

Let $v$ be a highest weight vector of $W$, $vR(y)^2 = 0$, $vH = mv$, $m \in \mathbb{Z}_{\geq 0}$.

Lemma 3.2. If $WU(x, y)^2 \neq (0)$, then $\tilde{W}$ is an irreducible $D_t$-bimodule.

Proof. We showed above that $W + WU(x, y)$ is a direct sum of two isomorphic irreducible $sl_2(k)$-modules. Let $W'$ be a nonzero $R_0$-submodule of $W + WU(x, y)$, $w_1 + w_2U(x, y) \in W'$, $w_1, w_2 \in W$. Clearly, $w_1 = (w_1 + w_2U(x, y))R(e_1) \in W'$, $w_2U(x, y) = (w_1 + w_2U(x, y))R(e_2) \in W'$. If $w_1 \neq 0$, then $W' \subseteq W'$ and $WU(x, y) \subseteq W'$, hence $W' = W + WU(x, y) \neq 0$. If $w_2U(x, y) \neq 0$, then $0 \neq w_2U(x, y)^2 \in W'$ and we argue as above.

We proved that $W + WU(x, y)$ is an irreducible $R_0$-module. By Lemma 2.10(b) the bimodule $\tilde{W}$ is irreducible. Lemma 3.2 is proved.

Similarly, if $W \subseteq \{ e_2, V_0, e_2 \}$ is an irreducible $sl_2(k)$-module and $WU(x, y)^2 \neq (0)$, then $W$ is an irreducible $D_t$-bimodule.

Lemma 3.3. If $W \subseteq \{ e_1, V_0, e_1 \}$ is an irreducible $sl_2(k)$-module of highest weight $m$ and $WU(x, y)^2 = (0)$, then $t = -\frac{m}{m+2}$ or $-\frac{m+2}{m}$, $m \in \mathbb{Z}_{\geq 0}$.
Proof. Let \( w \in W \) be a vector of maximal weight, \( wF = 0, wH = mw \). Hence \( w(R(x) \cdot R(y)) = -\alpha mw \), with \( \alpha = \frac{1+t}{m} \).

Then
\[
wU(x, y)^2 = w(2R(x)R(y) - R(x)R(y) - R(e_1 + te_2))(R(x)R(y) - 2R(y)R(x) - t)
\]
\[
= w(2R(x)R(y) + \alpha m - 1)(-\alpha m - 2R(xy)R(x) - t)
\]
\[
= -4\alpha wR(y)R(x) - 2(\alpha m - 1)wR(y)R(x)
\]
\[
- 2(\alpha m + t)wR(x)R(y) - (\alpha m + t)(\alpha m - 1)w
\]
\[
= (\alpha m + t)(\alpha m + 1)w.
\]

So \( wU(x, y)^2 = 0 \), implies that \( \alpha m + t = 0 \) or \( \alpha m + 1 = 0 \), that is, \( t = -\frac{m+2}{m} \) or \( t = -\frac{m}{m+2} \).

**Definition 3.1.** An element \( v \) of a unital \( D_t \)-bimodule is said to be a highest weight element if \( vR(y) = 0, vH = \lambda v \) for some \( \lambda \in k \) and \( v \) lies in some Peirce component with respect to \( e_1, e_2 \).

**Lemma 3.4.** An arbitrary finite dimensional unital \( D_t \)-bimodule, \( t \neq 0, 1 \), is generated by its highest weight elements.

**Proof.** Let \( V \) be a nonzero unital \( D_t \)-bimodule, \( t \neq 0, 1 \). Let \( \tilde{V} \) be a sub-bimodule generated by all highest weight elements of \( V \). Let \( W \subseteq \{e_1, V_0, e_1\} \) be an irreducible \( sl_2(k) \)-submodule with a highest weight element \( v \), that is, \( vH = mv, m \in \mathbb{Z}_{\geq 0}, vR(y)^2 = 0, v \) generates \( W \).

If \( WU(x, y)^2 \neq (0) \), then by Lemma 3.2 the bimodule \( \tilde{W} = W + WU(x, y) + WR(x) + WR(y) \) is irreducible, hence \( \tilde{W} \subseteq \tilde{V} \).

Suppose that \( WU(x, y)^2 = (0) \) and therefore \( t = -\frac{m}{m+2} \) or \( t = -\frac{m+2}{m} \).

Let \( v' = vU(x, y) \). Then \( v'H = mv', v'R(y)^2 = 0 \) and \( v'U(x, y) = 0 \). We have \( v'R(y) \in \tilde{V} \).

\[
v'R(y)R(x) = \frac{1}{2}v'(-U(x, y) - \frac{1+t}{2}H + R(e_1 + te_2)) = \frac{1}{2}(-\frac{1+t}{2}m + t)v'.
\]

The element \( v' \) lies in \( \tilde{V} \) unless \( -\frac{1+t}{2}m + t = 0 \), which is equivalent to \( t = -\frac{m-2}{m} \).

The latter contradicts our assumption that \( t = -(\frac{m+2}{m})^{\pm 1} \). We proved that \( v' \not\in \tilde{V} \).

The element \( vR(y) \) also lies in \( \tilde{V} \). We have
\[
vR(y)R(x) = \frac{1}{2}v'(-U(x, y) - \frac{1+t}{2}H + R(e_1 + te_2)) = \frac{1}{2}(-\frac{1+t}{2}m + 1)v
\]
mod \( \tilde{V} \).

Hence \( v \not\in \tilde{V} \) unless \( -\frac{1+t}{2}m + 1 = 0 \), which is equivalent to \( t = -\frac{m+2}{m} \neq -(\frac{m+2}{m})^{\pm 1} \). Hence \( v \in \tilde{V} \).

We proved that \( \{e_1, V_0, e_1\} \subseteq \tilde{V} \). Similarly, \( \{e_2, V_0, e_2\} \subseteq \tilde{V} \). The even part of the bimodule \( \{e_1, V_0, e_1\} + \{e_2, V_0, e_2\} + \{e_1, V_1, e_2\} \) lies in \( \tilde{V} \). By Lemma 2.10(a) \( \{e_1, V_1, e_2\} \subseteq \tilde{V} \). Passing to opposites, we get
\[
\{e_1, V_1, e_1\} + \{e_2, V_1, e_2\} + \{e_1, V_0, e_2\} \subseteq \tilde{V}.
\]

Hence \( \tilde{V} = V \). Lemma 3.4 is proved.
Theorem 3.1. Suppose that \( t \) is not of the type \(-\frac{m}{m+2}, -\frac{m+2}{m} \), \( 0, 1, m \in \mathbb{Z}_{>0} \). Then every finite dimensional unital bimodule \( V \) over \( D_t \) is completely reducible.

Proof. An arbitrary finite dimensional highest weight bimodule over \( D_t \) is a homomorphic image of some bimodule \( W(\sigma, \frac{1}{2}, m) \), which was shown to be irreducible. Hence \( V \) is a sum of irreducible sub-bimodules. Theorem 3.1 is proved.

Theorem 3.2. If \( t = -\frac{m+1}{m} \) or \( t = -\frac{m+1}{m+1} \), \( m \geq 2 \), then \( W(\sigma, \frac{1}{2}, m), \sigma = 0 \) or \( \bar{1} \), are the only finite dimensional indecomposable \( D_t \)-bimodules, which are not irreducible.

Proof. Let \( t = -\frac{m+1}{m}, m \geq 2 \). We have proved that

\[
0 \rightarrow \text{Irr}(0, 1, m - 1) \rightarrow W(1, \frac{1}{2}, m) \rightarrow \text{Irr}(1, \frac{1}{2}, m) \rightarrow 0
\]

is an exact sequence. Let us show that \( W(1, \frac{1}{2}, m) \) is not isomorphic to \( \text{Irr}(0, 1, m - 1) \oplus \text{Irr}(1, \frac{1}{2}, m) \). Indeed, in both bimodules the eigenspaces that correspond to the eigenvalue \( m \) are one-dimensional. However, in \( W(1, \frac{1}{2}, m) \) this eigenspace is not killed by \( R(x)R(e_1) \), whereas in \( \text{Irr}(0, 1, m - 1) \oplus \text{Irr}(1, \frac{1}{2}, m) \) it is killed by \( R(x)R(e_1) \). Hence \( W(1, \frac{1}{2}, m) \) is indecomposable. Similarly, \( W(0, \frac{1}{2}, m) \) is indecomposable.

Now let \( V \) be an indecomposable \( D_t \)-bimodule. By Lemma 3.4 \( V \) is a sum of highest weight bimodules, \( V = \sum_{i=1}^{s} V_i \). We showed above that all these bimodules \( V_i \) are either irreducible or isomorphic to \( W(\sigma, \frac{1}{2}, m) \).

If at least one bimodule, say \( V_i \), is irreducible, then either \( V = (\sum_{i=0}^{s-1} V_i) \oplus V_s \), which contradicts indecomposability of \( V \).

Suppose therefore that all summands are of the types \( W(0, \frac{1}{2}, m), W(1, \frac{1}{2}, m) \). Let \( V_i \simeq W(0, \frac{1}{2}, m), 1 \leq i \leq k; V_i \simeq W(1, \frac{1}{2}, m), k + 1 \leq i \leq s \).

The sub-bimodule \( \sum_{i=1}^{k} V_i \) contains only irreducible sub-bimodules of type \( \text{Irr}(1, 1, m - 1) \), whereas the sub-bimodule \( \sum_{i=k+1}^{s} V_i \) contains only irreducible sub-bimodules of type \( \text{Irr}(0, 1, m - 1) \). The bimodules \( \text{Irr}(1, 1, m - 1) \), \( \text{Irr}(0, 1, m - 1) \) are not isomorphic.

Hence \( V = (\sum_{i=1}^{k} V_i) \oplus (\sum_{i=k+1}^{s} V_i) \) is a direct sum, a contradiction.

Now suppose that all summands \( V_i \) are of the type \( W(1, \frac{1}{2}, m) \). Let \( v_i \in V_i \) be a highest weight element of the bimodule \( V_i \). If \( v_i \cap \sum_{i=1}^{s-1} V_i \neq (0) \), then \( v_i R(x)R(e_1) \in \sum_{i=1}^{s-1} V_i, v_i R(x)R(e_1) = \sum_{i=1}^{s-1} \alpha_i v_i R(x)R(e_1), \alpha_i \in k \). We have \( (v_i - \sum_{i=1}^{s-1} \alpha_i v_i) R(x)R(e_1) = 0 \).

Hence either \( v_i - \sum_{i=1}^{s-1} \alpha_i v_i = 0 \) or the sub-bimodule \( V_i \) generated by \( v_i - \sum_{i=1}^{s-1} \alpha_i v_i \) is isomorphic to \( \text{Irr}(1, \frac{1}{2}, m) \). Hence either \( V = (\sum_{i=1}^{s-1} V_i) \oplus V_i \).

We proved that \( V \simeq W(1, \frac{1}{2}, m) \). The case of \( t = -\frac{m+1}{m+1}, m \geq 2 \) is treated similarly. Theorem 3.2 is proved.

References


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