

## A GEOMETRIC CHARACTERIZATION OF INTERPOLATION IN $\hat{\mathcal{E}}'(\mathbb{R})$

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ABSTRACT. We give a geometric description of the interpolating varieties for the algebra of Fourier transforms of distributions (or Beurling ultradistributions) with compact support on the real line.

### 1. INTRODUCTION

Let  $\mathcal{E}(\mathbb{R})$  be the space of smooth functions in  $\mathbb{R}$  and let  $\mathcal{E}'(\mathbb{R})$  be its dual, the space of distributions with compact support on  $\mathbb{R}$ . It is well known that the space  $\hat{\mathcal{E}}'(\mathbb{R})$  of Fourier transforms of distributions in  $\mathcal{E}'(\mathbb{R})$  coincides with the algebra of entire functions  $f$  such that

$$|f(z)| \leq C(1 + |z|)^A e^{B|\operatorname{Im} z|},$$

where  $A, B, C > 0$  may depend on  $f$  (see [BrGa95, Theorem 1.4.15]).

A discrete sequence  $\Lambda \subset \mathbb{C}$  is called  $\hat{\mathcal{E}}'(\mathbb{R})$ -interpolating when the interpolation problem  $f(\lambda) = v_\lambda$ ,  $\lambda \in \Lambda$ , has a solution  $f \in \hat{\mathcal{E}}'(\mathbb{R})$  for every sequence of complex values  $\{v_\lambda\}_{\lambda \in \Lambda}$  having the characteristic growth of  $\hat{\mathcal{E}}'(\mathbb{R})$  on  $\Lambda$  (see the precise definition in Section 2).

The origin of the interest in  $\hat{\mathcal{E}}'(\mathbb{R})$ -interpolation lies in its relationship with convolution equations and, in particular, with the density of exponential families  $\{e^{i\lambda x}\}_{\lambda \in \Lambda}$  in the space of solutions  $g \in \mathcal{E}(\mathbb{R})$  of equations of type  $\mu \star g = 0$ ,  $\mu \in \mathcal{E}'(\mathbb{R})$ . Any solution  $g$  to the convolution equation is the limit of linear combinations of  $\{e^{i\lambda x}\}_{\lambda \in \Lambda}$  where  $\Lambda$  is the zero set of  $\hat{\mu}$ . If, moreover, the sequence  $\Lambda$  is  $\hat{\mathcal{E}}'(\mathbb{R})$ -interpolating, then the series that represents  $g$  enjoys better convergence properties. For more on this relationship see [EhMa74] or [BrGa95, Chapter 6] (in particular, Theorem 6.1.11).

The space  $\hat{\mathcal{E}}'(\mathbb{R})$  is a particular case of the algebras

$$A_p = \{f \in H(\mathbb{C}) : \log |f(z)| \leq A + Bp(z) \text{ for some } A, B > 0\}$$

associated to positive measurable weights  $p$ , obtained by taking

$$(1) \quad p(z) = |\operatorname{Im} z| + \log(1 + |z|^2).$$

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There exists an analytic characterization of interpolating sequences for general  $A_p$  spaces when  $p$  is subharmonic (see Theorem A below). However, a complete geometric description exists only for subharmonic weights  $p$  which are both radial ( $p(z) = p(|z|)$ ) and doubling ( $p(2z) \leq Cp(z)$  for some  $C > 0$ ); see [BrLi95, Corollary 4.8].

For the weight (1) Ehrenpreis and Malliavin gave a necessary geometric condition which turns out to be sufficient provided that  $\Lambda$  is a zero sequence of a slowly decreasing function (see [EhMa74, Theorem 4] and its proof). Later Squires, probably unaware of Ehrenpreis and Malliavin's result (which was stated in terms of solutions to convolution equations), proved the same result [Sq83, Theorem 2].

In this paper we give a geometric characterization for  $\hat{\mathcal{E}}'(\mathbb{R})$ -interpolating sequences (Theorem 1). The characterization shows in particular that the geometric condition given by Ehrenpreis & Malliavin and Squires is also sufficient whenever the sequence is contained in the region

$$|\operatorname{Im} z| \leq C \log(1 + |z|^2).$$

In general, however, their condition alone is not sufficient.

A similar characterization is obtained for the more general Beurling weights. These weights appear naturally in the context of convolution equations when one replaces distributions with compact support with Beurling-Björck ultradistributions of compact support (see [Bj66]). They are not necessarily subharmonic, but we will prove that they are equivalent to a subharmonic weight (see Lemma 8).

The paper is structured as follows. In Section 2 we give the precise definition of interpolating variety, discuss the background for the problem and state the main result. In Section 3 we prove that the geometric conditions of Theorem 1 are necessary, while in Section 4 we show that they are also sufficient.

A final remark about notation.  $C$  will always denote a positive constant and its actual value may change from one occurrence to the next.  $A = O(B)$  and  $A \lesssim B$  mean that  $A \leq cB$  for some  $c > 0$ , and  $A \simeq B$  is  $A \lesssim B \lesssim A$ .

## 2. PRELIMINARIES

For the following definition and general background on the problem we refer to [BrGa95, Chapter 2].

A measurable function  $p : \mathbb{C} \rightarrow \mathbb{R}_+$  is called a *weight* if for some  $C, D > 0$ :

- (a)  $\log(1 + |z|^2) \leq Cp(z)$  for all  $z \in \mathbb{C}$ .
- (b)  $p(\zeta) \leq Cp(z) + D$  if  $|\zeta - z| \leq 1$ .

The importance of these properties lies in their consequences for the ring  $A_p$  defined in the introduction: (a) implies that  $A_p$  contains all polynomials, and (b) that  $A_p$  is closed under differentiation.

The algebra  $A_p$  can be thought of as the union of the Hilbert spaces

$$A_{p,\alpha}^2 := \left\{ f \in H(\mathbb{C}) : \|f\|_{A_{p,\alpha}^2}^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-\alpha p(z)} dm(z) < \infty \right\}$$

for  $\alpha > 0$ , as well as the union of the Banach spaces

$$A_{p,\alpha}^\infty := \left\{ f \in H(\mathbb{C}) : \|f\|_{A_{p,\alpha}^\infty} = \sup_{z \in \mathbb{C}} |f(z)| e^{-\alpha p(z)} < \infty \right\}.$$

Also,  $A_p = \bigcup_{\alpha > 0} A_{p,\alpha}^\infty$  has the structure of an (LF)-space with the topology of the inductive limit.

**Definition.** Let  $\Lambda$  be a discrete sequence in  $\mathbb{C}$  and let  $\{m_\lambda\}_{\lambda \in \Lambda}$  be a sequence of natural numbers. The pair  $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$  is called an *interpolating variety* for the space  $A_p$  if for every sequence of values  $\{v_\lambda^l\}_{\lambda, l}$ ,  $\lambda \in \Lambda$ ,  $l = 0, \dots, m_\lambda - 1$ , with

$$(2) \quad \sup_{\lambda \in \Lambda} \left( \sum_{l=0}^{m_\lambda-1} |v_\lambda^l| \right) e^{-\alpha p(\lambda)} < \infty$$

for some  $\alpha > 0$ , there exists  $f \in A_p$  with

$$\frac{f^{(l)}(\lambda)}{l!} = v_\lambda^l, \quad \lambda \in \Lambda; \quad l = 0, \dots, m_\lambda - 1.$$

The choice of condition (2) on the values to be interpolated reflects the fact that for every  $f \in A_p$  there exists  $\alpha > 0$  such that

$$\sup_{z \in \mathbb{C}} \left( \sum_{l=0}^{\infty} \left| \frac{f^{(l)}(z)}{l!} \right| \right) e^{-\alpha p(z)} < \infty.$$

Thus, denoting by  $A_p(X)$  the space of sequences  $\{v_\lambda^l\}_{\lambda, l}$  satisfying (2) for some  $\alpha > 0$ , we can equivalently define interpolating varieties  $X$  as those such that the restriction operator

$$\begin{aligned} \mathcal{R}_X : A_p(\mathbb{C}) &\longrightarrow A_p(X) \\ f &\longmapsto \left\{ \frac{f^{(l)}(\lambda)}{l!} \right\}_{\lambda, l} \end{aligned}$$

is onto.

There exists an analytic characterization of  $A_p$ -interpolating varieties for general subharmonic weights  $p$  (see [BrLi95, Corollary 3.5]). Results for  $p(z) = |z|$  and the weight (1) were previously obtained respectively by Leont'ev [Le72] and Squires [Sq81].

Given a holomorphic function  $f$  let  $\mathcal{Z}(f)$  denote its zero variety, i.e., the set of pairs  $(z, m_z) \in \mathbb{C} \times \mathbb{N}$  such that  $f(z) = 0$  with multiplicity  $m_z$ .

**Theorem A.** *A variety  $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$  is  $A_p$ -interpolating if and only if there exists  $f \in A_p$  such that  $X \subset \mathcal{Z}(f)$  and for some constants  $\delta, C > 0$ ,*

$$\left| \frac{f^{(m_\lambda)}(\lambda)}{m_\lambda!} \right| \geq \delta e^{-Cp(\lambda)}, \quad \lambda \in \Lambda.$$

We would like to give a geometric description of  $A_p$ -interpolating varieties for the non-isotropic Beurling weights

$$p(z) = |\operatorname{Im} z| + \omega(|z|),$$

where  $\omega(t)$  is a subadditive increasing continuous function, normalized with  $\omega(0) = 0$  and such that:

- (c)  $\log(1+t) \lesssim \omega(t)$  for  $t > 1$ .
- (d)  $\int_0^\infty \frac{\omega(t)}{1+t^2} dt < \infty$ .

Canonical examples of such weights are given by  $\omega(t) = \log(1+t^2)$  and  $\omega(t) = t^\gamma$ ,  $\gamma \in (0, 1)$ .

Beurling weights satisfy the following additional properties:

- (e) For every  $c > 0$  there exists  $C > 0$  such that  $p(\zeta) \leq Cp(z)$  if  $\zeta \in D(z, cp(z))$ .

- (f) For  $\varepsilon > 0$  small enough, there exists  $C(\varepsilon) > 0$  such that if  $z \in D(\zeta, \varepsilon p(\zeta))$ , then  $p(\zeta) \leq C(\varepsilon)p(z)$ . Also,  $C(\varepsilon)$  tends to 1 as  $\varepsilon$  goes to 0.
- (g) For  $x \in \mathbb{R}^+$  big enough, the function  $\omega(x)$  does not oscillate too much. More precisely, for fixed  $C > 0$ , if  $y \in (x - C\omega(x), x + C\omega(x))$ , then  $1/2 \leq \omega(y)/\omega(x) \leq 2$  for  $x$  big enough.

Properties (e) and (f) follow easily from the subadditivity of  $\omega$ . Property (g) follows from the subadditivity and the fact that  $\omega(x) = o(|x|/\log|x|)$  (see [Bj66, Lemma 1.2.8]): for any  $y \in (x - C\omega(x), x + C\omega(x))$ ,

$$\begin{aligned} \omega(x - C\omega(x)) &\leq \omega(y) \leq \omega(x + C\omega(x)) \leq \omega(x - C\omega(x)) + \omega(2C\omega(x)) \\ &\leq \omega(x - C\omega(x)) + \omega(2Cx/\log x) \leq 2\omega(x - C\omega(x)). \end{aligned}$$

In order to state the geometric conditions on a variety  $X$  as above, we consider the counting function  $n(z, r) = \sum_{\lambda \in D(z, r)} m_\lambda$  and the integrated version

$$N(z, r) = \int_0^r \frac{n(z, t) - n(z, 0)}{t} dt + n(z, 0) \log r.$$

In case we want to specify the variety  $X$  to which the functions  $n$  and  $N$  refer, we will use the notation  $n(z, r, X)$  and  $N(z, r, X)$  respectively.

We are ready to state our main result.

**Theorem 1.** *A variety  $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$  is  $A_p$ -interpolating if and only if:*

- (i) *There is  $C > 0$  such that*

$$N(\lambda, p(\lambda), X) \leq Cp(\lambda) \quad \forall \lambda \in \Lambda.$$

- (ii) *The following Carleson-type condition holds:*

$$\sup_{x \in \mathbb{R}} \sum_{\lambda: |\operatorname{Im} \lambda| > \omega(|\lambda|)} m_\lambda \frac{|\operatorname{Im} \lambda|}{|x - \lambda|^2} < \infty.$$

Since the Poisson kernel at  $\lambda$  in the corresponding half-plane (upper half-plane if  $\operatorname{Im} \lambda > 0$  and lower half-plane when  $\operatorname{Im} \lambda < 0$ ) is  $P(\lambda, x) = \frac{|\operatorname{Im} \lambda|}{|x - \lambda|^2}$ , a restatement of condition (ii) is that the measure  $\sum_{\lambda: |\operatorname{Im} \lambda| > \omega(|\lambda|)} m_\lambda \delta_\lambda$  has bounded Poisson balayage.

*Remark 2.* Notice that for sequences  $\Lambda$  within the region  $|\operatorname{Im} z| \leq \omega(|z|)$ , condition (i) (shown to be necessary by Ehrenpreis & Malliavin and Squires) provides a complete characterization. However, this is not the case in general, i.e. condition (ii) does not follow from (i), as it is shown in the following example. Take the sequence  $\Lambda$  contained in the sector  $\mathcal{A} = \{z \in \mathbb{C}; |\operatorname{Re} z| < \operatorname{Im} z\}$  and having in each segment  $\{\operatorname{Im} z = 2^n\} \cap \mathcal{A}$  exactly  $2^n$  equispaced points. Then  $\Lambda$  satisfies condition (i) (basically  $n(\lambda, t) \leq t$  for  $t \leq p(\lambda)$ ), but it does not satisfy (ii) (it is not even a Blaschke sequence).

### 3. NECESSARY CONDITIONS

A standard feature of the spaces  $A_p$  is that the interpolation can be performed in a stable way. This consequence of the open mapping theorem for (LF)-spaces applied to the restriction mapping  $\mathcal{R}_X$  defined in Section 2 (see [BrGa95, Lemma 2.2.6]) is stated precisely in the following lemma.

**Lemma 3.** *If  $X$  is an interpolating variety, there exist  $C > 0$ ,  $M \in \mathbb{N}$  such that for every  $\lambda \in \Lambda$  there are functions  $f_\lambda, g_\lambda \in A_p$  with bounded norms  $\|f_\lambda\|_{A_{p,M}^\infty}, \|g_\lambda\|_{A_{p,M}^\infty} \leq C$  and*

$$\begin{aligned} f_\lambda^{(l)}(\lambda')/l! &= \delta_{\lambda\lambda'}\delta_{l0}, \\ g_\lambda^{(l)}(\lambda')/l! &= \delta_{\lambda\lambda'}\delta_{l(m_\lambda-1)} \quad \forall \lambda, \lambda' \in \Lambda, \quad 0 \leq l \leq m_\lambda. \end{aligned}$$

An application of Jensen’s Formula to the functions  $f_\lambda, g_\lambda$  in the disk  $D(\lambda, p(\lambda))$  gives the following result (see [EhMa74, Theorem 4] or [Sq83, Theorem 1]).

**Theorem 4.** *If  $X$  is  $A_p$ -interpolating, then condition (i) of Theorem 1 holds.*

The necessity of condition (ii) is an immediate consequence of the following result. Assume that  $\Lambda \cap \mathbb{R} = \emptyset$ ; otherwise move the horizontal line so that it does not touch any of the points in  $\Lambda$ . Let  $\mathbb{H}$  denote the upper half-plane.

**Proposition 5.** *Let  $X$  be  $A_p$ -interpolating. There exist  $C > 0$  such that*

$$\sum_{\substack{\lambda' \in \Lambda \cap \mathbb{H} \\ \lambda' \neq \lambda}} m_{\lambda'} \log \left| \frac{\lambda - \lambda'}{\lambda - \bar{\lambda}'} \right|^{-1} \leq Cp(\lambda) \quad \text{for all } \lambda \in \Lambda \cap \mathbb{H}.$$

Of course, an analogous result could be given for any upper ( $\{z : \text{Im} z > a\}$ ) or lower ( $\{z : \text{Im} z < a\}$ ) half-plane.

Before giving the proof of Proposition 5, we show that it implies condition (ii) of Theorem 1. Define  $\Lambda_+ = \Lambda \cap \{\text{Im} z > \omega(|z|)\}$ . Given  $x \in \mathbb{R}$  consider  $\lambda \in \Lambda_+$  such that  $|x - \lambda| = \inf_{\Lambda_+} |x - \lambda|$ . Then

$$|\lambda - \bar{\lambda}'| \leq |\lambda - x| + |x - \bar{\lambda}'| = |\lambda - x| + |x - \lambda'| \leq 2|x - \lambda'|,$$

and therefore

$$\sum_{\lambda' \in \Lambda_+} m_{\lambda'} \frac{|\text{Im } \lambda'|}{|x - \bar{\lambda}'|^2} \leq 2 \sum_{\lambda' \in \Lambda_+} m_{\lambda'} \frac{|\text{Im } \lambda'|}{|\lambda - \bar{\lambda}'|^2}.$$

The estimate  $\log t^{-1} \geq 1 - t$  for  $t \in (0, 1)$  shows that

$$\sum_{\substack{\lambda' \in \Lambda_+ \\ \lambda' \neq \lambda}} m_{\lambda'} \frac{|\text{Im } \lambda| |\text{Im } \lambda'|}{|\lambda - \bar{\lambda}'|^2} \leq \sum_{\substack{\lambda' \in \Lambda_+ \\ \lambda' \neq \lambda}} m_{\lambda'} \log \left| \frac{\lambda - \lambda'}{\lambda - \bar{\lambda}'} \right|^{-1}.$$

Since  $p(\lambda) \simeq |\text{Im } \lambda|$  for  $\lambda \in \Lambda_+$ , it is clear that this implies condition (ii) of Theorem 1.

*Remark 6.* The necessary condition of Proposition 5 can be seen as a Carleson-type condition; it can be rewritten as

$$|B_\lambda(\lambda)| \geq \delta e^{-Cp(\lambda)}, \quad \lambda \in \Lambda \cap \mathbb{H},$$

where  $B$  denotes the Blaschke product in  $\mathbb{H}$  of  $\{(\lambda, m_\lambda)\}_{\lambda \in \Lambda \cap \mathbb{H}}$ , and

$$B_\lambda(z) = B(z) \left( \frac{z - \bar{\lambda}}{z - \lambda} \right)^{m_\lambda}.$$

It can also be seen as density conditions for the counting function associated to the hyperbolic metric in the half-plane. Letting  $\nu = \sum_{\lambda \in \Lambda \cap \mathbb{H}} m_\lambda \delta_\lambda$  and using the distribution function we have

$$\sum_{\lambda \in \Lambda \cap \mathbb{H}} m_\lambda \log \left| \frac{z - \lambda}{z - \bar{\lambda}} \right|^{-1} = \int_{\mathbb{H}} \log \left| \frac{z - \zeta}{z - \bar{\zeta}} \right|^{-1} d\nu(\zeta) = \int_0^1 \frac{n_{\mathbb{H}}(z, t)}{t} dt,$$

where

$$D_{\mathbb{H}}(z, t) = \left\{ \zeta : \left| \frac{z - \zeta}{z - \bar{\zeta}} \right| < t \right\}, \quad \text{and} \quad n_{\mathbb{H}}(z, t) := \nu(D_{\mathbb{H}}(z, t))$$

is the number of points of  $\Lambda$  in the pseudohyperbolic disk of “center”  $z$  and “radius”  $t$  (actually the true disk of center  $\operatorname{Re}z + i\frac{1+t^2}{1-t^2}\operatorname{Im}z$  and radius  $\frac{2t}{1-t^2}\operatorname{Im}z$ ).

*Proof of Proposition 5.* Let  $z = x + iy$  and consider the Poisson transform of  $\omega(|t|)$ :

$$u(z) := P[\omega](z) = \int_{\mathbb{R}} \frac{y\omega(|t|)}{(x-t)^2 + y^2} dt,$$

which converges by (d). Define  $H = \exp(u + i\tilde{u})$ , where  $\tilde{u}$  is a harmonic conjugate of  $u$ .

Given  $\lambda \in \Lambda \cap \mathbb{H}$ , take the function  $f_\lambda$  given by Lemma 3 and define

$$h_\lambda(z) = \frac{f_\lambda(z)e^{iM_1z}}{(H(z))^{M_2}},$$

with  $M_1, M_2$  to be chosen. It is clear that  $h_\lambda$  is holomorphic in  $\mathbb{H}$ . On the other hand, for all  $z$  in the upper half-plane  $|\log |H(z)| - \omega(|\operatorname{Re}z|)| \leq A + B|\operatorname{Im}z|$ ; see [Bj66, Lemma 1.3.11]. Moreover,  $|\omega(|\operatorname{Re}z|) - \omega(|z|)| \leq \omega(|\operatorname{Im}z|) \leq A + B|\operatorname{Im}z|$ , thus  $|\log |H(z)| - \omega(|z|)| \leq A + B|\operatorname{Im}z|$ . Therefore, if  $M_1$  and  $M_2$  are big enough,  $h_\lambda$  is bounded in  $\mathbb{H}$  by a constant which does not depend on  $\lambda$ :

$$|h_\lambda(z)| \leq Ce^{Mp(z) - M_1\operatorname{Im}z - M_2\log |H(z)|} \lesssim 1.$$

Also,

$$|h_\lambda(\lambda)| = e^{-M_1\operatorname{Im}\lambda - M_2\log |H(\lambda)|} \geq e^{-Cp(\lambda)}.$$

Now apply Jensen’s formula in the half-plane to the function  $h_\lambda$ :

$$\log |h_\lambda(\lambda)| = \int_{\mathbb{R}} P(\lambda, x) \log |h_\lambda(x)| dx - \int_{\mathbb{H}} G(\lambda, \zeta) \Delta \log |h_\lambda(\zeta)|,$$

where  $P(\lambda, x)$  denotes the Poisson kernel and  $G(\lambda, \zeta) = \log \left| \frac{\lambda - \zeta}{\lambda - \bar{\zeta}} \right|^{-1}$  is the Green function in  $\mathbb{H}$  with pole in  $\lambda$ .

Since  $h_\lambda$  vanishes on  $\Lambda \setminus \{\lambda\}$ , Jensen’s Formula and the estimates above yield

$$\sum_{\substack{\lambda' \in \Lambda \cap \mathbb{H} \\ \lambda' \neq \lambda}} m_{\lambda'} \log \left| \frac{\lambda - \lambda'}{\lambda - \bar{\lambda}'} \right|^{-1} \leq \sup_{\mathbb{R}} \log |h_\lambda| - \log |h_\lambda(\lambda)| \lesssim p(\lambda).$$

□

#### 4. SUFFICIENT CONDITIONS

We split the sequence into three pieces, according to the non-isotropy of the weight  $p$ . Consider the regions

$$\Omega_0 = \{z \in \mathbb{C} : |\operatorname{Im}z| \leq \omega(|z|)\},$$

$$\Omega_+ = \{z \in \mathbb{C} : \operatorname{Im}z > \omega(|z|)\},$$

$$\Omega_- = \{z \in \mathbb{C} : \operatorname{Im}z < -\omega(|z|)\},$$

and define  $\Lambda_0 = \Lambda \cap \Omega_0$ ,  $\Lambda_+ = \Lambda \cap \Omega_+$  and  $\Lambda_- = \Lambda \cap \Omega_-$ . Let also  $X_0 = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda_0}$ ,  $X_+ = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda_+}$  and  $X_- = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda_-}$ .

It is enough to prove that each piece  $X_+, X_-, X_0$  of the variety  $X$  is  $A_p$ -interpolating. This is so because  $X$  is weakly separated (see Lemma 7(i) below), and

a weakly separated union of a finite number of  $A_p$ -interpolating varieties is also  $A_p$ -interpolating [Ou03, Theorem II.1]. It is also clear that the varieties  $X^+$  and  $X^-$  can be dealt with similarly.

We start with some easy consequences of condition (i) of Theorem 1.

**Lemma 7.** *If condition (i) in Theorem 1 holds, then*

- (i)  *$X$  is weakly separated: there exist  $\delta, C > 0$  such that the disks  $D_\lambda = D(\lambda, \delta e^{-C \frac{p(\lambda)}{m_\lambda}})$  are pairwise disjoint, i.e.*

$$|\lambda - \lambda'| \geq 2\delta \max[e^{-C \frac{p(\lambda)}{m_\lambda}}, e^{-C \frac{p(\lambda')}{m_{\lambda'}}}], \quad \lambda \neq \lambda'.$$

- (ii) *There exist  $\varepsilon, C > 0$  such that  $n(z, \varepsilon p(z), X) \leq Cp(z), \forall z \in \mathbb{C}$ .*

*Proof.* (i) If there exists  $\lambda'$  such that  $|\lambda' - \lambda| < 1$ , then

$$N(\lambda, p(\lambda), X) \geq \int_{|\lambda' - \lambda|}^1 \frac{n(\lambda, t) - m_\lambda}{t} dt \geq \int_{|\lambda' - \lambda|}^1 \frac{m_{\lambda'}}{t} dt = \log\left(\frac{1}{|\lambda' - \lambda|}\right)^{m_{\lambda'}}.$$

Using condition (i) in Theorem 1 and reversing the roles of  $\lambda$  and  $\lambda'$  we obtain the desired estimate.

- (ii) When  $z = \lambda \in \Lambda$ , this is immediate from the estimate

$$\int_{1/2p(\lambda)}^{p(\lambda)} \frac{n(\lambda, 1/2p(\lambda)) - 1}{t} dt \leq N(\lambda, p(\lambda)).$$

When  $z \notin \Lambda$ , then let  $\varepsilon > 0$  be such that  $\zeta \in D(z, \varepsilon p(z))$  implies

$$D(z, \varepsilon p(z)) \subset D(\zeta, 1/2p(\zeta)),$$

which exists by property (f) of the weight. Take  $\lambda \in D(z, \varepsilon p(z))$  (if there is no such  $\lambda$  the estimate is obviously true). Then, by the previous case and property (e) of the weight

$$n(z, \varepsilon p(z)) \leq n(\lambda, 1/2p(\lambda)) \lesssim p(\lambda) \lesssim p(z).$$

□

**4.1. Case  $\Lambda_0$ .** We would like to prove that  $X_0 = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda_0}$  is  $A_p$ -interpolating using a  $\bar{\partial}$ -scheme. This is easier if we can regularize the weight in the following way.

**Lemma 8.** *There exists  $\tilde{p}$  subharmonic in  $\mathbb{C}$  such that  $p(z) \simeq \tilde{p}(z)$  and*

$$(3) \quad 1/\tilde{p}(z) \lesssim \Delta \tilde{p}(z) \quad \text{if} \quad |\text{Im}z| \leq 2\omega(|z|).$$

The fact that  $p \simeq \tilde{p}$  clearly implies that  $A_p = A_{\tilde{p}}$  and the interpolating varieties for  $A_p$  and  $A_{\tilde{p}}$  are the same.

*Proof.* We will construct  $\tilde{p}(z) = |\text{Im}z| + r(z)$ , where  $r$  satisfies the following properties:

- (i)  $r \geq 0$  and  $\tilde{p}$  is subharmonic in  $\mathbb{C}$ .
- (ii)  $r(z) = 0$  if  $|\text{Im}z| \geq 10\omega(|z|)$ .
- (iii)  $1/p(z) \lesssim \Delta \tilde{p}(z)$  and  $r(z) \simeq \omega(|z|)$  if  $|\text{Im}z| \leq 2\omega(|z|)$ .

In order to construct  $r$ , we partition the real line into intervals  $I_n$  defined in the following way.

Let  $x_1 > 1$ ,  $x_{n+1} = x_n + \omega(x_n)$  for  $n \geq 1$  and  $x_n = -x_{-n}$  for  $n \leq -1$ . Set  $I_0 = [x_{-1}, x_1]$ ,  $I_n = [x_n, x_{n+1}]$  for  $n \geq 1$  and  $I_n = [x_{n-1}, x_n]$  for  $n \leq -1$ . Denote by  $\omega_n$  the length of  $I_n$ .

We consider two measures in  $\mathbb{C}$ . The first one is the usual length measure  $d\nu$  in  $\mathbb{R}$ , which we split  $d\nu = \sum_n d\nu_n$ , with  $d\nu_n = dx|_{I_n}$ . The second one is defined as a sum of convolutions of the  $d\nu_n$ 's: let

$$d\mu_n(z) = \left( \frac{1}{100\pi\omega_n^2} \int_{I_n} \chi_{D_n}(z-x) dx \right) dm(z),$$

where  $D_n = D(0, 10\omega_n)$ , and define  $d\mu = \sum_n d\mu_n$ .

Notice that when  $z$  is at a distance of  $I_n$  smaller than  $2\omega_n$ , we can use property (g) of the Beurling weights to deduce that  $d\mu(z) \simeq 1/\omega(|z|) \simeq 1/p(z)$ . Hence  $d\mu(z) \simeq dm(z)/p(z)$ .

Define

$$r(z) = \int_{\mathbb{C}} \log |z-w| (d\mu(w) - d\nu(w)).$$

Since  $\Delta|\operatorname{Im} z| = d\nu$ , we have  $\Delta\tilde{p} = d\mu \geq 0$ .

Let  $S_n$  denote the support of  $\mu_n$ . Let

$$r_n(z) := \int_{\mathbb{C}} \log |z-w| (d\mu_n(w) - d\nu_n(w)) = \int_{S_n} \log |z-w| d\mu_n(w) - \int_{I_n} \log |z-x| dx.$$

Using the definition of  $\mu_n$  and reversing the order of integration we get

$$r_n(z) = \int_{I_n} M(x) dx,$$

where

$$M(x) = \frac{1}{100\pi\omega_n^2} \int_{D(x, 10\omega_n)} \log |z-w| dm(w) - \log |z-x| \geq 0.$$

In particular,  $r$  is non-negative in  $\mathbb{C}$ .

If  $z \notin S_n$  and  $x \in I_n$ ,  $\log |z-w|$  is harmonic in  $D(x, 10\omega_n)$ , hence  $r_n(z) = 0$ .

Suppose now  $z \in D(x_n, 3\omega_n)$ . Then, for each  $x \in I_n$ ,  $|z-x| \leq 4\omega_n$  and

$$M(x) \geq \frac{1}{100\pi\omega_n^2} \int_{9\omega_n \leq |w-x| \leq 10\omega_n} \log \frac{|z-w|}{|z-x|} dm(w) \gtrsim 1.$$

Thus,  $r_n(z) \gtrsim \omega_n \gtrsim \omega(|z|)$ .

If  $z \in S_n$ , using that  $\mu_n$  and  $\nu_n$  have the same mass  $\omega(x_n)$ , we obtain

$$\begin{aligned} \int_{\mathbb{C}} \log |z-w| (d\mu_n(w) - d\nu_n(w)) &\leq \int_{\mathbb{C}} \left| \log \frac{|z-w|}{\omega(x_n)} \right| (d\mu_n(w) + d\nu_n(w)) \\ &\lesssim \int_{\mathbb{C}} \left| \log \frac{|x_n-w|}{\omega(x_n)} \right| (d\mu_n(w) + d\nu_n(w)) \lesssim \omega(|z|). \end{aligned}$$

Since  $|\operatorname{Im} z| \leq 2\omega(|z|)$ ,  $z$  belongs at most to a finite number of  $S_n$ 's and at least to one  $D(x_n, 10\omega_n)$ , by property (g) of the Beurling weights, we are done.  $\square$

Let us prove now that  $X_0$  is  $A_{\tilde{p}}$ -interpolating. In view of Lemma 8, we assume that  $1/p \lesssim \Delta\tilde{p}$  on  $|\operatorname{Im} z| \leq 2\omega(|z|)$ .

Consider the separation radii  $\delta_\lambda := \delta e^{-Cp(\lambda)}$  given by Lemma 7(i).

Given a sequence of values  $\{v_\lambda^\ell\}_{\lambda,\ell}$  satisfying (2), define the smooth interpolating function

$$F(z) = \sum_{\lambda \in \Lambda_0} p_\lambda(z) \mathcal{X}\left(\frac{|z - \lambda|^2}{\delta_\lambda^2}\right),$$

where  $p_\lambda(z) = \sum_{l=0}^{m_\lambda-1} v_\lambda^l (z - \lambda)^l$  and  $\mathcal{X}$  is a smooth cut-off function with  $|\mathcal{X}'| \lesssim 1$ ,  $\mathcal{X}(x) = 1$  if  $|x| \leq 1$  and  $\mathcal{X}(x) = 0$  if  $|x| \geq 2$ .

It is clear that  $F^{(l)}(\lambda)/l! = v_\lambda^l$ , and that  $F$  has the characteristic growth of  $A_p$  functions; the support of  $F$  is contained in  $\bigcup_\lambda D_\lambda$  and for  $z \in D_\lambda$ ,

$$|F(z)| \leq \sum_{l=0}^{m_\lambda-1} |v_\lambda^l| \leq C e^{\alpha p(\lambda)} \lesssim e^{Kp(z)}.$$

There is also a good estimate on  $\bar{\partial}F$ . Its support is the union of the annuli

$$C_\lambda = \{z \in \mathbb{C} : \delta_\lambda \leq |z - \lambda| \leq 2\delta_\lambda\},$$

and for  $z \in C_\lambda$ ,

$$\left| \frac{\partial F}{\partial \bar{z}}(z) \right| \lesssim \sum_{l=0}^{m_\lambda-1} |v_\lambda^l| |\mathcal{X}'| \frac{1}{\delta_\lambda} \lesssim e^{Cp(\lambda)} \lesssim e^{Kp(z)},$$

for  $K$  big enough.

Altogether, there exists  $\gamma > 0$  such that

$$(4) \quad \int_{\mathbb{C}} |F(z)|^2 e^{-\gamma p(z)} < \infty, \quad \int_{\mathbb{C}} |\bar{\partial}F(z)|^2 e^{-\gamma p(z)} < \infty.$$

Now, when looking for a holomorphic interpolating function of the form  $f = F - u$ , we are led to the  $\bar{\partial}$ -problem

$$\bar{\partial}u = \bar{\partial}F,$$

which we solve using Hörmander's theorem [Ho94, Theorem 4.2.1]: given a (pluri)-subharmonic function  $\psi$  in  $\mathbb{C}$ , there exists a solution  $u$  to the above equation such that

$$2 \int_{\mathbb{C}} |u|^2 \frac{e^{-\psi}}{(1 + |z|^2)^2} dm \leq \int_{\mathbb{C}} |\bar{\partial}F|^2 e^{-\psi} dm.$$

We apply Hörmander's theorem with

$$\psi_\beta(z) = \beta p(z) + v(z),$$

where  $\beta > 0$  will be chosen later on and

$$v(z) = \sum_{\lambda \in \Lambda_0} m_\lambda \left[ \log |z - \lambda|^2 - \frac{1}{\pi \varepsilon^2 p^2(\lambda)} \int_{D(\lambda, \varepsilon p(\lambda))} \log |z - \zeta|^2 dm(\zeta) \right].$$

Here  $\varepsilon$  is a fixed small constant to be determined later on.

Integrating by parts the equality

$$\int_0^{2\pi} \log |a - r e^{i\theta}|^2 \frac{d\theta}{2\pi} = \begin{cases} \log |a|^2 & \text{if } |a| > r, \\ \log r^2 & \text{if } |a| \leq r, \end{cases}$$

one sees that for  $a \in \mathbb{C}$  and  $r > 0$ ,

$$\log |a|^2 - \frac{1}{\pi r^2} \int_{D(a,r)} \log |\zeta|^2 dm(\zeta) = \begin{cases} \log \left| \frac{a}{r} \right|^2 + 1 - \left| \frac{a}{r} \right|^2 & \text{if } |a| \leq r, \\ 0 & \text{if } |a| > r. \end{cases}$$

Thus

$$v(z) = \sum_{\lambda:|\lambda-z|\leq \varepsilon p(\lambda)} m_\lambda \left[ \log \frac{|z-\lambda|^2}{\varepsilon^2 p^2(\lambda)} + 1 - \frac{|z-\lambda|^2}{\varepsilon^2 p^2(\lambda)} \right].$$

In particular,  $v \leq 0$  and  $\Delta v(z) = 0$  if  $z \notin \bigcup_\lambda D(\lambda, \varepsilon p(\lambda))$ . For  $z \in \bigcup_\lambda D(\lambda, \varepsilon p(\lambda))$  we have  $|\operatorname{Im} z| \leq 2\omega(|z|)$  and

$$\Delta v(z) \geq \sum_{\lambda:|\lambda-z|\leq \varepsilon p(\lambda)} \frac{-m_\lambda}{\varepsilon^2 p^2(\lambda)} \gtrsim \sum_{\lambda:|\lambda-z|\leq C(\varepsilon)p(z)} \frac{-m_\lambda}{p^2(z)} = -\frac{n(z, C(\varepsilon)p(z))}{p^2(z)}.$$

As observed in Lemma 7(ii), with  $\varepsilon$  small enough  $n(z, C(\varepsilon)p(z)) \lesssim p(z)$ , thus  $\Delta v(z) \gtrsim -1/p(z)$ . This and (3) show that  $\psi_\beta$  is subharmonic if  $\beta$  is chosen big enough.

Also, we deduce from (c) that for any  $\beta' > \beta$ ,

$$\int_{\mathbb{C}} |u|^2 e^{-\beta' p} dm \lesssim \int_{\mathbb{C}} |u|^2 \frac{e^{-\psi_\beta}}{(1+|z|^2)^2} dm \lesssim \int_{\mathbb{C}} |\bar{\partial} F|^2 e^{-\psi_\beta} dm.$$

We need to control  $\psi_\beta$  on the support of  $\bar{\partial} F$ . For  $z \in C_\lambda$ ,

$$\begin{aligned} |\psi_\beta(z) - \beta p(z)| &\leq \sum_{\lambda:|\lambda-z|\leq \varepsilon p(\lambda)} m_\lambda \log \frac{\varepsilon^2 p^2(\lambda)}{|z-\lambda|^2} \\ &\simeq m_\lambda \log \frac{\varepsilon^2 p^2(\lambda)}{|z-\lambda|^2} + \sum_{\substack{\lambda':|z-\lambda'|\leq \varepsilon p(\lambda') \\ \lambda' \neq \lambda}} m_{\lambda'} \log \frac{\varepsilon^2 p^2(\lambda')}{|z-\lambda'|^2} \\ &\lesssim p(\lambda) + \sum_{\substack{\lambda':|\lambda'-z|\leq C(\varepsilon)p(z) \\ \lambda' \neq \lambda}} m_{\lambda'} \log \frac{C(\varepsilon)^2 p^2(z)}{|z-\lambda'|^2} \\ &\lesssim p(z) + N(z, C(\varepsilon)p(z)). \end{aligned}$$

**Claim 9.** For  $\varepsilon$  small enough  $N(z, C(\varepsilon)p(z)) \lesssim p(z)$  for all  $z \in \operatorname{supp}(\bar{\partial} F)$ .

Assuming the claim we have  $|\psi_\beta(z) - \beta p(z)| \leq Kp(z)$  on  $\operatorname{supp}(\bar{\partial} F)$ . Therefore, for  $\beta$  big enough

$$\int_{\mathbb{C}} |u|^2 e^{-\beta' p} dm \lesssim \int_{\mathbb{C}} |\bar{\partial} F|^2 e^{-\psi_\beta} dm \leq \int_{\mathbb{C}} |\bar{\partial} F|^2 e^{-\gamma p} dm < \infty.$$

This shows that  $f := F - u \in A_p$ . Since  $e^{-\psi_\beta} \simeq |z-\lambda|^{-2m_\lambda}$  around each  $\lambda$ , also  $u^{(l)}(\lambda) = 0$  for all  $\lambda \in \Lambda$ ,  $l = 0, \dots, m_\lambda - 1$ , and therefore  $f^{(l)}(\lambda)/l! = F^{(l)}(\lambda)/l! = v_\lambda^l$ , as required.

*Proof of the claim.* Assume  $z \in C_\lambda$  and observe that  $n(z, t) = 0$  for  $t < \delta_\lambda$  and that  $n(z, t) \leq m_\lambda$  for  $\delta_\lambda \leq t < 2\delta_\lambda$ . Since  $D(z, t) \subset D(\lambda, t + 2\delta_\lambda)$  and  $|z| < |\lambda| + 2\delta_\lambda$ , we have (changing into  $s = t + 2\delta_\lambda$ )

$$\begin{aligned} N(z, C(\varepsilon)p(z)) &\leq \int_{\delta_\lambda}^{2\delta_\lambda} \frac{m_\lambda}{t} dt + \int_{2\delta_\lambda}^{C(\varepsilon)p(z)} \frac{n(z, t) - m_\lambda}{t} dt \\ &\leq p(\lambda) + \int_{4\delta_\lambda}^{C(\varepsilon)p(z)+2\delta_\lambda} \frac{n(\lambda, s) - m_\lambda}{s - 2\delta_\lambda} ds \\ &\lesssim p(\lambda) + \int_{4\delta_\lambda}^{C(\varepsilon)p(z)+2\delta_\lambda} \frac{n(\lambda, s) - m_\lambda}{s/2} ds \lesssim p(\lambda) + N(\lambda, C'(\varepsilon)p(\lambda)). \end{aligned}$$

From the properties of the weight and the hypothesis we have finally that for  $\varepsilon$  small  $N(z, C(\varepsilon)p(z)) \lesssim p(\lambda) \lesssim p(z)$ .

4.2. **Case  $\Lambda^+$ .** According to Theorem A, it is enough to construct a function  $G \in A_p$  such that  $X_+ \subset \mathcal{Z}(G)$  and

$$\frac{|G^{(m_\lambda)}(\lambda)|}{m_\lambda!} \geq \varepsilon e^{-Kp(\lambda)}, \quad \lambda \in \Lambda_+,$$

for some constants  $\varepsilon, k > 0$ . In fact, the hypotheses of Theorem A require the weight  $p$  to be subharmonic, and our weights are not necessarily so. Nevertheless, by Lemma 8, there exists a subharmonic weight  $\tilde{p}$  equivalent to  $p$ , and we may apply Theorem A to  $\tilde{p}$ .

Take any entire function  $F$  such that  $\mathcal{Z}(F) = X_+$ . Since the necessary conditions imply that  $X_+$  satisfies the Blaschke condition in  $\mathbb{H}$ , we can consider also the Blaschke product

$$B(z) = \prod_{\lambda \in \Lambda_+} \left( \frac{z - \lambda}{z - \bar{\lambda}} \right)^{m_\lambda}, \quad z \in \mathbb{H}.$$

Define

$$\phi(z) = \begin{cases} \log \left| \frac{F(z)}{B(z)} \right|, & \text{Im } z > 0, \\ \log |F(z)|, & \text{Im } z \leq 0. \end{cases}$$

**Lemma 10.**  *$\phi$  is harmonic outside the real axis, subharmonic on  $\mathbb{C}$  and its Laplacian is uniformly bounded.*

*Proof.* It is clear, by definition, that  $\phi$  is harmonic on  $\mathbb{C} \setminus \mathbb{R}$ . In order to prove that  $\phi$  is subharmonic on  $\mathbb{C}$ , it is enough to check the mean inequality for  $x \in \mathbb{R}$ . We have

$$\phi(x) = \log |F(x)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |F(x + re^{i\theta})| d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(x + re^{i\theta}) d\theta.$$

Since  $\Delta \log |F| \equiv 0$  around  $\mathbb{R}$ , it is enough to compute the Laplacian of

$$\psi(z) = \begin{cases} \log \frac{1}{|B(z)|}, & \text{Im } z > 0, \\ 0, & \text{Im } z \leq 0. \end{cases}$$

Being

$$\log \frac{1}{|B(z)|} = \frac{1}{2} \sum_{\lambda \in \Lambda^+} m_\lambda \log \left| \frac{z - \bar{\lambda}}{z - \lambda} \right|^2,$$

it will be enough to compute the Laplacian of each term

$$\psi_\lambda(z) = \begin{cases} \log \left| \frac{z - \bar{\lambda}}{z - \lambda} \right|^2, & \text{Im } z > 0, \\ 0, & \text{Im } z \leq 0. \end{cases}$$

It is clear that  $\partial\psi_\lambda/\partial x = 0$  on  $\mathbb{R}$ , hence  $\Delta\psi_\lambda = \partial^2\psi_\lambda/\partial y^2$ . Since  $\psi_\lambda$  is continuous around  $\mathbb{R}$ , this Laplacian has a magnitude equivalent to the jump of the first derivative of  $\psi_\lambda$ . The derivative of the Green function on the half-plane with respect to the normal direction  $y$  is the Poisson kernel:

$$\frac{\partial}{\partial y} \log \left| \frac{z - \bar{\lambda}}{z - \lambda} \right|^2 \Big|_{y=0} = \frac{4\text{Im } \lambda}{|x - \lambda|^2}.$$

Therefore,

$$\Delta\phi(x) = 4 \sum_{\lambda \in \Lambda^+} m_\lambda \frac{\operatorname{Im} \lambda}{|x - \lambda|^2} dx,$$

which is bounded by the hypothesis. □

Define

$$\Psi(z) = N|\operatorname{Im} z| - \phi(z).$$

Observe that  $\Delta\Psi(z) = N dx - \Delta\phi(x) dx$ , thus according to the previous lemma  $\Delta\Psi \simeq dx$  when  $N \in \mathbb{N}$  is big enough. In this situation, according to [OrSe99, Lemma 3], there exists a multiplier associated to  $\Psi$ , i.e., an entire function  $h$  such that:

- (a)  $\mathcal{Z}(h)$  is a separated sequence contained in  $\mathbb{R}$ .
- (b) Given any  $\varepsilon > 0$ ,  $|h(z)| \simeq \exp(\Psi(z))$  for all points  $z$  such that  $d(z, \mathcal{Z}(h)) > \varepsilon$ .

Define now  $G = hF$ . It is clear that  $G \in A_p$ :

$$|G(z)| \lesssim e^{\Psi(z) + \log |F(z)|} \leq e^{\Psi(z) + \phi(z)} \leq e^{Np(z)}, \quad z \in \mathbb{C}.$$

It is also clear that  $X_+ \subset \mathcal{Z}(G)$ , since  $X_+ \subset \mathcal{Z}(F)$ .

In order to prove that there exist  $\varepsilon, C > 0$  such that

$$(5) \quad \left| \frac{G^{(m_\lambda)}(\lambda)}{m_\lambda!} \right| \geq \varepsilon e^{-Cp(\lambda)}$$

consider then the disjoint disks  $D_\lambda = D(\lambda, \delta_\lambda)$ ,  $\delta_\lambda = \delta e^{-C\frac{p(\lambda)}{m_\lambda}}$  given by Lemma 7(i). Since  $\Lambda_+$  is far from  $\mathcal{Z}(h)$ , the estimate

$$|G(z)| = |h(z)|e^{\phi(z)}|B(z)| \simeq e^{N|\operatorname{Im} z|}|B(z)|, \quad z \in \partial D_\lambda,$$

holds.

**Claim 11.** *There exists  $C > 0$  such that  $|B(z)| \geq \varepsilon e^{-Cp(z)}$ ,  $z \in \partial D_\lambda$ .*

Assuming this we have  $|G(z)| \gtrsim e^{-Cp(z)}$  for all  $z \in \partial D_\lambda$ . Define then  $g(z) = G(z)/(z - \lambda)^{m_\lambda}$ . It is clear that  $g$  is holomorphic, non-vanishing in  $D_\lambda$ , and  $|g(z)| \gtrsim e^{-cp(\lambda)}$  for  $z \in \partial D_\lambda$ . By the minimum principle

$$\left| \frac{G^{(m_\lambda)}(\lambda)}{m_\lambda!} \right| = |g(0)| \gtrsim e^{-cp(\lambda)},$$

as desired.

*Proof of the claim.* As observed in Remark 6(b), the estimate we want to prove is equivalent to

$$\int_0^1 \frac{n_{\mathbb{H}}(z, t)}{t} dt \lesssim p(z), \quad z \in \partial D_\lambda.$$

This is proved like Claim 9, except we replace the Euclidean disks by the hyperbolic ones. We have

$$\int_0^1 \frac{n_{\mathbb{H}}(z, t)}{t} dt \lesssim \int_{\delta_\lambda}^{2\delta_\lambda} \frac{m_\lambda}{t} dt + \int_{2\delta_\lambda}^1 \frac{n_{\mathbb{H}}(z, t) - m_\lambda}{t} dt.$$

The first term is controlled by  $p(\lambda)$ . In order to control the second term observe that  $D_{\mathbb{H}}(z, t) \subset D_{\mathbb{H}}(\lambda, \frac{t+\delta_\lambda}{1+t\delta_\lambda})$ ; hence changing the variable into  $s = \frac{t+\delta_\lambda}{1+t\delta_\lambda}$  we get

$$\int_{2\delta_\lambda}^1 \frac{n_{\mathbb{H}}(z, t) - m_\lambda}{t} dt \leq \int_{\frac{3\delta_\lambda}{1+2\delta_\lambda^2}}^1 \frac{n_{\mathbb{H}}(\lambda, s) - m_\lambda}{s - \delta_\lambda} \frac{1 - \delta_\lambda^2}{(1 - \delta_\lambda)^2} ds.$$

There is no restriction in assuming that  $\delta_\lambda < 1/2$ . Then  $\frac{3\delta_\lambda}{1+2\delta_\lambda^2} > 2\delta_\lambda$  and therefore  $s - \delta_\lambda > s/2$ . With this and condition (ii) in Theorem 1 we obtain

$$\int_0^1 \frac{n_{\mathbb{H}}(z, t)}{t} dt \lesssim p(\lambda) + \int_0^1 \frac{n_{\mathbb{H}}(\lambda, s) - m_\lambda}{s} ds.$$

Since  $p(\lambda) \lesssim p(z)$ , we will be done as soon as we prove that

$$\int_0^1 \frac{n_{\mathbb{H}}(\lambda, s) - m_\lambda}{s} ds \lesssim p(\lambda).$$

There exists  $\delta > 0$  (independent of  $\lambda$ ) such that  $D_{\mathbb{H}}(\lambda, \delta) \subset D(\lambda, p(\lambda))$ . Then

$$\begin{aligned} \int_0^\delta \frac{n_{\mathbb{H}}(\lambda, s) - m_\lambda}{s} ds &= \sum_{0 < |\frac{\lambda - \lambda'}{\lambda - \lambda'}| < \delta} m_{\lambda'} \log \frac{\delta}{|\frac{\lambda - \lambda'}{\lambda - \lambda'}|} \leq \sum_{0 < |\frac{\lambda - \lambda'}{\lambda - \lambda'}| < \delta} m_{\lambda'} \log \frac{p(\lambda)}{|\lambda - \lambda'|} \\ &\lesssim \sum_{0 < |\lambda - \lambda'| < p(\lambda)} m_{\lambda'} \log \frac{p(\lambda)}{|\lambda - \lambda'|} \leq N(\lambda, p(\lambda)) \lesssim p(\lambda). \end{aligned}$$

For the remaining part we use condition (ii) in Theorem 1 and the estimate  $\log t^{-1} \simeq 1 - t$  for  $\delta < t < 1$ . Taking  $x = \operatorname{Re} \lambda$  we have

$$\begin{aligned} \int_\delta^1 \frac{n_{\mathbb{H}}(\lambda, s) - m_\lambda}{s} ds &\lesssim \sum_{\lambda \neq \lambda'} m_{\lambda'} \frac{|\operatorname{Im} \lambda| |\operatorname{Im} \lambda'|}{|\lambda - \bar{\lambda}'|^2} \\ &\lesssim \sum_{\lambda \neq \lambda'} m_{\lambda'} \frac{|\operatorname{Im} \lambda| |\operatorname{Im} \lambda'|}{|x - \lambda'|^2} \lesssim |\operatorname{Im} \lambda| \simeq p(\lambda). \end{aligned}$$

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