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FUNCTIONAL DISTRIBUTION OF $L(s, \chi_d)$ WITH REAL CHARACTERS AND DENSENESS OF QUADRATIC CLASS NUMBERS

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Abstract. We investigate the functional distribution of L-functions $L(s, \chi_d)$ with real primitive characters χ_d on the region $1/2 < \operatorname{Re} s < 1$ as d varies over fundamental discriminants. Actually we establish the so-called *universality theorem* for $L(s, \chi_d)$ in the d-aspect. From this theorem we can, of course, deduce some results concerning the value distribution and the non-vanishing. As another corollary, it follows that for any fixed a, b with $1/2 < a < b < 1$ and positive integers r', m, there exist infinitely many d such that for every $r = 1, 2, \cdots, r'$ the r-th derivative $L^{(r)}(s, \chi_d)$ has at least m zeros on the interval $[a, b]$ in the real axis. We also study the value distribution of $L(s, \chi_d)$ for fixed s with $\operatorname{Re} s = 1$ and variable d, and obtain the denseness result concerning class numbers of quadratic fields.

1. Introduction

Throughout this paper, for a discriminant d let χ_d denote the real character modulo $|\,d\,|$ defined by the Kronecker symbol $\left(\frac{d}{\cdot}\right)$, and $L(s, \chi_d)$ the Dirichlet L-function associated with χ_d. Various results concerning the value distribution of $L(s, \chi_d)$ as d varies over discriminants have been obtained by many authors (see e.g. [CE], [El], [GS], [J1], [Li], [St]). The main purpose of this paper is to investigate the functional distribution of $L(s, \chi_d)$ on D as d varies. Here and in the following, D denotes the region $\{s \in \mathbb{C} \mid 1/2 < \operatorname{Re} s < 1\}$ in the complex plane.

Before stating our theorems, we recall some related results on the Riemann zeta-function $\zeta(s)$. The study of the value distribution of $\zeta(\sigma + it)$ for fixed σ and variable $t \in \mathbb{R}$ was initiated by H. Bohr. Bohr and Courant [BC] have shown that the set $\{\zeta(\sigma + it) \mid t \in \mathbb{R}\}$ is dense in $\mathbb{C}$ for any fixed $\sigma \in \mathbb{R}$ with $1/2 < \sigma \leq 1$ (see also [Bo]). Further, Bohr and other authors studied the value distribution of $\zeta(\sigma + it)$ in terms of weak convergence of probability measures. Beyond the value distribution, in 1975 Voronin ([Vo], [KV]) obtained the remarkable result called the *universality theorem* for $\zeta(s)$, which concerns the functional distribution of $\zeta(s)$. One current form of the universality theorem for $\zeta(s)$ is the following. *Suppose that $h(s)$ is a holomorphic function on D which has no zeros. Let K be a compact*

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set in D. Let $\varepsilon > 0$. Then

$$\liminf_{T\to\infty} \frac{1}{T}\, m\left(\{0 < t < T \mid \max_{s\in K} |\, \zeta(s+it) - h(s)\,| < \varepsilon\}\right) \;>\; 0, \tag{1.1}$$

where m denotes the Lebesgue measure on $\mathbb{R}$.

We note that this form is equivalent to that of [La, Theorem 6.5.2], by virtue of Mergelyan's theorem ([Ru, Theorem 20.5]). See also the argument in [La, pp. 231–232]. The universality theorem for the Dirichlet L-function $L(s,\chi)$ with a character χ mod q was established by Bagchi [B2], Gonek [Go] and Voronin (see [KV, Chapter VII, Section 3]) independently; in fact, the *joint* universality theorem for $L(s,\chi)$'s, which is a stronger result, was established.

There are other types of universality theorems, in which parameters other than t as in (1.1) vary. In Bagchi's paper [B1, p. 154] it is shown that for any $h(s), K, \varepsilon$ as above, there is a constant $c > 0$ such that for all sufficiently large prime q, at least cq of the Dirichlet characters χ mod q satisfy $\max_{s\in K} |\, L(s,\chi) - h(s)\,| < \varepsilon$. See also [Go, Chapter V]. Recently, the universality theorem for $GL(2)/\mathbb{Q}$-automorphic L-functions $L(s,f)$ as f varies over a certain family of automorphic forms has been established in the second author's paper [Na], by using Selberg's trace formulas and so on.

We shall next prepare some notation. Throughout this paper, as usual we denote by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ the set of all complex numbers, real numbers, positive real numbers, negative real numbers, rational numbers, integers, and positive integers, respectively. Henceforth, p will denote a prime, and d a fundamental discriminant, so that χ_d is a real primitive character modulo $|\, d\,|$.

Let the letter γ stand for the plus $+$ or the minus $-$. For $\gamma \in \{+,-\}$ we define $\mathcal{D}^\gamma$ to be the set of positive square-free integers $n \neq 1$ with $n \equiv 1 (\bmod 8)$ if γ is $+$, and the set of negative square-free integers n with $n \equiv 1 (\bmod 8)$ if γ is $-$. Then every integer in $\mathcal{D}^\gamma$ is a fundamental discriminant. Note that $\chi_d(2) = 1$ for $d \in \mathcal{D}^\gamma$ by the definition of Kronecker's symbol. Further, for $X > 0$ we define the set

$$\mathcal{D}^\gamma_X := \begin{cases} \{d \in \mathcal{D}^+ \mid d \le X\} & \text{if } \gamma \text{ is } +, \\ \{d \in \mathcal{D}^- \mid d \ge -X\} & \text{if } \gamma \text{ is } -. \end{cases}$$

Our main result is Theorem 1.1, which is the universality theorem for $L(s,\chi_d)$ in the d-aspect. From this theorem we can, of course, deduce the denseness result on values of $L(s,\chi_d)$ for fixed $s \in D$ and variable $d \in \mathcal{D}^\gamma$ (Corollary 1.2). Besides, it follows that a positive proportion of d's satisfy $L(s,\chi_d) \neq 0$ uniformly on K, and more strongly that for any $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, a positive proportion of d's satisfy $\alpha < |\, L(s,\chi_d)\,| < \beta$ uniformly on K (Corollary 1.3). See the paper [CS] for the non-vanishing of $L(s,\chi_d)$'s uniformly on the interval $[1/2, 1] \subset \mathbb{R}$, for a certain proportion of d's. As another corollary of Theorem 1.1, we can obtain a result concerning the horizontal distribution of zeros of $L^{(r)}(s,\chi_d)$ on the segment $(1/2, 1)$ (see Corollary 1.4), noting that any $L(s,\chi_d)$ is $\mathbb{R}$-valued on this segment differently from $L(s,\chi)$ for a general Dirichlet character χ mod q. For a quadratic field $F/\mathbb{Q}$ we denote by $\zeta_F(s)$ the Dedekind zeta-function of F. Then we have Corollary 1.5 and Corollary 1.6 which concern the functional distribution of $\zeta_F(s)$ as F varies.

Theorem 1.1. *Let $\gamma \in \{+,-\}$. Let Ω be a simply connected region in D which is symmetric with respect to the real axis. Suppose that $h(s)$ is a holomorphic*

function on Ω which has no zeros on Ω and is $\mathbb{R}^+$-valued on the set $\Omega\cap\mathbb{R}$. Let K be a compact set in Ω. Let $\varepsilon>0$. Then there exist infinitely many $d\in\mathcal{D}^\gamma$ such that $\max_{s\in K}|\,L(s,\chi_d)-h(s)\,|<\varepsilon$. More precisely, we have

$$\liminf_{X\to\infty}\frac{1}{\#\mathcal{D}_X^\gamma}\#\{d\in\mathcal{D}_X^\gamma\mid \max_{s\in K}|\,L(s,\chi_d)-h(s)\,|<\varepsilon\}\ >\ 0. \tag{1.2}$$

A typical example of $h(s)$ in Theorem 1.1 is the function $e^{p(s)}$, where $p(s)$ is a polynomial in s whose coefficients are real numbers.

Corollary 1.2. (1) *Let any $s_0\in D$ be fixed with $\operatorname{Im}s_0\neq 0$. Then the set $\{L(s_0,\chi_d)|\ d\in\mathcal{D}^\gamma\}$ is dense in $\mathbb{C}$. In fact, for arbitrary $z_0\in\mathbb{C}$ and $\varepsilon>0$ we have*

$$\liminf_{X\to\infty}\frac{1}{\#\mathcal{D}_X^\gamma}\#\{d\in\mathcal{D}_X^\gamma\mid |\,L(s_0,\chi_d)-z_0\,|<\varepsilon\}\ >\ 0. \tag{1.3}$$

(2) *Let $1/2<\sigma_0<1$ be fixed. Then the set $\{L(\sigma_0,\chi_d)\,|\,d\in\mathcal{D}^\gamma\}$ is dense in $\mathbb{R}^+$. In fact, for arbitrary $x_0\in\mathbb{R}^+$ and $\varepsilon>0$ we have*

$$\liminf_{X\to\infty}\frac{1}{\#\mathcal{D}_X^\gamma}\#\{d\in\mathcal{D}_X^\gamma\mid |\,L(\sigma_0,\chi_d)-x_0\,|<\varepsilon\}\ >\ 0.$$

Corollary 1.3. *Let α,β be any positive real numbers with $\alpha<\beta$. Let K be a compact set in D. Then*

$$\liminf_{X\to\infty}\frac{1}{\#\mathcal{D}_X^\gamma}\#\{d\in\mathcal{D}_X^\gamma\mid \alpha<|\,L(s,\chi_d)\,|<\beta \text{ uniformly for } s\in K\}\ >\ 0.$$

Corollary 1.4. *Let $a,b\in\mathbb{R}$ with $1/2<a<b<1$ and $r',m\in\mathbb{N}$. Then there exist infinitely many $d\in\mathcal{D}^\gamma$ such that for every $r\in\mathbb{N}$ with $1\le r\le r'$, the r-th derivative $L^{(r)}(s,\chi_d)$ has at least m zeros on the interval $[a,b]\subset\mathbb{R}$. More precisely,*

$$\liminf_{X\to\infty}\frac{1}{\#\mathcal{D}_X^\gamma}\#\{d\in\mathcal{D}_X^\gamma\mid L^{(r)}(s,\chi_d) \text{ has at least } m \text{ zeros on } [a,b] \text{ for every } r=1,2,\dots,r'\}\ >\ 0.$$

Corollary 1.5. *Let Ω be as in Theorem 1.1. Assume that the Riemann zeta-function $\zeta(s)$ has no zeros on Ω. Let $g(s)$ be a holomorphic function on Ω which has no zeros on Ω and is $\mathbb{R}^-$-valued on $\Omega\cap\mathbb{R}$. Let K be a compact set in Ω, and $\varepsilon>0$. Then there exist infinitely many real quadratic fields F such that $\max_{s\in K}|\,\zeta_F(s)-g(s)\,|<\varepsilon$, and infinitely many imaginary quadratic fields F' such that $\max_{s\in K}|\,\zeta_{F'}(s)-g(s)\,|<\varepsilon$. More precisely,*

$$\liminf_{X\to\infty}\frac{1}{\#\mathcal{D}_X^\gamma}\#\{d\in\mathcal{D}_X^\gamma\mid \max_{s\in K}\left|\,\zeta_{\mathbb{Q}(\sqrt{d})}(s)-g(s)\,\right|<\varepsilon\}\ >\ 0.$$

Since $\zeta(s)$ has no zeros, for example, on the real line segment $(1/2,1)$, we can for the present obtain the following without any assumption.

Corollary 1.6. *Let $1/2<a<b<1$ and $\varepsilon>0$. Let $g(x)$ be a $\mathbb{R}^-$-valued continuous function on the interval $[a,b]\subset\mathbb{R}$. Then*

$$\liminf_{X\to\infty}\frac{1}{\#\mathcal{D}_X^\gamma}\#\{d\in\mathcal{D}_X^\gamma\mid \max_{x\in[a,b]}\left|\,\zeta_{\mathbb{Q}(\sqrt{d})}(x)-g(x)\,\right|<\varepsilon\}\ >\ 0.$$

We shall also investigate the value distribution of $L(s,\chi_d)$ for fixed $s\in\mathbb{C}$ with $\operatorname{Re}s=1$ and variable $d\in\mathcal{D}^\gamma$. Our results are Theorem 1.7 and Theorem 1.8, from which we obtain Corollary 1.9 by Dirichlet's class number formula. As a related

result to Theorem 1.8 and (2) of Corollary 1.2, the weaker result which asserts that the values $L(\sigma_0, \chi_d)$ for fixed $\sigma_0 \in \mathbb{R}$ with $3/4 < \sigma_0 \leq 1$ as d varies over all discriminants (not only fundamental discriminants) are dense in $\mathbb{R}^+$ is obtained in Chowla and Erdös' paper [CE]. It seems that there is no paper which explicitly gives the denseness result as in Corollary 1.9 about class numbers of quadratic fields. Note that as a related topic the moments of those class numbers have been investigated by some authors (see e.g. [J1], [Ba], [GS]).

Theorem 1.7. *Let $\gamma \in \{+,-\}$ and let $t \in \mathbb{R} - \{0\}$ be fixed. Then the set $\{L(1+it, \chi_d) \mid d \in \mathcal{D}^\gamma\}$ is dense in $\mathbb{C}$. More precisely, for any $z_0 \in \mathbb{C}$ and $\varepsilon > 0$ we have*

$$\liminf_{X\to\infty} \frac{1}{\#\mathcal{D}_X^\gamma} \#\{d \in \mathcal{D}_X^\gamma \mid |\, L(1+it, \chi_d) - z_0 \,| < \varepsilon\} \ > \ 0.$$

Theorem 1.8. *Let $\gamma \in \{+,-\}$. Then the set $\{L(1, \chi_d) \mid d \in \mathcal{D}^\gamma\}$ is dense in $\mathbb{R}^+$. More precisely, for any $x_0 \in \mathbb{R}^+$ and $\varepsilon > 0$ we have*

$$\liminf_{X\to\infty} \frac{1}{\#\mathcal{D}_X^\gamma} \#\{d \in \mathcal{D}_X^\gamma \mid |\, L(1, \chi_d) - x_0 \,| < \varepsilon\} \ > \ 0.$$

We denote by $h(d)$ the class number of the quadratic field $\mathbb{Q}(\sqrt{d})$ with $d \in \mathcal{D}^\gamma$, and by $\varepsilon(d)$ the fundamental unit of $\mathbb{Q}(\sqrt{d})$ with $d \in \mathcal{D}^+$. Note that the discriminant of $\mathbb{Q}(\sqrt{d})$ is equal to d if $d \in \mathcal{D}^\gamma$.

Corollary 1.9. *The sets $\{\frac{h(d)\log\varepsilon(d)}{\sqrt{d}} \mid d \in \mathcal{D}^+\}$ and $\{\frac{h(d)}{\sqrt{|\,d\,|}} \mid d \in \mathcal{D}^-\}$ are dense in $\mathbb{R}^+$.*

It seems that not only Theorems 1.1, 1.7, 1.8 but also all the above corollaries are new results.

Finally, we make a remark concerning the proofs of our theorems and the composition of this paper. Roughly speaking, one important key to proving the above results of Bohr, Voronin, etc., on $\zeta(s)$ is the fact that for fixed distinct primes $p_1, p_2, \ldots, p_n$, the set $(p_1^{-it}, p_2^{-it}, \ldots, p_n^{-it}) \in \prod_{j=1}^n \{s \in \mathbb{C} \mid |\, s \,| = 1\}$ behaves *randomly* as $t \in \mathbb{R}$ varies; actually the behavior is ergodic, since $\log p_1, \log p_2, \ldots, \log p_n$ are linearly independent over $\mathbb{Q}$. In contrast, the corresponding key in our case is the fact that for fixed distinct primes $p_1, p_2, \ldots, p_n$, the set $(\chi_d(p_1), \chi_d(p_2), \ldots, \chi_d(p_n)) \in \prod_{j=1}^n \{1, -1, 0\}$ behaves *randomly* as $d \in \mathcal{D}^\gamma$ varies (see Section 4). This fact is used more effectively in [El], [GS], etc., by constructing certain probability spaces. In Section 2 we prove by virtue of the theory of functional analysis that every function $g(s)$ on Ω satisfying a certain condition can be approximated by a function $\log \prod_{p\leq\nu} (1 - a_p p^{-s})^{-1}$ with $a_p \in \{1, -1\}$. In Section 3 it is shown that there are many d such that $L(s, \chi_d)$ is approximated by its finite Euler product on D. Combining Section 2, Section 3 and Section 4, we complete the proofs of Theorem 1.1 and its corollaries in Section 5. In Section 6 we study the value distribution of $L(s, \chi_d)$ for fixed s with $\operatorname{Re} s = 1$ and variable $d \in \mathcal{D}^\gamma$, and give the proofs of Theorem 1.7, Theorem 1.8 and Corollary 1.9.

2. General denseness lemma

The purpose of this section is to obtain Proposition 2.6. First we shall aim to establish Proposition 2.4 by using the theory of functional analysis.

Lemma 2.1. *Let H be a real Hilbert space with the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ and the norm $\|\cdot\|$. Let $u_1, u_2, \dots, u_n \in H$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ be real numbers in the interval $[-1,1]$. Then there exist $c_1, c_2, \dots, c_n \in \{1,-1\}$ such that*

$$\|\sum_{j=1}^{n}\lambda_j u_j - \sum_{j=1}^{n} c_j u_j\|^2 \le 4\sum_{j=1}^{n}\|u_j\|^2.$$

Proof. We prove the lemma by induction on n. For $n=1$, taking $c_1=1$ we have

$$\|\lambda_1 u_1 - c_1 u_1\|^2 = |\lambda_1 - c_1|^2\|u_1\|^2 \le 4\|u_1\|^2.$$

So we have the assertion when $n=1$.

Let $n \ge 1$, and assume that the assertion holds for n. Then we shall prove the assertion for $n+1$. Now we take $c_{n+1}=1$ if $\langle\sum_{j=1}^{n}(\lambda_j - c_j)u_j, u_{n+1}\rangle > 0$, and $c_{n+1}=-1$ if $\langle\sum_{j=1}^{n}(\lambda_j - c_j)u_j, u_{n+1}\rangle \le 0$. Then

$$(\lambda_{n+1} - c_{n+1})\left\langle \sum_{j=1}^{n}(\lambda_j - c_j)u_j, u_{n+1}\right\rangle \le 0, \tag{2.1}$$

since $\lambda_{n+1} \in [-1,1]$. We have, by (2.1) and the assumption,

$$\begin{aligned}
\left\|\sum_{j=1}^{n+1}\lambda_j u_j - \sum_{j=1}^{n+1} c_j u_j\right\|^2 &= \left\|\sum_{j=1}^{n}(\lambda_j - c_j)u_j + (\lambda_{n+1} - c_{n+1})u_{n+1}\right\|^2 \\
&= \left\|\sum_{j=1}^{n}(\lambda_j - c_j)u_j\right\|^2 + 2(\lambda_{n+1} - c_{n+1})\left\langle \sum_{j=1}^{n}(\lambda_j - c_j)u_j, u_{n+1}\right\rangle \\
&\quad + (\lambda_{n+1} - c_{n+1})^2\|u_{n+1}\|^2 \\
&\le \left\|\sum_{j=1}^{n}(\lambda_j - c_j)u_j\right\|^2 + 4\|u_{n+1}\|^2 \le 4\sum_{j=1}^{n}\|u_j\|^2 + 4\|u_{n+1}\|^2 = 4\sum_{j=1}^{n+1}\|u_j\|^2.
\end{aligned}$$

This completes the proof. □

Lemma 2.2. *Let H be a real Hilbert space with the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ and the norm $\|\cdot\|$. Let $\{u_n \mid n=1,2,\dots\}$ be a sequence in H satisfying*

(a) $\sum_{n=1}^{\infty}\|u_n\|^2 < \infty$,
(b) $\sum_{n=1}^{\infty}|\langle u_n, u\rangle| = \infty$ *for any $u \in H$ with $\|u\|=1$.*

Then for any $v \in H, \ell \in \mathbb{N}$ and $\varepsilon > 0$, there exist an integer $N \ge \ell$ and numbers $c_\ell, c_{\ell+1}, \dots, c_N \in \{1,-1\}$ such that

$$\left\|v - \sum_{n=\ell}^{N} c_n u_n\right\| < \varepsilon. \tag{2.2}$$

Proof. Let $\varepsilon > 0$ be arbitrary. In view of condition (a), we take a large integer $m \ge \ell$ such that

$$\sum_{n=m}^{\infty}\|u_n\|^2 < \frac{\varepsilon^2}{36}. \tag{2.3}$$

For this m let P_m be the set of all elements x in H of the form $x = \sum_{n=m}^{m'} \lambda_n u_n$, where $\lambda_m, \lambda_{m+1}, \ldots, \lambda_{m'} \in [-1, 1]$, $m' \in \mathbb{N}$ and $m' \geq m$. Then we see that P_m is a convex set in H, whence the closure $\overline{P_m}$ of P_m is a closed convex set.

We shall next prove $\overline{P_m} = H$. Suppose that it is false. Then according to Corollary 1 in [KV, p. 353], which is derived from the separation theorem and the Riesz representation theorem, there exists an element $e \in H$ with $\|e\| = 1$ such that

$$\sup_{x \in \overline{P_m}} \langle x, e \rangle < \infty. \tag{2.4}$$

By condition (b), for any $c > 0$ there exists an integer $M = M(c)$ such that $\sum_{n=m}^{M} |\langle u_n, e \rangle| > c$. For $n = m, m+1, \ldots, M$, set μ_n to be 1 if $\langle u_n, e \rangle \geq 0$, and -1 if $\langle u_n, e \rangle < 0$. Then

$$c < \sum_{n=m}^{M} |\langle u_n, e \rangle| = \sum_{n=m}^{M} \mu_n \langle u_n, e \rangle = \langle \sum_{n=m}^{M} \mu_n u_n, e \rangle.$$

Therefore, since $\sum_{n=m}^{M} \mu_n u_n \in \overline{P_m}$, we conclude that $\sup_{x \in \overline{P_m}} \langle x, e \rangle = \infty$. However, this contradicts (2.4), and hence we obtain $\overline{P_m} = H$.

Consequently, for any $w \in H$ there exist an integer $N \geq m$ and $\lambda_m, \lambda_{m+1}, \ldots, \lambda_N \in [-1, 1]$ such that

$$\left\| w - \sum_{n=m}^{N} \lambda_n u_n \right\| < \frac{\varepsilon}{3}. \tag{2.5}$$

By Lemma 2.1 there exist $c_m, c_{m+1}, \ldots, c_N \in \{1, -1\}$ for which

$$\left\| \sum_{n=m}^{N} \lambda_n u_n - \sum_{n=m}^{N} c_n u_n \right\|^2 \leq 4 \sum_{n=m}^{N} \|u_n\|^2. \tag{2.6}$$

Hence by (2.5), (2.6) and (2.3),

$$\begin{aligned} \left\| w - \sum_{n=m}^{N} c_n u_n \right\| &= \left\| w - \sum_{n=m}^{N} \lambda_n u_n + \sum_{n=m}^{N} \lambda_n u_n - \sum_{n=m}^{N} c_n u_n \right\| \\ &\leq \left\| w - \sum_{n=m}^{N} \lambda_n u_n \right\| + \left\| \sum_{n=m}^{N} \lambda_n u_n - \sum_{n=m}^{N} c_n u_n \right\| \\ &\leq \frac{\varepsilon}{3} + 2\sqrt{\sum_{n=m}^{N} \|u_n\|^2} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon. \end{aligned} \tag{2.7}$$

Taking $w = v - \sum_{n=\ell}^{m-1} u_n$ in (2.7), we obtain the assertion. □

Let Ω be a simply connected region in D symmetric with respect to the real axis, as in Theorem 1.1. Let U be a bounded, simply connected region in Ω which is symmetric with respect to the real axis and which satisfies $\overline{U} \subset \Omega$, where $\overline{U}$ denotes the closure of U. As usual, $L^2(U)$ denotes the set of all $\mathbb{C}$-valued measurable functions on U which are square-integrable with respect to the Lebesgue measure. In $L^2(U)$ we consider the inner product given by

$$\langle g_1(s), g_2(s) \rangle := \mathrm{Re} \int_U g_1(s) \overline{g_2(s)}\, d\sigma dt, \qquad s = \sigma + it \ (\sigma, t \in \mathbb{R}).$$

Note that the norm is given by

$$\|g(s)\| := \sqrt{\langle g(s), g(s)\rangle} = \left(\mathrm{Re}\int_U |\, g(s)\,|^2\, d\sigma dt\right)^{\frac{1}{2}} = \left(\int_U |\, g(s)\,|^2\, d\sigma dt\right)^{\frac{1}{2}}.$$

Then $L^2(U)$ is a real Hilbert space. If $g(s)$ is a holomorphic function on Ω, then it belongs to $L^2(U)$. Noting this, we define $H(U) = H(U,\Omega)$ to be the closure in $L^2(U)$ of the set

$$\{g(s) \mid g(s) \text{ is a holomorphic function on } \Omega \text{ which is } \mathbb{R}\text{-valued on the interval } \Omega\cap\mathbb{R}\ \}\ (\subset L^2(U)).$$

Then we see that $H(U)$ is a closed subspace of the real Hilbert space $L^2(U)$, so that $H(U)$ is also a real Hilbert space.

Next we shall prove Proposition 2.4 below by applying Lemma 2.2. In Lemma 2.2 we take $H = H(U)$ and $u_n = p_n^{-s}$, where p_n is the n-th prime. Then condition (a) in Lemma 2.2 is satisfied since

$$\sum_{n=1}^{\infty} \|u_n\|^2 \ll_U \sum_{n=1}^{\infty} \frac{1}{p_n^{2\sigma_1}} < \infty,$$

where $\sigma_1 := \min\{\mathrm{Re}\, s \mid s \in \overline{U}\} > \frac{1}{2}$. Condition (b) is satisfied by the following lemma, which is proved in [Na].

Lemma 2.3. *Suppose that $g(s)$ is a function in $H(U)$ with $\|g(s)\| = 1$. Then*

$$\sum_{n=1}^{\infty} \left| \mathrm{Re}\int_U p_n^{-s}\overline{g(s)}\, d\sigma dt \right| = \infty.$$

Therefore, we have obtained the following.

Proposition 2.4. *Let Ω and U be as above. Let $y > 0$ be fixed. Then the set*

$$\left\{ \sum_{y\le p\le \nu} \frac{c_p}{p^s} \;\middle|\; \nu \ge y,\ c_p \in \{1,-1\} \text{ for } y \le p \le \nu \right\}$$

is dense in $H(U)$.

The next lemma is a generalization of [Ti, p. 303, Lemma] (see also [Go, Lemma 2.5]).

Lemma 2.5. *Let U be a bounded region in $\mathbb{C}$. Let K be a compact set in $\mathbb{C}$ such that $K \subset U$. Let $A > 0$. Suppose that $f(s)$ is a holomorphic function on U satisfying*

$$\int_U |\, f(s)\,|^2\, d\sigma dt \le A.$$

Then

$$\max_{s\in K} |\, f(s)\,| \le a(U,K)\, A^{\frac{1}{2}},$$

where $a(U,K)$ is a certain positive constant depending only on U and K.

Proof. Let s_0 be any point in K. Set $R := \frac{1}{2}\min\{|\, s-z\,| \mid z \in \partial U, s \in K\}$. Then the disc $\{s \in \mathbb{C} \mid |\, s-s_0\,| < R\}$ is contained in U. Since the function $f^2(s)$ is holomorphic on U, we have for each r with $0 \le r \le R$,

$$f(s_0)^2 = \frac{1}{2\pi}\int_0^{2\pi} f(s_0+re^{iu})^2 du.$$

Therefore,

$$|\, f(s_0)\,|^2 \int_0^R r dr \le \frac{1}{2\pi}\int_0^R\int_0^{2\pi} \left|\, f(s_0+re^{iu})\,\right|^2 r\, du\, dr \le \frac{1}{2\pi}A$$

and the result follows. □

Proposition 2.6. *Let Ω be a simply connected region in D symmetric with respect to the real axis. Suppose that $g(s)$ is a holomorphic function on Ω which is $\mathbb{R}$-valued on $\Omega \cap \mathbb{R}$. Let K be a compact subset of Ω and $\nu_1 \ge 3$. Set $a_2 = 1$. Then for any $\varepsilon > 0$ there exist $\nu > \nu_1$ and $a_p \in \{1, -1\}$, for each prime p with $3 \le p \le \nu$, such that*

$$\max_{s\in K}\left| g(s) - \log\prod_{p\le\nu}\left(1-\frac{a_p}{p^s}\right)^{-1}\right| < \varepsilon,$$

where

$$\log\prod_{p\le\nu}\left(1-\frac{a_p}{p^s}\right)^{-1} = -\sum_{p\le\nu}\log\left(1-\frac{a_p}{p^s}\right) = \sum_{p\le\nu}\sum_{n=1}^{\infty}\frac{a_p^n}{np^{ns}}.$$

Proof. Take a bounded, simply connected region U in Ω which is symmetric with respect to the real axis and which satisfies $K \subset U$ and $\overline{U} \subset \Omega$. Set $\sigma_1 := \min\{\operatorname{Re} s | s \in \overline{U}\} > \frac{1}{2}$. Let $\varepsilon > 0$ be arbitrary. Fix a real number y satisfying $y > \nu_1$ and $y^{1-2\sigma_1}/(2\sigma_1 - 1) < \varepsilon$. Then we have

$$\begin{aligned}\sum_{p\ge y}\sum_{n=2}^{\infty}\frac{1}{np^{n\sigma_1}} &\le \sum_{p\ge y}\sum_{n=2}^{\infty}\frac{1}{p^{n\sigma_1}} = \sum_{p\ge y}\frac{p^{-2\sigma_1}}{1-p^{-\sigma_1}}\\ &\ll \sum_{p\ge y}\frac{1}{p^{2\sigma_1}} \ll \sum_{m\ge y, m\in\mathbb{N}}\frac{1}{m^{2\sigma_1}} \ll \frac{y^{1-2\sigma_1}}{2\sigma_1-1} < \varepsilon.\end{aligned} \tag{2.8}$$

From Proposition 2.4 it follows that there exist $\nu \ge y$ and $c_p \in \{1, -1\}$, for each prime p with $y \le p \le \nu$, such that

$$\int_U \left|\left(g(s) - \sum_{p<y}\sum_{n=1}^{\infty}\frac{1}{np^{ns}}\right) - \sum_{y\le p\le\nu}\frac{c_p}{p^s}\right|^2 d\sigma dt < \varepsilon^2.$$

This and Lemma 2.5 yield

$$\max_{s\in K}\left| g(s) - \sum_{p<y}\sum_{n=1}^{\infty}\frac{1}{np^{ns}} - \sum_{y\le p\le\nu}\frac{c_p}{p^s}\right| \ll_{U,K} \varepsilon. \tag{2.9}$$

For each prime p with $3 \le p \le \nu$ we set

$$a_p := \begin{cases} 1 & \text{if } 3 \le p < y,\\ c_p & \text{if } y \le p \le \nu.\end{cases}$$

Then we obtain, by (2.8) and (2.9),

$$\begin{aligned}
&\max_{s\in K}\left| g(s) - \log\prod_{p\le\nu}\left(1-\frac{a_p}{p^s}\right)^{-1}\right| \\
&= \max_{s\in K}\left| g(s) - \sum_{p<y}\sum_{n=1}^{\infty}\frac{1}{np^{ns}} - \sum_{y\le p\le\nu}\frac{c_p}{p^s} - \sum_{y\le p\le\nu}\sum_{n=2}^{\infty}\frac{c_p^n}{np^{ns}}\right| \\
&\le \max_{s\in K}\left| g(s) - \sum_{p<y}\sum_{n=1}^{\infty}\frac{1}{np^{ns}} - \sum_{y\le p\le\nu}\frac{c_p}{p^s}\right| + \max_{s\in K}\left|\sum_{y\le p\le\nu}\sum_{n=2}^{\infty}\frac{c_p^n}{np^{ns}}\right| \\
&\ll_{U,K} \varepsilon + \sum_{p\ge y}\sum_{n=2}^{\infty}\frac{1}{np^{n\sigma_1}} \ll \varepsilon,
\end{aligned}$$

which completes the proof. □

3. Approximation by the finite Euler product

The next lemma is obtained in [St, Lemma 8].

Lemma 3.1. *For large $X\in\mathbb{R}^+$, let R_X denote the set of complex numbers s defined by $1/2+(\log\log X)^{-1/2}\le \operatorname{Re} s\le 5/4$ and $|\operatorname{Im} s|\le\sqrt{\log X}$, and we set*

$$h_X = \exp\left((\log\log X)^{\frac{3}{4}}\right).$$

Then uniformly for $s\in R_X$,

$$\sum_{|d'|\le X}\left| L(s,\chi_{d'}) - \prod_{p\le h_X}\left(1-\frac{\chi_{d'}(p)}{p^s}\right)^{-1}\right|^2 \ll X\exp\left(-(\log\log X)^{\frac{1}{4}}\right), \tag{3.1}$$

where the sum $\sum_{|d'|\le X}$ is taken over all discriminants d' with $|d'|\le X$.

Proposition 3.2. *Let $\varepsilon>0$ and K be a compact set in the region $1/2<\operatorname{Re} s<5/4$. Define $\mathcal{A}_X^\gamma=\mathcal{A}_X^\gamma(\varepsilon,K)$ by*

$$\mathcal{A}_X^\gamma := \{d\in\mathcal{D}_X^\gamma \mid \max_{s\in K}\left| L(s,\chi_d) - \prod_{p\le h_X}\left(1-\frac{\chi_d(p)}{p^s}\right)^{-1}\right| < \varepsilon\}.$$

Then

$$\frac{\#\mathcal{A}_X^\gamma}{\#\mathcal{D}_X^\gamma} > 1-\varepsilon$$

if X is sufficiently large.

Proof. Take an open rectangle $U=\{s\in\mathbb{C}\mid \sigma_1<\operatorname{Re} s<\sigma_2, |\operatorname{Im} s|<A\}$ satisfying $1/2<\sigma_1<\min\{\operatorname{Re} s|s\in K\}\le\max\{\operatorname{Re} s|s\in K\}<\sigma_2<5/4$ and $\max\{|\operatorname{Im} s|\,|s\in K\}<A$. Then $K\subset U$. For large $X\in\mathbb{R}^+$ we define

$$\widetilde{\mathcal{A}}_X^\gamma := \{d\in\mathcal{D}_X^\gamma \mid \int_U\left| L(s,\chi_d) - \prod_{p\le h_X}\left(1-\frac{\chi_d(p)}{p^s}\right)^{-1}\right|^2 d\sigma dt < \frac{\varepsilon^2}{a(U,K)^2}\}, \tag{3.2}$$

where $a(U,K)$ is the constant in Lemma 2.5. Note that by Lemma 2.5,

$$\widetilde{\mathcal{A}}_X^\gamma \subset \mathcal{A}_X^\gamma. \tag{3.3}$$

From Lemma 3.1 and (4.2) in Lemma 4.1 below, we infer that for all large X,

$$\begin{aligned}
&\sum_{d\in\mathcal{D}_X^\gamma}\int_U \left| L(s,\chi_d) - \prod_{p\le h_X}\left(1-\frac{\chi_d(p)}{p^s}\right)^{-1}\right|^2 d\sigma dt \\
&\le \sum_{\substack{|d'|\le X \\ d':\text{discriminant}}} \int_U \left| L(s,\chi_{d'}) - \prod_{p\le h_X}\left(1-\frac{\chi_{d'}(p)}{p^s}\right)^{-1}\right|^2 d\sigma dt \qquad (3.4)\\
&\ll_U X\exp\left(-(\log\log X)^{\frac14}\right) \ll \#\mathcal{D}_X^\gamma \exp\left(-(\log\log X)^{\frac14}\right).
\end{aligned}$$

Since $\exp\left(-(\log\log X)^{\frac14}\right) \longrightarrow 0$ as $X\to\infty$, it follows from (3.4) that there exists a large number $X_0(\varepsilon,U,K)$ such that for all $X > X_0(\varepsilon,U,K)$,

$$\sum_{d\in\mathcal{D}_X^\gamma}\int_U \left| L(s,\chi_d) - \prod_{p\le h_X}\left(1-\frac{\chi_d(p)}{p^s}\right)^{-1}\right|^2 d\sigma dt < \frac{\varepsilon^3}{a(U,K)^2}\,\#\mathcal{D}_X^\gamma. \tag{3.5}$$

Now assume that there exists a real number $X > X_0(\varepsilon,U,K)$ such that

$$\#(\mathcal{D}_X^\gamma - \widetilde{\mathcal{A}}_X^\gamma) \ge \varepsilon\,\#\mathcal{D}_X^\gamma.$$

Then for this X we have, by (3.2),

$$\begin{aligned}
&\sum_{d\in\mathcal{D}_X^\gamma}\int_U \left| L(s,\chi_d) - \prod_{p\le h_X}\left(1-\frac{\chi_d(p)}{p^s}\right)^{-1}\right|^2 d\sigma dt \\
&\ge \sum_{d\in\mathcal{D}_X^\gamma-\widetilde{\mathcal{A}}_X^\gamma}\int_U \left| L(s,\chi_d) - \prod_{p\le h_X}\left(1-\frac{\chi_d(p)}{p^s}\right)^{-1}\right|^2 d\sigma dt \\
&\ge \varepsilon\,\#\mathcal{D}_X^\gamma\,\frac{\varepsilon^2}{a(U,K)^2} = \frac{\varepsilon^3}{a(U,K)^2}\#\mathcal{D}_X^\gamma.
\end{aligned}$$

However, this contradicts (3.5). Hence for any $X > X_0(\varepsilon,U,K)$ we have

$$\#(\mathcal{D}_X^\gamma - \widetilde{\mathcal{A}}_X^\gamma) < \varepsilon\,\#\mathcal{D}_X^\gamma,$$

that is, $\#\widetilde{\mathcal{A}}_X^\gamma/\#\mathcal{D}_X^\gamma > 1-\varepsilon$. This and (3.3) complete the proof. □

4. Results on real primitive characters χ_d

The aim of this section is to obtain Proposition 4.4. As before, the letter γ denotes the plus $+$ or the minus $-$. Throughout this section we set

$$\alpha = \alpha(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ is } +, \\ -1 & \text{if } \gamma \text{ is } -, \end{cases} \tag{4.1}$$

and for $X > 2$ define I_X to be the interval $[2,X]$ if γ is $+$, and $[-X,-2]$ if γ is $-$. Let $\mu(n)$ denote the Möbius function, and $\varphi(n)$ the Euler totient function, i.e., the number of positive integers not exceeding n which are relatively prime to n.

We can obtain the next lemma in the same fashion as the proof of [J2, Lemma 1].

Lemma 4.1. *Let $\gamma \in \{+,-\}$. For $X \geq 17$ we have*

$$\#\mathcal{D}_X^\gamma = \frac{1}{\pi^2}X + O(\sqrt{X}). \tag{4.2}$$

Lemma 4.2. *Let $\nu \geq 3$ be fixed and $a_p \in \{1,-1\}$ for each prime p with $3 \leq p \leq \nu$. Define $\mathcal{D}_{X,\nu}^\gamma = \mathcal{D}_{X,\nu}^\gamma(\{a_p\})$ by*

$$\mathcal{D}_{X,\nu}^\gamma := \{d \in \mathcal{D}_X^\gamma \mid \chi_d(p) = a_p \text{ for every prime } p \text{ with } 3 \leq p \leq \nu\}.$$

Then for all large X we have

$$\frac{\#\mathcal{D}_{X,\nu}^\gamma}{\#\mathcal{D}_X^\gamma} = C_\nu + O_\nu\left(X^{-\frac{1}{2}}\right),$$

where $C_\nu := \prod_{3\leq p\leq\nu} \frac{1}{2}\left(1+\frac{1}{p}\right)^{-1}$.

Proof. In general, for $q \in \mathbb{N}$ and $a \in \mathbb{Z}$, we denote by $[a]_q$ the set of all integers x such that $x \equiv a \pmod q$, that is, the residue class mod q which a belongs to.

Let p be an odd prime. Let $\mathcal{Q}_p$ be the set of all quadratic residue classes mod p other than the residue class $[0]_p$, $\mathcal{Q}_p'$ the set of all quadratic non-residue classes mod p. It is well known (see e.g. [Ap, Theorem 9.1]) that

$$\#\mathcal{Q}_p = \#\mathcal{Q}_p' = \frac{p-1}{2}. \tag{4.3}$$

According to the definitions of Kronecker's symbol and Legendre's symbol, it holds that for fixed p and $d \in \mathcal{D}^\gamma$, d satisfies $\chi_d(p) = \left(\frac{d}{p}\right) = a_p$ if and only if d belongs to one of the residue classes in $\mathcal{Q}_p$ if $a_p = 1$ and in $\mathcal{Q}_p'$ if $a_p = -1$. From this, (4.3) and the Chinese remainder theorem, it follows for a square-free integer m with $m \neq 0,1$ that m satisfies $m \equiv 1 \pmod 8$ (so $m \in \mathcal{D}^\gamma$) and $\chi_m(p) = a_p$ for every prime p with $3 \leq p \leq \nu$ if and only if m belongs to one of exactly $\prod_{3\leq p\leq\nu}\frac{p-1}{2}$ certain distinct residue classes mod Q, where $Q := 8\prod_{3\leq p\leq\nu} p$. We denote by $\mathcal{R}$ the set of these residue classes $[c]_Q$ mod Q, so that

$$\#\mathcal{R} = \prod_{3\leq p\leq\nu}\frac{p-1}{2}. \tag{4.4}$$

Thus we have

$$\begin{aligned}\mathcal{D}_{X,\nu}^\gamma =& \{m \in I_X \mid m : \text{square-free integer}, m \equiv 1 \bmod 8,\\ &\qquad \chi_m(p) = a_p \text{ for every prime } p \text{ with } 3 \leq p \leq \nu\ \}\\ =& \bigcup_{[c]_Q\in\mathcal{R}} \{m \in I_X \mid m : \text{square-free integer}, m \equiv c \bmod Q\}.\end{aligned} \tag{4.5}$$

Hence

$$\begin{aligned}\#\mathcal{D}_{X,\nu}^\gamma &= \sum_{[c]_Q\in\mathcal{R}} \sum_{\substack{m\in I_X\cap\mathbb{Z}\\ m:\text{square-free}, m\equiv c \bmod Q}} 1\\ &= \sum_{[c]_Q\in\mathcal{R}} \sum_{\substack{2\leq n\leq X\\ n:\text{square-free}, n\equiv\alpha c \bmod Q}} 1 = \sum_{[c]_Q\in\mathcal{R}} \sum_{\substack{2\leq n\leq X\\ n\equiv\alpha c \bmod Q}} \mu(n)^2,\end{aligned} \tag{4.6}$$

where α is as in (4.1). We see that if $[c]_Q \in \mathcal{R}$, then

$$(c, Q) = 1, \tag{4.7}$$

since $[0]_p \notin \mathcal{Q}_p$ and $[0]_p \notin \mathcal{Q}'_p$ for all primes p with $3 \le p \le \nu$.

By using the orthogonality relation for Dirichlet characters, the fact $\mu(n)^2 = \sum_{m:\, m^2|n} \mu(m)$ (see [HST, p. 87]) and (4.7), we infer that for $[c]_Q \in \mathcal{R}$,

$$\begin{aligned}\sum_{\substack{2\le n\le X\\ n\equiv \alpha c \bmod Q}} \mu(n)^2 &= \frac{X}{Q} L(2,\lambda_0)^{-1} + O\left(Q\sqrt{X}\right)\\ &= \frac{X}{Q}\zeta(2)^{-1}\prod_{p|Q}\left(1-\frac{1}{p^2}\right)^{-1} + O\left(Q\sqrt{X}\right),\end{aligned} \tag{4.8}$$

where λ_0 is the principal character $\bmod Q$. Note that the right-hand side of (4.8) is independent of $[c]_Q$. Therefore, from (4.8), (4.6) and (4.4) we obtain

$$\begin{aligned}\#\mathcal{D}^{\gamma}_{X,\nu} &= \left(\prod_{3\le p\le \nu}\frac{p-1}{2}\right)\left(\frac{X}{Q}\zeta(2)^{-1}\prod_{p|Q}\left(1-\frac{1}{p^2}\right)^{-1} + O\left(Q\sqrt{X}\right)\right)\\ &= \left(\prod_{3\le p\le \nu}\frac{p-1}{2}\right)\left(\frac{X}{\pi^2\prod_{3\le p\le\nu} p}\prod_{3\le p\le\nu}\left(1-\frac{1}{p^2}\right)^{-1} + O\left(Q\sqrt{X}\right)\right)\\ &= \frac{C_\nu}{\pi^2}X + O_\nu\left(\sqrt{X}\right).\end{aligned}$$

This and (4.2) give us

$$\frac{\#\mathcal{D}^{\gamma}_{X,\nu}}{\#\mathcal{D}^{\gamma}_{X}} = C_\nu + O_\nu\left(X^{-\frac{1}{2}}\right),$$

which completes the proof. □

Lemma 4.3. *Let $\nu \ge 3$ be fixed and $a_p \in \{1,-1\}$ for each prime p with $3 \le p \le \nu$. Let $\mathcal{D}^{\gamma}_{X,\nu}$ and C_ν be as in Lemma 4.2, $h_X = \exp\left((\log\log X)^{3/4}\right)$ as in Lemma 3.1, and $\sigma_1 > 1/2$. Then there exists $X_0(\nu) > 0$ such that for any $X > X_0(\nu)$ and uniformly for $s \in \mathbb{C}$ with $\operatorname{Re} s \ge \sigma_1$,*

$$\sum_{d\in\mathcal{D}^{\gamma}_{X,\nu}}\left|\sum_{\nu<p\le h_X}\frac{\chi_d(p)}{p^s}\right|^2 \ll \frac{\nu^{1-2\sigma_1}}{2\sigma_1-1}\, C_\nu\, \#\mathcal{D}^{\gamma}_{X}. \tag{4.9}$$

Proof. Let Q and $\mathcal{R}$ be as in the proof of Lemma 4.2. From (4.5) it follows that

$$\begin{aligned}\sum_{d\in\mathcal{D}^{\gamma}_{X,\nu}}\left|\sum_{\nu<p\le h_X}\frac{\chi_d(p)}{p^s}\right|^2 &= \sum_{[c]_Q\in\mathcal{R}}\ \sum_{\substack{m\in I_X\cap\mathbb{Z}\\ m:\,\text{square-free}, m\equiv c \bmod Q}}\left|\sum_{\nu<p\le h_X}\frac{\chi_m(p)}{p^s}\right|^2\\ &= \sum_{[c]_Q\in\mathcal{R}}\ \sum_{\substack{2\le n\le X\\ n:\,\text{square-free}, n\equiv \alpha c \bmod Q}}\left|\sum_{\nu<p\le h_X}\frac{\chi_{\alpha n}(p)}{p^s}\right|^2.\end{aligned} \tag{4.10}$$

For $[c]_Q \in \mathcal{R}$ we have

$$\begin{aligned}
&\sum_{\substack{2\le n\le X\\ n\,:\,\text{square-free}, n\equiv \alpha c \bmod Q}} \left| \sum_{\nu<p\le h_X} \frac{\chi_{\alpha n}(p)}{p^s} \right|^2 \\
&= \sum_{\nu<p\le h_X} \frac{1}{|\,p^s\,|^2} \sum_{\substack{2\le n\le X\\ n\,:\,\text{square-free}, n\equiv \alpha c \bmod Q}} |\,\chi_{\alpha n}(p)\,|^2 \\
&\quad + \sum_{\substack{p,\,q\,:\,\text{primes},\, p\ne q\\ \nu<p,q\le h_X}} \frac{1}{p^s\overline{q^s}} \sum_{\substack{2\le n\le X\\ n\,:\,\text{square-free}, n\equiv \alpha c \bmod Q}} \chi_{\alpha n}(p)\overline{\chi_{\alpha n}(q)} \\
&= S_1 + S_2, \quad \text{say.}
\end{aligned} \tag{4.11}$$

Using (4.8), we deduce that uniformly for $s \in \mathbb{C}$ with $\operatorname{Re} s \ge \sigma_1$,

$$\begin{aligned}
|\,S_1\,| &\le \sum_{\nu<p\le h_X} \frac{1}{p^{2\sigma_1}} \sum_{2\le n\le X, n\equiv \alpha c \bmod Q} \mu(n)^2 \\
&\ll \frac{\nu^{1-2\sigma_1}}{2\sigma_1-1} \left(\frac{X}{Q}\zeta(2)^{-1} \prod_{p|Q} \left(1-\frac{1}{p^2}\right)^{-1} + O\left(Q\sqrt{X}\right)\right).
\end{aligned} \tag{4.12}$$

Next we shall consider the sum S_2. Fix two distinct primes p and q such that $\nu < p \le h_X$ and $\nu < q \le h_X$. Then we have

$$\begin{aligned}
&\sum_{\substack{2\le n\le X\\ n\,:\,\text{square-free}, n\equiv \alpha c \bmod Q}} \chi_{\alpha n}(p)\overline{\chi_{\alpha n}(q)} \\
&= \sum_{2\le n\le X} \mu(n)^2 \left(\frac{\alpha n}{p}\right)\left(\frac{\alpha n}{q}\right) \frac{1}{\varphi(Q)} \sum_{\lambda \bmod Q} \lambda(n)\overline{\lambda(\alpha c)} \\
&= \frac{1}{\varphi(Q)} \left(\frac{\alpha}{p}\right)\left(\frac{\alpha}{q}\right) \sum_{\lambda \bmod Q} \overline{\lambda(\alpha c)} \left(\left(\sum_{1\le n\le X} \mu(n)^2 \left(\frac{n}{p}\right)\left(\frac{n}{q}\right)\lambda(n) \right) - 1 \right),
\end{aligned} \tag{4.13}$$

where $\sum_{\lambda \bmod Q}$ means the sum over all the Dirichlet characters $\lambda \bmod Q$, and $\left(\frac{\cdot}{p}\right)$ and $\left(\frac{\cdot}{q}\right)$ stand for Legendre's symbols. We note that for a Dirichlet character λ mod Q, the product $\left(\frac{\cdot}{p}\right)\left(\frac{\cdot}{q}\right)\lambda(\cdot)$ is a Dirichlet character mod pqQ. Further, since p, q and Q are relatively prime in pairs, we find from the Chinese remainder theorem that for any character $\lambda \bmod Q$ the product $\left(\frac{\cdot}{p}\right)\left(\frac{\cdot}{q}\right)\lambda(\cdot)$ is a non-principal character mod pqQ. This and the fact $\mu(n)^2 = \sum_{m:\, m^2|n} \mu(m)$ give us

$$\begin{aligned}
&\sum_{1\le n\le X} \mu(n)^2 \left(\frac{n}{p}\right)\left(\frac{n}{q}\right)\lambda(n) \\
&= \sum_{1\le m\le \sqrt{X}} \mu(m) \left(\frac{m}{p}\right)^2 \left(\frac{m}{q}\right)^2 \lambda(m)^2 \sum_{1\le a\le X/m^2} \left(\frac{a}{p}\right)\left(\frac{a}{q}\right)\lambda(a) \\
&= O\left(\sqrt{X}\right) O\left(pqQ\right) \ll \sqrt{X} h_X^2 Q.
\end{aligned} \tag{4.14}$$

Thus from (4.13) and (4.14) we infer

$$
\begin{aligned}
|\, S_2 \,| &\le \sum_{\substack{p,\, q\,:\,\text{primes},\, p\neq q \\ \nu<p,q\le h_X}} \frac{1}{p^{\sigma_1} q^{\sigma_1}} \left| \sum_{\substack{2\le n\le X \\ n\,:\,\text{square-free}, n\equiv \alpha c \bmod Q}} \chi_{\alpha n}(p)\overline{\chi_{\alpha n}(q)} \right| \\
&= \sum_{\substack{p,\, q\,:\,\text{primes},\, p\neq q \\ \nu<p,q\le h_X}} \frac{1}{p^{\sigma_1} q^{\sigma_1}} \left(O\left(\sqrt{X} h_X^2 Q\right) + O(1) \right) \qquad (4.15) \\
&\ll \left(\sum_{p\le h_X} \frac{1}{p^{\sigma_1}} \right)^2 O\left(\sqrt{X} h_X^2 Q\right) \ll \sqrt{X} h_X^4 Q.
\end{aligned}
$$

Consequently, from (4.11), (4.12) and (4.15) it follows that for fixed ν there exists $X_0(\nu)$ such that for any $X > X_0(\nu)$ and any $s \in \mathbb{C}$ with $\operatorname{Re} s \ge \sigma_1$,

$$
\sum_{\substack{2\le n\le X \\ n\,:\,\text{square-free}, n\equiv \alpha c \bmod Q}} \left| \sum_{\nu<p\le h_X} \frac{\chi_{\alpha n}(p)}{p^s} \right|^2 \ll \frac{\nu^{1-2\sigma_1}}{2\sigma_1 - 1} \frac{X}{Q} \prod_{p|Q} \left(1 - \frac{1}{p^2}\right)^{-1}. \tag{4.16}
$$

Note that the right-hand side of (4.16) is independent of c. Combining (4.16), (4.10), (4.4) and (4.2), we conclude that

$$
\begin{aligned}
\sum_{d\in\mathcal{D}^{\gamma}_{X,\nu}} \left| \sum_{\nu<p\le h_X} \frac{\chi_d(p)}{p^s} \right|^2 &\ll \left(\prod_{3\le p\le \nu} \frac{p-1}{2} \right) \frac{\nu^{1-2\sigma_1}}{2\sigma_1 - 1} \frac{\#\mathcal{D}^{\gamma}_X}{Q} \prod_{p|Q} \left(1 - \frac{1}{p^2}\right)^{-1} \\
&\ll \frac{\nu^{1-2\sigma_1}}{2\sigma_1 - 1} C_\nu \, \#\mathcal{D}^{\gamma}_X,
\end{aligned}
$$

which completes the proof. □

Proposition 4.4. *Let $\sigma_1 > 1/2$ and K be a compact subset of $\mathbb{C}$ such that $K \subset \{s \in \mathbb{C} \mid \operatorname{Re} s > \sigma_1\}$. Let $\varepsilon > 0$. Then there exists a large real number $\nu_0(\sigma_1, K, \varepsilon) > 3$ depending on σ_1, K and ε, and satisfying the following: Fix any real number $\nu > \nu_0(\sigma_1, K, \varepsilon)$, and let $a_p \in \{1, -1\}$ for each prime p with $3 \le p \le \nu$. Let $\mathcal{D}^{\gamma}_{X,\nu}, C_\nu$ and h_X be as in Lemma 4.3 for large X. Define $\mathcal{B}^{\gamma}_{X,\nu} = \mathcal{B}^{\gamma}_{X,\nu}(K, \varepsilon, \{a_p\})$ by*

$$
\mathcal{B}^{\gamma}_{X,\nu} := \left\{ d \in \mathcal{D}^{\gamma}_{X,\nu} \mid \max_{s\in K} \left| \sum_{\nu<p\le h_X} \frac{\chi_d(p)}{p^s} \right| < \varepsilon \right\}.
$$

Then for all X sufficiently large we have

$$
\frac{\#\mathcal{B}^{\gamma}_{X,\nu}}{\#\mathcal{D}^{\gamma}_X} > \frac{1}{2} C_\nu.
$$

Proof. Set $\sigma_2 = 1 + \sup\{\operatorname{Re} s \mid s \in K\}$ and $A = 1 + \sup\{|\operatorname{Im} s| \mid s \in K\}$. Let U be the open rectangle $\{s \in \mathbb{C} \mid \sigma_1 < \operatorname{Re} s < \sigma_2, |\operatorname{Im} s| < A\}$, and then $K \subset U$. Take a large real number $\nu_0 = \nu_0(\sigma_1, K, \varepsilon) > 3$ satisfying

$$
\left(\int_U 1 \, d\sigma dt \right) b \, \frac{\nu_0^{1-2\sigma_1}}{2\sigma_1 - 1} < \frac{\varepsilon^2}{4\, a(U, K)^2} \tag{4.17}
$$

where b is the absolute constant implied by the symbol $\ll$ in (4.9), and $a(U,K)$ is the constant in Lemma 2.5. Note that ν_0 is dependent only on σ_1, K and ε.

In the following we fix any $\nu > \nu_0$. For large X we define

$$\widetilde{\mathcal{B}}^{\gamma}_{X,\nu} := \{d \in \mathcal{D}^{\gamma}_{X,\nu} \mid \int_U \left| \sum_{\nu < p \le h_X} \frac{\chi_d(p)}{p^s} \right|^2 d\sigma dt < \frac{\varepsilon^2}{a(U,K)^2} \}. \tag{4.18}$$

By Lemma 2.5,

$$\widetilde{\mathcal{B}}^{\gamma}_{X,\nu} \subset \mathcal{B}^{\gamma}_{X,\nu}. \tag{4.19}$$

By Lemma 4.3 and (4.17), there exists a large number $X_0(\nu) > 0$ such that for all $X > X_0(\nu)$,

$$\begin{aligned} \sum_{d \in \mathcal{D}^{\gamma}_{X,\nu}} \int_U \left| \sum_{\nu < p \le h_X} \frac{\chi_d(p)}{p^s} \right|^2 d\sigma dt &\le \left(\int_U 1 \, d\sigma dt \right) b \, \frac{\nu^{1-2\sigma_1}}{2\sigma_1 - 1} \, C_\nu \, \#\mathcal{D}^{\gamma}_X \\ &< \left(\int_U 1 \, d\sigma dt \right) b \, \frac{{\nu_0}^{1-2\sigma_1}}{2\sigma_1 - 1} \, C_\nu \, \#\mathcal{D}^{\gamma}_X < \frac{\varepsilon^2}{4\, a(U,K)^2} \, C_\nu \, \#\mathcal{D}^{\gamma}_X. \end{aligned} \tag{4.20}$$

By the same argument as in Proposition 3.2, it follows from (4.18) and (4.20) that for any $X > X_0(\nu)$ we have $\#(\mathcal{D}^{\gamma}_{X,\nu} - \widetilde{\mathcal{B}}^{\gamma}_{X,\nu}) < \frac{1}{4} C_\nu \, \#\mathcal{D}^{\gamma}_X$, so

$$\frac{\#\widetilde{\mathcal{B}}^{\gamma}_{X,\nu}}{\#\mathcal{D}^{\gamma}_X} > \frac{\#\mathcal{D}^{\gamma}_{X,\nu}}{\#\mathcal{D}^{\gamma}_X} - \frac{1}{4} C_\nu. \tag{4.21}$$

Further, Lemma 4.2 implies that

$$\frac{\#\mathcal{D}^{\gamma}_{X,\nu}}{\#\mathcal{D}^{\gamma}_X} > \frac{3}{4} C_\nu \qquad \text{if } X \text{ is large enough.} \tag{4.22}$$

Combining (4.19), (4.21) and (4.22), we conclude that if X is large enough, then

$$\frac{\#\mathcal{B}^{\gamma}_{X,\nu}}{\#\mathcal{D}^{\gamma}_X} \ge \frac{\#\widetilde{\mathcal{B}}^{\gamma}_{X,\nu}}{\#\mathcal{D}^{\gamma}_X} > \frac{3}{4} C_\nu - \frac{1}{4} C_\nu = \frac{1}{2} C_\nu.$$

□

5. Proofs of Theorem 1.1 and its corollaries

In this section we finally prove Theorem 1.1 and its corollaries.

Proof of Theorem 1.1. We shall first prove the assertion that there exists a holomorphic function $g(s)$ on Ω such that $g(x) \in \mathbb{R}$ for any $x \in \Omega \cap \mathbb{R}$ and

$$h(s) = e^{g(s)}. \tag{5.1}$$

It is known that there exists a holomorphic function $g_0(s)$ on Ω for which $e^{g_0(s)} = h(s)$ (see [Ru, Theorem 13.11, (h)]). Fix $a \in \Omega \cap \mathbb{R}$. Since $h(x) \in \mathbb{R}^+$ for $x \in \Omega \cap \mathbb{R}$, and $g_0(s)$ and $h(s)$ are continuous functions on the connected open interval $\Omega \cap \mathbb{R}$, we find that there exists an integer n, depending on $a, g_0(a)$ and $h(a)$, such that $g_0(x) = \log h(x) + 2\pi i n$ uniformly for all $x \in \Omega \cap \mathbb{R}$. Define $g(s) := g_0(s) - 2\pi i n$ for $s \in \Omega$. Then this $g(s)$ gives the assertion.

Let $\varepsilon > 0$ be an arbitrary small number. Take a real number $\sigma_1 > 1/2$ such that $K \subset \{s \in \mathbb{C} \mid \operatorname{Re} s > \sigma_1\}$, and fix a large positive number ν_1 satisfying $\nu_1^{1-2\sigma_1}/(2\sigma_1 - 1) < \varepsilon$ and $\nu_1 > \nu_0(\sigma_1, K, \varepsilon)$, where $\nu_0(\sigma_1, K, \varepsilon)$ is the constant in Proposition

4.4. Now Proposition 2.6 implies that there exist $\nu > \nu_1$ and $a_p \in \{1, -1\}$, for each prime p with $3 \le p \le \nu$, such that

$$\max_{s\in K}\left| g(s) - \log \prod_{p\le\nu}\left(1-\frac{a_p}{p^s}\right)^{-1}\right| < \varepsilon, \tag{5.2}$$

where $a_2 = 1$.

For those a_p's we apply Proposition 4.4, from which it follows that for the above number ν and all large X, we have

$$\frac{\#\mathcal{B}^{\gamma}_{X,\nu}}{\#\mathcal{D}^{\gamma}_{X}} > \frac{1}{2}C_\nu. \tag{5.3}$$

Recall that for all $d \in \mathcal{D}^\gamma$ we have $\chi_d(2) = 1$, so that $\chi_d(2) = a_2$. Using this and the definition of $\mathcal{B}^{\gamma}_{X,\nu}$, we see that for all large X and any $d \in \mathcal{B}^{\gamma}_{X,\nu}$,

$$\begin{aligned}
&\max_{s\in K}\left| \log \prod_{p\le\nu}\left(1-\frac{a_p}{p^s}\right)^{-1} - \log \prod_{p\le h_X}\left(1-\frac{\chi_d(p)}{p^s}\right)^{-1}\right| \\
&= \max_{s\in K}\left| \sum_{\nu<p\le h_X}\frac{\chi_d(p)}{p^s} + \sum_{\nu<p\le h_X}\sum_{n=2}^{\infty}\frac{\chi_d(p^n)}{np^{ns}}\right| \\
&\le \max_{s\in K}\left| \sum_{\nu<p\le h_X}\frac{\chi_d(p)}{p^s}\right| + \max_{s\in K}\left| \sum_{\nu<p\le h_X}\sum_{n=2}^{\infty}\frac{\chi_d(p^n)}{np^{ns}}\right| \\
&\le \varepsilon + O(\varepsilon) \ll \varepsilon,
\end{aligned} \tag{5.4}$$

since

$$\left|\sum_{\nu<p\le h_X}\sum_{n=2}^{\infty}\frac{1}{np^{ns}}\right| \ll \sum_{\nu<p\le h_X}\frac{1}{p^{2\sigma_1}} \ll \frac{\nu^{1-2\sigma_1}}{2\sigma_1-1} < \frac{\nu_1^{1-2\sigma_1}}{2\sigma_1-1} < \varepsilon.$$

From (5.4) and (5.2) we deduce, for all large X and any $d \in \mathcal{B}^{\gamma}_{X,\nu}$,

$$\max_{s\in K}\left| g(s) - \log \prod_{p\le h_X}\left(1-\frac{\chi_d(p)}{p^s}\right)^{-1}\right| \ll \varepsilon$$

and therefore

$$\begin{aligned}
&\max_{s\in K}\left| \prod_{p\le h_X}\left(1-\frac{\chi_d(p)}{p^s}\right)^{-1} - h(s)\right| \\
&= \max_{s\in K}\left| h(s)\left(\frac{\prod_{p\le h_X}\left(1-\frac{\chi_d(p)}{p^s}\right)^{-1}}{h(s)} - 1\right)\right| \\
&\le \max_{s\in K}|\,h(s)\,|\ \max_{s\in K}\left| e^{\log\prod_{p\le h_X}\left(1-\frac{\chi_d(p)}{p^s}\right)^{-1} - g(s)} - 1\right| \ll_{K,h(s)} \varepsilon,
\end{aligned} \tag{5.5}$$

using (5.1) and the fact that $e^z - 1 \ll z$ if $|\,z\,|$ is small.

Let ε_1 be a small positive number such that $\varepsilon_1 < \min\{\varepsilon, \frac{C_\nu}{2}\}$. According to Proposition 3.2, if we put

$$\mathcal{A}_X^\gamma := \{d \in \mathcal{D}_X^\gamma \mid \max_{s\in K}\left| L(s,\chi_d) - \prod_{p\le h_X}\left(1-\frac{\chi_d(p)}{p^s}\right)^{-1}\right| < \varepsilon_1\}, \tag{5.6}$$

then for all large X,

$$\frac{\#\mathcal{A}_X^\gamma}{\#\mathcal{D}_X^\gamma} > 1-\varepsilon_1. \tag{5.7}$$

By (5.6) and (5.5), if X is large, then every $d \in \mathcal{A}_X^\gamma \cap \mathcal{B}_{X,\nu}^\gamma$ satisfies

$$\begin{aligned} &\max_{s\in K} |\, L(s,\chi_d) - h(s)\,| \\ &\le \max_{s\in K}\left| L(s,\chi_d) - \prod_{p\le h_X}\left(1-\frac{\chi_d(p)}{p^s}\right)^{-1}\right| + \max_{s\in K}\left|\prod_{p\le h_X}\left(1-\frac{\chi_d(p)}{p^s}\right)^{-1} - h(s)\right| \\ &\ll_{K,h(s)} \varepsilon. \end{aligned} \tag{5.8}$$

Furthermore, from (5.3) and (5.7) it follows that for the above number ν and all large X,

$$\begin{aligned} \#\left(\mathcal{A}_X^\gamma \cap \mathcal{B}_{X,\nu}^\gamma\right) &= \#\mathcal{A}_X^\gamma + \#\mathcal{B}_{X,\nu}^\gamma - \#\left(\mathcal{A}_X^\gamma \cup \mathcal{B}_{X,\nu}^\gamma\right) \\ &\ge \#\mathcal{A}_X^\gamma + \#\mathcal{B}_{X,\nu}^\gamma - \#\mathcal{D}_X^\gamma \ge \left(\frac{C_\nu}{2} - \varepsilon_1\right)\#\mathcal{D}_X^\gamma. \end{aligned} \tag{5.9}$$

Since $\frac{C_\nu}{2} - \varepsilon_1 > 0$, (5.8) and (5.9) yield (1.2). This completes the proof. □

Proof of Corollary 1.2. We shall first prove (1). It suffices to get (1.3) for any fixed $z_0 \in \mathbb{C} - \{0\}$, since the set $\mathbb{C} - \{0\}$ is dense in $\mathbb{C}$. Fix $z_0 \in \mathbb{C} - \{0\}$. We write $s_0 = \sigma_0 + it_0$, where $1/2 < \sigma_0 < 1$ and $t_0 \in \mathbb{R} - \{0\}$, and write $z_0 = re^{i\theta}$, where $r > 0$ and $\theta \in \mathbb{R}$. Now we consider the function $h(s) = re^{\frac{\theta}{t_0}(s-\sigma_0)}$, $s \in D$. Note that $h(s_0) = z_0$ and this function $h(s)$ satisfies the condition in Theorem 1.1 with $\Omega = D$. Thus Theorem 1.1, in which we choose $K = \{s_0\}$, yields (1.3).

Using the constant function $h(s) = x_0$, we also obtain (2) from Theorem 1.1. □

Proof of Corollary 1.3. Let $h(s)$ be the constant function on D whose value is $\frac{\alpha+\beta}{2}$. By the triangle inequality we have

$$\begin{aligned} &\{d \in \mathcal{D}_X^\gamma \mid \max_{s\in K} |\, L(s,\chi_d) - h(s)\,| < \frac{\beta-\alpha}{2}\} \\ &\subset \{d \in \mathcal{D}_X^\gamma \mid \alpha < |\, L(s,\chi_d)\,| < \beta \text{ uniformly for } s \in K\}. \end{aligned}$$

This and Theorem 1.1 complete the proof. □

Proof of Corollary 1.4. It suffices to prove that for a large integer $n > m + r' + 1$,

$$\begin{aligned} \liminf_{X\to\infty} \frac{1}{\#\mathcal{D}_X^\gamma} \#\{d \in \mathcal{D}_X^\gamma \mid L^{(r)}(s,\chi_d) &\text{ has at least } n-r-1 \text{ zeros on } [a,b] \\ &\text{for every } r = 1,2,\dots,r'\} > 0, \end{aligned} \tag{5.10}$$

since if $L^{(r)}(s,\chi_d)$ has at least $k+1$ zeros, then of course it has at least k zeros.

Fix an integer $n > m + r' + 1$. Let Ω be the open rectangle $\{s \in \mathbb{C} \,|\, 1/2 < \operatorname{Re} s < 1, |\operatorname{Im} s| < \frac{b-a}{n\pi}\}$. Consider the function

$$h(s) = 10 + \sin \frac{n\pi(s-a)}{b-a}, \qquad s \in \Omega.$$

Note that the function $h(s) - 10$ has $n-1$ zeros on the interval $(a,b) \subset \mathbb{R}$. Since $h(s)$ has no zeros on Ω and $h(s) \in \mathbb{R}^+$ for $s \in \Omega \cap \mathbb{R}$, Theorem 1.1 gives

$$\liminf_{X\to\infty} \frac{1}{\#\mathcal{D}_X^\gamma} \#\{d \in \mathcal{D}_X^\gamma \mid \max_{s\in[a,b]} |L(s,\chi_d) - h(s)| < \frac{1}{100}\} > 0. \tag{5.11}$$

Now assume that $d \in \mathcal{D}^\gamma$ satisfies

$$\max_{s\in[a,b]} |L(s,\chi_d) - h(s)| < \frac{1}{100}.$$

Then, noting that $L(s,\chi_d)$ is $\mathbb{R}$-valued on $[a,b]$, we find that the function $L(s,\chi_d) - 10$ has at least $n-1$ zeros on (a,b). Hence by Rolle's theorem, the function $\frac{d}{ds}(L(s,\chi_d) - 10) = L'(s,\chi_d)$ has at least $n-2$ zeros on (a,b). Making successive use of Rolle's theorem, we inductively deduce that $L^{(r)}(s,\chi_d)$ has at least $n-r-1$ zeros on (a,b) for any $r \in \mathbb{N}$ with $r < n-1$. Consequently,

$$\{d \in \mathcal{D}^\gamma \mid \max_{s\in[a,b]} |L(s,\chi_d) - h(s)| < \frac{1}{100}\} \subset \{d \in \mathcal{D}^\gamma \mid L^{(r)}(s,\chi_d) \text{ has at least } n-r-1 \text{ zeros on } (a,b) \text{ for every } r = 1,2,\dots,r'\}.$$

This and (5.11) yield (5.10), which completes the proof. □

Proof of Corollary 1.5. From the assumption on $\zeta(s)$, the fact that $\zeta(s)$ is holomorphic on D, and the condition on $g(s)$, it follows that the function $g(s)/\zeta(s)$ is holomorphic and has no zeros on D. Further, since $\zeta(s) < 0$ if s is real and $1/2 < s < 1$ (see e.g. [Ap, Theorem 13.11]), we have $g(s)/\zeta(s) \in \mathbb{R}^+$ for $1/2 < s < 1$.

For $d \in \mathcal{D}^\gamma$ we note that the discriminant of a quadratic field $\mathbb{Q}(\sqrt{d})$ is equal to d, and it is well known that the equality $\zeta_{\mathbb{Q}(\sqrt{d})}(s) = \zeta(s)L(s,\chi_d)$ holds. Hence $d \in \mathcal{D}^\gamma$ satisfies

$$\begin{aligned} \max_{s\in K} \left| \zeta_{\mathbb{Q}(\sqrt{d})}(s) - g(s) \right| &= \max_{s\in K} \left| \zeta(s) \left(L(s,\chi_d) - \frac{g(s)}{\zeta(s)} \right) \right| \\ &\ll_{K,\zeta(s)} \max_{s\in K} \left| L(s,\chi_d) - \frac{g(s)}{\zeta(s)} \right|. \end{aligned} \tag{5.12}$$

This and Theorem 1.1, in which we take $h(s) = g(s)/\zeta(s)$, complete the proof. □

Proof of Corollary 1.6. From the condition on $g(x)$ and the fact $\zeta(x) < 0$ for $1/2 < x < 1$ (see e.g. [Ap, Theorem 13.11]), it follows that $g(x)/\zeta(x)$ is a continuous function and is positive on $K = [a,b]$. According to the Weierstrass approximation theorem, for any small $\varepsilon > 0$ there exists a polynomial $p(x)$ with real coefficients such that $\max_{x\in K} |p(x) - \log(g(x)/\zeta(x))| < \varepsilon$. Hence

$$\begin{aligned} \max_{x\in K} \left| e^{p(x)} - \frac{g(x)}{\zeta(x)} \right| &= \max_{x\in K} \left| e^{\log(g(x)/\zeta(x))} \right| \left| e^{p(x) - \log(g(x)/\zeta(x))} - 1 \right| \\ &\ll_{K,g,\zeta} \max_{x\in K} |p(x) - \log(g(x)/\zeta(x))| < \varepsilon. \end{aligned} \tag{5.13}$$

As in (5.12), $d \in \mathcal{D}^\gamma$ satisfies

$$\begin{aligned}\max_{x\in K}\left|\zeta_{\mathbb{Q}(\sqrt{d})}(x)-g(x)\right| &\ll_{K,\zeta} \max_{x\in K}\left|L(x,\chi_d)-\frac{g(x)}{\zeta(x)}\right| \\ &\le \max_{x\in K}\left|L(x,\chi_d)-e^{p(x)}\right|+\max_{x\in K}\left|e^{p(x)}-\frac{g(x)}{\zeta(x)}\right|.\end{aligned} \tag{5.14}$$

Theorem 1.1, in which we choose $\Omega = D$ and $h(s) = e^{p(s)}$, shows that for any $\varepsilon > 0$,

$$\liminf_{X\to\infty}\frac{1}{\#\mathcal{D}_X^\gamma}\#\{d\in\mathcal{D}_X^\gamma \mid \max_{x\in K}\left|L(x,\chi_d)-e^{p(x)}\right|<\varepsilon\} > 0. \tag{5.15}$$

Combining (5.13), (5.14) and (5.15) gives the assertion. □

6. On the line $\operatorname{Re} s = 1$

In this section, we investigate the value distribution of $L(s,\chi_d)$ for fixed $s \in \mathbb{C}$ with $\operatorname{Re} s = 1$ and variable $d \in \mathcal{D}^\gamma$. First we shall consider the case $s \neq 1$.

Lemma 6.1. *Let $t \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$ be fixed. Then for any $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, there exist $\nu \ge y$ and $c_p \in \{1,-1\}$, for each prime p with $y \le p \le \nu$, such that*

$$\left|z_0-\sum_{y\le p\le\nu}\frac{c_p}{p^{1+it}}\right|<\varepsilon.$$

Proof. We prove this lemma by applying Lemma 2.2. Let $H = \mathbb{C}$. We consider H to be a real Hilbert space, by equipping H with the inner product $\operatorname{Re}\langle z_1, z_2\rangle$ ($z_1, z_2 \in H$), where $\langle z_1, z_2\rangle$ denotes the usual inner product attached to $\mathbb{C}$. Note that the norm $\|z\|$ of $z \in H$ is equal to the usual absolute value $|\,z\,|$, since $\|z\| := (\operatorname{Re}\langle z,z\rangle)^{1/2} = (\operatorname{Re}|\,z\,|^2)^{1/2} = (|\,z\,|^2)^{1/2} = |\,z\,|$. For $n \in \mathbb{N}$ let $u_n = 1/p_n^{1+it}$, where p_n is the n-th prime. Then Lemma 2.2 yields Lemma 6.1 if we verify the conditions (a) and (b) in Lemma 2.2 for H and u_n above.

In the first place, condition (a) holds since $\sum_p 1/p^2 < \infty$. Next, given any number $u = e^{2\pi i\theta}$ ($0 \le \theta < 2\pi$) with $|\,u\,| = 1$, we shall check condition (b), that is,

$$\sum_n |\operatorname{Re}\langle u_n, u\rangle| = \sum_n\left|\operatorname{Re}\frac{e^{-2\pi i\theta}}{p_n^{1+it}}\right| = \infty. \tag{6.1}$$

Fix $a \in \mathbb{R}$ such that $0 < a < \min\{t, 1/100\}$. For $m \in \mathbb{N}$ let $\alpha_m = 2\pi(m-\theta-a)/t$ and $\beta = 2\pi a/t$, and we define

$$\mathcal{P}_m = \{p \mid e^{\alpha_m} < p \le e^{\alpha_m+\beta},\ p \text{ is a prime}\}.$$

Then $p \in \mathcal{P}_m$ satisfies

$$-m \le -\theta-\frac{\log p}{2\pi}t < -m+a.$$

Hence for $p \in \mathcal{P}_m$ we have

$$\left|\operatorname{Re}\frac{e^{-2\pi i\theta}}{p^{1+it}}\right| > \frac{1}{p}\left|\operatorname{Re}e^{2\pi i(-m+a)}\right| > \frac{1}{2p} \ge \frac{1}{2e^{\alpha_m+\beta}}, \tag{6.2}$$

since $0 < a < 1/100$. Note that for $m, m' \in \mathbb{N}$ with $m \neq m'$,

$$\mathcal{P}_m \cap \mathcal{P}_{m'} = \varnothing \text{ (empty)}. \tag{6.3}$$

As usual, for $x > 0$ let $\pi(x)$ denote the number of primes not exceeding x. Since $0 < \beta < 2\pi$, $e^{\beta} - 1 > \beta$ and the prime number theorem $\pi(x) = \int_2^x \frac{1}{\log u} du + O\left(xe^{-c\sqrt{\log x}}\right)$ holds with an absolute positive constant c, we find that there exists a large integer m_0 such that for all $m \geq m_0$,

$$\begin{aligned}\#\mathcal{P}_m = \pi\left(e^{\alpha_m+\beta}\right) - \pi\left(e^{\alpha_m}\right) &= \int_{e^{\alpha_m}}^{e^{\alpha_m+\beta}} \frac{du}{\log u} + O\left(e^{\alpha_m+\beta}e^{-c\sqrt{\alpha_m}}\right) \\ &\geq \frac{e^{\alpha_m+\beta} - e^{\alpha_m}}{\alpha_m+\beta} + O\left(e^{\alpha_m+\beta}e^{-c\sqrt{\alpha_m}}\right) > \frac{1}{2}\,\frac{e^{\alpha_m+\beta} - e^{\alpha_m}}{\alpha_m+\beta} > \frac{\beta e^{\alpha_m}}{4\alpha_m}.\end{aligned}$$

This, (6.2) and (6.3) imply that for all large integers $M > m_0$,

$$\begin{aligned}\sum_{p \leq e^{2\pi(M-\theta)/t}} \left|\operatorname{Re}\frac{e^{-2\pi i\theta}}{p^{1+it}}\right| &\geq \sum_{m=m_0}^{M} \sum_{p\in\mathcal{P}_m} \left|\operatorname{Re}\frac{e^{-2\pi i\theta}}{p^{1+it}}\right| \geq \sum_{m=m_0}^{M} \sum_{p\in\mathcal{P}_m} \frac{1}{2\,e^{\alpha_m+\beta}} \\ &\gg \sum_{m=m_0}^{M} \frac{1}{2\,e^{\alpha_m+\beta}}\frac{\beta e^{\alpha_m}}{\alpha_m} \gg t\beta e^{-\beta} \sum_{m=m_0}^{M} \frac{1}{m} \gg t\beta e^{-\beta}\left(\log M + O(m_0)\right).\end{aligned}$$

Consequently,

$$\sum_{p \leq e^{2\pi(M-\theta)/t}} \left|\operatorname{Re}\frac{e^{-2\pi i\theta}}{p^{1+it}}\right| \longrightarrow \infty \quad \text{as } M \to \infty.$$

Thus we obtain (6.1) and complete the proof. □

Proposition 6.2. *Let $t \in \mathbb{R}^+$ be fixed. Let $z \in \mathbb{C}$ and $\nu_1 \geq 3$. Set $a_2 = 1$. Then for any $\varepsilon > 0$ there exist $\nu > \nu_1$ and $a_p \in \{1, -1\}$, for each prime p with $3 \leq p \leq \nu$, such that*

$$\left| z - \log \prod_{p \leq \nu} \left(1 - \frac{a_p}{p^{1+it}}\right)^{-1} \right| < \varepsilon,$$

where

$$\log \prod_{p \leq \nu} \left(1 - \frac{a_p}{p^{1+it}}\right)^{-1} = -\sum_{p \leq \nu} \log\left(1 - \frac{a_p}{p^{1+it}}\right) = \sum_{p \leq \nu} \sum_{n=1}^{\infty} \frac{a_p^n}{np^{n(1+it)}}.$$

Proof. The proof is similar to that of Proposition 2.6. Let $\varepsilon > 0$ be arbitrary. Take a large number $y > \nu_1$ such that $1/y < \varepsilon$. Then

$$\sum_{p \geq y} \sum_{n=2}^{\infty} \frac{1}{np^n} \ll \sum_{p \geq y} \frac{1}{p^2} \ll \frac{1}{y} < \varepsilon. \tag{6.4}$$

From Lemma 6.1 it follows that there exist $\nu \geq y$ and $c_p \in \{1, -1\}$, for each prime p with $y \leq p \leq \nu$, such that

$$\left| \left(z - \sum_{p<y} \sum_{n=1}^{\infty} \frac{1}{np^{n(1+it)}} \right) - \sum_{y \leq p \leq \nu} \frac{c_p}{p^{1+it}} \right| < \varepsilon. \tag{6.5}$$

For each prime p with $3 \leq p \leq \nu$ we set

$$a_p := \begin{cases} 1 & \text{if } 3 \leq p < y, \\ c_p & \text{if } y \leq p \leq \nu. \end{cases}$$

Then we obtain, by (6.4) and (6.5),

$$\begin{aligned}
&\left| z - \log \prod_{p\le\nu}\left(1-\frac{a_p}{p^{1+it}}\right)^{-1}\right| \\
&= \left| z - \sum_{p<y}\sum_{n=1}^{\infty}\frac{1}{np^{n(1+it)}} - \sum_{y\le p\le\nu}\frac{c_p}{p^{1+it}} - \sum_{y\le p\le\nu}\sum_{n=2}^{\infty}\frac{c_p^n}{np^{n(1+it)}}\right| \\
&\le \left| z - \sum_{p<y}\sum_{n=1}^{\infty}\frac{1}{np^{n(1+it)}} - \sum_{y\le p\le\nu}\frac{c_p}{p^{1+it}}\right| + \left|\sum_{y\le p\le\nu}\sum_{n=2}^{\infty}\frac{c_p^n}{np^{n(1+it)}}\right| \\
&< \varepsilon + \sum_{p\ge y}\sum_{n=2}^{\infty}\frac{1}{np^n} \ll \varepsilon,
\end{aligned}$$

which completes the proof. □

Proof of Theorem 1.7. The proof is similar to that of Theorem 1.1 in Section 5. Since $L(1+it,\chi_d) = \overline{L(1-it,\chi_d)}$, it suffices to verify the assertion in the case $t>0$. Moreover, it suffices to consider the case $z_0 \in \mathbb{C}-\{0\}$, since the set $\mathbb{C}-\{0\}$ is dense in $\mathbb{C}$.

Fix $z_0 \in \mathbb{C}-\{0\}$ and $t>0$. Take a complex number z such that $z_0 = e^z$. Let $\varepsilon>0$ be an arbitrary small number. Take $\sigma_1 \in \mathbb{R}$ with $1/2<\sigma_1<1$, and set $K=\{1+it\}$. Take $\nu_1 \in \mathbb{R}^+$ so large that $1/\nu_1<\varepsilon$ and $\nu_1>\nu_0(\sigma_1,K,\varepsilon)$, where $\nu_0(\sigma_1,K,\varepsilon)$ is the constant in Proposition 4.4. According to Proposition 6.2, there exist $\nu>\nu_1$ and $a_p\in\{1,-1\}$, for each prime p with $3\le p\le\nu$, such that

$$\left| z - \log \prod_{p\le\nu}\left(1-\frac{a_p}{p^{1+it}}\right)^{-1}\right| < \varepsilon, \tag{6.6}$$

where $a_2=1$.

For those a_p's we apply Proposition 4.4, from which it follows that for the above number ν and all large $X>0$, we have

$$\frac{\#\mathcal{B}^{\gamma}_{X,\nu}}{\#\mathcal{D}^{\gamma}_{X}} > \frac{1}{2}C_\nu. \tag{6.7}$$

Further, for all large X and all $d\in\mathcal{B}^{\gamma}_{X,\nu}$,

$$\begin{aligned}
&\left|\log\prod_{p\le\nu}\left(1-\frac{a_p}{p^{1+it}}\right)^{-1} - \log\prod_{p\le h_X}\left(1-\frac{\chi_d(p)}{p^{1+it}}\right)^{-1}\right| \\
&= \left|\sum_{\nu<p\le h_X}\frac{\chi_d(p)}{p^{1+it}} + \sum_{\nu<p\le h_X}\sum_{n=2}^{\infty}\frac{\chi_d(p^n)}{np^{n(1+it)}}\right| \le \varepsilon + O(\varepsilon) \ll \varepsilon,
\end{aligned} \tag{6.8}$$

since $\sum_{\nu<p\le h_X}\sum_{n=2}^{\infty}\frac{1}{np^n} \ll \sum_{\nu<p\le h_X}\frac{1}{p^2} \ll \nu^{-1} < \nu_1^{-1} < \varepsilon$. By (6.6) and (6.8), if $X>0$ is large, then every $d\in\mathcal{B}^{\gamma}_{X,\nu}$ satisfies

$$\left| z - \log\prod_{p\le h_X}\left(1-\frac{\chi_d(p)}{p^{1+it}}\right)^{-1}\right| \ll \varepsilon$$

and hence

$$\left| \prod_{p \le h_X} \left(1 - \frac{\chi_d(p)}{p^{1+it}}\right)^{-1} - z_0 \right| = |\, z_0 \,| \left| e^{\log \prod_{p \le h_X} \left(1 - \frac{\chi_d(p)}{p^{1+it}}\right)^{-1} - z} - 1 \right| \ll_{z_0} \varepsilon. \tag{6.9}$$

Let ε_1 be a small positive number such that $\varepsilon_1 < \min\{\varepsilon, \frac{C_\nu}{2}\}$. Proposition 3.2 implies that if we put

$$\mathcal{A}_X^\gamma := \{d \in \mathcal{D}_X^\gamma \mid \left| L(1+it, \chi_d) - \prod_{p \le h_X} \left(1 - \frac{\chi_d(p)}{p^{1+it}}\right)^{-1} \right| < \varepsilon_1\}, \tag{6.10}$$

then

$$\frac{\#\mathcal{A}_X^\gamma}{\#\mathcal{D}_X^\gamma} > 1 - \varepsilon_1 \tag{6.11}$$

for all large X. Hence by (6.9) and (6.10), if $X > 0$ is large, then every $d \in \mathcal{A}_X^\gamma \cap \mathcal{B}_{X,\nu}^\gamma$ satisfies

$$|\, L(1+it, \chi_d) - z_0 \,| \ll_{z_0} \varepsilon. \tag{6.12}$$

Furthermore, from (6.7) and (6.11) we see that for the above number ν and all X sufficiently large,

$$\#\left(\mathcal{A}_X^\gamma \cap \mathcal{B}_{X,\nu}^\gamma\right) \ge \left(\frac{C_\nu}{2} - \varepsilon_1\right) \#\mathcal{D}_X^\gamma. \tag{6.13}$$

Since $\frac{C_\nu}{2} - \varepsilon_1 > 0$, (6.12) and (6.13) complete the proof. □

Next we shall investigate the value distribution of $L(1, \chi_d)$ as d varies.

Lemma 6.3. *Let $y \in \mathbb{R}^+$. For any $x_0 \in \mathbb{R}$ and $\varepsilon > 0$ there exist $\nu \ge y$ and $c_p \in \{1, -1\}$, for each prime p with $y \le p \le \nu$, such that*

$$\left| x_0 - \sum_{y \le p \le \nu} \frac{c_p}{p} \right| < \varepsilon.$$

Proof. We prove this lemma by applying Lemma 2.2. In Lemma 2.2 we choose $H = \mathbb{R}$, which we equip with the usual inner product attached to $\mathbb{R}$, and choose $u_n = 1/p_n$, where p_n is the n-th prime.

In this setting, condition (a) in Lemma 2.2 holds since $\sum_p 1/p^2 < \infty$. Given any number $u \in \mathbb{R}$ with $|\, u \,| = 1$, we find that condition (b) holds, since

$$\sum_p \left| \left\langle \frac{1}{p}, u \right\rangle \right| = \sum_p \left| \frac{u}{p} \right| = \sum_p \frac{1}{p} = \infty.$$

Thus Lemma 2.2 gives the assertion. □

Proposition 6.4. *Let $x \in \mathbb{R}$. Let $\nu_1 \ge 3$ and $a_2 = 1$. Then for any $\varepsilon > 0$ there exist $\nu > \nu_1$ and $a_p \in \{1, -1\}$, for each prime p with $3 \le p \le \nu$, such that*

$$\left| x - \log \prod_{p \le \nu} \left(1 - \frac{a_p}{p}\right)^{-1} \right| < \varepsilon,$$

where

$$\log \prod_{p\le\nu}\left(1-\frac{a_p}{p}\right)^{-1} = -\sum_{p\le\nu}\log\left(1-\frac{a_p}{p}\right) = \sum_{p\le\nu}\sum_{n=1}^{\infty}\frac{a_p^n}{np^n}.$$

Proof. This proposition is proved similarly to the proof of Proposition 6.2, by using Lemma 6.3. □

Proof of Theorem 1.8. We can prove Theorem 1.8 by the same argument as in the proof of Theorem 1.7, using Propositions 6.4, 4.4, 3.2. □

Proof of Corollary 1.9. According to Dirichlet's class number formula, we have

$$L(1,\chi_d) = \begin{cases} \frac{2h(d)\log\varepsilon(d)}{\sqrt{d}} & \text{if } d\in\mathcal{D}^+, \\ \frac{\pi h(d)}{\sqrt{|\,d\,|}} & \text{if } d\in\mathcal{D}^-. \end{cases}$$

Hence by Theorem 1.8, the sets $\{\frac{2h(d)\log\varepsilon(d)}{\sqrt{d}} \mid d\in\mathcal{D}^+\}$ and $\{\frac{\pi h(d)}{\sqrt{|\,d\,|}} \mid d\in\mathcal{D}^-\}$ are dense in $\mathbb{R}^+$. From this we find that the sets $\{\frac{h(d)\log\varepsilon(d)}{\sqrt{d}} \mid d\in\mathcal{D}^+\}$ and $\{\frac{h(d)}{\sqrt{|\,d\,|}} \mid d\in\mathcal{D}^-\}$ are dense in $\mathbb{R}^+$. □

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