INSEPARABLE EXTENSIONS OF ALGEBRAS
OVER THE STEENROD ALGEBRA WITH APPLICATIONS
TO MODULAR INVARIANT THEORY OF FINITE GROUPS

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Abstract. We consider purely inseparable extensions \( H \hookrightarrow \mathfrak{P} \) of unstable Noetherian integral domains over the Steenrod algebra. It turns out that there exists a finite group \( G \leq \text{GL}(V) \) and a vector space decomposition \( V = W_0 \oplus W_1 \oplus \cdots \oplus W_e \) such that \( \mathfrak{P} = (F[W_0] \otimes F[W_1]^p \otimes \cdots \otimes F[W_e]^p)^G \) and \( \sqrt[\mathfrak{P}]{H} = F[V]^G \), where \( (\cdot) \) denotes the integral closure. Moreover, \( H \) is Cohen-Macaulay if and only if \( \sqrt[\mathfrak{P}]{H} \) is Cohen-Macaulay. Furthermore, \( H \) is polynomial if and only if \( \sqrt[\mathfrak{P}]{H} \) is polynomial, and \( \sqrt[\mathfrak{P}]{H} = F[h_1, \ldots, h_n] \) if and only if
\[
H = F[h_1, \ldots, h_{n_0}, h_{n_0+1}^p, \ldots, h_{n_1}^p, h_{n_1+1}^p, \ldots, h_{n_2}^p],
\]
where \( n_e = n \) and \( n_i = \dim_F(W_0 \oplus \cdots \oplus W_i) \).

1. Introduction and outline

Let \( K \hookrightarrow L \) be an algebraic extension of graded fields. Assume that the smaller field, \( K \), carries an action of the Steenrod algebra \( \mathfrak{P}^* \) of reduced powers. If the extension \( K \hookrightarrow L \) is separable, then the action of \( \mathfrak{P}^* \) can be uniquely extended to \( L \). In other words, the separable closure of \( K \) as a field over the Steenrod algebra coincides with the separable closure in the category of graded fields; see Proposition 2.2.2 in [3] and Proposition 2.2 in [7].

If the extension, however, is purely inseparable the situation is more delicate: Let \( p(X) \in K[X] \) be the minimal polynomial of \( l \in L \). Since the extension is purely inseparable, we have that
\[
p(X) = X^{p^e} - \kappa,
\]
so that \( b^{p^e} = \kappa \) for some \( \kappa \in K \). Of course, since our fields are graded, we obtain the following condition on the degrees:
\[
(*) \quad \deg(l) p^e = \deg(\kappa).
\]
However, the crucial issue is the following. If there were an extension of the $P^*$-action to the larger field $L$, then

$$\mathcal{P} \Delta^i(\kappa) = 0 \quad \forall i$$

because by equation (\star) the element $\kappa$ is a $p$th power. Thus, we need to define

(\star) \quad (\mathcal{P}^i(l))^{p^e} = \mathcal{P}^{ip^e}(\kappa) \in K.$

The problem is that it does not follow that $\mathcal{P}^i(l) \in L$. Nevertheless, as equation (\star) shows, the inseparable closures of $K$ as a graded field and as a field over the Steenrod algebra coincide. We denote this object by $\sqrt[p]{K}$.

This leads to the following question: Under which conditions can we extend the action of $\mathcal{P}^*$ from $K$ to $L$? Or equivalently, which intermediate fields $K \subseteq L \subseteq \sqrt[p]{K}$ are objects in the category of fields over the Steenrod algebra?

In this paper we study these questions in the more general framework of Noetherian integral domains $H$ over the Steenrod algebra.

In Sections 3 and 4 we start with the investigation of inseparable extensions $H \hookrightarrow \sqrt[p]{H}$, where $\sqrt[p]{H}$ is either the symmetric algebra, $F[V]$, on $V^*$ with $V = F^n$, or its field of fractions, $F(V)$. This has two reasons: for one, $F(V)$ and $F[V]$ are universal, in the sense that they are algebraically closed in our category. On the other hand, any unstable Noetherian integral domain $H$ can be embedded into $F[V]$ such that the inclusion

$$H \hookrightarrow F[V]$$

is finite; see the Embedding Theorem, Corollary 6.1.5 in [3]. Thus in Sections 3 and 4 we consider the diagram

$$
\begin{array}{c}
H \hookrightarrow \sqrt[p]{H} = F[V] \\
\downarrow \\
\mathbb{H} \hookrightarrow \sqrt[p]{\mathbb{H}} = F(V),
\end{array}
$$

where $\mathbb{H} = FF(H)$ is the field of fractions of $H$. In Section 3 we treat the case of purely inseparable extensions of exponent one, in Section 4 we look at extensions with higher exponents $e$. Denote by $(-)$ the integral closure. The results of these parts show that

$$\Pi = F[W_0] \otimes F[W_1]^p \otimes \cdots \otimes F[W_e]^p$$

and

$$\mathbb{H} = F(W_0) \otimes F(W_1)^p \otimes \cdots \otimes F(W_e)^p$$

for some vector space decomposition $V = W_0 \oplus W_1 \oplus \cdots \oplus W_e$; see Theorem 4.13 and Corollary 4.14. This reproves results in [3], Theorem II, and [3], Theorem 7.2.2. However, the proof presented here has the advantage that it gives precise information on the vector space dimensions of the $W_i$’s.

\footnote{1 All tensor products in this manuscript are tensor products over the ground field $F$.}

\footnote{2 We denote by $F[V]^p$ the algebra $F[x_1^p, \ldots, x_n^p]$ for $F[V] = F[x_1, \ldots, x_n]$.}
In Section \[5\] we come to the general case. By the Galois Embedding Theorem (Theorem 7.1.1 in [3]), we know that 
\[ \sqrt{H} \] is a ring of invariants of some finite group \( G \leq \text{GL}(V) \) acting linearly on \( \mathbb{F}[V] \). Thus
\[
\begin{align*}
\mathbb{H} \ &
\hookrightarrow \sqrt{\mathbb{H}} = \mathbb{F}[V]^G \\
\mathbb{H} \ &
\hookrightarrow \sqrt{\mathbb{H}} = \mathbb{F}(V)^G.
\end{align*}
\]
It turns out that there exists a vector space decomposition as above,
\[ V = W_0 \oplus W_1 \oplus \cdots \oplus W_e, \]
such that \( G \) acts on the flags
\[ W_0 \oplus W_1 \oplus \cdots \oplus W_i \]
for all \( i = 0, \ldots, e \). Moreover
\[
\begin{align*}
\mathbb{H} &= \mathbb{F}(W_0) \otimes \mathbb{F}(W_1)^p \otimes \cdots \otimes \mathbb{F}(W_e)^{p^e} \otimes \mathbb{F}(W_e)^{p^e} \otimes \mathbb{F}(W_e)^{p^e};
\end{align*}
\]
see Theorem 5.2. This extends results in [8], Theorem II, and [3], Theorem 7.2.2, in the sense that we are able to determine the vector space dimensions of the \( W_i \)'s and, more importantly, are able to prove that the group \( G \) in question remains unchanged. In particular this means that the group \( G \) must consist of flag matrices
\[
\begin{bmatrix}
A_0 & 0 & \cdots & 0 \\
* & A_1 & 0 & \cdots & 0 \\
* & \ddots & \ddots & \ddots & \vdots \\
\cdots & \ddots & \ddots & \ddots & 0 \\
* & \cdots & \cdots & * & A_e
\end{bmatrix},
\]
where \( A_i \) is an \( m_i \times m_i \)-matrix with \( m_i = \text{dim}(W_i) \). On the other hand, if \( V \) has no basis such that \( G \) consists of flag matrices, then the only purely inseparable extensions of exponent \( e \) are
\[
(\mathbb{F}[V]^{p^e})^G \subseteq \mathbb{F}[V]^G.
\]

In Section \[6\] we take a break from these constructive methods and look at homological properties of \( H \) and \( \sqrt{H} \). We show that \( H \) is Cohen-Macaulay if and only if \( \sqrt{H} \) is Cohen-Macaulay for any reduced Noetherian unstable algebra \( H \).

This motivates Section \[7\] where we look at polynomial rings. It turns out that \( \sqrt{H} \) is a polynomial algebra if and only if \( H \) is polynomial. Moreover, \( \sqrt{H} = \mathbb{F}[h_1, \ldots, h_n] \) if and only if
\[
H = \mathbb{F}[h_1, \ldots, h_{n_0}, \ldots, h_{n_0+1}, \ldots, h_{n_1}, \ldots, h_{n_1+1}, \ldots, h_{n_e}],
\]
where \( n_e = n \) and \( n_i = \text{dim}(W_i) \). Recall that an unstable \( \mathbb{P}^* \)-inseparably closed polynomial algebra over the Steenrod algebra is the ring of invariants \( \mathbb{F}[V]^G \) for some \( G \leq \text{GL}(n, \mathbb{F}) \) by the Galois Embedding Theorem (Theorem 7.1.1 in [3]). Combined with the results from Section \[5\] this means that if \( G \) consists of flag matrices, then \( H \) is polynomial if and only if \( \sqrt{H} \) is polynomial, and the generators are just \( p \)th powers/roots of one another. However, if \( G \) does not consist of flag
matrices, then there exists no unstable algebra \( H \hookrightarrow \mathbb{F}[V]^G \) such that \( \sqrt[p]{H} = \mathbb{F}[V]^G \). This solves a twenty-year-old conjecture due to Clarence Wilkerson; see Conjecture 5.1 in [3].

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2. Recollections and preliminaries

Let \( H \) be an unstable reduced algebra over the Steenrod algebra of reduced powers \( \mathcal{P}^* \). We denote the characteristic by \( p \), and the order of the ground field \( \mathbb{F} \) by \( q \). Recall that the Steenrod algebra contains an infinite sequence of derivations iteratively defined as

\[
\mathcal{P}^i = \mathcal{P}^{i-1} - \mathcal{P}^{i-1} \mathcal{P}^{i-1}
\]

for \( i \geq 2 \).

We set

\[
\mathcal{P}^{\Delta_0}(h) = \deg(h) \quad \forall h \in H.
\]

Note that \( \mathcal{P}^{\Delta_0} \) is not an element of the Steenrod algebra.

The algebra \( H \) is called \( \mathcal{P}^*\)-inseparably closed, if whenever \( h \in H \) and

\[
\mathcal{P}^i(h) = 0 \quad \forall i \geq 0,
\]

then there exists an element \( h' \in H \) such that

\[
(h')^p = h.
\]

The \( \mathcal{P}^*\)-inseparable closure of \( H \) is a \( \mathcal{P}^*\)-inseparably closed algebra \( \sqrt[p]{H} \) containing \( H \) such that the following universal property holds: Whenever we have a \( \mathcal{P}^*\)-inseparably closed algebra \( H' \) containing \( H \) there exists an embedding \( \varphi: \sqrt[p]{H} \hookrightarrow H' \).

The following method to construct the \( \mathcal{P}^*\)-inseparable closure of \( H \) is taken from Section 4.1 in [3]. Denote by \( C \subseteq H \) the subalgebra consisting of the \( \mathcal{P}^{\Delta} \)-constants for all \( i \geq 0 \), i.e.,

\[
C = \{ h \in H \mid \mathcal{P}^i(h) = 0 \ \forall i \geq 0 \}.
\]

It turns out that the subalgebra of constants \( C \) is an unstable algebra over the Steenrod algebra (Lemma 4.1.2 loc.cit.). Moreover, it is Noetherian whenever \( H \) is (Lemma 4.1.1 loc.cit.). By construction we have integral extensions

\[
H^p \hookrightarrow C \hookrightarrow H,
\]

where \( H^p = \{ h^p \mid h \in H \} \). Denote by \( \mathcal{S} \) a set of generators for \( C \) as a module over \( H^p \). Define an algebra

\[
H_1 = (H \otimes \mathbb{F}[\mathcal{S}]_s \mid s \in \mathcal{S}) / \mathcal{R}ad(\gamma^p_s - s \mid s \in \mathcal{S}),
\]

where \( \mathcal{R}ad(\cdot) \) denotes the radical of the ideal \( (\cdot) \). Note that this construction comes with a canonical inclusion

\[
\varphi_0: H \hookrightarrow H_1,
\]
by part (5) of Lemma 4.1.3 in [3]. Since the new algebra $H_1$ is again an unstable reduced algebra over the Steenrod algebra (see Lemma 4.1.4 loc.cit.), we can iterate the construction and obtain a nested sequence

$$H = H_0 \hookrightarrow H_1 \hookrightarrow \cdots \hookrightarrow H_i \hookrightarrow \cdots$$

of unstable reduced algebras over the Steenrod algebra. The colimit of this sequence is the $\mathcal{P}^*$-inseparable closure of $H$ (Proposition 4.1.5 loc.cit.). We recall the basic properties of $\mathcal{P}^*\sqrt{H}$ and the intermediate algebras $H_i$.

**Proposition 2.1.** Consider the chain of unstable reduced algebras

$$H = H_0 \hookrightarrow H_1 \hookrightarrow \cdots \hookrightarrow H_i \hookrightarrow \cdots \hookrightarrow \mathcal{P}^*\sqrt{H}.$$

Then the following statements hold.

1. If one of the algebras in this chain is an integral domain, then so are the others.
2. $H \hookrightarrow \mathcal{P}^*\sqrt{H}$ is an integral extension, and both algebras have the same Krull dimension.
3. If $H$ is integrally closed, then so is $\mathcal{P}^*\sqrt{H}$.
4. The following statements are equivalent.
   - $H$ is Noetherian.
   - $H_i$ is Noetherian.
   - $\mathcal{P}^*\sqrt{H}$ is Noetherian.
   - There exists an $r$ such that
     $$H_r = H_{r+1} = \cdots = \mathcal{P}^*\sqrt{H}.$$  

**Proof.** For (1)–(3) see Proposition 4.2.1 in [3]. For (4) see part (2) of Lemma 4.1.3, Lemma 4.2.2, Proposition 4.2.4, and Theorem 6.3.1 loc.cit. □

**Lemma 2.2.** Let $H$ be an integral domain. If $H$ is integrally closed, then the algebras $H_i$ are also integrally closed for all $i$.

**Proof.** It is shown in part (5) of Proposition 4.2.1 in [3] that $\mathcal{P}^*\sqrt{H}$ is integrally closed whenever $H$ is integrally closed. The same argument presented there can be used to show that also the algebras $H_i$ are also integrally closed. □

In the same way the $\mathcal{P}^*$-inseparable closure of a field $\mathbb{K}$ over the Steenrod algebra can be constructed. So we obtain a chain of fields over the Steenrod algebra

$$\mathbb{K} = \mathbb{K}_0 \hookrightarrow \mathbb{K}_1 \hookrightarrow \cdots \hookrightarrow \mathbb{K}_i \hookrightarrow \cdots \hookrightarrow \mathcal{P}^*\sqrt{\mathbb{K}}$$

by adjoining successively $p$th roots. Again the $p$th powers are detected by the vanishing of the derivations $\mathcal{P}^*\Delta$; see Section 2.3 in [3].

Let $H$ be an unstable integral domain over the Steenrod algebra. Denote by $\mathbb{H}$ its field of fractions. We have seen in Proposition 4.2.6 in [3] that

$$FF(\mathcal{P}^*\sqrt{\mathbb{H}}) = \mathcal{P}^*\sqrt{\mathbb{H}},$$

where $FF(-)$ denotes the field of fraction functor.

Our first goal is to refine this statement. For this we need the following result.

**Proposition 2.3.** Let $H$ be an unstable integral domain. Then

$$\mathcal{C}(\mathbb{H}) = FF(\mathcal{C}(H)).$$
Proof. Let $\frac{f_1}{f_2} \in \mathcal{C}(H)$, $f_1, f_2 \in H$. Then there exists an element $\frac{h_1}{h_2} \in {}^{2}\sqrt{H} = FF({}^{2}\sqrt{H})$, $h_1h_2 \in {}^{2}\sqrt{H}$, such that

$$\frac{h_1^p}{h_2^p} = \frac{f_1}{f_2}$$

for some $k \in \mathbb{N}_0$. Furthermore, since $h_1, h_2 \in {}^{2}\sqrt{H}$ we can choose $k$ such that $h_1^k, h_2^k \in H$. By construction, $h_1^k, h_2^k$ are in the subalgebra of constants, $\mathcal{C}(H)$. Thus

$$\frac{f_1}{f_2} = \frac{h_1^p}{h_2^p} \in FF(\mathcal{C}(H))$$

which shows that

$$\mathcal{C}(H) \subseteq FF(\mathcal{C}(H)).$$

Conversely, let $\frac{f_1}{f_2} \in FF(\mathcal{C}(H))$ with $f_1, f_2 \in \mathcal{C}(H)$. Then

$$\frac{f_1}{f_2} \in H$$

is a constant because

$$\mathcal{P}_\Delta(\frac{f_1}{f_2}) = \mathcal{P}_\Delta(f_1)f_2 - f_1 \mathcal{P}_\Delta(f_2) = 0$$

for all $i \in \mathbb{N}_0$. □

**Proposition 2.4.** Let $H$ be an unstable integral domain over the Steenrod algebra. Denote by $\mathbb{H}$ its field of fractions. Then for all $i \in \mathbb{N}_0$ we have

$$FF(H_i) = \mathbb{H}_i.$$  

Proof. By induction it is enough to show the statement for $i = 1$. If $\frac{h_1}{h_2} \in FF(H)_1$ for $h_1, h_2 \in H_1$, then $\frac{h_1}{h_2} \in FF(H_0) = \mathbb{H}_0 = \mathbb{H}$. Thus $\frac{h_1}{h_2} \in \mathbb{H}_1$, since $\mathbb{H}_1$ is obtained from $\mathbb{H}_0$ by adjoining all $p$th roots. Thus $FF(H_1) \subseteq \mathbb{H}_1$.

We prove the reverse inclusion. Let $h \in \mathbb{H}_1$. Then $h^p \in \mathbb{H} = FF(H)$. Thus by Proposition 2.3

$$h^p \in \mathcal{C}(\mathbb{H}) = FF(\mathcal{C}(H)).$$

Thus there exist elements $h_1, h_2 \in \mathcal{C}(H)$ such that

$$h^p = \frac{h_1}{h_2}.$$  

Moreover, since the elements $h_1, h_2$ are constants they have $p$th roots, say $f_1, f_2$, in $H_1$. Thus

$$h = \frac{f_1}{f_2} \in FF(H_1)$$

and we are done. □
Hence we obtain chains

\[
\begin{array}{c}
H = H_0 \hookrightarrow H_1 \hookrightarrow \cdots \hookrightarrow H_i \hookrightarrow \cdots \hookrightarrow \sqrt{\sqrt{H}} \\
FF(H_0) \hookrightarrow FF(H_1) \hookrightarrow \cdots \hookrightarrow FF(H_i) \hookrightarrow \cdots \hookrightarrow FF(\sqrt{\sqrt{H}}) \\
\mathbb{H}_0 \hookrightarrow \mathbb{H}_1 \hookrightarrow \cdots \hookrightarrow \mathbb{H}_i \hookrightarrow \cdots \hookrightarrow \sqrt{\sqrt{\mathbb{H}}}
\end{array}
\]

Let \( H \) be an unstable Noetherian integral domain over the Steenrod algebra. Then there exists an \( r \in \mathbb{N}_0 \) such that \( H_r = \sqrt{\sqrt{H}} \); see Theorem 6.1.3 and Proposition 4.2.4 in [3]. Also, there exists an \( s \in \mathbb{N}_0 \) such that \( H_s = \sqrt{\sqrt{\mathbb{H}}} \), loc.cit. Without loss of generality we assume that \( r \) and \( s \) are minimal with respect to this property. Then by Proposition 4.2.4 in [3] we have that \( r \geq s \). Thus for Noetherian unstable algebras we obtain finite chains

\[
\begin{array}{c}
H = H_0 \hookrightarrow H_1 \hookrightarrow \cdots \hookrightarrow H_s \hookrightarrow H_{s+1} \hookrightarrow \cdots \hookrightarrow H_r = \sqrt{\sqrt{H}} \\
FF(H_0) \hookrightarrow FF(H_1) \hookrightarrow \cdots \hookrightarrow FF(H_s) \hookrightarrow FF(H_{s+1}) \hookrightarrow \cdots \hookrightarrow FF(H_r) \\
\mathbb{H}_0 \hookrightarrow \mathbb{H}_1 \hookrightarrow \cdots \hookrightarrow \mathbb{H}_s \hookrightarrow \mathbb{H}_{s+1} \hookrightarrow \cdots \hookrightarrow \mathbb{H}_r = \sqrt{\sqrt{\mathbb{H}}}
\end{array}
\]

**Corollary 2.5.** Let \( H \) be an unstable Noetherian integral domain. Let \( H \) be integrally closed. Then with the preceding notation, \( r = s \).

**Proof.** By Proposition 2.4 \( FF(H_i) = \mathbb{H}_i \). Moreover, \( H_i \) is integrally closed for all \( i \) by Lemma 2.2. Thus for all \( i \) the unstable part of \( FF(H_i) \) is

\[
\text{un}(FF(H_i)) = H_i;
\]

see Theorem 2.4 in [4]. Thus

\[
H_s = \text{un}(FF(H_s)) = \text{un}(FF(H_r)) = H_r
\]

as desired. \( \square \)

3. **Inseparable Extensions of Exponent 1**

Let \( H \) be an unstable Noetherian reduced algebra over the Steenrod algebra. Define the \( H \)-module

\[
\text{Der}_H = \text{span}_H \{ \mathcal{P}^i \Delta \mid i \in \mathbb{N}_0 \}.
\]

Then \( \text{Der}_H \) is free as a module over \( H \); see Proposition 1.1.7 and Theorem 1.2.1 in [3]. Moreover it is a restricted Lie algebra of derivations acting on \( H \); cf. Section 2.4 in [3]. We denote by

\[
\mathcal{C}_{\text{Der}_H}(H) = \{ h \in H \mid \mathcal{P}^i \Delta (h) = 0 \ \forall i \} \subseteq H
\]

the subalgebra of constants with respect to the derivations in \( \text{Der}_H \).

\[\text{The module } \text{Der}_H \text{ is the module } \Delta(H) \text{ in this reference.}\]
Remark. In Section 2 we called the subalgebra of constants just $C = C(H)$. For what follows however, we need to keep track of the module of derivations that is used.

Clearly,

$$H^p \subseteq C_{\text{Der}_H}$$

as $\mathcal{P}^\Delta_i(h^p) = 0$ for all $h \in H$ and $i \in \mathbb{N}_0$. Since the extension

$$H^p \subseteq H$$

is purely inseparable of exponent one, so is the extension

$$H^p \subseteq C_{\text{Der}_H}(H).$$

Thus

$$\sqrt[p]{H^p} = \sqrt[p]{C_{\text{Der}_H}(H)} = \sqrt[p]{H}.$$

**Proposition 3.1.** Let $H$ be an unstable reduced algebra over the Steenrod algebra. Then $H$ is $\mathcal{P}^*$-inseparably closed if and only if

$$C_{\text{Der}_H}(H) = H^p.$$ 

**Proof.** Assume that $H$ is $\mathcal{P}^*$-inseparably closed. By what we have done so far we know that $H^p \subseteq C_{\text{Der}_H}(H)$. To prove the reverse inclusion, let $h \in C_{\text{Der}_H}(H)$. Then $\mathcal{P}^\Delta_i(h) = 0$ for all $i \in \mathbb{N}_0$. Thus there exists an element

$$f \in \sqrt[p]{C_{\text{Der}_H}(H)} = \sqrt[p]{H} = H$$

such that $f^{p^k} = h$. Since $f^{p^k} \in H^p$ for all $f \in H$, we have $h = f^{p^k} \in H^p$, and hence $H^p = C_{\text{Der}_H}(H)$. Conversely, assume that

$$H^p = C_{\text{Der}_H}(H) \subseteq H \subseteq \sqrt[p]{H}.$$

Let $h \in \sqrt[p]{H}$. Thus there exists a $k \in \mathbb{N}_0$ such that $h^{p^k} \in H$. Since $\text{Der}_H$ vanishes on $p$th powers we find that

$$h^{p^k} \in C_{\text{Der}_H}(H) = H^p.$$

Thus $h^{p^{k-1}} \in H$. Iteratively we find that $h \in H$, i.e., $H$ is $\mathcal{P}^*$-inseparably closed. □

**Corollary 3.2.** We have

$$C_{\text{Der}_H}([F[V]]) = F[V]^p = F[x_1^p, \ldots, x_n^p].$$

**Proof.** Since $F[V]$ is $\mathcal{P}^*$-inseparably closed (see Corollary 4.2.8 in [3]), this result is an immediate corollary of Proposition 3.1. □

We recall some facts about $\text{Der}_H$ and its action on $H$. First the Lie algebra structure is particularly simple, namely

$$[\mathcal{P}^\Delta_i, \mathcal{P}^\Delta_j] = \begin{cases} 0 & \text{if } i, j > 0, \\
\mathcal{P}^\Delta_i & \text{if } i \neq 0 \text{ and } j = 0, \\
-\mathcal{P}^\Delta_j & \text{if } i = 0 \text{ and } j \neq 0 \end{cases}$$

(see the remark on page 12 of [3]), and

$$\mathcal{P}^\Delta_i^p = 0$$

(\$\$)
The \( \Delta \)-length of \( H \) is defined to be the smallest integer \( \lambda \) such that the derivation

\[
h_0 \mathcal{P}^{\Delta 0} + \cdots + h_{\lambda} \mathcal{P}^{\Delta \lambda}
\]

vanishes on \( H \) for some \( h_0, \ldots, h_\lambda \in H \) and \( i_0, \ldots, i_\lambda \in \mathbb{N}_0 \) (see Section 1.2 in [3]). The \( \Delta \)-length \( \lambda_H \) is at most the Krull dimension of \( H \) over \( F \) (cf. Corollary 1.2.2 in [3]). Moreover, the coefficients can be chosen to be

\[
h_i = (-1)^i d_{\lambda, i}
\]

(up to a sign) the Dickson classes in dimension \( \lambda \) (see Theorems 5.1.9 and 5.2.1 in [3]). Note that by convention \( d_{\lambda, \lambda} = 1 \). Then the normalized equation

\[
(d_{\lambda, 0} \mathcal{P}^{\Delta 0} - \cdots + (-1)^\lambda d_{\lambda, \lambda} \mathcal{P}^{\Delta \lambda})(h) = 0 \quad \forall h \in H
\]

is called the \( \Delta \)-relation of \( H \). By abuse of notation, we also call the element

\[
d_H = d_{\lambda, 0} \mathcal{P}^{\Delta 0} - \cdots + (-1)^\lambda d_{\lambda, \lambda} \mathcal{P}^{\Delta \lambda} \in \text{Der}_H
\]

the \( \Delta \)-relation for \( H \). Finally we note that the \( \Delta \)-length of \( H \) is equal to its Krull dimension if \( H \) is \( P^\ast \)-inseparably closed (cf. Theorem 8.1.5 in [3]). The converse is not quite true as the following example taken from Section 7.4 in [3] shows.

**Example 3.3.** Consider the field \( F = \mathbb{F}_2 \) with two elements, and take a polynomial algebra in two linear generators \( x, y, F[x, y] \). The Dickson algebra in this case is

\[
D(2) = F[x^2 y + xy^2, x^2 + y^2 + xy] \rightarrow F[x, y].
\]

Define an intermediate algebra \( H \) by

\[
H = F[x^2 + y^2, xy, xy(x + y)]/((x^2 + y^2)(xy) + (xy(x + y))^2).
\]

Then \( H \) is an unstable integral domain, but it is not \( P^\ast \)-inseparably closed because

\[
\mathcal{P}^{\Delta i}(x^2 + y^2) = 0 \quad \forall i \geq 0, \text{ but } x + y \not\in H.
\]

However, its \( \Delta \)-relation

\[
d_{H} = d_{2, 0} \mathcal{P}^{\Delta 0} - d_{2, 1} \mathcal{P}^{\Delta 1} + \mathcal{P}^{\Delta 2}
\]

has length 2, which is equal to its Krull dimension. Note that the field of fractions of \( H \),

\[
F F(H) = F(x + y, xy),
\]

is inseparably closed. Therefore the algebra \( H \) is not integrally closed because \( x + y \not\in H \) (cf. Corollary 2.5).

**Proposition 3.4.** Let \( H \) be an unstable Noetherian integral domain. If the \( \Delta \)-length \( \lambda_H \) is equal to the Krull dimension \( n \) of \( H \), then

\[
P^\ast \sqrt{H} \subseteq \overline{H},
\]

where \( (\overline{\cdot}) \) denotes the integral closure.

---

4 If there is no possible confusion we will omit the subscript and just write \( \lambda = \lambda_H \).

5 If \( \lambda_H \in \mathbb{N}_0 \) exists, then \( H \) is called \( \Delta \)-finite. This is a weaker condition than Noetherianess. For example the polynomial algebra \( F[x_1^p, x_2^p, \ldots] \) in infinitely many generators has \( \Delta \)-length zero, but it is not Noetherian.

6 We will suppress the subscript, and write \( d \) for \( d_{H} \) if no confusion is possible.
Proof. Since the $\Delta$-length of $H$ is equal to its Krull dimension, we have integral extensions
\[ D(n) \hookrightarrow H \hookrightarrow \sqrt[\circ]{H} \hookrightarrow \mathbb{F}[V] \]
by the Little Imbedding Theorem (Theorem 7.4.4 in [3]) and the Embedding Theorem (Corollary 6.1.5 loc.cit.). The corresponding extensions of the field of fractions are Galois extensions, so in particular separable. Since $FF(H) \subseteq FF(\sqrt[\circ]{H})$ is also purely inseparable, we have $FF(\sqrt[\circ]{H}) = FF(H) = FF(H^G)$. Thus $FF(H) = \mathbb{F}(V)^G$ for some group $G \leq GL(V)$. Therefore, $H = \mathbb{F}(V)^G$ is inseparably closed. Hence $\sqrt[\circ]{H} \hookrightarrow H$ by the universal property of the inseparable closure.

Remark. Note that it follows from the preceding result that if $H$ is integrally closed and the $\Delta$-length is equal to its Krull dimension, then
\[ \sqrt[\circ]{H} = H. \]

We want to investigate purely inseparable extensions $H \hookrightarrow \mathbb{F}[V]$ of exponent one, i.e., we have
\[ \mathbb{F}[V]^P \hookrightarrow H \hookrightarrow \mathbb{F}[V]. \]
For this we turn our attention to the corresponding extensions of fields of fractions
\[ \mathbb{F}(V)^P \hookrightarrow H \hookrightarrow \mathbb{F}(V). \]
Let $\text{Der}_H$ be the vector space over $H$ generated by the elements $\mathcal{P}^\Delta_i$ for $i \in \mathbb{N}_0$. Since the relations $[\mathcal{P}]$ and $[\mathcal{H}]$ are intrinsic of the Steenrod algebra, the vector space $\text{Der}_H$ is also a restricted Lie algebra. Thus any vector subspace of $\text{Der}_H$ is a restricted Lie subalgebra and vice versa.

**Proposition 3.5.** Let $H$ be an unstable integral domain and $\mathbb{H}$ its field of fractions. The vector space $\text{Der}_H$ satisfies the following properties.

1. The elements $\mathcal{P}^\Delta_i$ are derivations on $\mathbb{H}$.
2. The $\Delta$-relation $\mathbb{H}$ is well defined, and coincides with the $\Delta$-relation on $H$. In particular, the $\Delta$-lengths are equal.

*Proof.* AD(1): The action of $\mathcal{P}^\Delta_i$ on $\mathbb{H}$ is given by the formula
\[ \mathcal{P}^\Delta_i \left( \frac{f_1}{f_2} \right) = \mathcal{P}^\Delta_i(f_1)f_2 - f_1 \mathcal{P}^\Delta_i(f_2) \]
for any $f_1, f_2 \in H$. Thus they are derivations on the field of fractions also.

AD(2): The set $\text{Der}_H$ is a vector space by construction. Let $\lambda_H$ be its dimension. Then any $\lambda_H + 1$ elements are linearly independent. Thus the $\Delta$-length is $\lambda_H$ with $\Delta$-relation
\[ d = f_0 \mathcal{P}^\Delta_0 + \cdots + f_\lambda \mathcal{P}^\Delta_\lambda. \]
Without loss of generality we can assume that the coefficients $f_i \in H$ for all $i$. Thus $\lambda_H$ is at least equal to the $\Delta$-length, $\lambda_H$, of $H$. On the other hand, if $d_H$ is a $\Delta$-relation for $H$, then by
\[ d_H \left( \frac{f_1}{f_2} \right) = \frac{d_H(f_1)f_2 - f_1 d_H(f_2)}{f_2^2} = 0 \]
$d_H$ vanishes also on $\mathbb{H}$. Thus $\lambda_H \leq \lambda_H$. Therefore $\lambda_H = \lambda_H$ and $d_H = d_H$. □

**Corollary 3.6.** Let $H$ be an unstable integral domain and $\mathbb{H}$ its integral closure. Then
\[ \lambda_H = \lambda_H \quad \text{and} \quad d_H = d_H. \]
Proof. By Proposition 5.3 part (2), $\Delta$-lengths, as well as the $\Delta$-relations of $H$ and its field of fractions, coincide. Since $H$ and $\mathbb{H}$ have the same field of fractions we are done. $\square$

Lemma 3.7. Let $H' \subseteq H$ be unstable reduced Noetherian algebras over the Steenrod algebra. Denote by $\lambda_{H'}$, resp. $\lambda_H$, the $\Delta$-length of $H'$, resp. $H$. Then $\lambda_{H'} \leq \lambda_H$.

Proof. By Corollary 3.6 the $\Delta$-length and $\Delta$-relation of an unstable algebra $H$ and its integral closure $\mathbb{H}$ are equal. Thus without loss of generality we assume that $H'$ and $H$ are integrally closed.

Denote by $\mathcal{D}(l)$ the Dickson algebra of dimension $l$. By Theorem 5.1.9 in [3]

$$\mathcal{D}(\lambda_H) \hookrightarrow \mathbb{H}$$

is a maximal Dickson algebra in $\mathbb{H}$. Applying the same theorem for $H'$ gives

$$\mathcal{D}(\lambda_H') \hookrightarrow \mathbb{H}' \hookrightarrow \mathbb{H}.$$

Since $\mathcal{D}(\lambda_H)$ is the maximal Dickson algebra in $\mathbb{H}$, we find that $\lambda_{H'} \leq \lambda_H$ as desired. $\square$

Lemma 3.8. Let $U$ and $W$ be finite dimensional vector spaces over $F$. We note that the $\Delta$-length $\lambda$ of $F[U] \otimes F[W]^p^t$, $t > 0$, is equal to the vector space dimension of $U$ with $\Delta$-relation

$$d = d_{\lambda,0} \mathcal{P}^{\Delta_0} - \cdots + (-1)^\lambda d_{\lambda,\lambda} \mathcal{P}^{\Delta_\lambda},$$

where $F[U]^{\text{GL}(\lambda,F)} = F[d_{\lambda,0}, \ldots,d_{\lambda,\lambda-1}]$.

Proof. The element $d$ is a $\Delta$-relation for $F[U]$ by Theorem 1.2.3 in [3]. Since $\mathcal{P}^{\Delta_i}$ vanishes on $p^t$th powers for all $i \in \mathbb{N}_0$, the element $d$ vanishes on $F[U] \otimes F[W]^p^t$. So, $\lambda \leq \dim_F(U)$.

On the other hand, $F[U] \hookrightarrow F[U] \otimes F[W]^p^t$. Therefore by Lemma 3.7 the $\Delta$-length is at least $\dim_F(U)$, and we are done. $\square$

Corollary 3.9. The $\Delta$-length $\lambda$ of $F(U) \otimes F(W)^p^t$ is equal to the vector space dimension of $U$, for $t \geq 1$. The subfield of constants is

$$C_{\text{Der}_F(U) \otimes F(W)^p^t}(F(U) \otimes F(W)^p^t) = F(U)^p \otimes F(W)^p^t.$$

Moreover, the $\Delta$-relation is

$$d = d_{\lambda,0} \mathcal{P}^{\Delta_0} - \cdots + (-1)^\lambda d_{\lambda,\lambda} \mathcal{P}^{\Delta_\lambda}.$$

Proof. This is immediate from part (2) of Proposition 3.5 Lemma 3.8 and Corollary 3.2. $\square$

Since the $\Delta$-relation of $F(V)$ has length $n = \dim_F(V)$, we have $\dim_{\mathbb{F}(V)}(\text{Der}_{\mathbb{F}(V)}) = n$ and

$$\text{Der}_{\mathbb{F}(V)} = \text{span}_{\mathbb{F}(V)}\{\mathcal{P}^{\Delta_0}, \ldots, \mathcal{P}^{\Delta_{n-1}}\}.$$ 

Moreover, the index over the subfield of constants is

$$[F(V) : F(V)^p] = p^n.$$ 

Thus we can apply the structure theorem for purely inseparable extensions of exponent one (see, e.g., Chapter IV, Section 8 in [1]). It tells us that

$$\mathbb{H} \subseteq F(V).$$
is a purely inseparable extension of exponent one if and only if there exists a restricted Lie subalgebra \( D \subseteq \text{Der}_F(V) \) such that
\[
\mathbb{H} = C_D(F(V)).
\]
So, take a subspace \( D \subseteq \text{Der}_F(V) \). We recall from Corollary 3.9 that
\[
F(V) = C_D(F(V)) \quad \text{for } D = 0
\]
and
\[
F(V)^p = C_D(F(V)) \quad \text{for } D = \text{Der}_F(V).
\]
Thus we are left to characterize those \( D \subseteq \text{Der}_F(V) \) such that \( \mathbb{H} = C_D(F(V)) \) carries a \( \mathcal{P} \)-module structure.

**Proposition 3.10.** Let \( K \hookrightarrow F(V) \) be a field over the Steenrod algebra. Let \( d \) be the \( \Delta \)-relation of \( K \) with \( \Delta \)-length \( \lambda \). Then the vector space of derivations vanishing on \( K \),
\[
D_K = \text{span}_F\{d \in \text{Der}_F(V) | d|_K = 0\},
\]
has dimension \( n - \lambda \), where \( n = \dim_F(V) \).

**Proof.** The \( \Delta \)-relation of \( K \) is
\[
d = d_{\lambda,0} \mathcal{P}^{\Delta_0} + \cdots + (-1)^\lambda d_{\lambda,\lambda} \mathcal{P}^{\Delta_\lambda}.
\]
By Proposition 1.1.7 in \( [3] \) any \( \lambda + 1 \) derivations in \( \text{Der}_F(V) \) are linearly dependent. Moreover by Lemma 1.1.8 loc.cit. we find that in particular the \( n - \lambda \) elements
\[
d_i = d_{\lambda,0}^{i_0} \mathcal{P}^{\Delta_i} + \cdots + (-1)^\lambda d_{\lambda,\lambda}^{i_\lambda} \mathcal{P}^{\Delta_\lambda+1}
\]
for \( i = 0, \ldots, n - \lambda - 1 \) vanish on \( K \). Since the \( d_i \)'s are linearly independent in \( \text{Der}_F(V) \) we have that
\[
\dim(D_K) \geq n - \lambda.
\]
On the other hand, if \( d \in D_K \), then
\[
d = f_0 \mathcal{P}^{\Delta_0} + \cdots + f_{n-1} \mathcal{P}^{\Delta_{n-1}}.
\]
Then there are \( k_0, \ldots, k_{n-1-\lambda} \) such that
\[
(*) \quad d - \sum_{i=0}^{n-1-\lambda} k_i d_i = f_0' \mathcal{P}^{\Delta_0} + \cdots + f_{\lambda-1}' \mathcal{P}^{\Delta_{\lambda-1}}
\]
for some \( f_0', \ldots, f_{\lambda-1}' \in K \). Thus if \( d \) were linearly independent of the \( d_i \)'s, then the expression (\( * \)) is not zero. This in turn means that there is a relation on \( K \) shorter than the \( \Delta \)-relation. This is a contradiction. Therefore \( \dim(D_K) = n - \lambda \). \( \square \)

**Theorem 3.11.** The extension \( \mathbb{H} \subseteq F(V) \) is a purely inseparable extension of exponent one of fields over the Steenrod algebra if and only if
\[
\mathbb{H} = F(x_1, \ldots, x_k, x_{k+1}^p, \ldots, x_n^p) = F(U) \otimes F(V/U)^p
\]
for some \( k \in \{1, \ldots, n\} \) and \( \dim(U) = k \). Furthermore, in this case
\[
\mathbb{H} = C_D(F(V))
\]
where \( D \) has vector space dimension \( n - k \). If \( k < n \), then \( D \) is generated by the \( \Delta \)-relation of \( \mathbb{H} \),
\[
d_{\mathbb{H}} = d_{k,0} \mathcal{P}^{\Delta_0} + \cdots + (-1)^k d_{k,k} \mathcal{P}^{\Delta_k}
\]
and its translates
\[ d_i = d_{k,0}^i \Delta_i + \cdots + (-1)^k d_{k,k}^i \Delta_{k+i} \]
for \( i = 1, \ldots, n-k-1 \).

**Proof.** If
\[ \mathbb{H} = F(x_1, \ldots, x_k, x_{k+1}^p, \ldots, x_n^p), \]
then it is clearly a field over the Steenrod algebra. Moreover,
\[ F(x_1, \ldots, x_k, x_{k+1}^p, \ldots, x_n^p) = C_D(F(V)) \]
for \( D \) generated by the \( \Delta \)-relation of \( \mathbb{H} \) and its translates \( d_i \) of length \( \lambda_H = k \) (see Corollary 3.9 and Proposition 3.10).

We prove the converse. Set \( \lambda = \lambda_H \). Let \( d \) be the \( \Delta \)-relation of \( \mathbb{H} \). Then
\[ d = d_{0,0}^\lambda \Delta_0 + \cdots + (-1)^\lambda d_{k,\lambda}^\lambda \Delta_\lambda \]
vanishes on \( \mathbb{H} \). Let \( U \subseteq V \) be a vector subspace of dimension \( \lambda \). We also note that the field \( F(U) \otimes F(V/U)^p \) has \( \Delta \)-relation \( d \) and \( \Delta \)-length \( \lambda \) by Corollary 3.9.

Certainly,
\[ F(V)^p \hookrightarrow F(U) \otimes F(V/U)^p \hookrightarrow F(V) \]
is a purely inseparable extension of exponent one. We show that \( F(U) \otimes F(V/U)^p \hookrightarrow \mathbb{H} \). Since \( \mathbb{H} \hookrightarrow F(V) \) is purely inseparable of exponent one, we have
\[ F(V/U)^p \hookrightarrow \mathbb{H}. \]

Since the coefficients of the \( \Delta \)-relation are the Dickson classes, we know that \( FF(D(\lambda)) \hookrightarrow \mathbb{H} \). Thus
\[ FF(D(\lambda)) \otimes F(V/U)^p \hookrightarrow \mathbb{H}. \]

Since \( \mathbb{H} \hookrightarrow F(V) \) is purely inseparable, we find that the separable closure of \( FF(D(\lambda)) \otimes F[V/U]^p \) is in \( \mathbb{H} \). This in turn is just
\[ F(U) \otimes F(V/U)^p \hookrightarrow \mathbb{H}. \]

Obviously \( |F(V)| : F(U) \otimes F(V/U)^p| = p^{n-\lambda} \). By Theorem 19 on page 186 of [1] we have that also
\[ |F(V)| : \mathbb{H}| = p^{n-\lambda} \]
because \( D_H \) has dimension \( n-\lambda \) (Proposition 3.10). Hence \( \mathbb{H} = F(U) \otimes F(V/U)^p \) as desired. \( \square \)

**Corollary 3.12.** Let \( H \subseteq F[V] \) be a purely inseparable extension of exponent one. Let \( H \) be integrally closed. Then \( H \) is an unstable algebra over the Steenrod algebra if and only if \( H = F[x_1, \ldots, x_\lambda, x_{\lambda+1}^p, \ldots, x_n^p] \), where \( \lambda = \lambda_H \) is the \( \Delta \)-length of \( H \).

**Proof.** If \( H \) is an unstable algebra over the Steenrod algebra, then \( \mathbb{H} \) is a field over the Steenrod algebra. Moreover, since \( H \hookrightarrow F[V] \) has exponent one, so has the extension \( \mathbb{H} \hookrightarrow F(V) \). Thus \( \mathbb{H} = F(U) \otimes F(V/U)^p \) for \( \dim_F(U) = \lambda \) by Theorem 3.11. Hence by Theorem 2.4 in [1]
\[ H = \mathbb{H} = \mathbb{H} \cap (F[U] \otimes F[V/U]^p). \]

Conversely, the algebra \( F[U] \otimes F[V/U]^p \) is certainly an unstable algebra over the Steenrod algebra. \( \square \)
Remark. For any unstable integral domain \( H \) its integral closure \( \overline{H} \) also carries an unstable \( P^* \)-module structure because \( H = \cup n(H) \) (see Theorem 2.4 in [4]). The converse is not true as we illustrate with the next example.

**Example 3.13.** Let \( F \) be the prime field of characteristic 2 and let \( A \) be the subalgebra of \( F[x, y] \) generated by \( x, xy, y^3 \). Then \( A \hookrightarrow F[x, y] \) is an integral extension. Moreover, \( F\overline{F}(A) = F[x, y] \). Therefore \( \overline{A} = F[x, y] \) is an unstable algebra over the Steenrod algebra. However \( A \) does not carry a \( P^* \)-module structure because \( P^1(xy) = x^2y + xy^2 \not\in A \), as the only elements of degree 3 in \( A \) are \( x^3, x^2y, y^3 \).

Thus the assumption \( H = \overline{H} \) cannot be dropped in the preceding result.

**Corollary 3.14.** Let \( U \leq V \). Denote \( m = \dim_F(U) \leq n = \dim_F(V) \). Then
\[
F[U] \otimes F[V/U]^p \hookrightarrow F[V]
\]
is the largest unstable subalgebra with \( \Delta \)-length equal to \( m \).

**Proof.** Certainly, \( F[U] \otimes F[V/U]^p \) has \( \Delta \)-length \( m \). Let
\[
(*) \quad F[U] \times F[V/U]^p \hookrightarrow H \hookrightarrow F[V]
\]
be an intermediate unstable algebra with \( \lambda_H = m \). Since the extension \( (*) \) is purely inseparable of exponent one, we have that
\[
H = F(U') \otimes F(V/U')^p
\]
for some \( U' \geq U \). But
\[
\dim_F(U) = m = \lambda_H = \dim_F(U')
\]
and therefore \( U = U' \). Hence
\[
F[U] \otimes F[V/U]^p \subseteq H \subseteq \cup n(H) = \overline{H} = F[U] \otimes F[V/U]^p
\]
gives the desired result. \( \square \)

4. **Purely inseparable extensions of arbitrary exponent**

In this section we proceed with the investigation of the purely inseparable extension
\[
H \hookrightarrow F[V].
\]
We consider the general case of exponent \( e \geq 1 \). Thus we need to detect \( p^s \)-th powers for \( s = 1, \ldots, e \). We introduce the following operators for \( s \in \mathbb{N}_0 \):
\[
P^{\Delta_s, 0} = \frac{1}{p^s} \deg(-)\text{id}(-),
\]
\[
P^{\Delta_s, 1} = P^s,
\]
\[
P^{\Delta_s, i} = P^s q^{i-1} P^{\Delta_{s-i}} - P^{\Delta_s} q^{i-1} P^s q^{i-1} \quad \text{for } i \geq 2.
\]

**Remark.** Note that for all \( s \in \mathbb{N}_0 \) we have \( P^{\Delta_s, i} \in P^s \) whenever \( i \neq 0 \).

**Remark.** Note also that the degree of \( P^{\Delta_s, i} \) is equal to \( q^i p^s - p^s \), for all \( i, s \in \mathbb{N}_0 \).
Proposition 4.1. The operators $\mathcal{P}^{\Delta_{s,i}}$ satisfy the following properties:

1. For all $i \in \mathbb{N}$ we have $\mathcal{P}^{\Delta_{s,i}}(h^p) = (\mathcal{P}^{\Delta_{s,-1,i}}(h))^{p^i}$ for $h \in H$ and $s \geq 1$.
2. For $i \geq 1$ and $k, s \geq 0$ we have
   $$[\mathcal{P}^{p^i}, \mathcal{P}^{\Delta_{s,i}}] = \mathcal{P}^{\Delta_{s,i+1}} \mathcal{P}^{p^i-p^i q^i}.$$ 
3. For $i, j \geq 1$ and $s \geq 0$ we have
   $$[\mathcal{P}^{\Delta_{s,i}}, \mathcal{P}^{\Delta_{s,j}}] = 0.$$
4. The $p$th iteration $\mathcal{P}^{\Delta_{s,1}} \ldots \mathcal{P}^{\Delta_{s,i}} = 0$ for all $i \geq 1$ and $s \geq 0$.

Proof. AD(1): For any $i, j \geq 0$ and any linear form $l$ we have
   $$\mathcal{P}^i(l^p) = \binom{j}{i} P^i q^i - i.$$ 
   as it can be easily seen by induction. Thus for all $i \geq 0$ we have
   $$\mathcal{P}^i(l^p) = \binom{j}{i} P^i q^i - i.$$ 
   Since $(p^i) \equiv 0 \pmod{p}$ precisely when $i \notin \{p^i, 0\}$, we have
   $$\mathcal{P}^i(l^p) = \begin{cases} p^i(l^p) & \text{for } i = 0, \\ p^i(l^p) & \text{for } i = p^i, \\ 0 & \text{otherwise}. \end{cases}$$ 
   Therefore
   $$\mathcal{P}^i(l^p) = \begin{cases} h^p & \text{for } i = 0, \\ h^p & \text{for } i = p^i, \\ 0 & \text{otherwise}, \end{cases}$$ 
   for any $h \in H$ (cf. page 261 of [3]). Thus
   $$\mathcal{P}^{\Delta_{s,i}}(h^p) = \mathcal{P}^{p^i}(h^p) = (\mathcal{P}^{p^{i-1}}(h))^{p^i} = (\mathcal{P}^{\Delta_{s,-1,i}}(h))^p.$$ 
   Thus by induction on $i$ we find
   $$\mathcal{P}^{\Delta_{s,i}}(h^p) = \mathcal{P}^{p^i q^{i-1}} \mathcal{P}^{\Delta_{s,-1,i}}(h^p) - \mathcal{P}^{\Delta_{s,-1,i}} \mathcal{P}^{p^i q^{i-1}}(h^p)$$
   $$= \mathcal{P}^{p^i q^{i-1}} (\mathcal{P}^{\Delta_{s,-1,i+1}}(h))^{p^i} - \mathcal{P}^{\Delta_{s,-1,i}} (\mathcal{P}^{p^{i-1} q^{i-1}}(h))^{p^i}$$
   $$= (\mathcal{P}^{p^{i-1} q^{i-1}} \mathcal{P}^{\Delta_{s,-1,i+1}}(h) - \mathcal{P}^{\Delta_{s,-1,i-1}} \mathcal{P}^{p^{i-1} q^{i-1}}(h))^{p^i}$$
   as claimed.

AD(2) and (3): The result follows, because it is true for any linear form.

AD(4): From the Adem relations it follows that
   $$\mathcal{P}^{p^i} \ldots \mathcal{P}^{p^i} = 0.$$ 
   Thus the result follows by induction on $i$ with the help of the commutation rules of (2) (cf. Lemma A.1.1 in [3]).

Define the $H^p$-module
   $$\text{Der}_{H,s} = \text{span}_{H^p} \{ \mathcal{P}^{\Delta_{s,i}} \mid i \in \mathbb{N}_0 \}.$$ 
   By definition it follows that $\text{Der}_{H,0} = \text{Der}_H$. 

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Proposition 4.2. The module $\text{Der}_{H,s}$ has the following properties:

(1) $\text{Der}_{H,s}$ acts in $H^p$ as derivations.

(2) For $s, k \geq 0$ we obtain

$$[\mathcal{P}^s k, \mathcal{P}^{\Delta s}] = k \mathcal{P}^s.$$

(3) If $s \geq 0$, then

$$[\mathcal{P}^{\Delta s}, \mathcal{P}^{\Delta s}] = \begin{cases} \mathcal{P}^{\Delta s} & \text{if } i \neq 0 \text{ and } j = 0, \\ -\mathcal{P}^{\Delta s} & \text{if } i = 0 \text{ and } j \neq 0, \\ 0 & \text{otherwise}. \end{cases}$$

(4) The $p$th iteration gives $\mathcal{P}^{\Delta s}(h^p) = \mathcal{P}^{\Delta s}(h^p)$ for all $s \geq 0$ and $h \notin H$.

(5) Let $s \geq 0$. Then $\mathcal{P}^{\Delta s}(h^p) = 0$ for each $i$ if and only if $h$ is a $p^{s+1}$th power.

Proof. AD(1): Let $h^p \in H^p$. By Proposition 4.1, $\text{Der}_{H,s}$ acts on $H^p$ according to the following formulae:

$$\mathcal{P}^{\Delta s}(h^p) = \deg(h) h^p = \mathcal{P}^0(h)^p,$$

$$\mathcal{P}^{\Delta s+1}(h^p) = (\mathcal{P}^1(h))^p = (\mathcal{P}^{\Delta s}(h))^p,$$

$$\mathcal{P}^{\Delta s}(h^p) = (\mathcal{P}^{\Delta s}(h))^{p^s} = (\mathcal{P}^{\Delta s}(h))^{p^s}.$$

Since taking $p$th powers is additive in characteristic $p$, this establishes the statement.

AD(2): Let $h^p \in H^p$. We have

$$[\mathcal{P}^s k, \mathcal{P}^{\Delta s}](h^p) = \mathcal{P}^s k \mathcal{P}^{\Delta s}(h^p) - \mathcal{P}^{\Delta s} \mathcal{P}^s k(h^p)$$

$$= \deg(h) \mathcal{P}^s k(h^p) - \mathcal{P}^{\Delta s} \mathcal{P}^s k(h^p)$$

$$= \deg(h) \mathcal{P}^s k(h^p) - \deg(\mathcal{P}^s k(h^p))$$

$$= (\deg(h) - \deg(h) + k - qk) \mathcal{P}^s k(h^p)$$

$$= k \mathcal{P}^s k(h^p).$$

AD(3): Let $h^p \in H^p$. If $i, j \geq 1$, then

$$[\mathcal{P}^{\Delta s}, \mathcal{P}^{\Delta s}](h^p) = 0$$

by part (3) of Proposition 4.1. Otherwise we have

$$[\mathcal{P}^{\Delta s}, \mathcal{P}^{\Delta s}](h^p) = \mathcal{P}^{\Delta s} \mathcal{P}^{\Delta s}(h^p) - \mathcal{P}^{\Delta s} \mathcal{P}^{\Delta s}(h^p)$$

$$= (\deg(h) + q^i - 1) \mathcal{P}^{\Delta s}(h^p) - \deg(h) \mathcal{P}^{\Delta s}(h^p)$$

$$= -\mathcal{P}^{\Delta s}(h^p).$$

The relation for $j = 0$ can be established in the same way.

AD(4): Let $h^p \in H^p$. Since $\mathcal{P}^{\Delta s}(h^p) = \deg(h) h^p$ and $\deg(h)^p = \deg(h)$

the result follows.

AD(5): Let $h^p = H^p$ and $i \geq 1$. Then

$$\mathcal{P}^{\Delta s}(h^p) = (\mathcal{P}^{\Delta s}(h))^p$$

and

$$\mathcal{P}^{\Delta s}(h^p) = \deg(h) h^p = \mathcal{P}^{\Delta s}(h)^p.$$
are simultaneously zero for all $i$ if and only if

$$\mathcal{P}^{\Delta_i}(h) = 0$$

for all $i \geq 0$, i.e., if and only if $h$ is a $p$th power; hence precisely when $h^{p^i}$ is a $p^{i+1}$st power.

Thus $\text{Der}_{H,s}$ is a restricted Lie algebra of derivations acting on $H^{p^i}$ vanishing precisely on the $p^{i+1}$st powers. We need to have a look at the relation between $\text{Der}_{H,s}$ and $\text{Der}_H$.

**Proposition 4.3.** Let $H$ be an unstable reduced Noetherian algebra over the Steenrod algebra. The action of $\text{Der}_{H,s}$ on $H^{p^i}$ has the following properties:

1. For any $d_s \in \text{Der}_{H,s}$ there exists a $d \in \text{Der}_H$ such that

$$d_s(h^{p^i}) = d(h)^{p^i} \quad \forall h \in H.$$

2. If there are $m$ derivations in $\text{Der}_{H,s}$ that are linearly dependent, then so are any $m$ derivations.

**Proof.** AD(1): For any $d_s = h_0^{p^i}\mathcal{P}^{\Delta_{s,0}} + \cdots + h_l^{p^i}\mathcal{P}^{\Delta_{s,l}} \in \text{Der}_{H,s}$ we find that

$$d_s = (d)^{p^i} = (h_0^{p^i}\mathcal{P}^{\Delta_{s,0}} + \cdots + h_l^{p^i}\mathcal{P}^{\Delta_{s,l}})^{p^i}$$

by part (1) of Proposition 4.1. By construction $d \in \text{Der}_H$.

AD(2): Let $d_{s,1}, \ldots, d_{s,m} \in \text{Der}_{H,s}$ be linearly dependent. By part (1) we find $d_1, \ldots, d_m \in \text{Der}_H$ such that $(d_i(h))^{p^i} = d_{s,i}(h^{p^i})$ for all $h \in H$. Thus the elements $d_1, \ldots, d_m \in \text{Der}_H$ are linearly dependent. Therefore any $m$ elements, say $d_{s,1}', \ldots, d_{s,m}'$, of $\text{Der}_{H,s}$ are linearly dependent with a relation

$$h_1d_{s,1}' + \cdots + h_md_{s,m}' = 0.$$ 

Thus

$$h_1^{p^i}d_{s,1}' + \cdots + h_m^{p^i}d_{s,m}' = 0$$

is a relation in $\text{Der}_{H,s}$. □

Thus the minimal $l_s$ such that $l_s + 1$ elements of $\text{Der}_{H,s}$ are linearly dependent is uniquely defined. We call $l_s$ the $\Delta_s$-length of $H^{p^i}$, denoted by $\lambda_{H,s}$ or if no confusion can arise by $\lambda_s$. If $\lambda_s \in \mathbb{N}_0$, we call the algebra $H^{p^i}$ $\Delta_s$-finite. Note that by construction $H$ is $\Delta$-finite if and only if $H^{p^i}$ is $\Delta_s$-finite.

**Proposition 4.4.** Let $H$ be an unstable reduced $\Delta$-finite algebra over the Steenrod algebra. Let $\lambda$ be its $\Delta$-length and $\lambda_s$ the $\Delta_s$-length of $H^{p^i}$. Then

1. $\lambda_s = \lambda$.
2. We have a relation of the form

$$d_s = d_{\lambda,s}^{\Delta} + \cdots + d_{\lambda_0}^{\Delta}$$

on $H^{p^i}$.

**Proof.** AD(1): Let

$$d = d_{\lambda,0}^{\Delta_0} + \cdots + (-1)^\lambda d_{\lambda,s}^{\Delta_s} = \text{Der}_H$$
be the ∆-relation of H. Then the element \( d_s(h^{p^s}) = (d(h))^{p^s} \) of \( \text{Der}_{H,s} \) vanishes on \( H^{p^s} \). Thus \( \lambda_s \leq \lambda \). Conversely, if 
\[
d_s = h_0^{p^s} \vartheta^{\Delta,0} + \cdots + h_{\lambda_s}^{p^s} \vartheta^{\Delta,\lambda_s} \in \text{Der}_{H,s}
\]
is a \( \Delta_s \)-relation of \( H^{p^s} \), then the element 
\[
d = h_0 \vartheta^{\Delta,0} + \cdots + h_{\lambda_s} \vartheta^{\Delta,\lambda_s} \in \text{Der}_H
\]
vanishes on H, by part (1) of Proposition 4.3. Thus, also, \( \lambda \leq \lambda_s \).

By part (4) of Proposition 4.3 we call the relation \( d_s \) of part (2) of this result the ∆-relation of \( H^{p^s} \).

We obtain analogously to Proposition 3.5 that \( \dim(\text{Der}_s(H)) = \dim(\text{Der}_{V,s}(H^{p^s})) \subseteq \text{Der}_H \).

**Proposition 4.5.** For \( s \geq 0 \) we have
\[
(\text{Der}_{H,s}(H))^{p^s} \subseteq \text{Der}_{H,s}(H^{p^s}) \subseteq \text{Der}_H(H).
\]

**Proof.** If \( h^{p^s} \in H^{p^s} \) is a \( \text{Der}_{H,s} \)-constant, then \( h \) is a \( p^{s+1} \)st power by part (5) of Proposition 4.2. Thus there exists an element \( k \in H_{p^s}^{p^s} = H^{p^{s-1}} \) such that \( k^{p^s} = h^{p^s} \).

Hence \( h^{p^s} = k^{p^s} \in H^{p^s} \subseteq H \) is a \( p^s \)th power, i.e., a \( \text{Der}_H \)-constant.

In order to prove the equality, let \( h \in H \) be a \( \text{Der}_H \)-constant, i.e., \( d(h) = 0 \) for all \( d \in \text{Der}_H \). For any \( d_s \in \text{Der}_{H,s} \) there is an element \( d \in \text{Der}_H \) such that \( d_s(h^{p^s}) = (d(h))^{p^s} = 0 \) by part (1) of Proposition 4.3. Thus \( (\text{Der}_{H,s}(H))^{p^s} \subseteq \text{Der}_{H,s}(H^{p^s}) \).

Conversely, if \( h^{p^s} \in \text{Der}_{H,s}(H^{p^s}) \), then \( h \in \text{Der}_H(H) \), by what we have proven so far. Thus \( h^{p^s} \in (\text{Der}_{H,s}(H^{p^s}))^{p^s} \) as desired. \( \square \)

**Remark.** We note that the preceding results imply that
\[
\dim(\text{Der}_{V,s}(H)) = \dim(\text{Der}_{V,s}(H^{p^s})) = \dim(\text{Der}_H(H)) = n.
\]

**Remark.** In Section 3 we defined \( \text{Der}_H \) for \( H = FF(V) \) for integral domains \( H \). In the same way we define
\[
\text{Der}_{H,s} = \text{span}_{\text{End}_H} \{ \vartheta^{\Delta,0} \mid i \in \mathbb{N}_0 \}.
\]

We obtain analogously to Proposition 3.5 that \( \dim(\text{Der}_{H,s}) = \dim(\text{Der}_{H,s}) \), Hence, \( \lambda_{H,s} = \lambda_{H,s} \) and \( \Delta_{H,s} = \Delta_{H,s} \). Therefore \( \text{Der}_{H,s} \) and \( \text{Der}_{H,s} \) have basis \( \{ \vartheta^{\Delta,0}, \ldots, \vartheta^{\Delta,\lambda_s} \} \), where \( \lambda_s = \lambda_{H,s} = \lambda_{H,s} \).

**Corollary 4.6.** The ∆-length of \( F[V]^{p^s} \), resp. \( F(V)^{p^s} \), is equal to the dimension of \( V \). Moreover, for \( D_s = 0 \)
\[
F[V]^{p^s} = C_{D_s}(F[V]^{p^s}), \quad \text{resp.} \quad F(V)^{p^s} = C_{D_s}(F(V)^{p^s}).
\]
Finally, the ∆-length of \( F[V]^{p^{s+1}} \), resp. \( F(V)^{p^{s+1}} \), is zero, and
\[
F[V]^{p^{s+1}} = C_{D_{F[V],s}}(F[V]^{p^s}), \quad \text{resp.} \quad F(V)^{p^{s+1}} = C_{D_{F(V),s}}(F(V)^{p^s}).
\]

**Proof.** The first statement follows from Lemma 3.8 and Proposition 4.5. The second statement follows from Corollaries 3.2 and 3.9 and Proposition 4.5. \( \square \)
We need a generalization of Lemma 3.7

**Lemma 4.7.** Let \( K^p \subseteq H^p \) be unstable reduced Noetherian algebras over the Steenrod algebra. Denote by \( \lambda_{K,s} \) resp. \( \lambda_{H,s} \), the \( \Delta \)-length of \( K^p \), resp. \( H^p \). Then \( \lambda_{K,s} \leq \lambda_{H,s} \).

**Proof.** The first statement follows from Proposition 4.4, Lemma 3.8, and Corollary 3.9. If \( H \subseteq \operatorname{Ker}(\partial) \) and \( \Delta \)-length of \( H \) is \( \leq s \), then by Lemma 4.7 its \( \Delta \)-length is \( \leq s \).

**Proposition 4.8.** Let \( t > s \) and \( W \leq V \). The \( \Delta \)-length of \( \mathbb{F}[W]^p \otimes \mathbb{F}[V/W]^p \) and \( \mathbb{F}[W]^p \) is \( \lambda_s = \dim_\mathbb{F}(W) \). Moreover, let \( H^p \subseteq \mathbb{F}[V]^p \). Then the \( \Delta \)-length of \( H^p \) is at most \( l \) if and only if

\[
\dim_\mathbb{F}(W) = l \text{ and } t > s.
\]

**Proof.** The first statement follows from Proposition 4.4, Lemma 3.8, and Corollary 3.9. If \( H^p \subseteq \mathbb{F}[W]^p \otimes \mathbb{F}[V/W]^p \), then by Lemma 4.7 its \( \Delta \)-length is the largest subalgebra of \( \mathbb{F}[W]^p \), which is \( l \) by what we proved so far.

Conversely, let the \( \Delta \)-length of \( H^p \) be \( \lambda_s \leq l \). Then \( H^p \subseteq \mathbb{F}[W]^p \otimes \mathbb{F}[V/W]^p \) and the result follows. \( \square \)

**Lemma 4.9.** We have

\[
FF(\mathcal{C}_{Der,Hs}(H^p)) = \mathcal{C}_{Der,Hs}(H^p).
\]

**Proof.** This follows from Propositions 4.3 and 2.3 \( \square \)

In order to be able to treat the general case, we need another preliminary result.

**Proposition 4.10.** Let \( V = W_0 \oplus \cdots \oplus W_e \) be a vector space decomposition. Consider the purely inseparable extension \( H^{p,s} \subseteq \mathbb{F}[W_0]^p \otimes \cdots \otimes \mathbb{F}[W_e]^p \). Let \( s_0 < \cdots < s_e \). Let \( H^{p,s} \) be an integrally closed unstable algebra over the Steenrod algebra. Then the \( \Delta_{s_0} \)-length of \( H^{p,s} \) is \( \lambda_{s_0} \) if and only if

\[
\mathbb{F}[U_0]^p \otimes \mathbb{F}[W_0/U_0]^p \otimes \cdots \otimes \mathbb{F}[W_e]^p
\]

for \( \dim_\mathbb{F}(U_0) = \lambda_{s_0} \), \( U_0 \leq W_0 \) and \( U_0 \) maximal with respect to this property.

**Proof.** Let the \( \Delta_{s_0} \)-length of \( H^{p,s} \) be \( \lambda_{s_0} \). Note that \( \lambda_{s_0} \leq \dim_\mathbb{F}(W_0) \) by Lemma 4.7. The \( \Delta \)-length of \( H \) is also \( \lambda_{s_0} \) by part (1) of Proposition 4.4. Moreover

\[
H \subseteq \mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_e]^p.
\]

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Let $\mathbf{d}$ be the $\Delta$-relation of $H$. Then $\mathbf{d}$ is also the $\Delta$-relation of its field of fractions, $\mathbb{H}$. Thus by construction we find for $D = D_H \subseteq \text{Der}_H$

\[
\mathbb{H} \hookrightarrow \mathcal{C}_D(F(W_0) \otimes F(W_1)^{p^1-r_0} \otimes \cdots \otimes F(W_r)^{p^r-r_0})
\]

\[
= F(U_0) \otimes F(W_0/U_0)^p \otimes F(W_1)^{p^1-r_0} \otimes \cdots \otimes F(W_r)^{p^r-r_0}
\]

for some $U_0 \leq W_0$ of dimension $\lambda_{s_0}$ by Corollary 3.9. Let $U$ be maximal with this property. The extension

\[
F(U_0) \cap \mathbb{H} \hookrightarrow F(U_0)
\]

is purely inseparable, since

\[
\mathbb{H} \hookrightarrow F(U_0) \otimes F(W_0/U_0)^p \otimes F(W_1)^{p^1-r_0} \otimes \cdots \otimes F(W_r)^{p^r-r_0}
\]

is purely inseparable and all elements in $F(U_0) \otimes F(W_0/U_0)^p \otimes F(W_1)^{p^1-r_0} \otimes \cdots \otimes F(W_r)^{p^r-r_0}$ that are algebraic over $F(U_0)$ are inseparable. By maximality of $\lambda_{s_0}$ all elements in $F(U_0)$ are separable over $\mathbb{H}$. Thus

\[
F(U_0) \cap \mathbb{H} \hookrightarrow F(U_0)
\]

is also separable. Hence

\[
F(U_0) = F(U_0) \cap \mathbb{H} \hookrightarrow \mathbb{H}.
\]

Therefore

\[
F[U_0]^{p^{s_0}} \hookrightarrow H^{p^{s_0}} \hookrightarrow F[U_0]^{p^{s_0}} \otimes F[W_0/U_0]^{p^{s_0+1}} \otimes \cdots \otimes F[W_r]^{p^r}
\]

as desired.

To prove the converse, assume that there exists a vector space $U_0 \leq W_0$ of $\dim_F(U_0) = \lambda_{s_0}$ such that

\[
F[U_0]^{p^{s_0}} \hookrightarrow H^{p^{s_0}} \hookrightarrow F[U_0]^{p^{s_0}} \otimes F[W_0/U_0]^{p^{s_0+1}} \otimes \cdots \otimes F[W_r]^{p^r}.
\]

Assume furthermore that $U_0$ is maximal with this property. Then the $\Delta_{s_0}$-length of $F[U_0]^{p^{s_0}}$ is $\lambda_{s_0}$ by Corollary 3.9. Equally, the $\Delta_{s_0}$-length of $F[U_0]^{p^{s_0}} \otimes F[W_0/U_0]^{p^{s_0+1}} \otimes \cdots \otimes F[W_r]^{p^r}$ is $\lambda_{s_0}$ by Proposition 4.8. Therefore the $\Delta_{s_0}$-length of $H^{p^{s_0}}$ is $\lambda_{s_0}$ by Lemma 4.7.

*Remark.* Note that any element in $H^{p^{s_0}}$ that is algebraic over $F[U_0]^{p^{s_0}}$ is separable over $F[U_0]^{p^{s_0}}$.

**Theorem 4.11.** Let $V = W_0 \oplus \cdots \oplus W_r$ be a vector space decomposition. Consider the purely inseparable extension $H \hookrightarrow F[W_0] \otimes F[W_1]^p \otimes \cdots \otimes F[W_r]^{p^r}$ of exponent one. Let $H$ be integrally closed. Then $H$ is an unstable algebra over the Steenrod algebra if and only if

\[
H = F[U_0] \otimes F[U_1]^p \otimes \cdots \otimes F[U_{e+1}]^{p^{e+1}}
\]

for some vector space decomposition

\[
V = U_0 \oplus \cdots \oplus U_{e+1}
\]

with

\[
U_0 \oplus \cdots \oplus U_i \leq W_0 \oplus \cdots \oplus W_i
\]

and $\dim(U_0 \oplus \cdots \oplus U_i)$ is the $\Delta$-length of $H_i$, $i = 0, \ldots, e + 1$. 
Proof: The “if” part of the statement is clear by Proposition 4.10. We need to prove the “only if” part.

Let \( d \in \text{Der}_H \) be the \( \Delta \)-relation on \( H \) of length \( \lambda_0 \). Then \( F[U_0] \hookrightarrow H \) for some vector space \( U_0 \) of dimension \( \lambda_0 \) by Proposition 4.10. Hence \( \lambda_0 \leq \dim(W_0) \), \( U_0 \leq W_0 \), and we have

\[
F[U_0] \hookrightarrow H \hookrightarrow F[W_0] \otimes F[W_1] \otimes \cdots \otimes F[W_e]^{p^e}.
\]

We consider the chain

\[
F[U_0] \hookrightarrow H_1 \hookrightarrow (F[W_0] \otimes F[W_1] \otimes \cdots \otimes F[W_e]^{p^e})_1 = F[W_0 \oplus W_1] \otimes F[W_2]^{p} \otimes \cdots \otimes F[W_e]^{p^{e-1}}.
\]

By Proposition 4.10 the \( \Delta \)-length of \( H_1 \) is at most the dimension of \( W_0 \oplus W_1 \) and

\[
F[U_0] \otimes F[U_1] \hookrightarrow H_1
\]

for a suitable \( U_0 \oplus U_1 \leq W_0 \oplus W_1 \). Since \( F[U_0] \hookrightarrow H_1 \), and \( U_0 \) is the maximal vector subspace with this property, we have

\[
F[U_0] \otimes F[U_1]^{p} \hookrightarrow H.
\]

Proceeding inductively gives an extension

\[
F[U_0] \otimes F[U_1]^{p} \otimes \cdots \otimes F[U_{e+1}]^{p^{e+1}} \hookrightarrow H,
\]

which is separable, because it is algebraic (cf. the remark after Proposition 4.10). This extension is also purely inseparable because

\[
F[U_0] \otimes F[U_1]^{p} \otimes \cdots \otimes F[U_{e+1}]^{p^{e+1}} \hookrightarrow F[W_0] \otimes F[W_1]^{p} \otimes \cdots \otimes F[W_{e+1}]^{p^{e+1}}
\]

is purely inseparable. Thus

\[
H = F[U_0] \otimes F[U_1]^{p} \otimes \cdots \otimes F[U_{e+1}]^{p^{e+1}}
\]

as desired. \( \square \)

Corollary 4.12. Let \( V = W_0 \oplus \cdots \oplus W_e \) be a vector space decomposition. Let \( H \hookrightarrow F(W_0) \otimes F(W_1)^{p} \otimes \cdots \otimes F(W_e)^{p^e} \) be a purely inseparable extension of exponent one. Then \( H \) is a field over the Steenrod algebra if and only if

\[
H = F(U_0) \otimes F(U_1)^{p} \otimes \cdots \otimes F(U_{e+1})^{p^{e+1}}
\]

for some vector space decomposition

\[
V = U_0 \oplus \cdots \oplus U_{e+1}
\]

with

\[
U_0 \oplus \cdots \oplus U_i \leq W_0 \oplus \cdots \oplus W_i
\]

and \( \dim(U_0 \oplus \cdots \oplus U_i) \) is the \( \Delta \)-length of \( H_i \), \( i = 0, \ldots, e + 1 \).

Proof. Since

\[
\text{Un}(H) \hookrightarrow F[U_0] \otimes F[U_1]^{p} \otimes \cdots \otimes F[U_{e}]^{p^e}
\]

is integrally closed and \( FF(\text{Un}(H)) = H \), the result follows from Theorem 4.11. \( \square \)
**Theorem 4.13.** $H \hookrightarrow \mathbb{F}(V)$ is a purely inseparable extension of exponent $e$ of fields over the Steenrod algebra if and only if
\[ H = \mathbb{F}(W_0) \otimes \mathbb{F}(W_1)^p \otimes \cdots \otimes \mathbb{F}(W_e)^p \]
for some vector space decomposition
\[ V = W_0 \oplus \cdots \oplus W_e, \]
where $\dim(W_0 \oplus \cdots \oplus W_i)$ is the $\Delta$-length of $H_i$, $i = 0, \ldots, e$.

**Proof.** The “if” part is clear by Corollary 4.12. We show the “only if” part.

We proceed by induction on $e$. The case $e = 1$ has been treated in Theorem 4.11. Thus assume that $e > 1$.

We have a chain of purely inseparable extensions of exponent one
\[ H = H_0 \hookrightarrow H_1 \hookrightarrow \cdots \hookrightarrow H_e = \mathbb{F}(V) \]
which is obtained by adjoining successively $p$th roots. Note that all $H_i$’s are fields over the Steenrod algebra.

By the induction hypothesis we can assume that
\[ H \hookrightarrow H_1 = \mathbb{F}(W_0) \otimes \mathbb{F}(W_1)^p \otimes \cdots \otimes \mathbb{F}(W_{e-1})^p \]
for a vector space decomposition
\[ V = W_0 \oplus \cdots \oplus W_{e-1}. \]
By Corollary 4.12 we are done. \qed

At the level of algebras we obtain the following result as an obvious corollary.

**Corollary 4.14.** Let $H$ be integrally closed. Let $H \hookrightarrow \mathbb{F}[V]$ be a purely inseparable extension of exponent $e$. Then $H$ is an algebra over the Steenrod algebra if and only if
\[ H = \mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_e]^p \]
for some vector space decomposition
\[ V = W_0 \oplus \cdots \oplus W_e, \]
where $\dim(W_0 \oplus \cdots \oplus W_i)$ is the $\Delta$-length of $H_i$, $i = 0, \ldots, e$. \qed

**Remark.** Note that Corollary 4.14 has been proven in Theorem 7.2.2 of [3] as well as in [7], Theorem II, without, however, the precise statement on the dimension of $W_0 \oplus \cdots \oplus W_i$.

### 5. Purely inseparable extensions, the general case

Let $H$ be an unstable Noetherian integral domain over the Steenrod algebra. Assume that the canonical inclusion
\[ H \hookrightarrow \sqrt[p]{H} \]
is purely inseparable of exponent $e$.

If $H$ is integrally closed, then so is $\sqrt[p]{H}$ by part (3) of Proposition 4.11. Then
\[ \sqrt[p]{H} \hookrightarrow \mathbb{F}[V] \]
is a Galois extension with Galois group $G \leq \text{GL}(n, \mathbb{F})$, where $n$ is the Krull dimension of $H$ (see the Galois Embedding Theorem, Theorem 7.1.1 in [3]). Thus
\[ \sqrt[p]{H} = \mathbb{F}[V]^G. \]
On the other hand we can take the separable closure first: The separable closure of $H \hookrightarrow F[V]$ denoted by $\mathbb{H}^{sep}$ is again an unstable algebra over the Steenrod algebra by the Separable Extension Lemma (Proposition 2.2.2 in [3]), since $\mathbb{H}^{sep} = \cup n(\mathbb{H}^{sep})$. Thus we obtain a purely inseparable extension of exponent $e$

$$\mathbb{H}^{sep} \hookrightarrow F[V].$$

Therefore, by Corollary 4.14

$$\mathbb{H}^{sep} = F[W_0] \otimes \cdots \otimes F[W_e]^p,$$

for some vector space decomposition $V = W_0 \oplus \cdots \oplus W_e$.

We need a technical lemma.

**Lemma 5.1.** Let $H$ be an unstable Noetherian integral domain over the Steenrod algebra. Then for all $i \in \mathbb{N}_0$ we have

$$(\mathbb{H}^{sep})_i = (H_1)^{sep}.$$

**Proof.** By induction on $i$, we need to prove the statement only for $i = 1$. By assumption we have the diagram

$$
\begin{array}{ccc}
\mathbb{H}^{sep} & \hookrightarrow & (\mathbb{H}_1)^{sep} \\
\downarrow & & \downarrow \\
H & \hookrightarrow & H_1 \quad \hookrightarrow (H_1)^{sep}
\end{array}
$$

If $h \in H_1$, then $h^p \in H \subseteq \mathbb{H}^{sep}$. Thus $h \in (\mathbb{H}^{sep})_1$. Thus

$$H_1 \hookrightarrow (\mathbb{H}^{sep})_1.$$ 

We note that $\mathbb{H}^{sep} = F[W_0] \otimes \cdots \otimes F[W_e]^p$ by Corollary 4.14 Therefore

$$(\mathbb{H}^{sep})_1 = F[W_0 \oplus W_1] \otimes F[W_2]^p \otimes \cdots \otimes F[W_e]^{p-1}.$$ 

Hence $(\mathbb{H}^{sep})_1$ is separably closed, and thus

$$(H_1)^{sep} \hookrightarrow (\mathbb{H}^{sep})_1.$$ 

Moreover, this extension is by the universal property of the separable closure purely inseparable. Next we show that this extension has exponent at most one. To this end, take $h \in (\mathbb{H}^{sep})_1$. Then $h^p \in \mathbb{H}^{sep}$ is separable over $H$, hence over $H_1$. Therefore $h^p \in (H_1)^{sep}$.

Denote the inseparable closure of $H_1$ inside $(\mathbb{H}^{sep})_1$ by $K$. Then $H_1 \hookrightarrow K$ has exponent at most one, and since $\mathbb{H}^{sep} \hookrightarrow (\mathbb{H}^{sep})_1$ has exponent one, the extension $H \hookrightarrow K$ also has exponent at most one. Since $H_1$ is the largest algebra such that $H \hookrightarrow K$ has exponent one, we have that $H_1 = K$ and $H_1 \hookrightarrow (\mathbb{H}^{sep})_1$ is separable.

Therefore, $(H_1)^{sep} \hookrightarrow (\mathbb{H}^{sep})_1$ is also separable. Since we already saw that this extension is purely inseparable, this means that

$$(H_1)^{sep} = (\mathbb{H}^{sep})_1$$

as claimed. $\square$

So, in what follows we can write $H_1^{sep}$ for $(H_1)^{sep} = (\mathbb{H}^{sep})_1$ without ambiguity.
**Theorem 5.2.** Let \( H \) be an integrally closed unstable Noetherian integral domain over the Steenrod algebra of Krull dimension \( n \). Set \( \dim_F(V) = n \). Let \( V = W_0 \oplus \cdots \oplus W_e \), and let
\[
\overline{H}^{sep} = F[W_0] \otimes \cdots \otimes F[W_e]^{p^{s}}.
\]
Then there exists a group \( G \leq \text{GL}(V) \) acting on the flags \( W_0 \oplus \cdots \oplus W_i \) for \( i = 0, \ldots, e \) such that \( \sqrt[p]{\overline{H}} = F[V]^G \) and
\[
H = (F[W_0] \otimes \cdots \otimes F[W_e]^{p^e})^G.
\]
Furthermore, \( \dim_F(W_0 \oplus \cdots \oplus W_i) \) is the \( \Delta \)-length of \( H_i \).

**Proof.** By assumption we have a diagram
\[
\begin{array}{ccc}
H & \to & F[V]^G \\
\downarrow & & \downarrow \\
\overline{H}^{sep} & \to & F[V]
\end{array}
\]
where the horizontal extensions are purely inseparable and the vertical are separable. Recall that
\[
\overline{H}^{sep} = F[W_0] \otimes \cdots \otimes F[W_e]^{p^e}
\]
where \( \dim_F(W_0 \oplus \cdots \oplus W_i) \) is the \( \Delta \)-length of \( H_i \) by Corollary 4.14. Consider the corresponding diagram of the respective field of fractions
\[
\begin{array}{ccc}
H & \to & F(V)^G \\
\downarrow & & \downarrow \\
\overline{H}^{sep} & \to & F(V).
\end{array}
\]
Recall from the Imbedding Theorem (Theorem 8.1.5 in [3]) that \( H \), and hence \( \overline{H}^{sep} \), contains a fractal of the Dickson algebra in dimension \( n = \dim_F(V) \). Thus
\[
\Delta(n)^q \to H \to \overline{H}^{sep} \to F(V)
\]
for some \( s \in \mathbb{N}_0 \). Therefore, the polynomial
\[
\Delta(X) = \prod_{l \in V} (X - l)^{q^e} = d_{0,0}^{q^e} X^{q^e} - d_{0,1}^{q^e} X^{q^e+1} + \cdots + (-1)^n d_{n,n}^{q^e} X^{q^e+n}
\]
has coefficients in \( \overline{H} \) (cf. Section 5.1 in [3]). Its roots are by construction the linear forms in \( F[V] \). Thus \( F(V) \) is the splitting field of \( \Delta(X) \). Hence, the field extension \( \overline{H} \to F(V) \) is normal\(^7\). Since \( \overline{H}^{sep} \to F(V) \) is purely inseparable, it follows from the structure theorem for finite dimensional normal field extensions that the extension
\[
\overline{H} \to \overline{H}^{sep}
\]
is Galois with some Galois group \( G' \). We have
\[
|G'| = |\overline{H}^{sep} : H| = |F(V) : F(V)^G| = |G|.
\]
Since
\[
H = (\overline{H}^{sep})^G = \overline{H}^{sep} \cap F[V]^G
\]
\(^7\)Note that this means that \( F(V) \) is algebraically closed in the category of fields over the Steenrod algebra; cf. Section 3.2 in [3].
it follows that $G' \geq G$. Thus $G' = G$. Finally we show that the $\Delta$-length of an unstable algebra $H$ coincides with the $\Delta$-length of its separable closure ($\overline{H}^{sep}$). Together with Lemma 5.1 this gives the result.

To this end, let $l_i = \dim_F(W_i)$. Since $G$ acts on $\overline{H}^{sep}$, the group $G$ consists of matrices of the form

$$(+): \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ * & A_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ * & \cdots & \cdots & * & A_e \end{bmatrix}$$

where $A_i$ is an $n_i \times n_i$-matrix with $n_i = \dim(W_i)$. Denote by $\hat{G}$ the subgroup of $GL(n, F)$ consisting of all matrices of the form $$(+)$$ Denote by $x_1, \ldots, x_n$ a basis for $W_0 \oplus \cdots \oplus W_e$. Then

$$(F[W_0] \otimes \cdots \otimes F[W_e]^{p^r})^\hat{G} = \mathcal{D}(n_0) \otimes F[c_{top}(x_{n_0+1}^p), \ldots, c_{top}(x_{n_1+1}^p), c_{top}(x_{n_1+1}^p), \ldots, c_{top}(x_{n_0+1}^p)]$$

where $c_{top}(\cdot)$ denotes the top orbit Chern class of the element $\cdot$ (cf. Section 4.1 in [6]). By construction, the top orbit Chern classes of $p$th powers are $p$th powers. Thus the $\Delta$-length of the ring of invariants $\hat{G}$ is equal to $n_0$. Therefore we have

$$(F[W_0] \otimes \cdots \otimes F[W_e]^{p^r})^\hat{G} \hookrightarrow H = (F[W_0] \otimes \cdots \otimes F[W_e]^{p^r})^G \hookrightarrow \overline{H}^{sep} = F[W_0] \otimes \cdots \otimes F[W_e]^{p^r}.$$ 

The smallest algebra, as well as the largest algebra in this chain, has $\Delta$-length $n_0$. Thus by Lemma 5.7 we are done. $\square$

**Remark.** Since $G$ acts on $\overline{H}^{sep}$, the group $G$ consists of matrices of the form given in $(+)$. So, if there exists no basis such that $G$ consists of flag matrices like above, then the only unstable algebras $H \hookrightarrow \sqrt[p^r]{H} = F[V]^G$ are the $p^s$th powers

$$H = (F[V]^G)^{p^s},$$

i.e., we have the trivial vector space decomposition $V = W_e$.

**Remark.** Note carefully that the proof shows that the $\Delta$-length of $H$ and the $\Delta$-length of any separable extension $H \hookrightarrow K$ coincide.

**Remark.** In Theorem 7.2.2 in [3] as well as in Theorem II in [8] it has been proven that

$$H = (F[W_0] \otimes \cdots \otimes F[W_e]^{p^r})^G.$$ 

However, the precise statement on the dimensions of $W_0 \oplus \cdots \oplus W_e$ is missing. Also the connection between the two Galois groups of $H \hookrightarrow F[W_0] \otimes \cdots \otimes F[W_e]^{p^r}$, resp. $\sqrt[p^r]{H} \hookrightarrow F[V]$, is not made.

We conclude this section with an example.

**Example 5.3.** Consider the regular representation of the cyclic group of order 2, $\mathbb{Z}/2$, over a field of characteristic 2 afforded by the matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$
Set $F[V] = F[x, y]$. Its ring of invariants is
\[ F[x, y]^{\mathbb{Z}/2} = F[x, y^2 - xy]. \]
Then $\mathbb{Z}/2$ acts on $F[x, y^2]$ with invariant ring
\[ F[x, y^2]^{\mathbb{Z}/2} = F[x, (y^2 - xy)^2]. \]
On the other hand, $\mathbb{Z}/2$ does not act on $F[x^2, y]$. Indeed
\[ F[x, y]^{\mathbb{Z}/2} \cap F[x^2, y] = F[x^2, (y^2 - xy)^2] = F[x^2, y^2]^{\mathbb{Z}/2}. \]
Furthermore we could consider the purely inseparable field extension
\[ F(x^2, y^2 - xy) \hookrightarrow F(x, y^2 + xy) \]
of degree 2. Note that
\[ (\ast) \quad \mathcal{O}_1(y^2 + xy) = x^2y + xy^2 \not\in F(x^2, y^2 - xy) \]
since our field contains only elements of even degree. Thus $F(x^2, y^2 - xy)$ is not a field over the Steenrod algebra. Its separable closure
\[ F(x^2, y^2, yx) = F\left(\frac{x}{y}, y^2\right) \]
is a Galois extension with the same Galois group $\mathbb{Z}/2$. However, the same calculation as above shows that it is also not closed under the action of the Steenrod algebra, as predicted in the previous result. Indeed, $F(x, y)$ is the smallest overfield of $F(x^2, y^2, yx)$, say $\mathbb{K}$, closed under the action of the Steenrod algebra as we see next:
\[ \mathcal{O}_1(xy) = x^2y + xy^2 = xy(x + y) \in \mathbb{K} \Rightarrow x + y \in \mathbb{K}. \]
Since $\mathbb{K}$ must have the form $F(W) \otimes F(V/W)^2$ for some $W \leq V$ we find that $\text{span}_F\{x + y\} \subseteq W$. The minimal polynomial of $x + y \in \mathbb{K}$ over $F(x^2, y^2, xy)$,
\[ p(X) = X^2 + (x^2 + y^2), \]
is inseparable of degree 2. Therefore
\[ 2 = |F(x, y) : F(x^2, y^2, xy)| = |F(x, y) : \mathbb{K}| |\mathbb{K} : F(x^2, y^2, xy)| = 2|F(x, y) : \mathbb{K}|, \]
and hence $F(x, y) = \mathbb{K}$ as claimed.

On the other hand, the largest subfield, call in $\mathbb{L}$, of $F(x^2, y^2, xy)$ that is closed under the Steenrod algebra is $F(x^2, y^2)$: by Equation (\ast) the field $\mathbb{L}$ does not contain $xy$. Since $xy$ is the root of
\[ p(X) = X^2 + (xy)^2 \in F(x^2, y^2)[X] \]
we find that
\[ 2 = |F(x^2, y^2, xy) : F(x^2, y^2)| = |F(x^2, y^2, xy) : \mathbb{L}| |\mathbb{L} : F(x^2, y^2)| = 2|\mathbb{L} : F(x^2, y^2)| \]
and hence $\mathbb{L} = F(x^2, y^2)$. 
6. Projective dimension

The goal of this section is to prove that a Noetherian reduced unstable algebra $H$ is Cohen-Macaulay if and only if its inseparable closure $^\sqrt{H}$ is Cohen-Macaulay.

Let $H$ be an unstable algebra over the Steenrod algebra $\mathcal{P}$. An ideal $I \subseteq H$ is called $\mathcal{P}$-invariant if it is closed under the action of the Steenrod algebra.

**Lemma 6.1.** Let $H$ be an unstable algebra over the Steenrod algebra. For any $s \in \mathbb{N}_0$, the canonical inclusion
\[
\psi: H^p_s \hookrightarrow H
\]
induces a bijection
\[
\psi^*: \text{Proj}(H) \rightarrow \text{Proj}(H^p_s)
\]
between the spaces of homogeneous $\mathcal{P}$-invariant prime ideals.

**Proof.** Since $\psi$ is an integral extension, the Lying-Over Theorem holds. Thus $\psi^*$ is surjective.

To prove injectivity take two homogeneous $\mathcal{P}$-invariant prime ideals $p_1, p_2 \subseteq H$, such that
\[
p_1 \cap H^p_s = p_2 \cap H^p_s.
\]
Thus for any $h \in p_1$ it follows that
\[
h^p_s \in p_1 \cap H^p_s = p_2 \cap H^p_s.
\]
Therefore
\[
h^p_s \in (\psi(p_2 \cap H^p_s)) \subseteq p_2.
\]
Since $p_2$ is prime, we find that $h \in p_2$. Interchanging the roles of $p_1$ and $p_2$ gives the result. \qed

This result could have been proven also by observing that the $s$th iteration of the Frobenius map
\[
F^s: H \rightarrow H^p_s
\]
hands us an isomorphism of unstable algebras of degree $p^s$ if $H$ is reduced. This in turn also implies the following result.

**Lemma 6.2.** Let $H$ be an unstable reduced algebra over the Steenrod algebra. For any $s \in \mathbb{N}_0$ we find that
\[
\text{depth}(H) = \text{depth}(H^p_s).
\]

We observe that $H$ is Noetherian of Krull dimension $n$ if and only if $H^p_s$ is. We find the following lemma.

**Lemma 6.3.** Let $H$ be Noetherian and reduced of Krull dimension $n$. Let $S = F[h_1^p, \ldots, h_n^p]$ be a system of parameters in $H^p_s$. Then
\[
\text{proj} - \text{dim}_S(H) = \text{proj} - \text{dim}_S(H^p_s) < \infty.
\]

**Proof.** Since $H^p_s \subseteq H$ is a finite integral extension, $S$ is also a system of parameters for $H$. Thus both projective dimensions are finite. Moreover, by the Auslander-Buchsbaum formula we have
\[
\text{proj} - \text{dim}_S(H^p_s) = \text{dim}(H^p_s) - \text{depth}(H^p_s) = \text{dim}(H) - \text{depth}(H) = \text{proj} - \text{dim}_S(H)
\]
by Lemma [6.2] \qed
We come to the desired result about a Noetherian unstable algebra \( H \) and its \( P^* \)-inseparable closure \( \sqrt[\ast]{H} \).

**Proposition 6.4.** Let \( H \) be Noetherian and reduced of Krull dimension \( n \). Then \( H \) is Cohen-Macaulay if and only if \( \sqrt[\ast]{H} \) is Cohen-Macaulay.

**Proof.** Since \( H \) is Noetherian, its \( P^* \)-inseparable closure is also Noetherian by Theorem 6.1.3 in [3]. Therefore \( \sqrt[\ast]{H} = H_s \) for some \( s \in \mathbb{N}_0 \). Thus

\[
(\sqrt[\ast]{H})^p \hookrightarrow H \hookrightarrow \sqrt[\ast]{H} = H_s
\]

is a finite integral extension. By Lemma 6.1 we have a bijection

\[
\text{Proj}_p(\sqrt[\ast]{H})^p \rightarrow \text{Proj}_p(\sqrt[\ast]{H}).
\]

By Theorem 4.3.1 in [3] and Lemma 6.1

\[
\text{Proj}_p(\sqrt[\ast]{H})^p \rightarrow \text{Proj}_p(H) \rightarrow \text{Proj}_p(\sqrt[\ast]{H})
\]

is also bijective. Moreover, by Lemma 6.2 the left and the right algebra have the same depth. Thus by Theorem 2.1 in [5] the results follows (cf. Corollary 2.2 loc.cit.). \( \Box \)

7. **Polynomial rings**

Let \( H \) be an integrally closed unstable Noetherian integral domain over the Steenrod algebra. By Theorem 5.2 we have

\[
H = (\overline{H}^{\text{sep}})^G \hookrightarrow \sqrt[\ast]{H} = \mathbb{F}[V]^G,
\]

where

\[
\overline{H}^{\text{sep}} = \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^s}
\]

for some vector space decomposition \( V = W_0 \oplus \cdots \oplus W_e \). By Proposition 6.4 we know that \( H \) is Cohen-Macaulay if and only if \( \sqrt[\ast]{H} \) is polynomial. Moreover, the algebra generators of \( H \) are just suitable \( p^s \)th powers of the algebra generators of \( \sqrt[\ast]{H} \) (for a minimal generating set).

Let \( G \) act on \( V = W_0 \oplus \cdots \oplus W_e \) such that

\[
gw_i \in W_0 \oplus \cdots \oplus W_i
\]

for all \( w_i \in W_i \), i.e., \( G \) consists of flag matrices of the form

\[
\begin{bmatrix}
A_0 & 0 & \cdots & 0 \\
& A_1 & 0 & \cdots & 0 \\
& & \ddots & \vdots \\
& & \ddots & 0 \\
& & & \ast & \cdots & A_e
\end{bmatrix}
\]

where \( A_i \) is an \( m_i \times m_i \)-matrix with \( m_i = \dim(W_i) \). For every \( i = 0, \ldots, e \) we have a group epimorphism

\[
\text{pr}_i : G \rightarrow G_i,
\]

\[
\begin{bmatrix}
A_0 & 0 & \cdots & 0 \\
& A_1 & 0 & \cdots & 0 \\
& & \ddots & \vdots \\
& & \ddots & 0 \\
& & & \ast & \cdots & A_e
\end{bmatrix}
\]

\[
\begin{bmatrix}
A_0 & 0 & \cdots & 0 \\
& A_1 & 0 & \cdots & 0 \\
& & \ddots & \vdots \\
& & \ddots & 0 \\
& & & \ast & \cdots & A_i
\end{bmatrix}
\]

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Lemma 7.1. With the preceding notation we have
\[ F[W_0 \oplus \cdots \oplus W_i]^G_i = F[V]^G \cap F[W_0 \oplus \cdots \oplus W_i] \subseteq F[V]^G. \]

Proof. The kernel of the projection \( \text{pr}_i \), \( \text{ker}(\text{pr}_i) \), consists of matrices of the form
\[
\begin{bmatrix}
I_0 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots \\
\vdots & 0 & \ddots & \ddots \\
* & * & \ddots & A_{i+1} \\
* & \cdots & * & \cdots
\end{bmatrix},
\]
where the \( I_j \)'s are identity matrices. Thus \( F[V]^{\ker(\text{pr}_i)} \supseteq F[W_0 \oplus \cdots \oplus W_i] \), and hence
\[
F[V]^G = (F[V]^{\ker(\text{pr}_i)})^G_i \supseteq F[W_0 \oplus \cdots \oplus W_i]^G_i.
\]
Since \( F[W_0 \oplus \cdots \oplus W_i]^G_i \subseteq F[W_0 \oplus \cdots \oplus W_i] \) we find
\[
F[W_0 \oplus \cdots \oplus W_i]^G_i \subseteq F[V]^G \cap F[W_0 \oplus \cdots \oplus W_i] \subseteq F[V]^G.
\]
Conversely, since \( F[V]^{\ker(\text{pr}_i)} \supseteq F[W_0 \oplus \cdots \oplus W_i] \), we have
\[
F[V]^{\ker(\text{pr}_i)} \cap F[W_0 \oplus \cdots \oplus W_i] = F[W_0 \oplus \cdots \oplus W_i].
\]
Thus
\[
F[W_0 \oplus \cdots \oplus W_i] = (F[V]^{\ker(\text{pr}_i)} \cap F[W_0 \oplus \cdots \oplus W_i])^G_i.
\]
Finally, note that
\[
F[V]^G \cap F[W_0 \oplus \cdots \oplus W_i] \subseteq (F[V]^{\ker(\text{pr}_i)} \cap F[W_0 \oplus \cdots \oplus W_i])^G_i.
\]
To see this, take an element \( f \in F[V]^G \cap F[W_0 \oplus \cdots \oplus W_i] \). Then \( f \in F[W_0 \oplus \cdots \oplus W_i] \) is invariant under the group \( G \). Thus \( f \) is also invariant under \( \ker(\text{pr}_i) \). Therefore,
\[
f \in F[V]^{\ker(\text{pr}_i)} \cap F[W_0 \oplus \cdots \oplus W_i].
\]
But \( f \) is also \( G \)-invariant, i.e.,
\[
f \in (F[V]^{\ker(\text{pr}_i)} \cap F[W_0 \oplus \cdots \oplus W_i])^G_i \subseteq (F[V]^{\ker(\text{pr}_i)} \cap F[W_0 \oplus \cdots \oplus w_i])^G_i
\]
as desired. \( \square \)

Let \( h_1, \ldots, h_m \in F[V]^G \) be a minimal generating set. Without loss of generality we assume that they are sorted such that
\[
h_1, \ldots, h_{n_0} \in F[W_0],
\]
\[
h_{n_0+1}, \ldots, h_{n_1} \in F[W_0 \oplus W_1],
\]
\[
\ldots
\]
\[
h_{n_{e-1}+1}, \ldots, h_{n_e} = h_m \in F[W_0 \oplus \cdots \oplus W_e].
\]
We assume that \( n_0, \ldots, n_e \) are maximal with this property. Thus by construction
\[
F[h_0, \ldots, h_{n_i}] \subseteq F[W_0 \oplus \cdots \oplus W_i]^G_i \subseteq F[V]^G
\]
for all \( i = 0, \ldots, e \).
Proposition 7.2. If \( n_e = \dim_{F}(V) \), i.e., if the ring of invariants
\[
F[W_0 \oplus \cdots \oplus W_e]^G
\]
is polynomial, then \( n_i = \dim_{F}(W_0 \oplus \cdots \oplus W_i) \).

Proof. Consider the integral extension
\[
F[W_0 \oplus \cdots \oplus W_i]^G_i \hookrightarrow F[W_0 \oplus \cdots \oplus W_i].
\]
The maximal ideal \( m_i \) of \( F[W_0 \oplus \cdots \oplus W_i]^G_i \) lies over the maximal ideal in \( F[W_0 \oplus \cdots \oplus W_e] \). Furthermore, \( m_i \) extends to a prime ideal \( p_i \subseteq F[W_0 \oplus \cdots \oplus W_i] \). By construction \( p_i \) is generated by all linear forms in \( F[W_0 \oplus \cdots \oplus W_i] \). Thus \( p_i \) is regular and prime of height equal to \( \dim_{F}(W_0 \oplus \cdots \oplus W_i) \). Hence, its contraction to the ring of invariants
\[
p_i^c = p_i \cap F[W_0 \oplus \cdots \oplus W_e]^G
\]
is also prime of height equal to \( \dim_{F}(W_0 \oplus \cdots \oplus W_i) \). Furthermore, \( p_i^c \) contains by construction
\[
(h_1, \ldots, h_{n_i}) \subseteq p_i^c.
\]
Thus the quotient
\[
F[W_0 \oplus \cdots \oplus W_i]^G/p_i^c = F[\overline{T}_{n_i+1}, \ldots, \overline{T}_n] \hookrightarrow F[\overline{T}_{i+1} \oplus \cdots \oplus W_e]
\]
is integral, and
\[
n - n_i = \dim_{F}(W_{i+1} \oplus \cdots \oplus W_e)
\]
for all \( i = 0, \ldots, e - 1 \). \( \square \)

Theorem 7.3. With the above notation, if
\[
(F[W_0] \otimes \cdots \otimes F[W_e])^G = F[h_1, \ldots, h_n]
\]
is polynomial, then for suitable \( s_1, \ldots, s_n \in \mathbb{N}_0 \)
\[
(F[U_0] \otimes \cdots \otimes F[U_f])^{p_i} = F[h_1^{p_1 s_1}, \ldots, h_n^{p_n s_n}]
\]
is polynomial for any subflag
\[
U_0 \oplus \cdots \oplus U_f \leq W_0 \oplus \cdots \oplus W_i
\]
that admits an action of \( G \).

Proof. To simplify notation we assume that the extension
\[
F[U_0] \otimes \cdots \otimes F[U_f]^{p_f} \hookrightarrow F[W_0] \otimes \cdots \otimes F[W_e]^{p_e}
\]
is purely inseparable of exponent one. The general case follows then inductively.
Since \( G \) acts on the flag \( W_0 \oplus \cdots \oplus W_e \) the algebra generator for the ring of invariants can be sorted such that
\[
h_1, \ldots, h_{n_i} \in F[W_0 \oplus \cdots \oplus W_i]
\]
with \( n_i - n_{i-1} = \dim(W_i) \), \( n_0 = \dim_{F}(W_0) \), by Proposition 7.2.
Since \( G \) acts also on the subflag \( U_0 \oplus \cdots \oplus U_f \) the algebra generator for the ring of invariants can be sorted such that
\[
h_1, \ldots, h_{m_i} \in F[U_0 \oplus \cdots \oplus U_i]
\]
with $m_i - m_{i-1} = \dim(U_i)$, $m_0 = \dim_{\mathbb{F}}(U_0)$, and $m_f = n_e = n$. Thus $n_i \geq m_i$.

Consider the algebra
\[ A = \mathbb{F}[h_1, \ldots, h_m, h_{m_0+1}, \ldots, h_{n_0+1}, \ldots, h_{m_1}, h_{m_1+1}, \ldots, h_{n_1}, \ldots, h_{m_e}, h_{m_e+1}, \ldots, h_{m_f}] \]
\[ \hookrightarrow \mathbb{F}[U_0] \otimes \cdots \otimes \mathbb{F}[U_f]^{p^{e_i}}. \]

Since $A$ consists of invariant polynomials it is contained in the ring of invariants
\[ (\mathbb{F}[U_0] \otimes \cdots \otimes \mathbb{F}[U_f])^G. \]

The diagram
\[ A \xrightarrow{\sim} (\mathbb{F}[U_0] \otimes \cdots \otimes \mathbb{F}[U_f])^G \xrightarrow{\sim} \mathbb{F}[U_0] \otimes \cdots \otimes \mathbb{F}[U_f]^{p^{e_i}} \]
\[ \mathbb{F}[h_1, \ldots, h_n] \xrightarrow{\sim} \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^{e_i}} \]

has by construction purely inseparable vertical extensions of degree $p^{\sum (m_i - m_{i-1})}$. Since the degree of
\[ \mathbb{F}[h_1, \ldots, h_n] \hookrightarrow \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^{e_i}} \]
is the group order $|G|$, the degree of
\[ A \hookrightarrow (\mathbb{F}[U_0] \otimes \cdots \otimes \mathbb{F}[U_f])^{p^{e_i}} \]
is also the group order. Thus $A$ is the desired ring of invariants as claimed. \( \square \)

The following result settles a twenty-year-old conjecture due to Clarence W. Wilkerson (see Conjecture 5.1 in [8]).

**Theorem 7.4.** Let $H$ be an integrally closed Noetherian unstable integral domain over the Steenrod algebra. Then $H$ is polynomial if and only if $^p\sqrt{H}$ is polynomial. Furthermore,
\[ ^p\sqrt{H} = \mathbb{F}[h_1, \ldots, h_n] \]
if and only if there are $s_1, \ldots, s_n \in \mathbb{N}_0$ such that
\[ H = \mathbb{F}[h_1^{p^{s_1}}, \ldots, h_n^{p^{s_n}}]. \]

**Proof.** By Theorem 5.2 there exist a group $G$ and a flag $V = W_0 \oplus \cdots \oplus W_e$ such that
\[ H = (\mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^{e}})^G \hookrightarrow ^p\sqrt{H} = \mathbb{F}[V]^G. \]

If $^p\sqrt{H}$ is polynomial, then so is $H$ by Theorem 7.3. Note that the same result also gives the precise statement on the respective algebra generators.

On the other hand, $(^p\sqrt{H})^{p^{e}} = (\mathbb{F}[V]^{p^{e}})^G \hookrightarrow H$ is the ring of invariants on the subflag $V \leq W_0 \oplus \cdots \oplus W_e$ for some large enough $e$. Therefore if $H$ is polynomial, then $(^p\sqrt{H})^{p^{e}} \hookrightarrow H$ is polynomial by the same Theorem 7.3. Thus $^p\sqrt{H}$ is polynomial since it is isomorphic as an algebra to $(^p\sqrt{H})^{p^{e}}$. \( \square \)

Thus we have the following corollary.

**Corollary 7.5.** Let $H$ be an unstable polynomial algebra over the Steenrod algebra. Set $H = \mathbb{F}[h_1, \ldots, h_n]$. Then $H$ is $p^{\ast}$-inseparably closed if and only if the polynomial generators $h_1, \ldots, h_n$ are no $p$th powers. \( \square \)
The example given at the end of Section 5 illustrates these results. We want to close with an example that shows that a simple generalization of Theorem 7.4 to nonpolynomial invariants is not true.

**Example 7.6.** Let $p$ be odd and let $\mathbb{F}$ be the prime field of characteristic $p$. Consider the four-dimensional modular representation $\mathbb{Z}/p \hookrightarrow \text{GL}(4, \mathbb{F})$ afforded by the matrix

$$
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}.
$$

Its ring of invariants turns out to be a hypersurface

$$\mathbb{F}[x_1, y_1, x_2, y_2]^{\mathbb{Z}/p} = \mathbb{F}[c_1, y_1, c_2, y_2, q]/(r),$$

where $c_i = x_i^p - x_i y_i^{p-1}$ are the top orbit Chern classes of $x_i$, $i = 1, 2$, and $q = x_1 y_2 - x_1 y_1$ is an invariant quadratic form. The relation is given by

$$r = q^p - c_1 y_2^p + c_2 y_1^p + q y_1^{p-1} y_2^{p-1}$$

(see Theorem 2.1 in [2]). Certainly, $\mathbb{Z}/p$ also acts on $\mathbb{F}[x_1, y_1] \otimes \mathbb{F}[x_2^p, y_2^p]$ and we find that

$$A = \mathbb{F}[c_1, y_1, c_2^p, y_2^p, q] \hookrightarrow (\mathbb{F}[x_1, y_1] \otimes \mathbb{F}[x_2^p, y_2^p])^{\mathbb{Z}/p}.$$

However, the new ring of invariants contains an invariant that is not in the algebra $A$, namely

$$q' = x_1 y_2^p - x_2^p y_1.$$

Indeed, with the methods presented in Theorem 2.1 of [2] it is not hard to see that

$$(\mathbb{F}[x_1, y_1] \otimes \mathbb{F}[x_2^p, y_2^p])^{\mathbb{Z}/p} = \mathbb{F}[c_1, y_1, c_2^p, y_2^p, q]/(r'),$$

where $r' = (q')^p - c_1 y_2^{p^2} + c_2 y_1^p - q y_1^{p-1} y_2^{p(p-1)}$. Interesting enough though, it transpires that this ring is again a hypersurface.

**References**


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