

**INSEPARABLE EXTENSIONS OF ALGEBRAS  
 OVER THE STEENROD ALGEBRA WITH APPLICATIONS  
 TO MODULAR INVARIANT THEORY OF FINITE GROUPS**

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*Dedicated to Clarence W. Wilkerson on the occasion of his 60th birthday*

**ABSTRACT.** We consider purely inseparable extensions  $H \hookrightarrow {}^{\mathcal{P}^*}\sqrt{H}$  of unstable Noetherian integral domains over the Steenrod algebra. It turns out that there exists a finite group  $G \leq \text{GL}(V)$  and a vector space decomposition  $V = W_0 \oplus W_1 \oplus \cdots \oplus W_e$  such that  $\overline{H} = (\mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e})^G$  and  ${}^{\mathcal{P}^*}\sqrt{H} = \mathbb{F}[V]^G$ , where  $\overline{(-)}$  denotes the integral closure. Moreover,  $H$  is Cohen-Macaulay if and only if  ${}^{\mathcal{P}^*}\sqrt{H}$  is Cohen-Macaulay. Furthermore,  $\overline{H}$  is polynomial if and only if  ${}^{\mathcal{P}^*}\sqrt{H}$  is polynomial, and  ${}^{\mathcal{P}^*}\sqrt{H} = \mathbb{F}[h_1, \dots, h_n]$  if and only if

$$H = \mathbb{F}[h_1, \dots, h_{n_0}, h_{n_0+1}^p, \dots, h_{n_1}^p, h_{n_1+1}^{p^2}, \dots, h_{n_e}^{p^e}],$$

where  $n_e = n$  and  $n_i = \dim_{\mathbb{F}}(W_0 \oplus \cdots \oplus W_i)$ .

1. INTRODUCTION AND OUTLINE

Let  $\mathbb{K} \hookrightarrow \mathbb{L}$  be an algebraic extension of graded fields. Assume that the smaller field,  $\mathbb{K}$ , carries an action of the Steenrod algebra  $\mathcal{P}^*$  of reduced powers. If the extension  $\mathbb{K} \hookrightarrow \mathbb{L}$  is separable, then the action of  $\mathcal{P}^*$  can be uniquely extended to  $\mathbb{L}$ . In other words, the separable closure of  $\mathbb{K}$  as a field over the Steenrod algebra coincides with the separable closure in the category of graded fields; see Proposition 2.2.2 in [3] and Proposition 2.2 in [7].

If the extension, however, is purely inseparable the situation is more delicate: Let  $p(X) \in \mathbb{K}[X]$  be the minimal polynomial of  $l \in \mathbb{L}$ . Since the extension is purely inseparable, we have that

$$p(X) = X^{p^e} - \kappa,$$

so that  $l^{p^e} = \kappa$  for some  $\kappa \in \mathbb{K}$ . Of course, since our fields are graded, we obtain the following condition on the degrees:

$$(*) \quad \deg(l)p^e = \deg(\kappa).$$

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However, the crucial issue is the following. If there were an extension of the  $\mathcal{P}^*$ -action to the larger field  $\mathbb{L}$ , then

$$\mathcal{P}^{\Delta_i}(\kappa) = 0 \quad \forall i$$

because by equation (\*) the element  $\kappa$  is a  $p$ th power. Thus, we need to define

$$(*) \quad (\mathcal{P}^i(l))^{p^e} = \mathcal{P}^{ip^e}(\kappa) \in \mathbb{K}.$$

The problem is that it does not follow that  $\mathcal{P}^i(l) \in \mathbb{L}$ . Nevertheless, as equation (\*) shows, the inseparable closures of  $\mathbb{K}$  as a graded field and as a field over the Steenrod algebra coincide. We denote this object by  ${}^{\mathcal{P}^*}\sqrt{\mathbb{K}}$ .

This leads to the following question: Under which conditions can we extend the action of  $\mathcal{P}^*$  from  $\mathbb{K}$  to  $\mathbb{L}$ ? Or equivalently, which intermediate fields  $\mathbb{K} \subseteq \mathbb{L} \subseteq {}^{\mathcal{P}^*}\sqrt{\mathbb{K}}$  are objects in the category of fields over the Steenrod algebra?

In this paper we study these questions in the more general framework of Noetherian integral domains  $H$  over the Steenrod algebra.

In Section 2 we recall the construction of inseparable closures over the Steenrod algebra and its basic properties. To this list we add a few more that will be of use later.

In Sections 3 and 4 we start with the investigation of inseparable extensions  $H \hookrightarrow {}^{\mathcal{P}^*}\sqrt{H}$ , where  ${}^{\mathcal{P}^*}\sqrt{H}$  is either the symmetric algebra,  $\mathbb{F}[V]$ , on  $V^*$  with  $V = \mathbb{F}^n$ , or its field of fractions,  $\mathbb{F}(V)$ . This has two reasons: for one,  $\mathbb{F}(V)$  and  $\mathbb{F}[V]$  are universal, in the sense that they are algebraically closed in our category. On the other hand, any unstable Noetherian integral domain  $H$  can be embedded into  $\mathbb{F}[V]$  such that the inclusion

$$H \hookrightarrow \mathbb{F}[V]$$

is finite; see the Embedding Theorem, Corollary 6.1.5 in [3]. Thus in Sections 3 and 4 we consider the diagram

$$\begin{array}{ccc} H & \hookrightarrow & {}^{\mathcal{P}^*}\sqrt{H} = \mathbb{F}[V] \\ \downarrow & & \downarrow \\ \mathbb{H} & \hookrightarrow & {}^{\mathcal{P}^*}\sqrt{\mathbb{H}} = \mathbb{F}(V), \end{array}$$

where  $\mathbb{H} = FF(H)$  is the field of fractions of  $H$ . In Section 3 we treat the case of purely inseparable extensions of exponent one, in Section 4 we look at extensions with higher exponents  $e$ . Denote by  $\overline{(-)}$  the integral closure. The results of these parts show that<sup>1</sup>

$$\begin{aligned} \overline{H} &= \mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e} \quad \text{and} \\ \mathbb{H} &= \mathbb{F}(W_0) \otimes \mathbb{F}(W_1)^p \otimes \cdots \otimes \mathbb{F}(W_e)^{p^e} \end{aligned}$$

for some vector space decomposition  $V = W_0 \oplus W_1 \oplus \cdots \oplus W_e$ ; see Theorem 4.13 and Corollary 4.14.<sup>2</sup> This reproves results in [8], Theorem II, and [3], Theorem 7.2.2. However, the proof presented here has the advantage that it gives precise information on the vector space dimensions of the  $W_i$ 's.

<sup>1</sup>All tensor products in this manuscript are tensor products over the ground field  $\mathbb{F}$ .  
<sup>2</sup>We denote by  $\mathbb{F}[V]^p$  the algebra  $\mathbb{F}[x_1^p, \dots, x_n^p]$  for  $\mathbb{F}[V] = \mathbb{F}[x_1, \dots, x_n]$ .

In Section 5 we come to the general case. By the Galois Embedding Theorem (Theorem 7.1.1 in [3]), we know that  $\overline{{}^p\sqrt{\mathbb{H}}}$  is a ring of invariants of some finite group  $G \leq GL(V)$  acting linearly on  $\mathbb{F}[V]$ . Thus

$$\begin{array}{ccc} \overline{\mathbb{H}} & \hookrightarrow & \overline{{}^p\sqrt{\mathbb{H}}} = \mathbb{F}[V]^G \\ \downarrow & & \downarrow \\ \mathbb{H} & \hookrightarrow & {}^p\sqrt{\mathbb{H}} = \mathbb{F}(V)^G. \end{array}$$

It turns out that there exists a vector space decomposition as above,

$$V = W_0 \oplus W_1 \oplus \dots \oplus W_e,$$

such that  $G$  acts on the flags

$$W_0 \oplus W_1 \oplus \dots \oplus W_i$$

for all  $i = 0, \dots, e$ . Moreover

$$\overline{\mathbb{H}} = (\mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \dots \otimes \mathbb{F}[W_e]^{p^e})^G \quad \text{and}$$

$$\mathbb{H} = (\mathbb{F}(W_0) \otimes \mathbb{F}(W_1)^p \otimes \dots \otimes \mathbb{F}(W_e)^{p^e})^G;$$

see Theorem 5.2. This extends results in [8], Theorem II, and [3], Theorem 7.2.2, in the sense that we are able to determine the vector space dimensions of the  $W_i$ 's and, more importantly, are able to prove that the group  $G$  in question remains unchanged. In particular this means that the group  $G$  must consist of flag matrices

$$\begin{bmatrix} A_0 & 0 & \dots & 0 \\ * & A_1 & 0 & \dots & 0 \\ & * & \ddots & & \vdots \\ \dots & & \ddots & & 0 \\ * & \dots & * & A_e \end{bmatrix},$$

where  $A_i$  is an  $m_i \times m_i$ -matrix with  $m_i = \dim(W_i)$ . On the other hand, if  $V$  has no basis such that  $G$  consists of flag matrices, then the only purely inseparable extensions of exponent  $e$  are

$$(\mathbb{F}[V]^{p^e})^G \subseteq \mathbb{F}[V]^G.$$

In Section 6 we take a break from these constructive methods and look at homological properties of  $\mathbb{H}$  and  ${}^p\sqrt{\mathbb{H}}$ . We show that  $\mathbb{H}$  is Cohen-Macaulay if and only if  ${}^p\sqrt{\mathbb{H}}$  is Cohen-Macaulay for any reduced Noetherian unstable algebra  $\mathbb{H}$ .

This motivates Section 7, where we look at polynomial rings. It turns out that  ${}^p\sqrt{\mathbb{H}}$  is a polynomial algebra if and only if  $\mathbb{H}$  is polynomial. Moreover,  ${}^p\sqrt{\mathbb{H}} = \mathbb{F}[h_1, \dots, h_n]$  if and only if

$$\mathbb{H} = \mathbb{F}[h_1, \dots, h_{n_0}, h_{n_0+1}^0, \dots, h_{n_1}^p, h_{n_1+1}^{p^2}, \dots, h_{n_e}^{p^e}],$$

where  $n_e = n$  and  $n_i = \dim_{\mathbb{F}}(W_0 \oplus \dots \oplus W_i)$ . Recall that an unstable  $\mathcal{P}^*$ -inseparably closed polynomial algebra over the Steenrod algebra is the ring of invariants  $\mathbb{F}[V]^G$  for some  $G \leq GL(n, \mathbb{F})$  by the Galois Embedding Theorem (Theorem 7.1.1 in [3]). Combined with the results from Section 5, this means that if  $G$  consists of flag matrices, then  $\mathbb{H}$  is polynomial if and only if  ${}^p\sqrt{\mathbb{H}}$  is polynomial, and the generators are just  $p$ th powers/roots of one another. However, if  $G$  does not consist of flag

matrices, then there exists no unstable algebra  $H \hookrightarrow \mathbb{F}[V]^G$  such that  ${}^{\mathcal{P}^*}\sqrt{H} = \mathbb{F}[V]^G$ . This solves a twenty-year-old conjecture due to Clarence Wilkerson; see Conjecture 5.1 in [8].

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## 2. RECOLLECTIONS AND PRELIMINARIES

Let  $H$  be an unstable reduced algebra over the Steenrod algebra of reduced powers  $\mathcal{P}^*$ . We denote the characteristic by  $p$ , and the order of the ground field  $\mathbb{F}$  by  $q$ . Recall that the Steenrod algebra contains an infinite sequence of derivations iteratively defined as

$$\begin{aligned}\mathcal{P}^{\Delta_1} &= \mathcal{P}^1, \\ \mathcal{P}^{\Delta_i} &= \mathcal{P}^{\Delta_{i-1}} \mathcal{P}^{q^{i-1}} - \mathcal{P}^{q^{i-1}} \mathcal{P}^{\Delta_{i-1}} \quad \text{for } i \geq 2.\end{aligned}$$

We set

$$\mathcal{P}^{\Delta_0}(h) = \deg(h)h \quad \forall h \in H.$$

Note that  $\mathcal{P}^{\Delta_0}$  is not an element of the Steenrod algebra.

The algebra  $H$  is called  $\mathcal{P}^*$ -inseparably closed, if whenever  $h \in H$  and

$$\mathcal{P}^{\Delta_i}(h) = 0 \quad \forall i \geq 0,$$

then there exists an element  $h' \in H$  such that

$$(h')^p = h.$$

The  $\mathcal{P}^*$ -inseparable closure of  $H$  is a  $\mathcal{P}^*$ -inseparably closed algebra  ${}^{\mathcal{P}^*}\sqrt{H}$  containing  $H$  such that the following universal property holds: Whenever we have a  $\mathcal{P}^*$ -inseparably closed algebra  $H'$  containing  $H$  there exists an embedding  $\varphi: {}^{\mathcal{P}^*}\sqrt{H} \hookrightarrow H'$ .

The following method to construct the  $\mathcal{P}^*$ -inseparable closure of  $H$  is taken from Section 4.1 in [3]. Denote by  $\mathcal{C} \subseteq H$  the subalgebra consisting of the  $\mathcal{P}^{\Delta_i}$ -constants for all  $i \geq 0$ , i.e.,

$$\mathcal{C} = \mathcal{C}(H) = \{h \in H \mid \mathcal{P}^{\Delta_i}(h) = 0 \forall i \geq 0\}.$$

It turns out that the subalgebra of constants  $\mathcal{C}$  is an unstable algebra over the Steenrod algebra (Lemma 4.1.2 loc.cit.). Moreover, it is Noetherian whenever  $H$  is (Lemma 4.1.1 loc.cit.). By construction we have integral extensions

$$H^p \hookrightarrow \mathcal{C} \hookrightarrow H,$$

where  $H^p = \{h^p \mid h \in H\}$ . Denote by  $\mathcal{S}$  a set of generators for  $\mathcal{C}$  as a module over  $H^p$ . Define an algebra

$$H_1 = (H \otimes_{\mathbb{F}} \mathbb{F}[\gamma_s \mid s \in \mathcal{S}]) / \mathcal{R}ad(\gamma_s^p - s \mid s \in \mathcal{S}),$$

where  $\mathcal{R}ad(-)$  denotes the radical of the ideal  $(-)$ . Note that this construction comes with a canonical inclusion

$$\varphi_0: H \hookrightarrow H_1,$$

by part (5) of Lemma 4.1.3 in [3]. Since the new algebra  $H_1$  is again an unstable reduced algebra over the Steenrod algebra (see Lemma 4.1.4 loc.cit.), we can iterate the construction and obtain a nested sequence

$$H = H_0 \hookrightarrow H_1 \hookrightarrow \dots \hookrightarrow H_i \hookrightarrow \dots \hookrightarrow$$

of unstable reduced algebras over the Steenrod algebra. The colimit of this sequence is the  $\mathcal{P}^*$ -inseparable closure of  $H$  (Proposition 4.1.5 loc.cit.). We recall the basic properties of  ${}^{\mathcal{P}^*}\sqrt{H}$  and the intermediate algebras  $H_i$ .

**Proposition 2.1.** *Consider the chain of unstable reduced algebras*

$$H = H_0 \hookrightarrow H_1 \hookrightarrow \dots \hookrightarrow H_i \hookrightarrow \dots \hookrightarrow {}^{\mathcal{P}^*}\sqrt{H}.$$

*Then the following statements hold.*

- (1) *If one of the algebras in this chain is an integral domain, then so are the others.*
- (2)  *$H \hookrightarrow {}^{\mathcal{P}^*}\sqrt{H}$  is an integral extension, and both algebras have the same Krull dimension.*
- (3) *If  $H$  is integrally closed, then so is  ${}^{\mathcal{P}^*}\sqrt{H}$ .*
- (4) *The following statements are equivalent.*
  - *$H$  is Noetherian.*
  - *$H_i$  is Noetherian.*
  - *${}^{\mathcal{P}^*}\sqrt{H}$  is Noetherian.*
  - *There exists an  $r$  such that*

$$H_r = H_{r+1} = \dots = {}^{\mathcal{P}^*}\sqrt{H}.$$

*Proof.* For (1)–(3) see Proposition 4.2.1 in [3]. For (4) see part (2) of Lemma 4.1.3, Lemma 4.2.2, Proposition 4.2.4, and Theorem 6.3.1 loc.cit. □

**Lemma 2.2.** *Let  $H$  be an integral domain. If  $H$  is integrally closed, then the algebras  $H_i$  are also integrally closed for all  $i$ .*

*Proof.* It is shown in part (5) of Proposition 4.2.1 in [3] that  ${}^{\mathcal{P}^*}\sqrt{H}$  is integrally closed whenever  $H$  is integrally closed. The same argument presented there can be used to show that also the algebras  $H_i$  are also integrally closed. □

In the same way the  $\mathcal{P}^*$ -inseparable closure of a field  $\mathbb{K}$  over the Steenrod algebra can be constructed. So we obtain a chain of fields over the Steenrod algebra

$$\mathbb{K} = \mathbb{K}_0 \hookrightarrow \mathbb{K}_1 \hookrightarrow \dots \hookrightarrow \mathbb{K}_i \hookrightarrow \dots \hookrightarrow {}^{\mathcal{P}^*}\sqrt{\mathbb{K}}$$

by adjoining successively  $p$ th roots. Again the  $p$ th powers are detected by the vanishing of the derivations  $\mathcal{P}^{\Delta_i}$ ; see Section 2.3 in [3].

Let  $H$  be an unstable integral domain over the Steenrod algebra. Denote by  $\mathbb{H}$  its field of fractions. We have seen in Proposition 4.2.6 in [3] that

$$FF({}^{\mathcal{P}^*}\sqrt{H}) = {}^{\mathcal{P}^*}\sqrt{\mathbb{H}},$$

where  $FF(-)$  denotes the field of fraction functor.

Our first goal is to refine this statement. For this we need the following result.

**Proposition 2.3.** *Let  $H$  be an unstable integral domain. Then*

$$\mathcal{C}(\mathbb{H}) = FF(\mathcal{C}(H)).$$

*Proof.* Let  $\frac{f_1}{f_2} \in \mathcal{C}(\mathbb{H})$ ,  $f_1, f_2 \in \mathbb{H}$ . Then there exists an element  $\frac{h_1}{h_2} \in \sqrt[p^*]{\mathbb{H}} = FF(\sqrt[p^*]{\mathbb{H}})$ ,  $h_1, h_2 \in \sqrt[p^*]{\mathbb{H}}$ , such that

$$\frac{h_1^{p^k}}{h_2^{p^k}} = \frac{f_1}{f_2}$$

for some  $k \in \mathbb{N}_0$ . Furthermore, since  $h_1, h_2 \in \sqrt[p^*]{\mathbb{H}}$  we can choose  $k$  such that  $h_1^{p^k}, h_2^{p^k} \in \mathbb{H}$ . By construction,  $h_1^{p^k}, h_2^{p^k}$  are in the subalgebra of constants,  $\mathcal{C}(\mathbb{H})$ . Thus

$$\frac{f_1}{f_2} = \frac{h_1^{p^k}}{h_2^{p^k}} \in FF(\mathcal{C}(\mathbb{H}))$$

which shows that

$$\mathcal{C}(\mathbb{H}) \subseteq FF(\mathcal{C}(\mathbb{H})).$$

Conversely, let  $\frac{f_1}{f_2} \in FF(\mathcal{C}(\mathbb{H}))$  with  $f_1, f_2 \in \mathcal{C}(\mathbb{H})$ . Then

$$\frac{f_1}{f_2} \in \mathbb{H}$$

is a constant because

$$\mathcal{P}^{\Delta_i} \left( \frac{f_1}{f_2} \right) = \frac{\mathcal{P}^{\Delta_i}(f_1)f_2 - f_1\mathcal{P}^{\Delta_i}(f_2)}{f_2^2} = 0$$

for all  $i \in \mathbb{N}_0$ . □

**Proposition 2.4.** *Let  $\mathbb{H}$  be an unstable integral domain over the Steenrod algebra. Denote by  $\mathbb{H}$  its field of fractions. Then for all  $i \in \mathbb{N}_0$  we have*

$$FF(\mathbb{H}_i) = \mathbb{H}_i.$$

*Proof.* By induction it is enough to show the statement for  $i = 1$ . If  $\frac{h_1}{h_2} \in FF(\mathbb{H})_1$  for  $h_1, h_2 \in \mathbb{H}_1$ , then  $\frac{h_1^p}{h_2^p} \in FF(\mathbb{H}_0) = \mathbb{H}_0 = \mathbb{H}$ . Thus  $\frac{h_1}{h_2} \in \mathbb{H}_1$ , since  $\mathbb{H}_1$  is obtained from  $\mathbb{H}_0$  by adjoining all  $p$ th roots. Thus  $FF(\mathbb{H}_1) \subseteq \mathbb{H}_1$ .

We prove the reverse inclusion. Let  $h \in \mathbb{H}_1$ . Then  $h^p \in \mathbb{H} = FF(\mathbb{H})$ . Thus by Proposition 2.3

$$h^p \in \mathcal{C}(\mathbb{H}) = FF(\mathcal{C}(\mathbb{H})).$$

Thus there exist elements  $h_1, h_2 \in \mathcal{C}(\mathbb{H})$  such that

$$h^p = \frac{h_1}{h_2}.$$

Moreover, since the elements  $h_1, h_2$  are constants they have  $p$ th roots, say  $f_1, f_2$ , in  $\mathbb{H}_1$ . Thus

$$h = \frac{f_1}{f_2} \in FF(\mathbb{H}_1)$$

and we are done. □

Hence we obtain chains

$$\begin{array}{ccccccc}
 H = H_0 & \hookrightarrow & H_1 & \hookrightarrow & \dots & \hookrightarrow & H_i & \hookrightarrow & \dots & \hookrightarrow & \sqrt[p^*]{H} \\
 \downarrow & & \downarrow & & & & \downarrow & & & & \downarrow \\
 FF(H_0) & \hookrightarrow & FF(H_1) & \hookrightarrow & \dots & \hookrightarrow & FF(H_i) & \hookrightarrow & \dots & \hookrightarrow & FF(\sqrt[p^*]{H}) \\
 \parallel & & \parallel & & & & \parallel & & & & \parallel \\
 \mathbb{H}_0 & \hookrightarrow & \mathbb{H}_1 & \hookrightarrow & \dots & \hookrightarrow & \mathbb{H}_i & \hookrightarrow & \dots & \hookrightarrow & \sqrt[p^*]{\mathbb{H}}
 \end{array}$$

Let  $H$  be an unstable Noetherian integral domain over the Steenrod algebra. Then there exists an  $r \in \mathbb{N}_0$  such that  $H_r = \sqrt[p^*]{H}$ ; see Theorem 6.1.3 and Proposition 4.2.4 in [3]. Also, there exists an  $s \in \mathbb{N}_0$  such that  $\mathbb{H}_s = \sqrt[p^*]{\mathbb{H}}$ , loc.cit. Without loss of generality we assume that  $r$  and  $s$  are minimal with respect to this property. Then by Proposition 4.2.4 in [3] we have that  $r \geq s$ . Thus for Noetherian unstable algebras we obtain *finite* chains

$$\begin{array}{ccccccc}
 H = H_0 & \hookrightarrow & H_1 & \hookrightarrow & \dots & \hookrightarrow & H_s & \hookrightarrow & H_{s+1} & \hookrightarrow & \dots & \hookrightarrow & H_r = \sqrt[p^*]{H} \\
 \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
 FF(H_0) & \hookrightarrow & FF(H_1) & \hookrightarrow & \dots & \hookrightarrow & FF(H_s) & = & FF(H_{s+1}) & = & \dots & = & FF(H_r) \\
 \parallel & & \parallel & & & & \parallel & & \parallel & & & & \parallel \\
 \mathbb{H}_0 & \hookrightarrow & \mathbb{H}_1 & \hookrightarrow & \dots & \hookrightarrow & \mathbb{H}_s & = & \mathbb{H}_{s+1} & = & \dots & = & \mathbb{H}_r = \sqrt[p^*]{\mathbb{H}}
 \end{array}$$

**Corollary 2.5.** *Let  $H$  be an unstable Noetherian integral domain. Let  $H$  be integrally closed. Then with the preceding notation,  $r = s$ .*

*Proof.* By Proposition 2.4,  $FF(H_i) = \mathbb{H}_i$ . Moreover,  $H_i$  is integrally closed for all  $i$  by Lemma 2.2. Thus for all  $i$  the unstable part of  $FF(H_i)$  is

$$\mathcal{U}_n(FF(H_i)) = H_i;$$

see Theorem 2.4 in [4]. Thus

$$H_s = \mathcal{U}_n(FF(H_s)) = \mathcal{U}_n(FF(H_r)) = H_r$$

as desired. □

### 3. INSEPARABLE EXTENSIONS OF EXPONENT 1

Let  $H$  be an unstable Noetherian reduced algebra over the Steenrod algebra. Define the  $H$ -module

$$\text{Der}_H = \text{span}_H \{ \mathcal{P}^{\Delta_i} \mid i \in \mathbb{N}_0 \}.$$

Then  $\text{Der}_H$  is free as a module over  $H$ ; see Proposition 1.1.7 and Theorem 1.2.1 in [3].<sup>3</sup> Moreover it is a restricted Lie algebra of derivations acting on  $H$ ; cf. Section 2.4 in [3]. We denote by

$$\mathcal{C}_{\text{Der}_H}(H) = \{ h \in H \mid \mathcal{P}^{\Delta_i}(h) = 0 \ \forall i \} \subseteq H$$

the subalgebra of constants with respect to the derivations in  $\text{Der}_H$ .

<sup>3</sup>The module  $\text{Der}_H$  is the module  $\Delta(H)$  in this reference.

*Remark.* In Section 2 we called the subalgebra of constants just  $\mathcal{C} = \mathcal{C}(H)$ . For what follows however, we need to keep track of the module of derivations that is used.

Clearly,

$$H^p \subseteq \mathcal{C}_{\text{Der}_H}$$

as  $\mathcal{P}^{\Delta_i}(h^p) = 0$  for all  $h \in H$  and  $i \in \mathbb{N}_0$ . Since the extension

$$H^p \subseteq H$$

is purely inseparable of exponent one, so is the extension

$$H^p \subseteq \mathcal{C}_{\text{Der}_H}(H).$$

Thus

$${}^{\mathcal{P}^*}\sqrt{H^p} = {}^{\mathcal{P}^*}\sqrt{\mathcal{C}_{\text{Der}_H}(H)} = {}^{\mathcal{P}^*}\sqrt{H}.$$

**Proposition 3.1.** *Let  $H$  be an unstable reduced algebra over the Steenrod algebra. Then  $H$  is  $\mathcal{P}^*$ -inseparably closed if and only if*

$$\mathcal{C}_{\text{Der}_H}(H) = H^p.$$

*Proof.* Assume that  $H$  is  $\mathcal{P}^*$ -inseparably closed. By what we have done so far we know that  $H^p \subseteq \mathcal{C}_{\text{Der}_H}(H)$ . To prove the reverse inclusion, let  $h \in \mathcal{C}_{\text{Der}_H}(H)$ . Then  $\mathcal{P}^{\Delta_i}(h) = 0$  for all  $i \in \mathbb{N}_0$ . Thus there exists an element

$$f \in {}^{\mathcal{P}^*}\sqrt{\mathcal{C}_{\text{Der}_H}(H)} = {}^{\mathcal{P}^*}\sqrt{H} = H$$

such that  $f^{p^k} = h$ . Since  $f^{p^k} \in H^p$  for all  $f \in H$ , we have  $h = f^{p^k} \in H^p$ , and hence  $H^p = \mathcal{C}_{\text{Der}_H}(H)$ . Conversely, assume that

$$H^p = \mathcal{C}_{\text{Der}_H}(H) \subseteq H \subseteq {}^{\mathcal{P}^*}\sqrt{H}.$$

Let  $h \in {}^{\mathcal{P}^*}\sqrt{H}$ . Thus there exists a  $k \in \mathbb{N}_0$  such that  $h^{p^k} \in H$ . Since  $\text{Der}_H$  vanishes on  $p$ th powers we find that

$$h^{p^k} \in \mathcal{C}_{\text{Der}_H}(H) = H^p.$$

Thus  $h^{p^{k-1}} \in H$ . Iteratively we find that  $h \in H$ , i.e.,  $H$  is  $\mathcal{P}^*$ -inseparably closed.  $\square$

**Corollary 3.2.** *We have*

$$\mathcal{C}_{\text{Der}_{\mathbb{F}[V]}}(\mathbb{F}[V]) = \mathbb{F}[V]^p = \mathbb{F}[x_1^p, \dots, x_n^p].$$

*Proof.* Since  $\mathbb{F}[V]$  is  $\mathcal{P}^*$ -inseparably closed (see Corollary 4.2.8 in [3]), this result is an immediate corollary of Proposition 3.1.  $\square$

We recall some facts about  $\text{Der}_H$  and its action on  $H$ . First the Lie algebra structure is particularly simple, namely

$$(\ddagger) \quad [\mathcal{P}^{\Delta_i}, \mathcal{P}^{\Delta_j}] = \begin{cases} 0 & \text{if } i, j > 0, \\ \mathcal{P}^{\Delta_i} & \text{if } i \neq 0 \text{ and } j = 0, \\ -\mathcal{P}^{\Delta_j} & \text{if } i = 0 \text{ and } j \neq 0 \end{cases}$$

(see the remark on page 12 of [3]), and

$$(\S) \quad (\mathcal{P}^{\Delta_i})^p = 0$$

(see Section 2.4 in [3]). The  $\Delta$ -length of  $H$  is defined to be the smallest integer<sup>4</sup>  $\lambda_H$  such that the derivation

$$h_0 \mathcal{P}^{\Delta_{i_0}} + \dots + h_\lambda \mathcal{P}^{\Delta_{i_\lambda}}$$

vanishes on  $H$  for some  $h_0, \dots, h_\lambda \in H$  and  $i_0, \dots, i_\lambda \in \mathbb{N}_0$  (see Section 1.2 in [3]).<sup>5</sup> In this case, any  $\lambda + 1$  derivations are linearly dependent over  $H$  (Proposition 1.1.7 in [3]). The  $\Delta$ -length  $\lambda_H$  is at most the Krull dimension of  $H$  over  $\mathbb{F}$  (cf. Corollary 1.2.2 in [3]). Moreover, the coefficients can be chosen to be

$$h_i = (-1)^i \mathbf{d}_{\lambda,i}$$

(up to a sign) the Dickson classes in dimension  $\lambda$  (see Theorems 5.1.9 and 5.2.1 in [3]). Note that by convention  $\mathbf{d}_{\lambda,\lambda} = 1$ . Then the *normalized* equation

$$(\mathbf{d}_{\lambda,0} \mathcal{P}^{\Delta_0} - \dots + (-1)^\lambda \mathbf{d}_{\lambda,\lambda} \mathcal{P}^{\Delta_\lambda})(h) = 0 \quad \forall h \in H$$

is called *the  $\Delta$ -relation* of  $H$ . By abuse of notation, we also call the element

$$\mathbf{d}_H = \mathbf{d}_{\lambda,0} \mathcal{P}^{\Delta_0} - \dots + (-1)^\lambda \mathbf{d}_{\lambda,\lambda} \mathcal{P}^{\Delta_\lambda} \in \text{Der}_H$$

*the  $\Delta$ -relation* for  $H$ .<sup>6</sup> Finally we note that the  $\Delta$ -length of  $H$  is equal to its Krull dimension if  $H$  is  $\mathcal{P}^*$ -inseparably closed (cf. Theorem 8.1.5 in [3]). The converse is not quite true as the following example taken from Section 7.4 in [3] shows.

**Example 3.3.** Consider the field  $\mathbb{F} = \mathbb{F}_2$  with two elements, and take a polynomial algebra in two linear generators  $x, y, \mathbb{F}[x, y]$ . The Dickson algebra in this case is

$$\mathcal{D}(2) = \mathbb{F}[x^2y + xy^2, x^2 + y^2 + xy] \hookrightarrow \mathbb{F}[x, y].$$

Define an intermediate algebra  $H$  by

$$H = \mathbb{F}[x^2 + y^2, xy, xy(x + y)] / ((x^2 + y^2)(xy) + (xy(x + y))^2).$$

Then  $H$  is an unstable integral domain, but it is not  $\mathcal{P}^*$ -inseparably closed because

$$\mathcal{P}^{\Delta_i}(x^2 + y^2) = 0 \quad \forall i \geq 0, \text{ but } x + y \notin H.$$

However, its  $\Delta$ -relation

$$\mathbf{d}_H = \mathbf{d}_{2,0} \mathcal{P}^{\Delta_0} - \mathbf{d}_{2,1} \mathcal{P}^{\Delta_1} + \mathcal{P}^{\Delta_2}$$

has length 2, which is equal to its Krull dimension. Note that the field of fractions of  $H$ ,

$$FF(H) = \mathbb{F}(x + y, xy),$$

is inseparably closed. Therefore the algebra  $H$  is not integrally closed because  $x + y \notin H$  (cf. Corollary 2.5).

**Proposition 3.4.** *Let  $H$  be an unstable Noetherian integral domain. If the  $\Delta$ -length  $\lambda_H$  is equal to the Krull dimension  $n$  of  $H$ , then*

$$\sqrt[\mathcal{P}^*]{\overline{H}} \subseteq \overline{H},$$

where  $\overline{(-)}$  denotes the integral closure.

<sup>4</sup>If there is no possible confusion we will omit the subscript and just write  $\lambda = \lambda_H$ .

<sup>5</sup>If  $\lambda_H \in \mathbb{N}_0$  exists, then  $H$  is called  *$\Delta$ -finite*. This is a weaker condition than Noetherianity. For example the polynomial algebra  $\mathbb{F}[x_1^p, x_2^p, \dots]$  in infinitely many generators has  $\Delta$ -length zero, but it is not Noetherian.

<sup>6</sup>We will suppress the subscript, and write  $\mathbf{d}$  for  $\mathbf{d}_H$  if no confusion is possible.

*Proof.* Since the  $\Delta$ -length of  $H$  is equal to its Krull dimension, we have integral extensions

$$\mathcal{D}(n) \hookrightarrow H \hookrightarrow {}^{p^*}\sqrt{H} \hookrightarrow \mathbb{F}[V]$$

by the Little Imbedding Theorem (Theorem 7.4.4 in [3]) and the Embedding Theorem (Corollary 6.1.5 loc.cit.). The corresponding extensions of the field of fractions are Galois extensions, so in particular separable. Since  $FF(H) \subseteq FF({}^{p^*}\sqrt{H})$  is also purely inseparable, we have  $FF({}^{p^*}\sqrt{H}) = FF(H) = FF(\overline{H})$ . Thus  $FF(H) = \mathbb{F}(V)^G$  for some group  $G \leq GL(V)$ . Therefore,  $\overline{H} = \mathbb{F}[V]^G$  is inseparably closed. Hence  ${}^{p^*}\sqrt{H} \hookrightarrow \overline{H}$  by the universal property of the inseparable closure.  $\square$

*Remark.* Note that it follows from the preceding result that if  $H$  is integrally closed and the  $\Delta$ -length is equal to its Krull dimension, then

$${}^{p^*}\sqrt{H} = \overline{H} = H.$$

We want to investigate purely inseparable extensions  $H \hookrightarrow \mathbb{F}[V]$  of exponent one, i.e., we have

$$\mathbb{F}[V]^p \hookrightarrow H \hookrightarrow \mathbb{F}[V].$$

For this we turn our attention to the corresponding extensions of fields of fractions

$$\mathbb{F}(V)^p \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{F}(V).$$

Let  $\text{Der}_{\mathbb{H}}$  be the vector space over  $\mathbb{H}$  generated by the elements  $\mathcal{P}^{\Delta_i}$  for  $i \in \mathbb{N}_0$ . Since the relations  $(\blackboxtimes)$  and  $(\blackboxplus)$  are intrinsic of the Steenrod algebra, the vector space  $\text{Der}_{\mathbb{H}}$  is also a restricted Lie algebra. Thus any vector subspace of  $\text{Der}_{\mathbb{H}}$  is a restricted Lie subalgebra and vice versa.

**Proposition 3.5.** *Let  $H$  be an unstable integral domain and  $\mathbb{H}$  its field of fractions. The vector space  $\text{Der}_{\mathbb{H}}$  satisfies the following properties.*

- (1) *The elements  $\mathcal{P}^{\Delta_i}$  are derivations on  $\mathbb{H}$ .*
- (2) *The  $\Delta$ -relation  $\mathbb{H}$  is well defined, and coincides with the  $\Delta$ -relation on  $H$ . In particular, the  $\Delta$ -lengths are equal.*

*Proof.* AD(1): The action of  $\mathcal{P}^{\Delta_i}$  on  $\mathbb{H}$  is given by the formula

$$\mathcal{P}^{\Delta_i} \left( \frac{f_1}{f_2} \right) = \frac{\mathcal{P}^{\Delta_i}(f_1)f_2 - f_1\mathcal{P}^{\Delta_i}(f_2)}{f_2^2}$$

for any  $f_1, f_2 \in \mathbb{H}$ . Thus they are derivations on the field of fractions also.

AD(2): The set  $\text{Der}_{\mathbb{H}}$  is a vector space by construction. Let  $\lambda_{\mathbb{H}}$  be its dimension. Then any  $\lambda_{\mathbb{H}} + 1$  elements are linearly independent. Thus the  $\Delta$ -length is  $\lambda_{\mathbb{H}}$  with  $\Delta$ -relation

$$d = f_0\mathcal{P}^{\Delta_0} + \dots + f_{\lambda}\mathcal{P}^{\Delta_{\lambda}}.$$

Without loss of generality we can assume that the coefficients  $f_i \in \mathbb{H}$  for all  $i$ . Thus  $\lambda_{\mathbb{H}}$  is at least equal to the  $\Delta$ -length,  $\lambda_H$ , of  $H$ . On the other hand, if  $\mathbf{d}_H$  is a  $\Delta$ -relation for  $H$ , then by

$$\mathbf{d}_H \left( \frac{f_1}{f_2} \right) = \frac{\mathbf{d}_H(f_1)f_2 - f_1\mathbf{d}_H(f_2)}{f_2^2} = 0$$

$\mathbf{d}_H$  vanishes also on  $\mathbb{H}$ . Thus  $\lambda_{\mathbb{H}} \leq \lambda_H$ . Therefore  $\lambda_{\mathbb{H}} = \lambda_H$  and  $\mathbf{d}_{\mathbb{H}} = \mathbf{d}_H$ .  $\square$

**Corollary 3.6.** *Let  $H$  be an unstable integral domain and  $\overline{H}$  its integral closure. Then*

$$\lambda_H = \lambda_{\overline{H}} \quad \text{and} \quad \mathbf{d}_H = \mathbf{d}_{\overline{H}}.$$

*Proof.* By Proposition 3.5, part (2),  $\Delta$ -lengths, as well as the  $\Delta$ -relations of  $H$  and its field of fractions, coincide. Since  $H$  and  $\overline{H}$  have the same field of fractions we are done.  $\square$

**Lemma 3.7.** *Let  $H' \subseteq H$  be unstable reduced Noetherian algebras over the Steenrod algebra. Denote by  $\lambda_{H'}$ , resp.  $\lambda_H$ , the  $\Delta$ -length of  $H'$ , resp.  $H$ . Then  $\lambda_{H'} \leq \lambda_H$ .*

*Proof.* By Corollary 3.6 the  $\Delta$ -length and  $\Delta$ -relation of an unstable algebra  $H$  and its integral closure  $\overline{H}$  are equal. Thus without loss of generality we assume that  $H'$  and  $H$  are integrally closed.

Denote by  $\mathcal{D}(l)$  the Dickson algebra of dimension  $l$ . By Theorem 5.1.9 in [3]

$$\mathcal{D}(\lambda_H) \hookrightarrow \mathbb{H}$$

is a maximal Dickson algebra in  $\mathbb{H}$ . Applying the same theorem for  $H'$  gives

$$\mathcal{D}(\lambda_{H'}) \hookrightarrow \mathbb{H}' \hookrightarrow \mathbb{H}.$$

Since  $\mathcal{D}(\lambda_H)$  is the maximal Dickson algebra in  $\mathbb{H}$ , we find that  $\lambda_{H'} \leq \lambda_H$  as desired.  $\square$

**Lemma 3.8.** *Let  $U$  and  $W$  be finite dimensional vector spaces over  $\mathbb{F}$ . We note that the  $\Delta$ -length  $\lambda$  of  $\mathbb{F}[U] \otimes \mathbb{F}[W]^{p^t}$ ,  $t > 0$ , is equal to the vector space dimension of  $U$  with  $\Delta$ -relation*

$$\mathbf{d} = \mathbf{d}_{\lambda,0} \mathcal{P}^{\Delta_0} - \dots + (-1)^\lambda \mathbf{d}_{\lambda,\lambda} \mathcal{P}^{\Delta_\lambda},$$

where  $\mathbb{F}[U]^{\text{GL}(\lambda, \mathbb{F})} = \mathbb{F}[\mathbf{d}_{\lambda,0}, \dots, \mathbf{d}_{\lambda,\lambda-1}]$ .

*Proof.* The element  $\mathbf{d}$  is a  $\Delta$ -relation for  $\mathbb{F}[U]$  by Theorem 1.2.3 in [3]. Since  $\mathcal{P}^{\Delta_i}$  vanishes on  $p$ th powers for all  $i \in \mathbb{N}_0$ , the element  $\mathbf{d}$  vanishes on  $\mathbb{F}[U] \otimes \mathbb{F}[W]^{p^t}$ . So,  $\lambda \leq \dim_{\mathbb{F}}(U)$ .

On the other hand,  $\mathbb{F}[U] \hookrightarrow \mathbb{F}[U] \otimes \mathbb{F}[W]^{p^t}$ . Therefore by Lemma 3.7 the  $\Delta$ -length is at least  $\dim_{\mathbb{F}}(U)$ , and we are done.  $\square$

**Corollary 3.9.** *The  $\Delta$ -length  $\lambda$  of  $\mathbb{F}(U) \otimes \mathbb{F}(W)^{p^t}$  is equal to the vector space dimension of  $U$ , for  $t \geq 1$ . The subfield of constants is*

$$\mathcal{C}_{\text{Der}_{\mathbb{F}(U) \otimes \mathbb{F}(W)^{p^t}}}(\mathbb{F}(U) \otimes \mathbb{F}(W)^{p^t}) = \mathbb{F}(U)^p \otimes \mathbb{F}(W)^{p^t}.$$

Moreover, the  $\Delta$ -relation is

$$\mathbf{d} = \mathbf{d}_{\lambda,0} \mathcal{P}^{\Delta_0} + \dots + (-1)^\lambda \mathbf{d}_{\lambda,\lambda} \mathcal{P}^{\Delta_\lambda}.$$

*Proof.* This is immediate from part (2) of Proposition 3.5, Lemma 3.8, and Corollary 3.2.  $\square$

Since the  $\Delta$ -relation of  $\mathbb{F}(V)$  has length  $n = \dim_{\mathbb{F}}(V)$ , we have  $\dim_{\mathbb{F}(V)}(\text{Der}_{\mathbb{F}(V)}) = n$  and

$$\text{Der}_{\mathbb{F}(V)} = \text{span}_{\mathbb{F}(V)}\{\mathcal{P}^{\Delta_0}, \dots, \mathcal{P}^{\Delta_{n-1}}\}.$$

Moreover, the index over the subfield of constants is

$$[\mathbb{F}(V) : \mathbb{F}(V)^p] = p^n.$$

Thus we can apply the structure theorem for purely inseparable extensions of exponent one (see, e.g., Chapter IV, Section 8 in [1]). It tells us that

$$\mathbb{H} \subseteq \mathbb{F}(V)$$

is a purely inseparable extension of exponent one if and only if there exists a restricted Lie subalgebra  $D \subseteq \text{Der}_{\mathbb{F}}(V)$  such that

$$\mathbb{H} = \mathcal{C}_D(\mathbb{F}(V)).$$

So, take a subspace  $D \subseteq \text{Der}_{\mathbb{F}}(V)$ . We recall from Corollary 3.9 that

$$\mathbb{F}(V) = \mathcal{C}_D(\mathbb{F}(V)) \quad \text{for } D = 0$$

and

$$\mathbb{F}(V)^p = \mathcal{C}_D(\mathbb{F}(V)) \quad \text{for } D = \text{Der}_{\mathbb{F}}(V).$$

Thus we are left to characterize those  $D \subseteq \text{Der}_{\mathbb{F}}(V)$  such that  $\mathbb{H} = \mathcal{C}_D(\mathbb{F}(V))$  carries a  $\mathcal{P}^*$ -module structure.

**Proposition 3.10.** *Let  $\mathbb{K} \hookrightarrow \mathbb{F}(V)$  be a field over the Steenrod algebra. Let  $\mathbf{d}$  be the  $\Delta$ -relation of  $\mathbb{K}$  with  $\Delta$ -length  $\lambda$ . Then the vector space of derivations vanishing on  $\mathbb{K}$ ,*

$$D_{\mathbb{K}} = \text{span}_{\mathbb{F}}\{d \in \text{Der}_{\mathbb{F}}(V) \mid d|_{\mathbb{K}} = 0\},$$

*has dimension  $n - \lambda$ , where  $n = \dim_{\mathbb{F}}(V)$ .*

*Proof.* The  $\Delta$ -relation of  $\mathbb{K}$  is

$$\mathbf{d} = \mathbf{d}_{\lambda,0} \mathcal{P}^{\Delta_0} + \dots + (-1)^\lambda \mathbf{d}_{\lambda,\lambda} \mathcal{P}^{\Delta_\lambda}.$$

By Proposition 1.1.7 in [3] any  $\lambda + 1$  derivations in  $\text{Der}_{\mathbb{F}}(V)$  are linearly dependent. Moreover by Lemma 1.1.8 loc.cit. we find that in particular the  $n - \lambda$  elements

$$\mathbf{d}_i = \mathbf{d}_{\lambda,0}^{q^i} \mathcal{P}^{\Delta_i} + \dots + (-1)^\lambda \mathbf{d}_{\lambda,\lambda}^{q^i} \mathcal{P}^{\Delta_{\lambda+1}}$$

for  $i = 0, \dots, n - \lambda - 1$  vanish on  $\mathbb{K}$ . Since the  $\mathbf{d}_i$ 's are linearly independent in  $\text{Der}_{\mathbb{F}}(V)$  we have that

$$\dim(D_{\mathbb{K}}) \geq n - \lambda.$$

On the other hand, if  $d \in D_{\mathbb{K}}$ , then

$$d = f_0 \mathcal{P}^{\Delta_0} + \dots + f_{n-1} \mathcal{P}^{\Delta_{n-1}}.$$

Then there are  $k_0, \dots, k_{n-1-\lambda}$  such that

$$(*) \quad d - \sum_{i=0}^{n-1-\lambda} k_i \mathbf{d}_i = f'_0 \mathcal{P}^{\Delta_0} + \dots + f'_{\lambda-1} \mathcal{P}^{\Delta_{\lambda-1}}$$

for some  $f'_0, \dots, f'_{\lambda-1} \in \mathbb{K}$ . Thus if  $d$  were linearly independent of the  $\mathbf{d}_i$ 's, then the expression (\*) is not zero. This in turn means that there is a relation on  $\mathbb{K}$  shorter than the  $\Delta$ -relation. This is a contradiction. Therefore  $\dim(D_{\mathbb{K}}) = n - \lambda$ .  $\square$

**Theorem 3.11.** *The extension  $\mathbb{H} \subseteq \mathbb{F}(V)$  is a purely inseparable extension of exponent one of fields over the Steenrod algebra if and only if*

$$\mathbb{H} = \mathbb{F}(x_1, \dots, x_k, x_{k+1}^p, \dots, x_n^p) = \mathbb{F}(U) \otimes \mathbb{F}(V/U)^p$$

*for some  $k \in \{1, \dots, n\}$  and  $\dim(U) = k$ . Furthermore, in this case*

$$\mathbb{H} = \mathcal{C}_D(\mathbb{F}(V))$$

*where  $D$  has vector space dimension  $n - k$ . If  $k < n$ , then  $D$  is generated by the  $\Delta$ -relation of  $\mathbb{H}$ ,*

$$\mathbf{d}_{\mathbb{H}} = \mathbf{d}_{k,0} \mathcal{P}^{\Delta_0} + \dots + (-1)^k \mathbf{d}_{k,k} \mathcal{P}^{\Delta_k}$$

and its translates

$$\mathbf{d}_i = \mathbf{d}_{k,0}^{q^i} \mathcal{P}^{\Delta_i} + \dots + (-1)^k \mathbf{d}_{k,k}^{q^i} \mathcal{P}^{\Delta_{k+i}}$$

for  $i = 1, \dots, n - k - 1$ .

*Proof.* If

$$\mathbb{H} = \mathbb{F}(x_1, \dots, x_k, x_{k+1}^p, \dots, x_n^p),$$

then it is clearly a field over the Steenrod algebra. Moreover,

$$\mathbb{F}(x_1, \dots, x_k, x_{k+1}^p, \dots, x_n^p) = \mathcal{C}_D(\mathbb{F}(V))$$

for  $D$  generated by the  $\Delta$ -relation of  $\mathbb{H}$  and its translates  $\mathbf{d}_i$  of length  $\lambda_{\mathbb{H}} = k$  (see Corollary 3.9 and Proposition 3.10).

We prove the converse. Set  $\lambda = \lambda_{\mathbb{H}}$ . Let  $\mathbf{d}$  be the  $\Delta$ -relation of  $\mathbb{H}$ . Then

$$\mathbf{d} = \mathbf{d}_{\lambda,0} \mathcal{P}^{\Delta_0} + \dots + (-1)^\lambda \mathbf{d}_{\lambda,\lambda} \mathcal{P}^{\Delta_\lambda}$$

vanishes on  $\mathbb{H}$ . Let  $U \leq V$  be a vector subspace of dimension  $\lambda$ . We also note that the field  $\mathbb{F}(U) \otimes \mathbb{F}(V/U)^p$  has  $\Delta$ -relation  $\mathbf{d}$  and  $\Delta$ -length  $\lambda$  by Corollary 3.9. Certainly,

$$\mathbb{F}(V)^p \hookrightarrow \mathbb{F}(U) \otimes \mathbb{F}(V/U)^p \hookrightarrow \mathbb{F}(V)$$

is a purely inseparable extension of exponent one. We show that  $\mathbb{F}(U) \otimes \mathbb{F}(V/U)^p \hookrightarrow \mathbb{H}$ . Since  $\mathbb{H} \hookrightarrow \mathbb{F}(V)$  is purely inseparable of exponent one, we have

$$\mathbb{F}(V/U)^p \hookrightarrow \mathbb{H}.$$

Since the coefficients of the  $\Delta$ -relation are the Dickson classes, we know that  $FF(\mathcal{D}(\lambda)) \hookrightarrow \mathbb{H}$ . Thus

$$FF(\mathcal{D}(\lambda)) \otimes \mathbb{F}(V/U)^p \hookrightarrow \mathbb{H}.$$

Since  $\mathbb{H} \hookrightarrow \mathbb{F}(V)$  is purely inseparable, we find that the separable closure of  $FF(\mathcal{D}(\lambda)) \otimes \mathbb{F}[V/U]^p$  is in  $\mathbb{H}$ . This in turn is just

$$\mathbb{F}(U) \otimes \mathbb{F}(V/U)^p \hookrightarrow \mathbb{H}.$$

Obviously  $|\mathbb{F}(V) : \mathbb{F}(U) \otimes \mathbb{F}(V/U)^p| = p^{n-\lambda}$ . By Theorem 19 on page 186 of [1] we have that also

$$|\mathbb{F}(V) : \mathbb{H}| = p^{n-\lambda}$$

because  $\mathcal{D}_{\mathbb{H}}$  has dimension  $n - \lambda$  (Proposition 3.10). Hence  $\mathbb{H} = \mathbb{F}(U) \otimes \mathbb{F}(V/U)^p$  as desired. □

**Corollary 3.12.** *Let  $H \subseteq \mathbb{F}[V]$  be a purely inseparable extension of exponent one. Let  $H$  be integrally closed. Then  $H$  is an unstable algebra over the Steenrod algebra if and only if  $H = \mathbb{F}[x_1, \dots, x_\lambda, x_{\lambda+1}^p, \dots, x_n^p]$ , where  $\lambda = \lambda_H$  is the  $\Delta$ -length of  $H$ .*

*Proof.* If  $H$  is an unstable algebra over the Steenrod algebra, then  $\mathbb{H}$  is a field over the Steenrod algebra. Moreover, since  $H \hookrightarrow \mathbb{F}[V]$  has exponent one, so has the extension  $\mathbb{H} \hookrightarrow \mathbb{F}(V)$ . Thus  $\mathbb{H} = \mathbb{F}(U) \otimes \mathbb{F}(V/U)^p$  for  $\dim_{\mathbb{F}}(U) = \lambda$  by Theorem 3.11. Hence by Theorem 2.4 in [4]

$$H = \overline{H} = \mathcal{U}n(\mathbb{H}) = \mathbb{F}[U] \otimes \mathbb{F}[V/U]^p.$$

Conversely, the algebra  $\mathbb{F}[U] \otimes \mathbb{F}[V/U]^p$  is certainly an unstable algebra over the Steenrod algebra. □

*Remark.* For any unstable integral domain  $H$  its integral closure  $\overline{H}$  also carries an unstable  $\mathcal{P}^*$ -module structure because  $\overline{H} = \mathcal{U}n(\mathbb{H})$  (see Theorem 2.4 in [4]). The converse is not true as we illustrate with the next example.

**Example 3.13.** Let  $\mathbb{F}$  be the prime field of characteristic 2 and let  $A$  be the subalgebra of  $\mathbb{F}[x, y]$  generated by  $x, xy, y^3$ . Then  $A \hookrightarrow \mathbb{F}[x, y]$  is an integral extension. Moreover,  $FF(A) = \mathbb{F}(x, y)$ . Therefore  $\overline{A} = \mathbb{F}[x, y]$  is an unstable algebra over the Steenrod algebra. However  $A$  does not carry a  $\mathcal{P}^*$ -module structure because

$$\mathcal{P}^1(xy) = x^2y + xy^2 \notin A,$$

as the only elements of degree 3 in  $A$  are  $x^3, x^2, y^3$ .

Thus the assumption  $H = \overline{H}$  cannot be dropped in the preceding result.

**Corollary 3.14.** *Let  $U \leq V$ . Denote  $m = \dim_{\mathbb{F}}(U) \leq n = \dim_{\mathbb{F}}(V)$ . Then*

$$\mathbb{F}[U] \otimes \mathbb{F}[V/U]^p \hookrightarrow \mathbb{F}[V]$$

*is the largest unstable subalgebra with  $\Delta$ -length equal to  $m$ .*

*Proof.* Certainly,  $\mathbb{F}[U] \otimes \mathbb{F}[V/U]^p$  has  $\Delta$ -length  $m$ . Let

$$(\star) \quad \mathbb{F}[U] \times \mathbb{F}[V/U]^p \hookrightarrow H \hookrightarrow \mathbb{F}[V]$$

be an intermediate unstable algebra with  $\lambda_H = m$ . Since the extension  $(\star)$  is purely inseparable of exponent one, we have that

$$H = \mathbb{F}(U') \otimes \mathbb{F}(V/U')^p$$

for some  $U' \geq U$ . But

$$\dim_{\mathbb{F}}(U) = m = \lambda_H = \dim_{\mathbb{F}}(U')$$

and therefore  $U = U'$ . Hence

$$\mathbb{F}[U] \otimes \mathbb{F}[V/U]^p \subseteq H \subseteq \mathcal{U}n(H) = \overline{H} = \mathbb{F}[U] \otimes \mathbb{F}[V/U]^p$$

gives the desired result. □

#### 4. PURELY INSEPARABLE EXTENSIONS OF ARBITRARY EXPONENT

In this section we proceed with the investigation of the purely inseparable extension

$$H \hookrightarrow \mathbb{F}[V].$$

We consider the general case of exponent  $e \geq 1$ . Thus we need to detect  $p^s$ th powers for  $s = 1, \dots, e$ . We introduce the following operators for  $s \in \mathbb{N}_0$ :

$$\begin{aligned} \mathcal{P}^{\Delta_{s,0}} &= \frac{1}{p^s} \text{deg}(-)\text{id}(-), \\ \mathcal{P}^{\Delta_{s,1}} &= \mathcal{P}^{p^s}, \\ \mathcal{P}^{\Delta_{s,i}} &= \mathcal{P}^{p^s} q^{i-1} \mathcal{P}^{\Delta_{s,i-1}} - \mathcal{P}^{\Delta_{s,i-1}} \mathcal{P}^{p^s} q^{i-1} \quad \text{for } i \geq 2. \end{aligned}$$

*Remark.* Note that for all  $s \in \mathbb{N}_0$  we have  $\mathcal{P}^{\Delta_{s,i}} \in \mathcal{P}^*$  whenever  $i \neq 0$ .

*Remark.* Note also that the degree of  $\mathcal{P}^{\Delta_{s,i}}$  is equal to  $q^i p^s - p^s$ , for all  $i, s \in \mathbb{N}_0$ .

**Proposition 4.1.** *The operators  $\mathcal{P}^{\Delta_{s,i}}$  satisfy the following properties:*

- (1) *For all  $i \in \mathbb{N}$  we have  $\mathcal{P}^{\Delta_{s,i}}(h^p) = (\mathcal{P}^{\Delta_{s-1,i}}(h))^p$  for  $h \in H$  and  $s \geq 1$ .*
- (2) *For  $i \geq 1$  and  $k, s \geq 0$  we have*

$$[\mathcal{P}^{p^s k}, \mathcal{P}^{\Delta_{s,i}}] = \mathcal{P}^{\Delta_{s,i+1}} \mathcal{P}^{p^s k - p^s q^i}.$$

- (3) *For  $i, j \geq 1$  and  $s \geq 0$  we have*

$$[\mathcal{P}^{\Delta_{s,i}}, \mathcal{P}^{\Delta_{s,j}}] = 0.$$

- (4) *The  $p$ th iteration  $\mathcal{P}^{\Delta_{s,i}} \dots \mathcal{P}^{\Delta_{s,i}} = 0$  for all  $i \geq 1$  and  $s \geq 0$ .*

*Proof.* AD(1): For any  $i, j \geq 0$  and any linear form  $l$  we have

$$\mathcal{P}^i(l^j) = \binom{j}{i} l^{iq+j-i}$$

as it can be easily seen by induction. Thus for all  $i \geq 0$  we have

$$\mathcal{P}^i(l^{p^s}) = \binom{p^s}{i} l^{iq+p^s-i}.$$

Since  $\binom{p^s}{i} \equiv 0 \pmod{p}$  precisely when  $i \notin \{p^s, 0\}$ , we have

$$\mathcal{P}^i(l^{p^s}) = \begin{cases} \mathcal{P}^0(l)^{p^s} & \text{for } i = 0, \\ \mathcal{P}^1(l)^{p^s} & \text{for } i = p^s, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\mathcal{P}^i(h^{p^s}) = \begin{cases} h^{p^s} & \text{for } i = 0, \\ \mathcal{P}^k(h)^{p^s} & \text{for } i = kp^s, \\ 0 & \text{otherwise,} \end{cases}$$

for any  $h \in H$  (cf. page 261 of [6]). Thus

$$\mathcal{P}^{\Delta_{s,1}}(h^p) = \mathcal{P}^{p^s}(h^p) = (\mathcal{P}^{p^{s-1}}(h))^p = (\mathcal{P}^{\Delta_{s-1,1}}(h))^p.$$

Thus by induction on  $i$  we find

$$\begin{aligned} \mathcal{P}^{\Delta_{s,i}}(h^p) &= \mathcal{P}^{p^s q^{i-1}} \mathcal{P}^{\Delta_{s,i-1}}(h^p) - \mathcal{P}^{\Delta_{s,i-1}} \mathcal{P}^{p^s q^{i-1}}(h^p) \\ &= \mathcal{P}^{p^s q^{i-1}} (\mathcal{P}^{\Delta_{s-1,i-1}}(h))^p - \mathcal{P}^{\Delta_{s,i-1}} (\mathcal{P}^{p^{s-1} q^{i-1}}(h))^p \\ &= (\mathcal{P}^{p^{s-1} q^{i-1}} \mathcal{P}^{\Delta_{s-1,i-1}}(h) - \mathcal{P}^{\Delta_{s-1,i-1}} \mathcal{P}^{p^{s-1} q^{i-1}}(h))^p \\ &= (\mathcal{P}^{\Delta_{s-1,i}}(h))^p, \end{aligned}$$

as claimed.

AD(2) and (3): The result follows, because it is true for any linear form.

AD(4): From the Adem relations it follows that

$$\mathcal{P}^{p^s} \dots \mathcal{P}^{p^s} = 0.$$

Thus the result follows by induction on  $i$  with the help of the commutation rules of (2) (cf. Lemma A.1.1 in [3]). □

Define the  $H^{p^s}$ -module

$$\text{Der}_{H,p^s} = \text{span}_{H^{p^s}} \{ \mathcal{P}^{\Delta_{s,i}} \mid i \in \mathbb{N}_0 \}.$$

By definition it follows that  $\text{Der}_{H,0} = \text{Der}_H$ .

**Proposition 4.2.** *The module  $\text{Der}_{H,s}$  has the following properties:*

- (1)  $\text{Der}_{H,s}$  acts in  $H^{p^s}$  as derivations.
- (2) For  $s, k \geq 0$  we obtain

$$[\mathcal{P}^{p^s k}, \mathcal{P}^{\Delta_{s,0}}] = k \mathcal{P}^{k p^s}.$$

- (3) If  $s \geq 0$ , then

$$[\mathcal{P}^{\Delta_{s,i}}, \mathcal{P}^{\Delta_{s,j}}] = \begin{cases} \mathcal{P}^{\Delta_{s,i}} & \text{if } i \neq 0 \text{ and } j = 0, \\ -\mathcal{P}^{\Delta_{s,j}} & \text{if } i = 0 \text{ and } j \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

- (4) The  $p$ th iteration gives  $\mathcal{P}^{\Delta_{s,0}} \dots \mathcal{P}^{\Delta_{s,0}}(h^{p^s}) = \mathcal{P}^{\Delta_{s,0}}(h^{p^s})$  for all  $s \geq 0$  and  $h \in H$ .

- (5) Let  $s \geq 0$ . Then  $\mathcal{P}^{\Delta_{s,i}}(h^{p^s}) = 0 \forall i$  if and only if  $h$  is a  $p^{s+1}$ st power.

*Proof.* AD(1): Let  $h^{p^s} \in H^{p^s}$ . By Proposition 4.1  $\text{Der}_{H,s}$  acts on  $H^{p^s}$  according to the following formulae:

$$\begin{aligned} \mathcal{P}^{\Delta_{s,0}}(h^{p^s}) &= \text{deg}(h)h^{p^s} = \mathcal{P}^{\Delta_0}(h)^{p^s}, \\ \mathcal{P}^{\Delta_{s,1}}(h^{p^s}) &= (\mathcal{P}^1(h))^{p^s} = (\mathcal{P}^{\Delta_1}(h))^{p^s}, \\ \mathcal{P}^{\Delta_{s,i}}(h^{p^s}) &= (\mathcal{P}^{q^{i-1}} \mathcal{P}^{\Delta_{i-1}} - \mathcal{P}^{\Delta_{i-1}} \mathcal{P}^{q^{i-1}}(h))^{p^s} = (\mathcal{P}^{\Delta_i}(h))^{p^s}. \end{aligned}$$

Since taking  $p$ th powers is additive in characteristic  $p$ , this establishes the statement.

AD(2): Let  $h^{p^s} \in H^{p^s}$ . We have

$$\begin{aligned} [\mathcal{P}^{p^s k}, \mathcal{P}^{\Delta_{s,0}}](h^{p^s}) &= \mathcal{P}^{p^s k} \mathcal{P}^{\Delta_{s,0}}(h^{p^s}) - \mathcal{P}^{\Delta_{s,0}} \mathcal{P}^{p^s k}(h^{p^s}) \\ &= \text{deg}(h) \mathcal{P}^{p^s k}(h^{p^s}) - \mathcal{P}^{\Delta_{s,0}}(\mathcal{P}^k(h))^{p^s} \\ &= \text{deg}(h) \mathcal{P}^{p^s k}(h^{p^s}) - \text{deg}(\mathcal{P}^k(h)) \mathcal{P}^{p^s k}(h^{p^s}) \\ &= (\text{deg}(h) - \text{deg}(h) + k - kq) \mathcal{P}^{p^s k}(h^{p^s}) \\ &= k \mathcal{P}^{p^s k}(h^{p^s}). \end{aligned}$$

AD(3): Let  $h^{p^s} \in H^{p^s}$ . If  $i, j \geq 1$ , then

$$[\mathcal{P}^{\Delta_{s,0}}, \mathcal{P}^{\Delta_{s,j}}] = 0$$

by part (3) of Proposition 4.1. Otherwise we have

$$\begin{aligned} [\mathcal{P}^{\Delta_{s,0}}, \mathcal{P}^{\Delta_{s,j}}](h^{p^s}) &= \mathcal{P}^{\Delta_{s,0}} \mathcal{P}^{\Delta_{s,j}}(h^{p^s}) - \mathcal{P}^{\Delta_{s,j}} \mathcal{P}^{\Delta_{s,0}}(h^{p^s}) \\ &= (\text{deg}(h) + q^j - 1) \mathcal{P}^{\Delta_{s,j}}(h^{p^s}) - \text{deg}(h) \mathcal{P}^{\Delta_{s,j}}(h^{p^s}) \\ &= -\mathcal{P}^{\Delta_{s,j}}(h^{p^s}). \end{aligned}$$

The relation for  $j = 0$  can be established in the same way.

AD(4): Let  $h^{p^s} \in H^{p^s}$ . Since  $\mathcal{P}^{\Delta_{s,0}}(h^{p^s}) = \text{deg}(h)h^{p^s}$  and  $\text{deg}(h)^{p^s} = \text{deg}(h)$  the result follows.

Ad(5): Let  $h^{p^s} = H^{p^s}$  and  $i \geq 1$ . Then

$$\mathcal{P}^{\Delta_{s,i}}(h^{p^s}) = (\mathcal{P}^{\Delta_i}(h))^{p^s}$$

and

$$\mathcal{P}^{\Delta_{s,0}}(h^{p^s}) = \text{deg}(h)h^{p^s} = \mathcal{P}^{\Delta_0}(h)^{p^s}$$

are simultaneously zero for all  $i$  if and only if

$$\mathcal{P}^{\Delta_i}(h) = 0$$

for all  $i \geq 0$ , i.e., if and only if  $h$  is a  $p$ th power; hence precisely when  $h^{p^s}$  is a  $p^{s+1}$ st power.  $\square$

Thus  $\text{Der}_{H,s}$  is a restricted Lie algebra of derivations acting on  $H^{p^s}$  vanishing precisely on the  $p^{s+1}$ st powers. We need to have a look at the relation between  $\text{Der}_{H,s}$  and  $\text{Der}_H$ .

**Proposition 4.3.** *Let  $H$  be an unstable reduced Noetherian algebra over the Steenrod algebra. The action of  $\text{Der}_{H,s}$  on  $H^{p^s}$  has the following properties:*

(1) *For any  $d_s \in \text{Der}_{H,s}$  there exists a  $d \in \text{Der}_H$  such that*

$$d_s(h^{p^s}) = d(h)^{p^s} \quad \forall h \in H.$$

(2) *If there are  $m$  derivations in  $\text{Der}_{H,s}$  that are linearly dependent, then so are any  $m$  derivations.*

*Proof.* AD(1): For any

$$d_s = h_0^{p^s} \mathcal{P}^{\Delta_{s,i_0}} + \dots + h_l^{p^s} \mathcal{P}^{\Delta_{s,i_l}} \in \text{Der}_{H,s}$$

we find that

$$d_s = (d)^{p^s} = (h_0 \mathcal{P}^{\Delta_{i_0}} + \dots + h_l \mathcal{P}^{\Delta_{i_l}})^{p^s}$$

by part (1) of Proposition 4.1. By construction  $d \in \text{Der}_H$ .

AD(2): Let  $d_{s,1}, \dots, d_{s,m} \in \text{Der}_{H,s}$  be linearly dependent. By part (1) we find  $d_1, \dots, d_m \in \text{Der}_H$  such that  $(d_i(h))^{p^s} = d_{s,i}(h^{p^s})$  for all  $h \in H$ . Thus the elements  $d_1, \dots, d_m \in \text{Der}_H$  are linearly dependent. Therefore any  $m$  elements, say  $d'_1, \dots, d'_m$ , of  $\text{Der}_H$  are linearly dependent with a relation

$$h_1 d'_1 \dots + h_m d'_m = 0.$$

Thus

$$h_1^{p^s} d'_{s,1} + \dots + h_m^{p^s} d'_{s,m} = 0$$

is a relation in  $\text{Der}_{H,s}$ .  $\square$

Thus the minimal  $l_s$  such that  $l_s + 1$  elements of  $\text{Der}_{H,s}$  are linearly dependent is uniquely defined. We call  $l_s$  the  $\Delta_s$ -length of  $H^{p^s}$ , denoted by  $\lambda_{H,s}$  or if no confusion can arise by  $\lambda_s$ . If  $\lambda_s \in \mathbb{N}_0$ , we call the algebra  $H^{p^s}$   $\Delta_s$ -finite. Note that by construction  $H$  is  $\Delta$ -finite if and only if  $H^{p^s}$  is  $\Delta_s$ -finite.

**Proposition 4.4.** *Let  $H$  be an unstable reduced  $\Delta$ -finite algebra over the Steenrod algebra. Let  $\lambda$  be its  $\Delta$ -length and  $\lambda_s$  the  $\Delta_s$ -length of  $H^{p^s}$ . Then*

(1)  $\lambda_s = \lambda$ .

(2) *We have a relation of the form*

$$\mathbf{d}_s = \mathbf{d}_{\lambda_s,0}^{p^s} \mathcal{P}^{\Delta_{s,0}} + \dots + \mathbf{d}_{\lambda_s,\lambda_s}^{p^s} \mathcal{P}^{\Delta_{s,\lambda_s}}$$

on  $H^{p^s}$ .

*Proof.* AD(1): Let

$$d = \mathbf{d}_{\lambda,0} \mathcal{P}^{\Delta_0} + \dots + (-1)^\lambda \mathbf{d}_{\lambda,\lambda} \mathcal{P}^{\Delta_\lambda} \in \text{Der}_H$$

be the  $\Delta$ -relation of  $H$ . Then the element  $d_s(h^{p^s}) = (d(h))^{p^s}$  of  $\text{Der}_{H,s}$  vanishes on  $H^{p^s}$ . Thus  $\lambda_s \leq \lambda$ . Conversely, if

$$d_s = h_0^{p^s} \mathcal{P}^{\Delta_{s,0}} + \dots + h_{\lambda_s}^{p^s} \mathcal{P}^{\Delta_{s,\lambda_s}} \in \text{Der}_{H,s}$$

is a  $\Delta_s$ -relation of  $H^{p^s}$ , then the element

$$d = h_0 \mathcal{P}^{\Delta_0} + \dots + h_{\lambda_s} \mathcal{P}^{\Delta_{\lambda_s}} \in \text{Der}_H$$

vanishes on  $H$ , by part (1) of Proposition 4.3. Thus, also,  $\lambda \leq \lambda_s$ .

AD(2): By (1)  $\lambda_s$  is the  $\Delta$ -length of  $H$ . Therefore

$$\mathbf{d} = \mathbf{d}_{\lambda_s,0} \mathcal{P}^{\Delta_0} + \dots + \mathbf{d}_{\lambda_s,\lambda_s} \mathcal{P}^{\Delta_{\lambda_s}}$$

is the  $\Delta$ -relation on  $H$ . Thus the result follows by part (1) of Proposition 4.3.  $\square$

We call the relation  $\mathbf{d}_s$  of part (2) of this result the  $\Delta_s$ -relation of  $H^{p^s}$ .

Let  $H^{p^s} \hookrightarrow \mathbb{F}[V]^{p^s}$  with  $\Delta_s$ -length  $\lambda_s$ . Then  $H = H^{p^s} \hookrightarrow \mathbb{F}[V]$ , since  $H^{p^s}$  arises from  $H^{p^s}$  by adjoining all  $p^s$ th roots. Let  $H$  have  $\Delta$ -length  $\lambda$ . Thus  $\lambda = \lambda_s \leq n$  by Proposition 4.4, part (1).

**Proposition 4.5.** *For  $s \geq 0$  we have*

$$(\mathcal{C}_{\text{Der}_H(H)})^{p^s} = \mathcal{C}_{\text{Der}_{H,s}(H^{p^s})} \subseteq \mathcal{C}_{\text{Der}_H(H)}.$$

*Proof.* If  $h^{p^s} \in H^{p^s}$  is a  $\text{Der}_{H,s}$ -constant, then  $h$  is a  $p^{s+1}$ st power by part (5) of Proposition 4.2. Thus there exists an element  $k \in H_1^{p^s} = H^{p^{s-1}}$  such that  $k^p = h^{p^s}$ . Hence  $h^{p^s} = k^p \in H_1^{p^s} \subseteq H$  is a  $p$ th power, i.e., a  $\text{Der}_H$ -constant.

In order to prove the equality, let  $h \in H$  be a  $\text{Der}_H$ -constant, i.e.,  $d(h) = 0$  for all  $d \in \text{Der}_H$ . For any  $d_s \in \text{Der}_{H,s}$  there is an element  $d \in \text{Der}_H$  such that  $d_s(h^{p^s}) = (d(h))^{p^s} = 0$  by part (1) of Proposition 4.3. Thus  $(\mathcal{C}_{\text{Der}_H(H)})^{p^s} \subseteq \mathcal{C}_{\text{Der}_{H,s}(H^{p^s})}$ . Conversely, if  $h^{p^s} \in \mathcal{C}_{\text{Der}_{H,s}(H^{p^s})}$ , then  $h \in \mathcal{C}_{\text{Der}_H(H)}$ , by what we have proven so far. Thus  $h^{p^s} \in (\mathcal{C}_{\text{Der}_H(H)})^{p^s}$  as desired.  $\square$

*Remark.* We note that the preceding results imply that

$$\dim(\text{Der}_{\mathbb{F}[V],s}) = \dim(\text{Der}_{\mathbb{F}[V]}) = \dim_{\mathbb{F}}(V) = n.$$

*Remark.* In Section 3 we defined  $\text{Der}_{\mathbb{H}}$  for  $\mathbb{H} = FF(H)$  for integral domains  $H$ . In the same way we define

$$\text{Der}_{\mathbb{H},s} = \text{span}_{\mathbb{H}^{p^s}} \{ \mathcal{P}^{\Delta_{i,s}} \mid i \in \mathbb{N}_0 \}.$$

We obtain analogously to Proposition 3.5 that  $\dim(\text{Der}_{\mathbb{H},s}) = \dim(\text{Der}_{H,s})$ . Hence,  $\lambda_{\mathbb{H},s} = \lambda_{H,s}$  and  $\mathbf{d}_{\mathbb{H},s} = \mathbf{d}_{H,s}$ . Therefore  $\text{Der}_{\mathbb{H},s}$  and  $\text{Der}_{H,s}$  have basis  $\{ \mathcal{P}^{\Delta_{s,0}}, \dots, \mathcal{P}^{\Delta_{s,\lambda_s}} \}$ , where  $\lambda_s = \lambda_{\mathbb{H},s} = \lambda_{H,s}$ .

**Corollary 4.6.** *The  $\Delta_s$ -length of  $\mathbb{F}[V]^{p^s}$ , resp.  $\mathbb{F}(V)^{p^s}$ , is equal to the dimension of  $V$ . Moreover, for  $D_s = 0$*

$$\mathbb{F}[V]^{p^s} = \mathcal{C}_{D_s}(\mathbb{F}[V]^{p^s}), \quad \text{resp. } \mathbb{F}(V)^{p^s} = \mathcal{C}_{D_s}(\mathbb{F}(V)^{p^s}).$$

*Finally, the  $\Delta_s$ -length of  $\mathbb{F}[V]^{p^{s+1}}$ , resp.  $\mathbb{F}(V)^{p^{s+1}}$ , is zero, and*

$$\mathbb{F}[V]^{p^{s+1}} = \mathcal{C}_{\text{Der}_{\mathbb{F}[V],s}}(\mathbb{F}[V]^{p^s}), \quad \text{resp. } \mathbb{F}(V)^{p^{s+1}} = \mathcal{C}_{\text{Der}_{\mathbb{F}(V),s}}(\mathbb{F}(V)^{p^s}).$$

*Proof.* The first statement follows from Lemma 3.8 and Proposition 4.5. The second statement follows from Corollaries 3.2 and 3.9 and Proposition 4.5.  $\square$

We need a generalization of Lemma 3.7.

**Lemma 4.7.** *Let  $K^{p^s} \subseteq H^{p^s}$  be unstable reduced Noetherian algebras over the Steenrod algebra. Denote by  $\lambda_{K,s}$ , resp.  $\lambda_{H,s}$ , the  $\Delta_s$ -length of  $K^{p^s}$ , resp.  $H^{p^s}$ . Then  $\lambda_{K,s} \leq \lambda_{H,s}$ .*

*Proof.* The proof works just as the one of Lemma 3.7. □

**Proposition 4.8.** *Let  $t > s$  and  $W \leq V$ . The  $\Delta_s$ -length of  $\mathbb{F}[W]^{p^s} \otimes \mathbb{F}[V/W]^{p^t}$  as well as of  $\mathbb{F}(W)^{p^s} \otimes \mathbb{F}(V/W)^{p^t}$ , is equal to  $\lambda_s = \dim_{\mathbb{F}}(W)$ . Moreover, let  $H^{p^s} \hookrightarrow \mathbb{F}[V]^{p^s}$ . Then the  $\Delta_s$ -length of  $H^{p^s}$  is at most  $l$  if and only if*

$$H^{p^s} \subseteq \mathbb{F}[W]^{p^s} \otimes \mathbb{F}[V/W]^{p^t},$$

where  $\dim_{\mathbb{F}}(W) = l$  and  $t > s$ .

*Proof.* The first statement follows from Proposition 4.4, Lemma 3.8, and Corollary 3.9. If  $H^{p^s} \subseteq \mathbb{F}[W]^{p^s} \otimes \mathbb{F}[V/W]^{p^t}$ , then by Lemma 4.7 its  $\Delta_s$ -length is at most the  $\Delta_s$ -length of  $\mathbb{F}[W]^{p^s} \otimes \mathbb{F}[V/W]^{p^t}$ , which is  $l$  by what we proved so far.

Conversely, let the  $\Delta_s$ -length of  $H^{p^s}$  be  $\lambda_s \leq l$ . Then  $H$  has  $\Delta$ -length  $\lambda = \lambda_s$  by Proposition 4.4. Thus its  $\Delta$ -relation is

$$\mathbf{d} = \mathbf{d}_{\lambda,0} \mathcal{P}^{\Delta_0} + \dots + \mathbf{d}_{\lambda,\lambda} \mathcal{P}^{\Delta_\lambda}$$

and therefore the Dickson algebra of dimension  $\lambda$  is contained in  $H$ .

$$\mathcal{D}(\lambda) \hookrightarrow H \hookrightarrow \mathbb{F}[U] \otimes \mathbb{F}[V/U]$$

for some  $U \leq V$  with  $\dim(U) = \lambda$ . But  $\mathbb{F}[U] \otimes \mathbb{F}[V/U]^p$  is the largest subalgebra of  $\mathbb{F}[V]$  with  $\Delta$ -length  $\lambda$  by Corollary 3.14. Thus

$$H \hookrightarrow \mathbb{F}[U] \otimes \mathbb{F}[V/U]^p \hookrightarrow \mathbb{F}[W] \otimes \mathbb{F}[V/W]^p$$

and the result follows. □

**Lemma 4.9.** *We have*

$$FF(\mathcal{C}_{\text{Der}_H,s}(H^{p^s})) = \mathcal{C}_{\text{Der}_{\mathbb{H}},s}(\mathbb{H}^{p^s}).$$

*Proof.* This follows from Propositions 4.5 and 2.3. □

In order to be able to treat the general case, we need another preliminary result.

**Proposition 4.10.** *Let  $V = W_0 \oplus \dots \oplus W_e$  be a vector space decomposition. Consider the purely inseparable extension  $H^{p^{s_0}} \hookrightarrow \mathbb{F}[W_0]^{p^{s_0}} \otimes \dots \otimes \mathbb{F}[W_e]^{p^{s_e}}$ . Let  $s_0 < \dots < s_e$ . Let  $H^{p^{s_0}}$  be an integrally closed unstable algebra over the Steenrod algebra. Then the  $\Delta_{s_0}$ -length of  $H^{p^{s_0}}$  is  $\lambda_{s_0}$  if and only if*

$$\mathbb{F}[U_0]^{p^{s_0}} \hookrightarrow H^{p^{s_0}} \hookrightarrow \mathbb{F}[U - 0]^{p^{s_0}} \otimes \mathbb{F}[W_0/U_0]^{p^{s_0+1}} \otimes \mathbb{F}[W_1]^{p^{s_1}} \otimes \dots \otimes \mathbb{F}[W_e]^{p^{s_e}}$$

for  $\dim_{\mathbb{F}}(U_0) = \lambda_{s_0}$ ,  $U_0 \leq W_0$  and  $U_0$  maximal with respect to this property.

*Proof.* Let the  $\Delta_{s_0}$ -length of  $H^{p^{s_0}}$  be  $\lambda_{s_0}$ . Note that  $\lambda_{s_0} \leq \dim_{\mathbb{F}}(W_0)$  by Lemma 4.7. The  $\Delta$ -length of  $H$  is also  $\lambda_{s_0}$  by part (1) of Proposition 4.4. Moreover

$$H \hookrightarrow \mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^{p^{s_1-s_0}} \otimes \dots \otimes \mathbb{F}[W_e]^{p^{s_e-s_0}}.$$

Let  $\mathbf{d}$  be the  $\Delta$ -relation of  $\mathbb{H}$ . Then  $\mathbf{d}$  is also the  $\Delta$ -relation of its field of fractions,  $\mathbb{H}$ . Thus by construction we find for  $D = D_{\mathbb{H}} \subseteq \text{Der}_{\mathbb{H}}$

$$\begin{aligned} \mathbb{H} &\hookrightarrow \mathcal{C}_D(\mathbb{F}(W_0) \otimes \mathbb{F}(W_1)^{p^{s_1-s_0}} \otimes \cdots \otimes \mathbb{F}(W_e)^{p^{s_e-s_0}}) \\ &= \mathbb{F}(U_0) \otimes \mathbb{F}(W_0/U_0)^p \otimes \mathbb{F}(W_1)^{p^{s_1-s_0}} \otimes \cdots \otimes \mathbb{F}(W_e)^{p^{s_e-s_0}} \end{aligned}$$

for some  $U_0 \leq W_0$  of dimension  $\lambda_{s_0}$  by Corollary 3.9. Let  $U$  be maximal with this property. The extension

$$\mathbb{F}(U_0) \cap \mathbb{H} \hookrightarrow \mathbb{F}(U_0)$$

is purely inseparable, since

$$\mathbb{H} \hookrightarrow \mathbb{F}(U_0) \otimes \mathbb{F}(W_0/U_0)^p \otimes \mathbb{F}(W_1)^{p^{s_1-s_0}} \otimes \cdots \otimes \mathbb{F}(W_e)^{p^{s_e-s_0}}$$

is purely inseparable and all elements in  $\mathbb{F}(U_0) \otimes \mathbb{F}(W_0/U_0)^p \otimes \mathbb{F}(W_1)^{p^{s_1-s_0}} \otimes \cdots \otimes \mathbb{F}(W_e)^{p^{s_e-s_0}}$  that are algebraic over  $\mathbb{F}(U_0)$  are inseparable. By maximality of  $\lambda_{s_0}$  all elements in  $\mathbb{F}(U_0)$  are separable over  $\mathbb{H}$ . Thus

$$\mathbb{F}(U_0) \cap \mathbb{H} \hookrightarrow \mathbb{F}(U_0)$$

is also separable. Hence

$$\mathbb{F}(U_0) = \mathbb{F}(U_0) \cap \mathbb{H} \hookrightarrow \mathbb{H}.$$

Therefore

$$\mathbb{F}[U_0]^{p^{s_0}} \hookrightarrow \mathbb{H}^{p^{s_0}} \hookrightarrow \mathbb{F}[U_0]^{p^{s_0}} \otimes \mathbb{F}[W_0/U_0]^{p^{s_0+1}} \otimes \cdots \otimes \mathbb{F}[W_e]^{p^{s_e}}$$

as desired.

To prove the converse, assume that there exists a vector space  $U_0 \leq W_0$  of  $\dim_{\mathbb{F}}(U_0) = \lambda_{s_0}$  such that

$$\mathbb{F}[U_0]^{p^{s_0}} \hookrightarrow \mathbb{H}^{p^{s_0}} \hookrightarrow \mathbb{F}[U_0]^{p^{s_0}} \otimes \mathbb{F}[W_0/U_0]^{p^{s_0+1}} \otimes \cdots \otimes \mathbb{F}[W_e]^{p^{s_e}}.$$

Assume furthermore that  $U_0$  is maximal with this property. Then the  $\Delta_{s_0}$ -length of  $\mathbb{F}[U_0]^{p^{s_0}}$  is  $\lambda_{s_0}$  by Corollary 4.6. Equally, the  $\Delta_{s_0}$ -length of  $\mathbb{F}[U_0]^{p^{s_0}} \otimes \mathbb{F}[W_0/U_0]^{p^{s_0+1}} \otimes \cdots \otimes \mathbb{F}[W_e]^{p^{s_e}}$  is  $\lambda_{s_0}$  by Proposition 4.8. Therefore the  $\Delta_{s_0}$ -length of  $\mathbb{H}^{p^{s_0}}$  is  $\lambda_{s_0}$  by Lemma 4.7.  $\square$

*Remark.* Note that any element in  $\mathbb{H}^{p^{s_0}}$  that is algebraic over  $\mathbb{F}[U_0]^{p^{s_0}}$  is separable over  $\mathbb{F}[U_0]^{p^{s_0}}$ .

**Theorem 4.11.** *Let  $V = W_0 \oplus \cdots \oplus W_e$  be a vector space decomposition. Consider the purely inseparable extension  $\mathbb{H} \hookrightarrow \mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}$  of exponent one. Let  $\mathbb{H}$  be integrally closed. Then  $\mathbb{H}$  is an unstable algebra over the Steenrod algebra if and only if*

$$\mathbb{H} = \mathbb{F}[U_0] \otimes \mathbb{F}[U_1]^p \otimes \cdots \otimes \mathbb{F}[U_{e+1}]^{p^{e+1}}$$

for some vector space decomposition

$$V = U_0 \oplus \cdots \oplus U_{e+1}$$

with

$$U_0 \oplus \cdots \oplus U_i \leq W_0 \oplus \cdots \oplus W_i$$

and  $\dim(U_0 \oplus \cdots \oplus U_i)$  is the  $\Delta$ -length of  $\mathbb{H}_i$ ,  $i = 0, \dots, e + 1$ .

*Proof.* The “if” part of the statement is clear by Proposition 4.10. We need to prove the “only if” part.

Let  $\mathbf{d} \in \text{Der}_{\mathbb{H}}$  be the  $\Delta$ -relation on  $\mathbb{H}$  of length  $\lambda_0$ . Then  $\mathbb{F}[U_0] \hookrightarrow \mathbb{H}$  for some vector space  $U_0$  of dimension  $\lambda_0$  by Proposition 4.10. Hence  $\lambda_0 \leq \dim(W_0)$ ,  $U_0 \leq W_0$ , and we have

$$\mathbb{F}[U_0] \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}.$$

We consider the chain

$$\begin{aligned} \mathbb{F}[U_0]_1 \hookrightarrow \mathbb{H}_1 &\hookrightarrow (\mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e})_1 \\ &= \mathbb{F}[W_0 \oplus W_1] \otimes \mathbb{F}[W_2]^p \otimes \cdots \otimes \mathbb{F}[W_e]^{p^{e-1}}. \end{aligned}$$

By Proposition 4.10 the  $\Delta$ -length of  $\mathbb{H}_1$  is at most the dimension of  $W_0 \oplus W_1$  and

$$\mathbb{F}[U_0] \otimes \mathbb{F}[U_1] \hookrightarrow \mathbb{H}_1$$

for a suitable  $U_0 \oplus U_1 \leq W_0 \oplus W_1$ . Since  $\mathbb{F}[U_0] \hookrightarrow \mathbb{H}_1$ , and  $U_0$  is the maximal vector subspace with this property, we have

$$\mathbb{F}[U_0] \otimes \mathbb{F}[U_1]^p \hookrightarrow \mathbb{H}.$$

Proceeding inductively gives an extension

$$\mathbb{F}[U_0] \otimes \mathbb{F}[U_1]^p \otimes \cdots \otimes \mathbb{F}[U_{e+1}]^{p^{e+1}} \hookrightarrow \mathbb{H},$$

which is separable, because it is algebraic (cf. the remark after Proposition 4.10). This extension is also purely inseparable because

$$\mathbb{F}[U_0] \otimes \mathbb{F}[U_1]^p \otimes \cdots \otimes \mathbb{F}[U_{e+1}]^{p^{e+1}} \hookrightarrow \mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_{e+1}]^{p^{e+1}}$$

is purely inseparable. Thus

$$\mathbb{H} = \mathbb{F}[U_0] \otimes \mathbb{F}[U_1]^p \otimes \cdots \otimes \mathbb{F}[U_{e+1}]^{p^{e+1}}$$

as desired. □

**Corollary 4.12.** *Let  $V = W_0 \oplus \cdots \oplus W_e$  be a vector space decomposition. Let  $\mathbb{H} \hookrightarrow \mathbb{F}(W_0) \otimes \mathbb{F}(W_1)^p \otimes \cdots \otimes \mathbb{F}(W_e)^{p^e}$  be a purely inseparable extension of exponent one. Then  $\mathbb{H}$  is a field over the Steenrod algebra if and only if*

$$\mathbb{H} = \mathbb{F}(U_0) \otimes \mathbb{F}(U_1)^p \otimes \cdots \otimes \mathbb{F}(U_{e+1})^{p^{e+1}}$$

for some vector space decomposition

$$V = U_0 \oplus \cdots \oplus U_{e+1}$$

with

$$U_0 \oplus \cdots \oplus U_i \leq W_0 \oplus \cdots \oplus W_i$$

and  $\dim(U_0 \oplus \cdots \oplus U_i)$  is the  $\Delta$ -length of  $\mathbb{H}_i$ ,  $i = 0, \dots, e + 1$ .

*Proof.* Since

$$\mathcal{U}n(\mathbb{H}) \hookrightarrow \mathbb{F}[U_0] \otimes \mathbb{F}[U_1]^p \otimes \cdots \otimes \mathbb{F}[U_e]^{p^e}$$

is integrally closed and  $FF(\mathcal{U}n(\mathbb{H})) = \mathbb{H}$ , the result follows from Theorem 4.11. □

**Theorem 4.13.**  $\mathbb{H} \hookrightarrow \mathbb{F}(V)$  is a purely inseparable extension of exponent  $e$  of fields over the Steenrod algebra if and only if

$$\mathbb{H} = \mathbb{F}(W_0) \otimes \mathbb{F}(W_1)^p \otimes \cdots \otimes \mathbb{F}(W_e)^{p^e}$$

for some vector space decomposition

$$V = W_0 \oplus \cdots \oplus W_e,$$

where  $\dim(W_0 \oplus \cdots \oplus W_i)$  is the  $\Delta$ -length of  $\mathbb{H}_i$ ,  $i = 0, \dots, e$ .

*Proof.* The “if” part is clear by Corollary 4.12. We show the “only if” part.

We proceed by induction on  $e$ . The case  $e = 1$  has been treated in Theorem 4.11. Thus assume that  $e > 1$ .

We have a chain of purely inseparable extensions of exponent one

$$\mathbb{H} = \mathbb{H}_0 \hookrightarrow \mathbb{H}_1 \hookrightarrow \cdots \hookrightarrow \mathbb{H}_e = \mathbb{F}(V)$$

which is obtained by adjoining successively  $p$ th roots. Note that all  $\mathbb{H}_i$ 's are fields over the Steenrod algebra.

By the induction hypothesis we can assume that

$$\mathbb{H} \hookrightarrow \mathbb{H}_1 = \mathbb{F}(W_0) \otimes \mathbb{F}(W_1)^p \otimes \cdots \otimes \mathbb{F}(W_{e-1})^{p^{e-1}}$$

for a vector space decomposition

$$V = W_0 \oplus \cdots \oplus W_{e-1}.$$

By Corollary 4.12 we are done.  $\square$

At the level of algebras we obtain the following result as an obvious corollary.

**Corollary 4.14.** Let  $\mathbb{H}$  be integrally closed. Let  $\mathbb{H} \hookrightarrow \mathbb{F}[V]$  be a purely inseparable extension of exponent  $e$ . Then  $\mathbb{H}$  is an algebra over the Steenrod algebra if and only if

$$\mathbb{H} = \mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}$$

for some vector space decomposition

$$V = W_0 \oplus \cdots \oplus W_e,$$

where  $\dim(W_0 \oplus \cdots \oplus W_i)$  is the  $\Delta$ -length of  $\mathbb{H}_i$ ,  $i = 0, \dots, e$ .  $\square$

*Remark.* Note that Corollary 4.14 has been proven in Theorem 7.2.2 of [3] as well as in [7], Theorem II, without, however, the precise statement on the dimension of  $W_0 \oplus \cdots \oplus W_i$ .

## 5. PURELY INSEPARABLE EXTENSIONS, THE GENERAL CASE

Let  $\mathbb{H}$  be an unstable Noetherian integral domain over the Steenrod algebra. Assume that the canonical inclusion

$$\mathbb{H} \hookrightarrow \sqrt[p^*]{\mathbb{H}}$$

is purely inseparable of exponent  $e$ .

If  $\mathbb{H}$  is integrally closed, then so is  $\sqrt[p^*]{\mathbb{H}}$  by part (3) of Proposition 2.1. Then

$$\sqrt[p^*]{\mathbb{H}} \hookrightarrow \mathbb{F}[V]$$

is a Galois extension with Galois group  $G \leq \mathrm{GL}(n, \mathbb{F})$ , where  $n$  is the Krull dimension of  $\mathbb{H}$  (see the Galois Embedding Theorem, Theorem 7.1.1 in [3]). Thus

$$\sqrt[p^*]{\mathbb{H}} = \mathbb{F}[V]^G.$$

On the other hand we can take the separable closure first: The separable closure of  $H \hookrightarrow \mathbb{F}[V]$  denoted by  $\overline{H}^{sep}$  is again an unstable algebra over the Steenrod algebra by the Separable Extension Lemma (Proposition 2.2.2 in [3]), since  $\overline{H}^{sep} = \text{Un}(\overline{\mathbb{H}}^{sep})$ . Thus we obtain a purely inseparable extension of exponent  $e$

$$\overline{H}^{sep} \hookrightarrow \mathbb{F}[V].$$

Therefore, by Corollary 4.14

$$\overline{H}^{sep} = \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e},$$

for some vector space decomposition  $V = W_0 \oplus \cdots \oplus W_e$ .

We need a technical lemma.

**Lemma 5.1.** *Let  $H$  be an unstable Noetherian integral domain over the Steenrod algebra. Then for all  $i \in \mathbb{N}_0$  we have*

$$(\overline{H}^{sep})_i = \overline{(H_i)}^{sep}.$$

*Proof.* By induction on  $i$ , we need to prove the statement only for  $i = 1$ . By assumption we have the diagram

$$\begin{array}{ccccc} H & \hookrightarrow & H_1 & \hookrightarrow & \overline{(H_1)}^{sep} \\ \downarrow & & & & \\ \overline{H}^{sep} & \hookrightarrow & & & (\overline{H}^{sep})_1. \end{array}$$

If  $h \in H_1$ , then  $h^p \in H \subseteq \overline{H}^{sep}$ . Thus  $h \in (\overline{H}^{sep})_1$ . Thus

$$H_1 \hookrightarrow (\overline{H}^{sep})_1.$$

We note that  $\overline{H}^{sep} = \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}$  by Corollary 4.14. Therefore

$$(\overline{H}^{sep})_1 = \mathbb{F}[W_0 \oplus W_1] \otimes \mathbb{F}[W_2]^p \otimes \cdots \otimes \mathbb{F}[W_e]^{p^{e-1}}.$$

Hence  $(\overline{H}^{sep})_1$  is separably closed, and thus

$$\overline{(H_1)}^{sep} \hookrightarrow (\overline{H}^{sep})_1.$$

Moreover, this extension is by the universal property of the separable closure purely inseparable. Next we show that this extension has exponent at most one. To this end, take  $h \in (\overline{H}^{sep})_1$ . Then  $h^p \in \overline{H}^{sep}$  is separable over  $H$ , hence over  $H_1$ . Therefore  $h^p \in \overline{(H_1)}^{sep}$ .

Denote the inseparable closure of  $H_1$  inside  $(\overline{H}^{sep})_1$  by  $K$ . Then  $H_1 \hookrightarrow K$  has exponent at most one, and since  $\overline{H}^{sep} \hookrightarrow (\overline{H}^{sep})_1$  has exponent one, the extension  $H \hookrightarrow K$  also has exponent at most one. Since  $H_1$  is the largest algebra such that  $H \hookrightarrow K$  has exponent one, we have that  $H_1 = K$  and  $H_1 \hookrightarrow (\overline{H}^{sep})_1$  is separable.

Therefore,  $\overline{(H_1)}^{sep} \hookrightarrow (\overline{H}^{sep})_1$  is also separable. Since we already saw that this extension is purely inseparable, this means that

$$(\overline{H}^{sep})_1 = \overline{(H_1)}^{sep}$$

as claimed. □

So, in what follows we can write  $\overline{H_1}^{sep}$  for  $\overline{(H_1)}^{sep} = (\overline{H}^{sep})_1$  without ambiguity.

**Theorem 5.2.** *Let  $H$  be an integrally closed unstable Noetherian integral domain over the Steenrod algebra of Krull dimension  $n$ . Set  $\dim_{\mathbb{F}}(V) = n$ . Let  $V = W_0 \oplus \cdots \oplus W_e$ , and let*

$$\overline{H}^{sep} = \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}.$$

*Then there exists a group  $G \leq \text{GL}(V)$  acting on the flags  $W_0 \oplus \cdots \oplus W_i$  for  $i = 0, \dots, e$  such that  ${}^p\sqrt{\overline{H}} = \mathbb{F}[V]^G$  and*

$$H = (\mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e})^G.$$

*Furthermore,  $\dim_{\mathbb{F}}(W_0 \oplus \cdots \oplus W_i)$  is the  $\Delta$ -length of  $H_i$ .*

*Proof.* By assumption we have a diagram

$$\begin{array}{ccc} H & \hookrightarrow & {}^p\sqrt{\overline{H}} = \mathbb{F}[V]^G \\ \downarrow & & \downarrow \\ \overline{H}^{sep} & \hookrightarrow & \mathbb{F}[V] \end{array}$$

where the horizontal extensions are purely inseparable and the vertical are separable. Recall that

$$\overline{H}^{sep} = \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}$$

where  $\dim_{\mathbb{F}}(W_0 \oplus \cdots \oplus W_i)$  is the  $\Delta$ -length of  $\overline{H}_i^{sep}$  by Corollary 4.14. Consider the corresponding diagram of the respective field of fractions

$$\begin{array}{ccc} \mathbb{H} & \hookrightarrow & \mathbb{F}(V)^G \\ \downarrow & & \downarrow \\ \overline{\mathbb{H}}^{sep} & \hookrightarrow & \mathbb{F}(V). \end{array}$$

Recall from the Imbedding Theorem (Theorem 8.1.5 in [3]) that  $H$ , and hence  $\mathbb{H}$ , contains a fractal of the Dickson algebra in dimension  $n = \dim_{\mathbb{F}}(V)$ . Thus

$$\mathcal{D}(n)^{q^s} \hookrightarrow H \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{F}(V)$$

for some  $s \in \mathbb{N}_0$ . Therefore, the polynomial

$$\Delta(X) = \prod_{l \in V^*} (X - l)^{q^s} = \mathbf{d}_{n,0}^{q^s} X^{q^s} - \mathbf{d}_{n,1}^{q^s} X^{q^s+1} + \cdots + (-1)^n \mathbf{d}_{n,n}^{q^s} X^{q^{n+s}}$$

has coefficients in  $\mathbb{H}$  (cf. Section 5.1 in [3]). Its roots are by construction the linear forms in  $\mathbb{F}[V]$ . Thus  $\mathbb{F}(V)$  is the splitting field of  $\Delta(X)$ . Hence, the field extension  $\mathbb{H} \hookrightarrow \mathbb{F}(V)$  is normal.<sup>7</sup> Since  $\overline{\mathbb{H}}^{sep} \hookrightarrow \mathbb{F}(V)$  is purely inseparable, it follows from the structure theorem for finite dimensional normal field extensions that the extension

$$\mathbb{H} \hookrightarrow \overline{\mathbb{H}}^{sep}$$

is Galois with some Galois group  $G'$ . We have

$$|G'| = |\overline{\mathbb{H}}^{sep} : \mathbb{H}| = |\mathbb{F}(V) : \mathbb{F}(V)^G| = |G|.$$

Since

$$\mathbb{H} = (\overline{\mathbb{H}}^{sep})^{G'} = \overline{\mathbb{H}}^{sep} \cap \mathbb{F}[V]^G$$

<sup>7</sup>Note that this means that  $\mathbb{F}(V)$  is algebraically closed in the category of fields over the Steenrod algebra; cf. Section 3.2 in [3].

it follows that  $G' \geq G$ . Thus  $G' = G$ . Finally we show that the  $\Delta$ -length of an unstable algebra  $H$  coincides with the  $\Delta$ -length of its separable closure  $(\overline{H}^{sep})$ . Together with Lemma 5.1 this gives the result.

To this end, let  $l_i = \dim_{\mathbb{F}}(W_i)$ . Since  $G$  acts on  $\overline{H}^{sep}$ , the group  $G$  consists of matrices of the form

$$(+)\quad \begin{bmatrix} A_0 & 0 & \dots & 0 \\ * & A_1 & 0 & \dots & 0 \\ & * & \ddots & & \vdots \\ \dots & & \ddots & & 0 \\ * & \dots & * & A_e \end{bmatrix}$$

where  $A_i$  is an  $n_i \times n_i$ -matrix with  $n_i = \dim(W_i)$ . Denote by  $\widehat{G}$  the subgroup of  $GL(n, \mathbb{F})$  consisting of all matrices of the form (+). Denote by  $x_1, \dots, x_n$  a basis for  $W_0 \oplus \dots \oplus W_e$ . Then

$$\begin{aligned} & (\mathbb{F}[W_0] \otimes \dots \otimes \mathbb{F}[W_e]^{p^e})^{\widehat{G}} \\ &= \mathcal{D}(n_0) \otimes \mathbb{F}[c_{\text{top}}(x_{n_0+1}^p), \dots, c_{\text{top}}(x_{n_1}^p), c_{\text{top}}(x_{n_1+1}^{p^2}), \dots, c_{\text{top}}(x_n^{p^e})], \end{aligned}$$

where  $c_{\text{top}}(-)$  denotes the top orbit Chern class of the element  $-$  (cf. Section 4.1 in [6]). By construction, the top orbit Chern classes of  $p$ th powers are  $p$ th powers. Thus the  $\Delta$ -length of the ring of invariants  $\widehat{G}$  is equal to  $n_0$ . Therefore we have

$$\begin{aligned} (\mathbb{F}[W_0] \otimes \dots \otimes \mathbb{F}[W_e]^{p^e})^{\widehat{G}} &\hookrightarrow H = (\mathbb{F}[W_0] \otimes \dots \otimes \mathbb{F}[W_e]^{p^e})^G \\ &\hookrightarrow \overline{H}^{sep} = \mathbb{F}[W_0] \otimes \dots \otimes \mathbb{F}[W_e]^{p^e}. \end{aligned}$$

The smallest algebra, as well as the largest algebra in this chain, has  $\Delta$ -length  $n_0$ . Thus by Lemma 3.7 we are done.  $\square$

*Remark.* Since  $G$  acts on  $\overline{H}^{sep}$ , the group  $G$  consists of matrices of the form given in (+). So, if there exists no basis such that  $G$  consists of flag matrices like above, then the only unstable algebras  $H \hookrightarrow {}^{p^*}\sqrt{H} = \mathbb{F}[V]^G$  are the  $p^s$ th powers

$$H = (\mathbb{F}[V]^G)^{p^s},$$

i.e., we have the trivial vector space decomposition  $V = W_e$ .

*Remark.* Note carefully that the proof shows that the  $\Delta$ -length of  $H$  and the  $\Delta$ -length of any separable extension  $H \hookrightarrow K$  coincide.

*Remark.* In Theorem 7.2.2 in [3] as well as in Theorem II in [8] it has been proven that

$$H = (\mathbb{F}[W_0] \otimes \dots \otimes \mathbb{F}[W_e]^{p^e})^G.$$

However, the precise statement on the dimensions of  $W_0 \oplus \dots \oplus W_i$  is missing. Also the connection between the two Galois groups of  $H \hookrightarrow \mathbb{F}[W_0] \otimes \dots \otimes \mathbb{F}[W_e]^{p^e}$ , resp.  ${}^{p^*}\sqrt{H} \hookrightarrow \mathbb{F}[V]$ , is not made.

We conclude this section with an example.

**Example 5.3.** Consider the regular representation of the cyclic group of order 2,  $\mathbb{Z}/2$ , over a field of characteristic 2 afforded by the matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Set  $\mathbb{F}[V] = \mathbb{F}[x, y]$ . Its ring of invariants is

$$\mathbb{F}[x, y]^{\mathbb{Z}/2} = \mathbb{F}[x, y^2 - yx].$$

Then  $\mathbb{Z}/2$  acts on  $\mathbb{F}[x, y^2]$  with invariant ring

$$\mathbb{F}[x, y^2]^{\mathbb{Z}/2} = \mathbb{F}[x, (y^2 - yx)^2].$$

On the other hand,  $\mathbb{Z}/2$  does not act on  $\mathbb{F}[x^2, y]$ . Indeed

$$\mathbb{F}[x, y]^{\mathbb{Z}/2} \cap \mathbb{F}[x^2, y] = \mathbb{F}[x^2, (y^2 - yx)^2] = \mathbb{F}[x^2, y^2]^{\mathbb{Z}/2}.$$

Furthermore we could consider the purely inseparable field extension

$$\mathbb{F}(x^2, y^2 - yx) \hookrightarrow \mathbb{F}(x, y^2 + xy)$$

of degree 2. Note that

$$(*) \quad \mathcal{P}^1(y^2 + xy) = x^2y + xy^2 \notin \mathbb{F}(x^2, y^2 - yx)$$

since our field contains only elements of even degree. Thus  $\mathbb{F}(x^2, y^2 - yx)$  is not a field over the Steenrod algebra. Its separable closure

$$\mathbb{F}(x^2, y^2, yx) = \mathbb{F}\left(\frac{x}{y}, y^2\right)$$

is a Galois extension with the same Galois group  $\mathbb{Z}/2$ . However, the same calculation as above shows that it is also not closed under the action of the Steenrod algebra, as predicted in the previous result. Indeed,  $\mathbb{F}(x, y)$  is the smallest overfield of  $\mathbb{F}(x^2, y^2, yx)$ , say  $\mathbb{K}$ , closed under the action of the Steenrod algebra as we see next:

$$\mathcal{P}^1(xy) = x^2y + xy^2 = xy(x + y) \in \mathbb{K} \Rightarrow x + y \in \mathbb{K}.$$

Since  $\mathbb{K}$  must have the form  $\mathbb{F}(W) \otimes \mathbb{F}(V/W)^2$  for some  $W \leq V$  we find that  $\text{span}_{\mathbb{F}}\{x + y\} \subseteq W$ . The minimal polynomial of  $x + y \in \mathbb{K}$  over  $\mathbb{F}(x^2, y^2, xy)$ ,

$$p(X) = X^2 + (x^2 + y^2),$$

is inseparable of degree 2. Therefore

$$2 = |\mathbb{F}(x, y) : \mathbb{F}(x^2, y^2, xy)| = |\mathbb{F}(x, y) : \mathbb{K}| |\mathbb{K} : \mathbb{F}(x^2, y^2, xy)| = 2|\mathbb{F}(x, y) : \mathbb{K}|,$$

and hence  $\mathbb{F}(x, y) = \mathbb{K}$  as claimed.

On the other hand, the largest subfield, call in  $\mathbb{L}$ , of  $\mathbb{F}(x^2, y^2, xy)$  that is closed under the Steenrod algebra is  $\mathbb{F}(x^2, y^2)$ : by Equation (\*) the field  $\mathbb{L}$  does not contain  $xy$ . Since  $xy$  is the root of

$$p(X) = X^2 + (xy)^2 \in \mathbb{F}(x^2, y^2)[X]$$

we find that

$$2 = |\mathbb{F}(x^2, y^2, xy) : \mathbb{F}(x^2, y^2)| = |\mathbb{F}(x^2, y^2, xy) : \mathbb{L}| |\mathbb{L} : \mathbb{F}(x^2, y^2)| = 2|\mathbb{L} : \mathbb{F}(x^2, y^2)|$$

and hence  $\mathbb{L} = \mathbb{F}(x^2, y^2)$ .

6. PROJECTIVE DIMENSION

The goal of this section is to prove that a Noetherian reduced unstable algebra  $H$  is Cohen-Macaulay if and only if its inseparable closure  ${}^{p^*}\sqrt{H}$  is Cohen-Macaulay.

Let  $H$  be an unstable algebra over the Steenrod algebra  $\mathcal{P}^*$ . An ideal  $I \subseteq H$  is called  $\mathcal{P}^*$ -invariant if it is closed under the action of the Steenrod algebra.

**Lemma 6.1.** *Let  $H$  be an unstable algebra over the Steenrod algebra. For any  $s \in \mathbb{N}_0$ , the canonical inclusion*

$$\psi: H^{p^s} \hookrightarrow H$$

*induces a bijection*

$$\psi^*: \text{Proj}_{\mathcal{P}^*}(H) \rightarrow \text{Proj}_{\mathcal{P}^*}(H^{p^s})$$

*between the spaces of homogeneous  $\mathcal{P}^*$ -invariant prime ideals.*

*Proof.* Since  $\psi$  is an integral extension, the Lying-Over Theorem holds. Thus  $\psi^*$  is surjective.

To prove injectivity take two homogeneous  $\mathcal{P}^*$ -invariant prime ideals  $\mathfrak{p}_1, \mathfrak{p}_2 \subseteq H$ , such that

$$\mathfrak{p}_1 \cap H^{p^s} = \mathfrak{p}_2 \cap H^{p^s}.$$

Thus for any  $h \in \mathfrak{p}_1$  it follows that

$$h^{p^s} \in \mathfrak{p}_1 \cap H^{p^s} = \mathfrak{p}_2 \cap H^{p^s}.$$

Therefore

$$h^{p^s} \in (\psi(\mathfrak{p}_2 \cap H^{p^s})) \subseteq \mathfrak{p}_2.$$

Since  $\mathfrak{p}_2$  is prime, we find that  $h \in \mathfrak{p}_2$ . Interchanging the roles of  $\mathfrak{p}_1$  and  $\mathfrak{p}_2$  gives the result.  $\square$

This result could have been proven also by observing that the  $s$ th iteration of the Frobenius map

$$F^s: H \rightarrow H^{p^s}$$

hands us an isomorphism of unstable algebras of degree  $p^s$  if  $H$  is reduced. This in turn also implies the following result.

**Lemma 6.2.** *Let  $H$  be an unstable reduced algebra over the Steenrod algebra. For any  $s \in \mathbb{N}_0$  we find that*

$$\text{depth}(H) = \text{depth}(H^{p^s}).$$

$\square$

We observe that  $H$  is Noetherian of Krull dimension  $n$  if and only if  $H^{p^s}$  is. We find the following lemma.

**Lemma 6.3.** *Let  $H$  be Noetherian and reduced of Krull dimension  $n$ . Let  $S = \mathbb{F}[h_1^{p^s}, \dots, h_n^{p^s}]$  be a system of parameters in  $H^{p^s}$ . Then*

$$\text{proj} - \dim_S(H) = \text{proj} - \dim_S(H^{p^s}) < \infty.$$

*Proof.* Since  $H^{p^s} \subseteq H$  is a finite integral extension,  $S$  is also a system of parameters for  $H$ . Thus both projective dimensions are finite. Moreover, by the Auslander-Buchsbaum formula we have

$$\text{proj} - \dim_S(H^{p^s}) = \dim(H^{p^s}) - \text{depth}(H^{p^s}) = \dim(H) - \text{depth}(H) = \text{proj} - \dim_S(H)$$

by Lemma 6.2.  $\square$

We come to the desired result about a Noetherian unstable algebra  $H$  and its  $\mathcal{P}^*$ -inseparable closure  ${}^{\mathcal{P}^*}\sqrt{H}$ .

**Proposition 6.4.** *Let  $H$  be Noetherian and reduced of Krull dimension  $n$ . Then  $H$  is Cohen-Macaulay if and only if  ${}^{\mathcal{P}^*}\sqrt{H}$  is Cohen-Macaulay.*

*Proof.* Since  $H$  is Noetherian, its  $\mathcal{P}^*$ -inseparable closure is also Noetherian by Theorem 6.1.3 in [3]. Therefore  ${}^{\mathcal{P}^*}\sqrt{H} = H_s$  for some  $s \in \mathbb{N}_0$ . Thus

$$({}^{\mathcal{P}^*}\sqrt{H})^{p^s} \hookrightarrow H \hookrightarrow {}^{\mathcal{P}^*}\sqrt{H} = H_s$$

is a finite integral extension. By Lemma 6.1 we have a bijection

$$\mathcal{P}roj_{\mathcal{P}^*}({}^{\mathcal{P}^*}\sqrt{H})^{p^s} \rightarrow \mathcal{P}roj_{\mathcal{P}^*}({}^{\mathcal{P}^*}\sqrt{H}).$$

By Theorem 4.3.1 in [3] and Lemma 6.1

$$\mathcal{P}roj_{\mathcal{P}^*}({}^{\mathcal{P}^*}\sqrt{H})^{p^s} \rightarrow \mathcal{P}roj_{\mathcal{P}^*}(H) \rightarrow \mathcal{P}roj_{\mathcal{P}^*}({}^{\mathcal{P}^*}\sqrt{H})$$

is also bijective. Moreover, by Lemma 6.2, the left and the right algebra have the same depth. Thus by Theorem 2.1 in [5] the results follows (cf. Corollary 2.2 loc.cit.).  $\square$

### 7. POLYNOMIAL RINGS

Let  $H$  be an integrally closed unstable Noetherian integral domain over the Steenrod algebra. By Theorem 5.2 we have

$$H = (\overline{H}^{sep})^G \hookrightarrow {}^{\mathcal{P}^*}\sqrt{H} = \mathbb{F}[V]^G,$$

where

$$\overline{H}^{sep} = \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}$$

for some vector space decomposition  $V = W_0 \oplus \cdots \oplus W_e$ . By Proposition 6.4 we know that  $H$  is Cohen-Macaulay if and only if  ${}^{\mathcal{P}^*}\sqrt{H}$  is polynomial. Moreover, the algebra generators of  $H$  are just suitable  $p^s$ th powers of the algebra generators of  ${}^{\mathcal{P}^*}\sqrt{H}$  (for a minimal generating set).

Let  $G$  act on  $V = W_0 \oplus \cdots \oplus W_e$  such that

$$gw_i \in W_0 \oplus \cdots \oplus W_i$$

for all  $w_i \in W_i$ , i.e.,  $G$  consists of flag matrices of the form

$$\begin{bmatrix} A_0 & 0 & \cdots & 0 \\ * & A_1 & 0 & \cdots & 0 \\ & * & \ddots & & \vdots \\ \cdots & & \ddots & & 0 \\ * & \cdots & * & A_e \end{bmatrix}$$

where  $A_i$  is an  $m_i \times m_i$ -matrix with  $m_i = \dim(W_i)$ . For every  $i = 0, \dots, e$  we have a group epimorphism

$$\text{pr}_i: G \rightarrow G_i, \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ * & A_1 & 0 & \cdots & 0 \\ & * & \ddots & & \vdots \\ \cdots & & \ddots & & 0 \\ * & \cdots & * & A_e \end{bmatrix} \mapsto \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ * & A_1 & 0 & \cdots & 0 \\ & * & \ddots & & \vdots \\ \cdots & & \ddots & & 0 \\ * & \cdots & * & A_i \end{bmatrix}.$$

**Lemma 7.1.** *With the preceding notation we have*

$$\mathbb{F}[W_0 \oplus \cdots \oplus W_i]^{G_i} = \mathbb{F}[V]^G \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i] \subseteq \mathbb{F}[V]^G.$$

*Proof.* The kernel of the projection  $\text{pr}_i$ ,  $\ker(\text{pr}_i)$ , consists of matrices of the form

$$\begin{bmatrix} I_0 & 0 & & \cdots & 0 \\ 0 & \ddots & & & 0 \\ \vdots & & I_i & 0 & \cdots & 0 \\ * & * & * & A_{i+1} & & \vdots \\ & & & * & \ddots & \\ * & \cdots & * & & & A_e \end{bmatrix},$$

where the  $I_j$ 's are identity matrices. Thus  $\mathbb{F}[V]^{\ker(\text{pr}_i)} \supseteq \mathbb{F}[W_0 \oplus \cdots \oplus W_i]$ , and hence

$$\mathbb{F}[V]^G = (\mathbb{F}[V]^{\ker(\text{pr}_i)})^{G_i} \supseteq \mathbb{F}[W_0 \oplus \cdots \oplus W_i]^{G_i}.$$

Since  $\mathbb{F}[W_0 \oplus \cdots \oplus W_i]^{G_i} \subseteq \mathbb{F}[W_0 \oplus \cdots \oplus W_i]$  we find

$$\mathbb{F}[W_0 \oplus \cdots \oplus W_i]^{G_i} \subseteq \mathbb{F}[V]^G \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i] \subseteq \mathbb{F}[V]^G.$$

Conversely, since  $\mathbb{F}[V]^{\ker(\text{pr}_i)} \supseteq \mathbb{F}[W_0 \oplus \cdots \oplus W_i]$ , we have

$$\mathbb{F}[V]^{\ker(\text{pr}_i)} \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i] = \mathbb{F}[W_0 \oplus \cdots \oplus W_i].$$

Thus

$$\mathbb{F}[W_0 \oplus \cdots \oplus W_i]^{G_i} = (\mathbb{F}[V]^{\ker(\text{pr}_i)} \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i])^{G_i}.$$

Finally, note that

$$\mathbb{F}[V]^G \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i] \subseteq (\mathbb{F}[V]^{\ker(\text{pr}_i)} \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i])^{G_i}.$$

To see this, take an element  $f \in \mathbb{F}[V]^G \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i]$ . Then  $f \in \mathbb{F}[W_0 \oplus \cdots \oplus W_i]$  is invariant under the group  $G$ . Thus  $f$  is also invariant under  $\ker(\text{pr}_i)$ . Therefore,

$$f \in \mathbb{F}[V]^{\ker(\text{pr}_i)} \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i].$$

But  $f$  is also  $G$ -invariant, i.e.,

$$f \in (\mathbb{F}[V]^{\ker(\text{pr}_i)} \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i])^G \subseteq (\mathbb{F}[V]^{\ker(\text{pr}_i)} \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i])^{G_i}$$

as desired. □

Let  $h_1, \dots, h_m \in \mathbb{F}[V]^G$  be a minimal generating set. Without loss of generality we assume that they are sorted such that

$$\begin{aligned} h_1, \dots, h_{n_0} &\in \mathbb{F}[W_0], \\ h_{n_0+1}, \dots, h_{n_1} &\in \mathbb{F}[W_0 \oplus W_1], \\ &\dots \\ h_{n_{e-1}+1}, \dots, h_{n_e} &= h_m \in \mathbb{F}[W_0 \oplus \cdots \oplus W_e]. \end{aligned}$$

We assume that  $n_0, \dots, n_e$  are maximal with this property. Thus by construction

$$\mathbb{F}[h_0, \dots, h_{n_i}] \subseteq \mathbb{F}[W_0 \oplus \cdots \oplus W_i]^{G_i} \subseteq \mathbb{F}[V]^G$$

for all  $i = 0, \dots, e$ .

**Proposition 7.2.** *If  $n_e = \dim_{\mathbb{F}}(V)$ , i.e., if the ring of invariants*

$$\mathbb{F}[W_0 \oplus \cdots \oplus W_e]^G$$

*is polynomial, then  $n_i = \dim_{\mathbb{F}}(W_0 \oplus \cdots \oplus W_i)$ .*

*Proof.* Consider the integral extension

$$\mathbb{F}[W_0 \oplus \cdots \oplus W_i]^{G_i} \hookrightarrow \mathbb{F}[W_0 \oplus \cdots \oplus W_i].$$

The maximal ideal  $\mathfrak{m}_i$  of  $\mathbb{F}[W_0 \oplus \cdots \oplus W_i]$  lies over the maximal ideal in  $\mathbb{F}[W_0 \oplus \cdots \oplus W_i]^{G_i}$ . Furthermore,  $\mathfrak{m}_i$  extends to a prime ideal  $\mathfrak{p}_i \subseteq \mathbb{F}[W_0 \oplus \cdots \oplus W_e]$ . By construction  $\mathfrak{p}_i$  is generated by all linear forms in  $\mathbb{F}[W_0 \oplus \cdots \oplus W_i]$ . Thus  $\mathfrak{p}_i$  is regular and prime of height equal to  $\dim_{\mathbb{F}}(W_0 \oplus \cdots \oplus W_i)$ . Hence, its contraction to the ring of invariants

$$\mathfrak{p}_i^c = \mathfrak{p}_i \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_e]^G$$

is also prime of height equal to  $\dim_{\mathbb{F}}(W_0 \oplus \cdots \oplus W_i)$ . Furthermore,  $\mathfrak{p}_i^c$  contains by construction

$$(h_1, \dots, h_{n_i}) \subseteq \mathfrak{p}_i^c.$$

Thus the quotient

$$\mathbb{F}[W_0 \oplus \cdots \oplus W_e]^G / \mathfrak{p}_i^c = \mathbb{F}[\bar{h}_{n_i+1}, \dots, \bar{h}_n] \hookrightarrow \mathbb{F}[W_{i+1} \oplus \cdots \oplus W_e]$$

is integral, and

$$n - n_i = \dim_{\mathbb{F}}(W_{i+1} \oplus \cdots \oplus W_e)$$

for all  $i = 0, \dots, e - 1$ . □

**Theorem 7.3.** *With the above notation, if*

$$(\mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e})^G = \mathbb{F}[h_1, \dots, h_n]$$

*is polynomial, then for suitable  $s_1, \dots, s_n \in \mathbb{N}_0$*

$$(\mathbb{F}[U_0] \otimes \cdots \otimes \mathbb{F}[U_f]^{p^f})^G = \mathbb{F}[h_1^{p^{s_1}}, \dots, h_n^{p^{s_n}}]$$

*is polynomial for any subflag*

$$U_0 \oplus \cdots \oplus U_j \leq W_0 \oplus \cdots \oplus W_i$$

*that admits an action of  $G$ .*

*Proof.* To simplify notation we assume that the extension

$$\mathbb{F}[U_0] \otimes \cdots \otimes \mathbb{F}[U_f]^{p^f} \hookrightarrow \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}$$

is purely inseparable of exponent one. The general case follows then inductively.

Since  $G$  acts on the flag  $W_0 \oplus \cdots \oplus W_e$  the algebra generator for the ring of invariants can be sorted such that

$$h_1, \dots, h_{n_i} \in \mathbb{F}[W_0 \oplus \cdots \oplus W_i]$$

with  $n_i - n_{i-1} = \dim(W_i)$ ,  $n_0 = \dim_{\mathbb{F}}(W_0)$ , by Proposition 7.2.

Since  $G$  acts also on the subflag  $U_0 \oplus \cdots \oplus U_f$  the algebra generator for the ring of invariants can be sorted such that

$$h_1, \dots, h_{m_i} \in \mathbb{F}[U_0 \oplus \cdots \oplus U_i]$$

with  $m_i - m_{i-1} = \dim(U_i)$ ,  $m_0 = \dim_{\mathbb{F}}(U_0)$ , and  $m_f = n_e = n$ . Thus  $n_i \geq m_i$ . Consider the algebra

$$A = \mathbb{F}[h_1, \dots, h_{m_0}, h_{m_0+1}^p, \dots, h_{n_0}^p, h_{n_0+1}, \dots, h_{m_1}, h_{m_1+1}^p, \dots, h_{n_1}^p, \dots, h_{m_e}, h_{m_e+1}^p, \dots, h_{m_f}^p] \\ \hookrightarrow \mathbb{F}[U_0] \otimes \dots \otimes \mathbb{F}[U_f]^{p^f}.$$

Since A consists of invariant polynomials it is contained in the ring of invariants

$$(\mathbb{F}[U_0] \otimes \dots \otimes \mathbb{F}[U_f]^{p^f})^G.$$

The diagram

$$\begin{array}{ccccc} A & \hookrightarrow & (\mathbb{F}[U_0] \otimes \dots \otimes \mathbb{F}[U_f]^{p^f})^G & \hookrightarrow & \mathbb{F}[U_0] \otimes \dots \otimes \mathbb{F}[U_f]^{p^f} \\ \downarrow & & & & \downarrow \\ \mathbb{F}[h_1, \dots, h_n] & \hookrightarrow & \mathbb{F}[W_0] \otimes \dots \otimes \mathbb{F}[W_e]^{p^e} & & \end{array}$$

has by construction purely inseparable vertical extensions of degree  $p^{\sum_i(n_i - m_i)}$ . Since the degree of

$$\mathbb{F}[h_1, \dots, h_n] \hookrightarrow \mathbb{F}[W_0] \otimes \dots \otimes \mathbb{F}[W_e]^{p^e}$$

is the group order  $|G|$ , the degree of

$$A \hookrightarrow \mathbb{F}[U_0] \otimes \dots \otimes \mathbb{F}[U_f]^{p^f}$$

is also the group order. Thus A is the desired ring of invariants as claimed.  $\square$

The following result settles a twenty-year-old conjecture due to Clarence W. Wilkerson (see Conjecture 5.1 in [8]).

**Theorem 7.4.** *Let H be an integrally closed Noetherian unstable integral domain over the Steenrod algebra. Then H is polynomial if and only if  ${}^{\mathcal{P}^*}\sqrt{H}$  is polynomial. Furthermore,*

$${}^{\mathcal{P}^*}\sqrt{H} = \mathbb{F}[h_1, \dots, h_n]$$

if and only if there are  $s_1, \dots, s_n \in \mathbb{N}_0$  such that

$$H = \mathbb{F}[h_1^{p^{s_1}}, \dots, h_n^{p^{s_n}}].$$

*Proof.* By Theorem 5.2 there exist a group  $G$  and a flag  $V = W_0 \oplus \dots \oplus W_e$  such that

$$H = (\mathbb{F}[W_0] \otimes \dots \otimes \mathbb{F}[W_e]^{p^e})^G \hookrightarrow {}^{\mathcal{P}^*}\sqrt{H} = \mathbb{F}[V]^G.$$

If  ${}^{\mathcal{P}^*}\sqrt{H}$  is polynomial, then so is H by Theorem 7.3. Note that the same result also gives the precise statement on the respective algebra generators.

On the other hand,  $({}^{\mathcal{P}^*}\sqrt{H})^{p^e} = (\mathbb{F}[V]^{p^e})^G \hookrightarrow H$  is the ring of invariants on the subflag  $V \leq W_0 \oplus \dots \oplus W_e$  for some large enough  $e$ . Therefore if H is polynomial, then  $({}^{\mathcal{P}^*}\sqrt{H})^{p^e} \hookrightarrow H$  is polynomial by the same Theorem 7.3. Thus  ${}^{\mathcal{P}^*}\sqrt{H}$  is polynomial since it is isomorphic as an algebra to  $({}^{\mathcal{P}^*}\sqrt{H})^{p^e}$ .  $\square$

Thus we have the following corollary.

**Corollary 7.5.** *Let H be an unstable polynomial algebra over the Steenrod algebra. Set  $H = \mathbb{F}[h_1, \dots, h_n]$ . Then H is  $\mathcal{P}^*$ -inseparably closed if and only if the polynomial generators  $h_1, \dots, h_n$  are no  $p$ th powers.  $\square$*

The example given at the end of Section 5 illustrates these results. We want to close with an example that shows that a simple generalization of Theorem 7.4 to nonpolynomial invariants is not true.

**Example 7.6.** Let  $p$  be odd and let  $\mathbb{F}$  be the prime field of characteristic  $p$ . Consider the four-dimensional modular representation  $\mathbb{Z}/p \hookrightarrow \mathrm{GL}(4, \mathbb{F})$  afforded by the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Its ring of invariants turns out to be a hypersurface

$$\mathbb{F}[x_1, y_1, x_2, y_2]^{\mathbb{Z}/p} = \mathbb{F}[c_1, y_1, c_2, y_2, q]/(r),$$

where  $c_i = x_i^p - x_i y_i^{p-1}$  are the top orbit Chern classes of  $x_i$ ,  $i = 1, 2$ , and  $q = x_1 y_2 - x_2 y_1$  is an invariant quadratic form. The relation is given by

$$r = q^p - c_1 y_2^p + c_2 y_1^p + q y_1^{p-1} y_2^{p-1}$$

(see Theorem 2.1 in [2]). Certainly,  $\mathbb{Z}/p$  also acts on  $\mathbb{F}[x_1, y_1] \otimes \mathbb{F}[x_2^p, y_2^p]$  and we find that

$$A = \mathbb{F}[c_1, y_1, c_2^p, y_2^p, q^p] \hookrightarrow (\mathbb{F}[x_1, y_1] \otimes \mathbb{F}[x_2^p, y_2^p])^{\mathbb{Z}/p}.$$

However, the new ring of invariants contains an invariant that is not in the algebra  $A$ , namely

$$q' = x_1 y_2^p - x_2^p y_1.$$

Indeed, with the methods presented in Theorem 2.1 of [2] it is not hard to see that

$$(\mathbb{F}[x_1, y_1] \otimes \mathbb{F}[x_2^p, y_2^{2p}])^{\mathbb{Z}/p} = \mathbb{F}[c_1, y_1, c_2^p, y_2^p, q']/(r'),$$

where  $r' = (q')^p - c_1 y_2^{p^2} + c_2^p y_1^p - q' y_1^{p-1} y_2^{p(p-1)}$ . Interesting enough though, it transpires that this ring is again a hypersurface.

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