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OPEN LOCI OF GRADED MODULES

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Abstract. Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be an excellent homogeneous Noetherian graded ring and let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a finitely generated graded A-module. We consider M as a module over A_0 and show that the (S_k)-loci of M are open in $\operatorname{Spec}(A_0)$. In particular, the Cohen-Macaulay locus $U^0_{CM} = \{\mathfrak{p} \in \operatorname{Spec}(A_0) \mid M_{\mathfrak{p}} \text{ is Cohen-Macaulay}\}$ is an open subset of $\operatorname{Spec}(A_0)$. We also show that the (S_k)-loci on the homogeneous parts M_n of M are eventually stable. As an application we obtain that for a finitely generated Cohen-Macaulay module M over an excellent ring A and for an ideal $I \subseteq A$ which is not contained in any minimal prime of M, the (S_k)-loci for the modules M/I^nM are eventually stable.

Introduction

A well-known theorem of Grothendieck states that if M is a finitely generated module over an excellent Noetherian ring A, then for all $k \in \mathbb{N}$ the (S_k)-locus of M

$$U_{S_k}(M) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid M_{\mathfrak{p}} \text{ satisfies } (S_k)\}$$

is an open subset of $\operatorname{Spec}(A)$. As usual, (S_k) denotes the Serre condition, that is, $M_{\mathfrak{p}}$ satisfies (S_k) if for all $\mathfrak{q} \in \operatorname{Spec}(A)$ with $\mathfrak{q} \subseteq \mathfrak{p}$ it holds that

$$\operatorname{depth}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}}) \geq \min(k, \dim(M_{\mathfrak{q}})).$$

It also follows that for such modules M the Cohen-Macaulay locus

$$U_{CM}(M) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid M_{\mathfrak{p}} \text{ is Cohen-Macaulay}\}$$

is an open subset of $\operatorname{Spec}(A)$.

Let $A = \bigoplus_{n \geq 0} A_n$ be a Noetherian graded excellent homogeneous ring and $M = \bigoplus_{i \in \mathbb{Z}} M_i$ a finitely generated graded A-module. Considered as a module over the base ring A_0, M is a direct sum of finitely generated A_0-modules. Moreover, if the base ring A_0 is local, the standard notion of depth is meaningful for the A_0-module M and we may consider its (S_k)-loci

$$U^0_{S_k}(M) = \{\mathfrak{p} \in \operatorname{Spec}(A_0) \mid M_{\mathfrak{p}} \text{ satisfies } S_k\},$$

where $M_{\mathfrak{p}}$ denotes the localization of M at the multiplicative set $A_0 \smallsetminus \mathfrak{p}$. In this paper we prove that under these assumptions the (S_k)-loci of the A_0-module M are open subsets of $\operatorname{Spec}(A_0)$. In particular, the Cohen-Macaulay locus of M (as

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an A_0-module)

$$U^0_{CM}(M) = \{\mathfrak{p} \in \operatorname{Spec}(A_0) \mid M_{\mathfrak{p}} \text{ is Cohen-Macaulay}\}$$

is an open subset of $\operatorname{Spec}(A_0)$.

The proof follows the main ideas of Grothendieck's proof. It is, however, not merely a copy of the proof in EGA and requires a number of modifications. For the benefit of the reader we have included complete proofs of the results. Our proof is based on the following two observations: First, if A is a polynomial ring over the base ring A_0, then every graded resolution of M by finitely generated graded free A-modules provides a free resolution of the A_0-module M which is finitely generated on the homogeneous parts. The second is a result by Hochster and Roberts which states for the A-module M that there is an element $a \in A_0 \setminus (0)$ so that M_a is a free $(A_0)_a$-module provided that the ring A_0 is a domain.

The paper is organized as follows:

The first section contains basic facts about graded rings and modules which are relevant for the rest of the paper. As a main result we obtain that the Auslander-Buchsbaum formula holds for the A_0-module M.

The second section shows that the codepth-loci of M are open in $\operatorname{Spec}(A_0)$. This is the main step in proving the openness of the (S_k)-loci which we present in the next section.

In Section 4 we consider the homogeneous parts of the graded module M. We show that the codepth-loci and (S_k)-loci of the homogeneous parts of M are eventually stable. This is applied in the last section to the case of a finitely generated module M over an excellent Noetherian ring A. If $I \subseteq A$ is an ideal we recover a well-known result by Kodiyalam [7], namely that for $k \geq k_0$

$$\operatorname{depth}(M/I^kM) = \operatorname{depth}(M/I^{k_0}M).$$

We also show that if M is a Cohen-Macaulay module over A and if $I \subseteq A$ is not contained in a minimal prime of M, then the codepth- and (S_k)-loci of M/I^nM are eventually stable.

1. Basic facts

In this paper we assume that $A = \bigoplus_{i \in \mathbb{N}} A_i$ is a Noetherian homogeneous graded ring and that $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a finitely generated A-module. As usual, we let A_+ denote the irrelevant ideal of A, that is, $A_+ = \bigoplus_{i \geq 1} A_i$.

If $\mathfrak{p} \in \operatorname{Spec}(A_0)$ is a prime ideal of A_0, then $M_{\mathfrak{p}}$ denotes the localization $S^{-1}M$ where $S = A_0 \setminus \mathfrak{p}$. Note that $M_{\mathfrak{p}}$ is a graded module over the graded ring $A_{\mathfrak{p}}$.

Our goal is to show that if A is excellent, then the codepth-loci and the (S_k)-loci of M, considered as a module over the base ring A_0, are open subsets of $\operatorname{Spec}(A_0)$.

1.1. **General remarks.** We begin our investigation with some well-known facts about graded modules. Since these results are frequently used throughout the paper, we include them together with their (short) proofs in this introductory section.

1.1.1. **Lemma.** *There exists an integer* t *so that* $\operatorname{ann}_{A_0}(M_t) = \operatorname{ann}_{A_0}(M_k)$ *for all* $k \geq t$.

Proof. For all $k \in \mathbb{Z}$ set $J_k = \operatorname{ann}_{A_0}(M_k)$. Since A is homogeneous and M is a finitely generated A-module, there exists $t_0 \in \mathbb{Z}$ such that

$$A_1 M_k = M_{k+1} \quad \text{for all} \quad k \geq t_0 \, .$$

We conclude $J_k \subseteq J_{k+1}$ for all $k \geq t_0$. Since A_0 is Noetherian, there then exists $t \geq t_0$ so that $J_k = J_t$ for all $k \geq t$. $\square$

1.1.2. **Lemma.** *The following two functions are well defined and surjective:*

(1) *The function* $\varphi\colon \operatorname{Supp}_A(M) \to \operatorname{Supp}_{A_0}(M)$ *defined by* $\varphi(P) = P \cap A_0$.
(2) *The function* $\psi\colon \operatorname{Ass}_A(M) \to \operatorname{Ass}_{A_0}(M)$ *defined by* $\psi(P) = P \cap A_0$.

Proof. (1) If $P \in \operatorname{Supp}_A(M)$, then $M_P \neq 0$ and in particular $M_{\mathfrak{p}} \neq 0$, where $\mathfrak{p} = P \cap A_0$. This shows that φ is well defined. Let $\mathfrak{p} \in \operatorname{Supp}_{A_0}(M)$. Then

$$M_{\mathfrak{p}} = \bigoplus_{i \in \mathbb{Z}} (M_i)_{\mathfrak{p}} \neq 0,$$

and we may consider $M_{\mathfrak{p}}$ as a graded module over the graded ring $A_{\mathfrak{p}}$. Note that $A_{\mathfrak{p}}$ is a *local ring with unique graded maximal ideal $\mathfrak{m} = \mathfrak{p}(A_0)_{\mathfrak{p}} \oplus (A_+)_{\mathfrak{p}}$. Since all minimal primes of $\operatorname{Supp}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ are graded, $\mathfrak{m} \in \operatorname{Supp}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$. Thus there is a prime $P \in \operatorname{Supp}_A(M)$ with $P \cap A_0 = \mathfrak{p}$.

(2) If $P \in \operatorname{Ass}_A(M)$, then there exists $y \in M$ so that $\operatorname{ann}_A(y) = P$. Thus $\operatorname{ann}_{A_0}(y) = P \cap A_0 = \mathfrak{p}$ and $\mathfrak{p} \in \operatorname{Ass}_{A_0}(M)$. Conversely, let $\mathfrak{p} \in \operatorname{Ass}_{A_0}(M)$. Consider again the graded $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$. There exists $z \in M_{\mathfrak{p}}$ so that $\operatorname{ann}_{(A_0)_{\mathfrak{p}}}(z) = \mathfrak{p}(A_0)_{\mathfrak{p}}$, and therefore

$$\mathfrak{p}(A_0)_{\mathfrak{p}} \subseteq \bigcup_{Q \in \operatorname{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})} Q.$$

Since $M_{\mathfrak{p}}$ is a finitely generated $A_{\mathfrak{p}}$-module, there exists $Q \in \operatorname{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ with $\mathfrak{p}(A_0)_{\mathfrak{p}} \subseteq Q$. Since $A_{\mathfrak{p}}$ is *local with unique graded maximal ideal $\mathfrak{p}(A_0)_{\mathfrak{p}} \oplus (A_+)_{\mathfrak{p}}$, we obtain $Q \cap (A_0)_{\mathfrak{p}} = \mathfrak{p}(A_0)_{\mathfrak{p}}$, and a preimage $P \in \operatorname{Spec}(A)$ of Q is an associated prime of the A-module M, with $P \cap A_0 = \mathfrak{p}$. $\square$

Lemma 1.1.2 shows in particular that M as an A_0-module has a finite set of associated primes.

1.1.3. **Lemma.** *Let A and M be as above and set* $I = \operatorname{ann}_{A_0}(M)$. *For any* $\mathfrak{p} \in \operatorname{Spec}(A_0)$ *the following hold:*

(1) *If* $M_{\mathfrak{p}} = 0$, *then there is an element* $a \in A_0 \setminus \mathfrak{p}$ *with* $M_a = 0$.
(2) $\operatorname{ann}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) = I(A_0)_{\mathfrak{p}}$.

Proof. (1) This is a basic fact about Noetherian modules using that M is a finitely generated module over A and $A_0 \setminus \mathfrak{p}$ is a multiplicative subset of A.

(2) Obviously, $I(A_0)_{\mathfrak{p}} \subseteq \operatorname{ann}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}})$. Let $x \in \operatorname{ann}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}})$ with $x = \frac{b}{s}$, where $b \in A_0$ and $s \in A_0 \setminus \mathfrak{p}$. Assume that $m_1, \ldots, m_r$ is a system of generators of the A-module M. Since $x\frac{m_i}{1} = 0$ for all $1 \leq i \leq r$ there is an element $t \in A_0 \setminus \mathfrak{p}$ with $tbm_i = 0$ for all $1 \leq i \leq r$. We have that $tb \in I$ and hence $x = \frac{b}{s} \in I(A_0)_{\mathfrak{p}}$. $\square$

1.2. **The Auslander-Buchsbaum formula.** Let $A = \bigoplus_{i \geq 0} A_i$ be a graded Noetherian homogeneous ring with $(A_0, \mathfrak{m}_0)$ local and let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a finitely generated A-module. Since M is (in general) not finitely generated as an A_0-module, we need to verify that the classical definition of A_0-depth works in the case of a finitely generated graded module. First note that an element $z \in \mathfrak{m}_0$ is regular on M if and only if z is regular on M_i for all $i \in \mathbb{Z}$ with $M_i \neq 0$. Let $x_1, \dots, x_s \in \mathfrak{m}_0$ and $y_1, \dots, y_t \in \mathfrak{m}_0$ be two maximal regular M-sequences (as an A_0-module). Then for all $i \in \mathbb{Z}$ with $M_i \neq 0$ the two sequences are regular on the A_0-module M_i, and the sets

$$\operatorname{Ass}_{A_0}(M/(x_1, \dots, x_s)M) = \bigcup_{i \in \mathbb{Z}} \operatorname{Ass}_{A_0}(M_i/(x_1, \dots, x_s)M_i),$$
$$\operatorname{Ass}_{A_0}(M/(y_1, \dots, y_t)M) = \bigcup_{i \in \mathbb{Z}} \operatorname{Ass}_{A_0}(M_i/(y_1, \dots, y_t)M_i)$$

are finite by Lemma 1.1.2. The maximality of the first sequences yields that there is an $i \in \mathbb{Z}$ with $M_i \neq 0$ and $\mathfrak{m}_0 \in \operatorname{Ass}_{A_0}(M_i/(x_1, \dots, x_s)M_i)$. Since the second sequence is also regular on M_i we have that $t \leq s$. A similar argument shows that $s \leq t$, and we obtain that two maximal regular sequences on M have the same length. Therefore the classical definition of depth is efficient and we put:

1.2.1. **Definition.** Let A and M be as above with $(A_0, \mathfrak{m}_0)$ local. We define the *depth* of M as an A_0-module to be the number

$$\operatorname{depth}_{A_0}(M) := \sup\{n \in \mathbb{N} \mid \exists \text{ an } M\text{-sequence of length } n\}.$$

In general, for a (not necessarily finitely generated) module M over a Noetherian local ring A, the depth of M is defined by means of Koszul homology (see [2, Definition 9.1.1]). In our setting, the definition above coincides with the one in [2].

The aim of this section is to prove the Auslander-Buchsbaum theorem for finitely generated graded modules M over *local graded Noetherian rings A when M is considered a module over the base ring A_0. There is a generalized version of the Auslander-Buchsbaum theorem which applies to our case (see [3, (12.2)] or [6, Theorem (2.1)]). For the convenience of the reader we include a proof of this theorem in the graded case, which only makes use of the classical definition of depth as given above.

1.2.2. **Lemma.** *Let A and M be as above and assume that $(A_0, \mathfrak{m}_0)$ is local. Then:*

(1) $\dim_{A_0}(M) = \sup\{\dim_{A_0}(M_i) \mid i \in \mathbb{Z}\}$,
(2) $\operatorname{depth}_{A_0}(M) = \inf\{\operatorname{depth}_{A_0}(M_i) \mid i \in \mathbb{Z} \text{ with } M_i \neq 0\}$,
(3) $\operatorname{projdim}_{A_0}(M) = \sup\{\operatorname{projdim}_{A_0}(M_i) \mid i \in \mathbb{Z}\}$.

Proof. (1) By Lemma 1.1.1 there is an integer $s \in \mathbb{Z}$ so that $\operatorname{ann}_{A_0}(M_k) = \operatorname{ann}_{A_0}(M_s)$ for all $k \geq s$. In particular, for all $k \geq s$, $\dim_{A_0}(M_k) = \dim_{A_0}(M_s)$ and

$$\dim_{A_0}(M) = \dim_{A_0}(M_r \oplus M_{r-1} \oplus \ldots \oplus M_{s-1} \oplus M_s),$$

where $r \in \mathbb{Z}$ is the smallest integer j with $M_j \neq 0$. The dimension of a finite direct sum of A_0-modules is the maximum of the dimensions of its summands.

(2) If $r_1, \dots, r_s \in A_0$ is a regular sequence on M, then $r_1, \dots, r_s$ is a regular sequence on M_i for all $i \in \mathbb{Z}$ with $M_i \neq 0$. Thus $\operatorname{depth}_{A_0}(M) \leq \operatorname{depth}_{A_0}(M_i)$ for all $i \in \mathbb{Z}$ with $M_i \neq 0$, and hence

$$\operatorname{depth}_{A_0}(M) \leq \inf\{\operatorname{depth}_{A_0}(M_i) \mid i \in \mathbb{Z} \text{ with } M_i \neq 0\}.$$

In order to show the other inequality we proceed by induction on $t = \operatorname{depth}_{A_0}(M)$.

Note that by Lemma 1.1.3, $\operatorname{Ass}_{A_0}(M)$ is a finite set.

If $t = 0$, then $\mathfrak{m}_0 \in \operatorname{Ass}_{A_0}(M)$ and there is an $i \in \mathbb{Z}$ so that $\mathfrak{m}_0 \in \operatorname{Ass}_{A_0}(M_i)$. Thus

$$\inf\{\operatorname{depth}_{A_0}(M_i) \mid i \in \mathbb{Z} \text{ with } M_i \neq 0\} = 0\,.$$

Now assume that $t = \operatorname{depth}_{A_0}(M) > 0$. This implies that

$$\bigcup_{\mathfrak{p}\in\operatorname{Ass}_{A_0}(M)} \mathfrak{p} \neq \mathfrak{m}_0\,.$$

Consider an element

$$r \in \mathfrak{m}_0 \smallsetminus \bigcup_{\mathfrak{p}\in\operatorname{Ass}_{A_0}(M)} \mathfrak{p}.$$

Since r is regular on M, and therefore is regular on M_i for all $i \in \mathbb{Z}$ with $M_i \neq 0$, we obtain

$$\operatorname{depth}_{A_0}(M/rM) = \operatorname{depth}_{A_0}(M) - 1,$$

and for all $i \in \mathbb{Z}$ with $M_i \neq 0$,

$$\operatorname{depth}_{A_0}(M_i/rM_i) = \operatorname{depth}_{A_0}(M_i) - 1\,.$$

By the induction hypothesis

$$\operatorname{depth}_{A_0}(M/rM) = \inf\{\operatorname{depth}_{A_0}(M_i/rM_i) \mid i \in \mathbb{Z} \text{ and } M_i/rM_i \neq 0\}\,.$$

The assertion follows.

(3) For all $i \in \mathbb{Z}$ let $F_\bullet^{(i)}$ be a finite free resolution of M_i. Then

$$F_\bullet = \bigoplus_{i\in\mathbb{Z}} F_\bullet^{(i)}$$

is a free resolution of the A_0-module M yielding

$$\operatorname{projdim}_{A_0}(M) \leq \sup\{\operatorname{projdim}_{A_0}(M_i) \mid i \in \mathbb{Z}\}\,.$$

In order to show the other inequality, assume that $\operatorname{projdim}_{A_0}(M) = r$ and consider for all $i \in \mathbb{Z}$ the rth syzygy $T_r^{(i)}$ of M_i and the exact sequence

$$0 \longrightarrow T_r^{(i)} \longrightarrow F_{r-1}^{(i)} \longrightarrow \ldots \longrightarrow F_0^{(i)} \longrightarrow M_i \longrightarrow 0\,.$$

By taking direct sums we see that

$$\bigoplus_{i\in\mathbb{Z}} T_r^{(i)}$$

is an rth syzygy of M and thus projective. Therefore every $T_r^{(i)}$ is a projective finitely generated A_0-module. Since A_0 is a local Noetherian ring, every $T_r^{(i)}$ is a free A_0-module and thus for all $i \in \mathbb{Z}$

$$\operatorname{projdim}_{A_0}(M_i) \leq r\,.$$

This shows (3). □

1.2.3. **Proposition.** *Let A and M be as above with $(A_0, \mathfrak{m}_0)$ a local ring. Then the Auslander-Buchsbaum formula holds for M as an A_0-module. That is, if* $\operatorname{projdim}_{A_0}(M)$ *is finite, then*

$$\operatorname{depth}_{A_0}(M) + \operatorname{projdim}_{A_0}(M) = \operatorname{depth}(A_0)\,.$$

Proof. Let $\operatorname{projdim}_{A_0}(M) = r < \infty$. Then by Lemma 1.2.2(2) there is an $i \in \mathbb{Z}$ with $\operatorname{projdim}_{A_0}(M) = \operatorname{projdim}_{A_0}(M_i)$, and for all $j \in \mathbb{Z}$

$$\operatorname{projdim}_{A_0}(M_j) \leq r.$$

The Auslander-Buchsbaum formula holds for finitely generated A_0-modules

$$\operatorname{depth}_{A_0}(M_j) + \operatorname{projdim}_{A_0}(M_j) = \operatorname{depth}_{A_0}(A_0) \quad \text{for all} \quad j \in \mathbb{Z},$$

and therefore

$$\operatorname{depth}_{A_0}(M_j) \geq \operatorname{depth}_{A_0}(M_i) \quad \text{for all} \quad j \in \mathbb{Z}.$$

Using Lemma 1.2.2(1), we conclude $\operatorname{depth}_{A_0}(M) = \operatorname{depth}_{A_0}(M_i)$. The Auslander-Buchsbaum formula for M_i then gives the desired formula. □

2. Openness of the codepth locus

Throughout this section we assume that $A = \bigoplus_{i \in \mathbb{N}_0} A_i$ is a graded Noetherian homogeneous ring and that $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a finitely generated A-module. Our aim is to generalize and/or modify existing theorems for finitely generated modules over Noetherian rings to the graded case where the module M is considered a module over the base ring A_0. We begin with a result on the flat locus of the A_0-module M.

2.1. The flat locus of M.

Our first result is a modification of [8, Theorem 24.3]. The proof follows the proof in Matsumura's book. A key observation is that for a finitely generated graded module M the localizations $M_\mathfrak{p}$ are I-adically separated for every ideal $I \subseteq (A_0)_\mathfrak{p}$.

Proposition. *Let A and M be as above. The flat locus of M as an A_0-module*

$$U^0(M) = \{\mathfrak{p} \in \operatorname{Spec}(A_0) \mid M_\mathfrak{p} \textit{ is flat over } A_0\}$$

is open in $\operatorname{Spec}(A_0)$.

Proof. According to Nagata's criterion on the openness of loci [8, Theorem 24.2] we have to show:

(a) If $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(A_0)$ with $\mathfrak{p} \in U^0(M)$ and $\mathfrak{q} \subseteq \mathfrak{p}$, then $\mathfrak{q} \in U^0(M)$.
(b) If $\mathfrak{p} \in U^0(M)$, then $U^0(M)$ contains a nonempty open subset of $V^0(\mathfrak{p}) = \{\mathfrak{n} \in \operatorname{Spec}(A_0) \mid \mathfrak{p} \subseteq \mathfrak{n}\}$.

(a) is trivial. Let $\mathfrak{p} \in U^0(M)$, that is, assume that $M_\mathfrak{p}$ is flat over A_0. Set $\bar{A}_0 = A_0/\mathfrak{p}$. By [8, Theorem 22.3] for every $\mathfrak{q} \in V^0(\mathfrak{p})$ the module $M_\mathfrak{q}$ is flat over A_0 if and only if $(M/\mathfrak{p}M)_\mathfrak{q}$ is flat over $\bar{A}_0$ and $\operatorname{Tor}_1^{A_0}(M_\mathfrak{q}, \bar{A}_0) = 0$. A similar argument as in the proof of [8, Theorem 23.2] shows that $\operatorname{Tor}_1^{A_0}(M, \bar{A}_0)$ is a finitely generated module over A. Therefore there is an element $a \in A_0 \setminus \mathfrak{p}$ so that $(\operatorname{Tor}_1^{A_0}(M, \bar{A}_0))_a = 0$. By applying [8, Theorem 24.1] to the $\bar{A}_0$-module $M/\mathfrak{p}M$ we obtain an element $b \in A_0 \setminus \mathfrak{p}$ so that $(M/\mathfrak{p}M)_b$ is a free $(\bar{A}_0)_b$-module. Set $D^0_{ab} = \{\mathfrak{q} \in \operatorname{Spec}(A_0) \mid ab \notin \mathfrak{q}\}$. Then for all $\mathfrak{q} \in V^0(\mathfrak{p}) \cap D^0_{ab}$ we have that $\operatorname{Tor}_1^{A_0}(M_\mathfrak{q}, \bar{A}_0) = 0$ and that $(M/\mathfrak{p}M)_\mathfrak{q}$ is flat over $(\bar{A}_0)_\mathfrak{q}$. Thus by [8, Theorem 22.3] the module $M_\mathfrak{q}$ is flat over $(A_0)_\mathfrak{q}$ and $M_\mathfrak{q}$ is flat over A_0. □

2.2. **A proposition by Auslander.** As before, let A be a Noetherian graded homogeneous ring and let M be a finitely generated A-module. The following Proposition is an extension of a proposition in EGA [4, (6.11.1) and (6.11.2)] to the (not finitely generated) A_0-module M.

Proposition. *The function* $\gamma : \operatorname{Spec}(A_0) \longrightarrow \mathbb{N}$ *defined by*

$$\gamma(\mathfrak{p}) = \operatorname{projdim}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) \quad \textit{for all} \quad \mathfrak{p} \in \operatorname{Spec}(A_0)$$

is upper semicontinuous. That is, for all $n \in \mathbb{N}$ *the set*

$$U_n^0(M) = \{\mathfrak{p} \in \operatorname{Spec}(A_0) \mid \operatorname{projdim}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) \leq n\}$$

is open in Spec(A_0).

Proof. Note that the ring A is the homomorphic image of the polynomial ring $B = A_0[x_1, \dots, x_t]$, and that, with the standard grading on the polynomial ring B, the graded B-module M is finitely generated. We may replace A by B and assume that A is a graded polynomial ring over A_0. Let $\mathfrak{p} \in \operatorname{Spec}(A_0)$ with $\operatorname{projdim}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) \leq n$.

Consider a graded finitely generated free resolution of the A-module M:

$$F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \xrightarrow{\varphi_1} F_1 \xrightarrow{\varphi_0} M \to 0,$$

where the F_i are finitely generated graded free A-modules and the φ_i are homogeneous A-linear maps. Let T be the nth syzygy of M, yielding an exact sequence of graded A-modules:

$$(*) \qquad 0 \to T \xrightarrow{\delta} F_{n-1} \xrightarrow{\varphi_{n-1}} \dots \xrightarrow{\varphi_1} F_1 \xrightarrow{\varphi_0} M \to 0\,.$$

Since all the homogeneous parts of F_i are free A_0-modules and since T is a graded A-module, we obtain for all $k \in \mathbb{Z}$ an exact sequence of A_0-modules

$$0 \to T_k \xrightarrow{(\delta)_k} (F_{n-1})_k \xrightarrow{(\varphi_{n-1})_k} \dots \xrightarrow{(\varphi_1)_k} (F_1)_k \xrightarrow{(\varphi_0)_k} M_k \to 0$$

with $(F_i)_k$ a finitely generated free A_0-module. Therefore by considering $(*)$ as an exact sequence of A_0-modules we obtain that every module F_i is free over A_0 and T is an nth syzygy of the A_0-module M. Localization at $\mathfrak{p}$ yields exact sequences:

$$0 \to T_\mathfrak{p} \xrightarrow{\delta_\mathfrak{p}} (F_{n-1})_\mathfrak{p} \xrightarrow{(\varphi_{n-1})_\mathfrak{p}} \dots \xrightarrow{(\varphi_1)_\mathfrak{p}} (F_1)_\mathfrak{p} \xrightarrow{(\varphi_0)_\mathfrak{p}} M_\mathfrak{p} \to 0\,.$$

Since $\operatorname{projdim}_{(A_0)p}(M_\mathfrak{p}) \leq n$, it follows that $T_\mathfrak{p}$ is a projective $(A_0)_\mathfrak{p}$-module. Therefore $T_\mathfrak{p}$ is a free $(A_0)_\mathfrak{p}$-module. Since T is a finitely generated graded A-module, it follows from Proposition 2.1 that the set

$$U^0(T) = \{\mathfrak{q} \in \operatorname{Spec}(A_0) \mid T_\mathfrak{q} \quad \text{is a flat over} \quad (A_0)_\mathfrak{q}\}$$

is an open subset of $\operatorname{Spec}(A_0)$. Since T is a finitely generated graded A-module,

$$T = \bigoplus_{i \in \mathbb{Z}} T_i,$$

we have for $\mathfrak{q} \in \operatorname{Spec}(A_0)$

$$T_\mathfrak{q} = \bigoplus_{i \in \mathbb{Z}} (T_i)_\mathfrak{q}\,.$$

If $T_{\mathfrak{q}}$ is flat over $(A_0)_{\mathfrak{q}}$, then, by [1, chapter 1, §2.3, Proposition 2], for all $i \in \mathbb{Z}$, $(T_i)_{\mathfrak{q}}$ is flat over $(A_0)_{\mathfrak{q}}$. Since every $(T_i)_{\mathfrak{q}}$ is a finitely generated $(A_0)_{\mathfrak{q}}$-module, each $(T_i)_{\mathfrak{q}}$ is a free $(A_0)_{\mathfrak{q}}$-module and

$$U^0(T) = \{\mathfrak{q} \in \operatorname{Spec}(A_0) \mid T_{\mathfrak{q}} \quad \text{is a free over} \quad (A_0)_{\mathfrak{q}}\}\,.$$

This shows that $\mathfrak{p} \in U^0(T)$ and

$$U^0(T) \subseteq \{\mathfrak{q} \in \operatorname{Spec}(A_0) \mid \operatorname{projdim}_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) \leq n\}\,.$$

The set $\{\mathfrak{q} \in \operatorname{Spec}(A_0) \mid \operatorname{projdim}_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) \leq n\}$ is thus open in $\operatorname{Spec}(A_0)$. ☐

2.3. **A dimension formula.**

Proposition. *Let A and M be as above. Assume that A_0 is catenary and let $\mathfrak{p}$ be a prime ideal in A_0 with $\mathfrak{p} \in \operatorname{Supp}_{A_0}(M)$. Then there is an open subset U in* $\operatorname{Spec}(A_0)$ *such that $\mathfrak{p} \in U$, and for all $\mathfrak{q} \in U \cap V^0(\mathfrak{p})$ we have*

$$\dim(M_{\mathfrak{q}}) = \dim(M_{\mathfrak{p}}) + \dim((A_0/\mathfrak{p})_{\mathfrak{q}})\,.$$

Proof. Set $S = A_0/\operatorname{ann}_{A_0}(M)$ and choose an element $a \in S \smallsetminus \mathfrak{p}$ so that the following equality on the set of minimal primes holds:

$$\operatorname{Min}(S_{\mathfrak{p}}) = \operatorname{Min}(S_a)\,.$$

Assume that $\dim(M_{\mathfrak{p}}) = \operatorname{ht}(\mathfrak{p}S) = t$ and choose elements $y_1, y_2, \ldots, y_t \in S$ so that

$$\begin{aligned} y_1 &\quad \text{not in a minimal prime of} \quad S_{\mathfrak{p}}, \\ y_2 &\quad \text{not in a minimal prime of} \quad y_1 S_{\mathfrak{p}}, \\ &\ldots \\ y_t &\quad \text{not in a minimal prime of} \quad (y_1, \ldots, y_{t-1})S_{\mathfrak{p}}. \end{aligned}$$

Then there is an element $b \in S \smallsetminus \mathfrak{p}$ so that

$$\begin{aligned} y_1 &\quad \text{not in a minimal prime of} \quad S_b, \\ y_2 &\quad \text{not in a minimal prime of} \quad y_1 S_b, \\ &\ldots \\ y_t &\quad \text{not in a minimal prime of} \quad (y_1, \ldots, y_{t-1})S_b\,. \end{aligned}$$

Let a, b also denote preimages of a and b in A_0 and put $U = D_{ab} = \{\mathfrak{q} \in \operatorname{Spec}(A_0) \mid ab \notin \mathfrak{q}\}$. Then for every $\mathfrak{q} \in U \cap V^0(\mathfrak{p})$ the elements $y_1, \ldots, y_t$ extend to a system of parameters of $S_{\mathfrak{q}}$. Since $S_{\mathfrak{p}}$ and $S_{\mathfrak{q}}$ have the same set of minimal primes and since S is catenary, we obtain that

$$\dim(S_{\mathfrak{q}}) = \dim(S_{\mathfrak{p}}) + \dim((S/\mathfrak{p})_{\mathfrak{q}})\,.$$

This is the same as

$$\dim(M_{\mathfrak{q}}) = \dim(M_{\mathfrak{p}}) + \dim((A_0/\mathfrak{p})_{\mathfrak{q}})\,.$$ ☐

2.4. **The special case of A_0 regular.** Let $(R, \mathfrak{m})$ be a local Noetherian ring and M an R-module. Then we define

$$\operatorname{codepth}_R(M) := \dim_R(M) - \operatorname{depth}_R(M)\,.$$

As usual the depth of the zero module is defined to be ∞, and the dimension of the zero module is $-\infty$, implying that the codepth of the zero module is $-\infty$.

The following proposition extends a result by Auslander [4, (6.11.2)] to the graded case.

Proposition. *Let A and M be as above and assume that A_0 is a homomorphic image of a regular ring. The function* $\varphi\colon \operatorname{Spec}(A_0) \longrightarrow \mathbb{N}$ *defined by*

$$\varphi(\mathfrak{p}) = \operatorname{codepth}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) \quad \textit{for all} \quad \mathfrak{p} \in \operatorname{Spec}(A_0)$$

is upper semicontinuous, that is, for all $n \in \mathbb{N}$, the set

$$U^0_{C_n}(M) = \{\mathfrak{p} \in \operatorname{Spec}(A_0) \mid \operatorname{codepth}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) \le n\}$$

is open in $\operatorname{Spec}(A_0)$.

Proof. If A_0 is a homomorphic image of a regular ring R_0, then the dimension and the depth of the R_0-module M are identical to the dimension and depth of M considered as an R_0-module. If we show that the set

$$\widetilde{U}^0_{C_n}(M) = \{\mathfrak{q} \in \operatorname{Spec}(R_0) \mid \operatorname{codepth}_{(R_0)_\mathfrak{q}}(M_\mathfrak{q}) \le n\}$$

is open in $\operatorname{Spec}(R_0)$ (where M is considered an R_0-module), then the corresponding set for the A_0-module M is given by

$$U^0_{C_n}(M) = \widetilde{U}^0_{C_n}(M) \cap V(J),$$

where $A_0 = R_0/J$. Thus we may assume that A_0 is a regular ring. We may also assume that A is a polynomial ring over A_0 equipped with the standard grading.

Let $\mathfrak{p} \in \operatorname{Spec}(A_0)$. By Proposition 1.2.3, the Auslander-Buchsbaum formula holds:

$$\operatorname{depth}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{depth}((A_0)_\mathfrak{p}) - \operatorname{projdim}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p})\,.$$

Let $I = \operatorname{ann}_{A_0}(M)$. By Lemma 1.1.3, $I_\mathfrak{p} = \operatorname{ann}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p})$, and we have that

$$\dim_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) = \dim((A_0)_\mathfrak{p}) - \operatorname{ht}(I(A_0)_\mathfrak{p})\,.$$

Suppose that $\mathfrak{p} \in \operatorname{Spec}(A_0)$ is such that

$$\operatorname{codepth}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) \le n\,.$$

If $M_\mathfrak{p} = 0$, then $\mathfrak{p} \not\supseteq I$. Take an element $a \in I \cap (A_0 \setminus \mathfrak{p})$. Then for all

$$\mathfrak{q} \in D_a = \{\mathfrak{w} \in \operatorname{Spec}(A_0) \mid a \notin \mathfrak{w}\}$$

we have that $M_\mathfrak{q} = 0$ and $\operatorname{codepth}_{(A_0)_\mathfrak{q}}(M_\mathfrak{q}) = -\infty \le n$.

If $M_\mathfrak{p} \ne 0$ pick an element $a_1 \in A_0 \setminus \mathfrak{p}$ so that $(A_0)_\mathfrak{p}$ and $(A_0)_{a_1}$ have the same minimal primes and put $U_1 = D_{a_1} = \{\mathfrak{w} \in \operatorname{Spec}(A_0) \mid a_1 \notin \mathfrak{w}\}$. Then for all $\mathfrak{q} \in U_1 \cap V^0(I)$,

$$\operatorname{ht}(I(A_0)_\mathfrak{q}) \ge \operatorname{ht}(I(A_0)_\mathfrak{p})\,.$$

Let $\operatorname{projdim}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) = t$. Then by Proposition 2.2 there is an open subset U_2 in $\operatorname{Spec}(A_0)$ so that

$$\operatorname{projdim}_{(A_0)_\mathfrak{q}}(M_\mathfrak{q}) \le t \quad \text{for all} \quad \mathfrak{q} \in U_2\,.$$

Using the Auslander-Buchsbaum formula and the fact that A_0 is regular, we obtain for all $\mathfrak{q} \in U_2 \cap U_1 \cap V^0(I)$:

$$\begin{aligned}\operatorname{codepth}_{(A_0)_\mathfrak{q}}(M_\mathfrak{q}) &= \dim_{(A_0)_\mathfrak{q}}(M_\mathfrak{q}) - \operatorname{depth}_{(A_0)_\mathfrak{q}}(M_\mathfrak{q}) \\ &= \dim((A_0)_\mathfrak{q}) - \operatorname{ht}(I(A_0)_\mathfrak{q}) - \dim((A_0)_\mathfrak{q}) + \operatorname{projdim}_{(A_0)_\mathfrak{q}}(M_\mathfrak{q}) \\ &= \operatorname{projdim}_{(A_0)_\mathfrak{q}}(M_\mathfrak{q}) - \operatorname{ht}(I(A_0)_\mathfrak{q}).\end{aligned}$$

This implies that for all $\mathfrak{q} \in U = U_1 \cap U_2$,

$$\operatorname{codepth}_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) \leq \operatorname{codepth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}),$$

and it follows that $U^0_{C_n}(M)$ is an open subset of $\operatorname{Spec}(A_0)$. □

2.5. **A local formula.** Using the fact that a complete local Noetherian ring is the homomorphic image of a regular local ring, we obtain a result similar to [4, (6.11.5)]:

Lemma. *Let A be a Noetherian graded homogeneous ring and let M be a finitely generated graded A-module. Then for all prime ideals $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(A_0)$ with $\mathfrak{p} \subseteq \mathfrak{q}$ we have that*

$$\operatorname{codepth}_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) \geq \operatorname{codepth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}})\,.$$

Proof. By replacing A_0 by $(A_0)_{\mathfrak{q}}$ (and A by $A_{\mathfrak{q}}$) we may assume that $(A_0, \mathfrak{m}_0)$ is a local ring. Then we have to show

$$\operatorname{codepth}_{A_0}(M) \geq \operatorname{codepth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}})\,.$$

Let $\widehat{\mathfrak{p}} \in \operatorname{Spec}(\widehat{A}_0)$ be a minimal prime ideal over $\mathfrak{p}\widehat{A}_0$. Then $\widehat{\mathfrak{p}} \cap A_0 = \mathfrak{p}$ and $(\widehat{A}_0)_{\widehat{\mathfrak{p}}}$ is flat over $(A_0)_{\mathfrak{p}}$ with trivial special fiber. Moreover,

$$\begin{aligned} M_{\mathfrak{p}} \otimes_{(A_0)_{\mathfrak{p}}} (\widehat{A}_0)_{\widehat{\mathfrak{p}}} &= (\textstyle\bigoplus_{i\in\mathbb{Z}}(M_i)_{\mathfrak{p}}) \otimes_{(A_0)_{\mathfrak{p}}} (\widehat{A}_0)_{\widehat{\mathfrak{p}}} \\ &= \textstyle\bigoplus_{i\in\mathbb{Z}}((M_i)_{\mathfrak{p}} \otimes_{(A_0)_{\mathfrak{p}}} (\widehat{A}_0)_{\widehat{\mathfrak{p}}}) \\ &\cong \textstyle\bigoplus_{i\in\mathbb{Z}}(\widehat{M}_i)_{\widehat{\mathfrak{p}}}, \end{aligned}$$

where $\widehat{M}_i \cong M_i \otimes_{A_0} \widehat{A}_0$. We have that

$$\operatorname{depth}_{A_0}(M) = \inf\{\operatorname{depth}_{A_0}(M_i) \mid M_i \neq 0\},$$
$$\dim_{A_0}(M) = \sup\{\dim_{A_0}(M_i) \mid i \in \mathbb{Z}\}.$$

By [8, Theorem 23.3], for all $i \in \mathbb{Z}$,

$$\begin{aligned} \operatorname{depth}_{(\widehat{A}_0)_{\widehat{\mathfrak{p}}}}((\widehat{M}_i)_{\widehat{\mathfrak{p}}}) &= \operatorname{depth}_{(A_0)_{\mathfrak{p}}}((M_i)_{\mathfrak{p}}) + \operatorname{depth}((\widehat{A}_0)_{\widehat{\mathfrak{p}}}/\mathfrak{p}(\widehat{A}_0)_{\widehat{\mathfrak{p}}}) \\ &= \operatorname{depth}_{(A_0)_{\mathfrak{p}}}((M_i)_{\mathfrak{p}}), \end{aligned}$$

and by [8, Theorem 15.1],

$$\begin{aligned} \dim_{(\widehat{A}_0)_{\widehat{\mathfrak{p}}}}((\widehat{M}_i)_{\widehat{\mathfrak{p}}}) &= \dim_{(A_0)_{\mathfrak{p}}}((M_i)_{\mathfrak{p}}) + \dim((\widehat{A}_0)_{\widehat{\mathfrak{p}}}/\mathfrak{p}(\widehat{A}_0)_{\widehat{\mathfrak{p}}}) \\ &= \dim_{(A_0)_{\mathfrak{p}}}((M_i)_{\mathfrak{p}}). \end{aligned}$$

Let

$$\widetilde{M} := \bigoplus_{i\in\mathbb{Z}} \widehat{M}_i \cong M \otimes_{A_0} \widehat{A}_0,$$

and note that $\widetilde{M}$ is a finitely generated graded module over the Noetherian homogeneous graded ring

$$\widetilde{A} := A \otimes_{A_0} \widehat{A}_0\,.$$

The computation above shows that

$$\operatorname{codepth}_{(\widehat{A}_0)_{\widehat{\mathfrak{p}}}}(\widetilde{M}_{\widehat{\mathfrak{p}}}) = \operatorname{codepth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) =: n\,.$$

Since $\widehat{A}_0$ is a homomorphic image of a regular local ring, by Proposition 2.3 the set $U^0_{C_{n-1}}(\widetilde{M})$ is open in $\operatorname{Spec}(\widehat{A}_0)$. This implies that

$$\operatorname{codepth}_{\widehat{A}_0}(\widetilde{M}) \geq \operatorname{codepth}_{(\widehat{A}_0)_{\widehat{\mathfrak{p}}}}(\widetilde{M}_{\widehat{\mathfrak{p}}})\,.$$

The same argument as above shows that

$$\operatorname{codepth}_{\widehat{A}_0}(\widetilde{M}) = \operatorname{codepth}_{A_0}(M),$$

which proves the claim

$$\operatorname{codepth}_{A_0}(M) \geq \operatorname{codepth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}})\,. \qquad \square$$

2.6. **Formulas for depth and codepth.** In this section we make the same assumption as at the beginning, namely, A is a positively graded Noetherian homogeneous ring and M is a finitely generated graded A-module. The following proposition is the graded version of [4, (6.10.6)]:

2.6.1. **Proposition.** *Let A and M be as above and assume that A is excellent. Then for every $\mathfrak{p} \in \operatorname{Spec}(A_0)$ there is an open subset $U^0 \subseteq \operatorname{Spec}(A_0)$ with $\mathfrak{p} \in U^0$ so that for all $\mathfrak{q} \in U^0 \cap V^0(\mathfrak{p})$,*

$$\operatorname{depth}_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) = \operatorname{depth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \operatorname{depth}((A_0)_{\mathfrak{q}}/\mathfrak{p}(A_0)_{\mathfrak{q}})\,.$$

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(A_0)$. Then by Lemma 2.5 for all $\mathfrak{q} \in V^0(\mathfrak{p})$,

$$\operatorname{codepth}_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) \geq \operatorname{codepth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}),$$

or equivalently,

$$(*) \qquad \dim_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) - \operatorname{depth}_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) \geq \dim_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \operatorname{depth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}})\,.$$

According to Proposition 2.3 there is an open subset $U_1 \subseteq \operatorname{Spec}(A_0)$ with $\mathfrak{p} \in U_1$ so that for all $\mathfrak{q} \in U_1 \cap V^0(\mathfrak{p})$,

$$\dim_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) = \dim_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim((A_0/\mathfrak{p})_{\mathfrak{q}})\,.$$

Since A_0 is excellent, there is an open subset $U_2 \subseteq \operatorname{Spec}(A_0)$ so that $\mathfrak{p} \in U_2$, and for all $\mathfrak{q} \in U_2 \cap V^0(\mathfrak{p})$ the local ring

$$(A_0/\mathfrak{p})_{\mathfrak{q}} \quad \text{is Cohen-Macaulay.}$$

There is also an open subset $U_3 \subseteq \operatorname{Spec}(A_0)$ so that $\mathfrak{p} \in U_3$, and for all $\mathfrak{q} \in U_3 \cap V^0(\mathfrak{p})$ we have equality on the set of minimal primes:

$$\operatorname{Min}_{(A_0)_{\mathfrak{q}}}(I(A_0)_{\mathfrak{q}}) = \operatorname{Min}_{(A_0)_{\mathfrak{p}}}(I(A_0)_{\mathfrak{p}}),$$

where $I := \operatorname{ann}_{A_0}(M)$ denotes the A_0-annihilator of M. In particular, for all $\mathfrak{q} \in U_3 \cap V^0(\mathfrak{p})$,

$$\operatorname{ht}(I(A_0)_{\mathfrak{q}}) = \operatorname{ht}(I(A_0)_{\mathfrak{p}})\,.$$

Put $\widetilde{U}_1 = U_1 \cap U_2 \cap U_3$; then for all $\mathfrak{q} \in \widetilde{U}_1 \cap V^0(\mathfrak{p})$,

$$\dim_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) = \dim((A_0/I)_{\mathfrak{q}}) \quad \text{and} \quad \dim_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \dim((A_0/I)_{\mathfrak{p}}).$$

Since A is excellent, the ring A_0 is universally catenary, and for all $\mathfrak{q} \in \widetilde{U}_1 \cap V^0(\mathfrak{p})$,

$$\dim((A_0/I)_{\mathfrak{q}}) - \dim((A_0/I)_{\mathfrak{p}}) = \dim((A_0/\mathfrak{p})_{\mathfrak{q}}) = \operatorname{depth}((A_0/\mathfrak{p})_{\mathfrak{q}})\,.$$

From $(*)$ we obtain

$$\operatorname{depth}_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) - \operatorname{depth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \operatorname{depth}((A_0/\mathfrak{p})_{\mathfrak{q}})$$

for all $\mathfrak{q} \in \widetilde{U}_1 \cap V^0(\mathfrak{p})$.

In order to prove the other inequality,

$$\operatorname{depth}_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) - \operatorname{depth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \operatorname{depth}((A_0/\mathfrak{p})_{\mathfrak{q}}),$$

assume that $\operatorname{depth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) = t$ and let $f_1, \ldots, f_t \in \mathfrak{p}$ be such that $f_1, \ldots, f_t$ is a regular sequence on $M_{\mathfrak{p}}$. A prime avoidance argument shows that there is an element $a \in A_0 \smallsetminus \mathfrak{p}$ so that $f_1, \ldots, f_t$ is a regular sequence on M_a. (The argument again makes use of the fact that the sets $\operatorname{Ass}_{A_0}(M)$ and $\operatorname{Ass}_{A_0}(M/(f_1, \ldots, f_i)M)$ for all $1 \leq i \leq t$ are finite.)

Put

$$\overline{M} := M/(f_1, \ldots, f_t)M,$$

and consider the associated graded module

$$\operatorname{gr}_{\mathfrak{p}}(\overline{M}) = \bigoplus_{i \in \mathbb{N}} \mathfrak{p}^i \overline{M}/\mathfrak{p}^{i+1}\overline{M}.$$

The module $\overline{M}$ is finitely generated over A, and $\operatorname{gr}_{\mathfrak{p}}(\overline{M})$ is a finitely generated $\operatorname{gr}_{\mathfrak{p}}(A)$-module. Also note that $\operatorname{gr}_{\mathfrak{p}}(A)$ is a finitely generated algebra over $A/\mathfrak{p}A$ and that $A/\mathfrak{p}A$ is a finitely generated algebra over $A_0/\mathfrak{p}$. Thus $\operatorname{gr}_{\mathfrak{p}}(A)$ is a finitely generated $A_0/\mathfrak{p}$-algebra. By [8, Theorem 24.1] there is an element $b \in A_0 \smallsetminus \mathfrak{p}$ so that the $(A_0/\mathfrak{p})_b$-module

$$\operatorname{gr}_{\mathfrak{p}}(\overline{M})_b = \bigoplus_{i \in \mathbb{N}} (\mathfrak{p}^i \overline{M}/\mathfrak{p}^{i+1}\overline{M})_b$$

is free. Set $\widetilde{U}_2 = D_b = \{\mathfrak{q} \in \operatorname{Spec}(A_0) \mid b \notin \mathfrak{q}\}$ and fix a prime ideal $\mathfrak{q} \in \widetilde{U}_2 \cap V^0(\mathfrak{p})$. Assume that

$$\operatorname{depth}((A_0/\mathfrak{p}))_{\mathfrak{q}} = s,$$

and let $g_1, \ldots, g_s \in \mathfrak{q}$ be such that $g_1, \ldots, g_s$ is a regular sequence on $(A_0/\mathfrak{p})_{\mathfrak{q}}$.

Claim 1. g_1 is a regular element on $\overline{M}_{\mathfrak{q}}$.

Claim 2. Set $N_1 := \overline{M}_{\mathfrak{q}}/g_1\overline{M}_{\mathfrak{q}}$; then $\operatorname{gr}_{\mathfrak{p}}(N_1) \cong \operatorname{gr}_{\mathfrak{p}}(\overline{M}_{\mathfrak{q}})/g_1 \operatorname{gr}_{\mathfrak{p}}(\overline{M}_{\mathfrak{q}})$.

Assuming the claims, we finish the proof. From the second claim it follows that $\operatorname{gr}_{\mathfrak{p}}(N_1)$ is a free $(A_0/(g_1, \mathfrak{p})A_0)_{\mathfrak{q}}$-module. Since g_2 is a regular element on $(A_0/(g_1, \mathfrak{p})A_0)_{\mathfrak{q}}$, we may apply Claims 1 and 2 to N_1. Note that N_1 is also a finitely generated graded $A_{\mathfrak{q}}$-module. This yields that g_2 is a regular element on N_1 and that with $N_2 = N_1/g_2N_1$,

$$\operatorname{gr}_{\mathfrak{p}}(N_2) \cong \operatorname{gr}_{\mathfrak{p}}(N_1)/g_2 \operatorname{gr}_{\mathfrak{p}}(N_1)\,.$$

An induction argument yields that $g_1, \ldots, g_s$ is a regular sequence on $\overline{M}_{\mathfrak{q}}$, and we have that

$$\operatorname{depth}_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) \geq \operatorname{depth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \operatorname{depth}((A_0/\mathfrak{p})_{\mathfrak{q}}\,.$$

This inequality holds for all $\mathfrak{q} \in \widetilde{U}_2 \cap V^0(\mathfrak{p})$. Assuming the claims the proposition is now proved with $U^0 = \widetilde{U}_1 \cap \widetilde{U}_2$.

In order to prove the claims, set $g = g_1$ and $N = N_1$.

Proof of Claim 1. Let $z \in \overline{M}_{\mathfrak{q}}$ with $gz = 0$. Consider the image $\bar{z}$ of z in $\overline{M}_{\mathfrak{q}}/\mathfrak{p}\overline{M}_{\mathfrak{q}}$. Since $\overline{M}_{\mathfrak{q}}/\mathfrak{p}\overline{M}_{\mathfrak{q}}$ is a free module over $(A_0/\mathfrak{p})_{\mathfrak{q}}$ and since g is regular on $(A_0/\mathfrak{p})_{\mathfrak{q}}$, we obtain that $\bar{z} = 0$ and $z \in \mathfrak{p}\overline{M}_{\mathfrak{q}}$. Now consider the image of z in $\mathfrak{p}\overline{M}_{\mathfrak{q}}/\mathfrak{p}^2\overline{M}_{\mathfrak{q}}$ and repeat the argument. This yields

$$z \in \bigcap_{j=0}^{\infty} \mathfrak{p}^j \overline{M}_{\mathfrak{q}}\,.$$

Note that

$$\overline{M}_{\mathfrak{q}} = \bigoplus_{i\in\mathbb{Z}} (\overline{M}_i)_{\mathfrak{q}} \quad \text{with} \quad (\overline{M}_i)_{\mathfrak{q}} = (M_i)_{\mathfrak{q}}/(f_1,\dots,f_t)(M_i)_{\mathfrak{q}}\,.$$

In particular,

$$\mathfrak{p}^j\overline{M}_{\mathfrak{q}} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{p}^j(\overline{M}_i)_{\mathfrak{q}},$$

and every $(\overline{M}_i)_{\mathfrak{q}}$ is a finitely generated $(A_0)_{\mathfrak{q}}$-module. This shows that $z = 0$.

Proof of Claim 2. By assumption, we have that $\operatorname{gr}_{\mathfrak{p}}(\overline{M}_{\mathfrak{q}})$ is a free $(A_0/\mathfrak{p})_{\mathfrak{q}}$- module and $\mathfrak{p}^j\overline{M}_{\mathfrak{q}}/\mathfrak{p}^{j+1}\overline{M}_{\mathfrak{q}}$ is a direct summand of $\operatorname{gr}_{\mathfrak{p}}(\overline{M}_{\mathfrak{q}})$. Thus $\mathfrak{p}^j\overline{M}_{\mathfrak{q}}/\mathfrak{p}^{j+1}\overline{M}_{\mathfrak{q}}$ is a free $(A_0/\mathfrak{p})_{\mathfrak{q}}$-module and g is regular on $(A_0/\mathfrak{p})_{\mathfrak{q}}$. Therefore

$$(**) \qquad \mathfrak{p}^j\overline{M}_{\mathfrak{q}} \cap g\overline{M}_{\mathfrak{q}} = g\mathfrak{p}^j\overline{M}_{\mathfrak{q}}$$

and thus

$$\begin{aligned} \mathfrak{p}^j\overline{M}_{\mathfrak{q}}/g\mathfrak{p}^j\overline{M}_{\mathfrak{q}} &\cong \mathfrak{p}^j\overline{M}_{\mathfrak{q}}/(\mathfrak{p}^j\overline{M}_{\mathfrak{q}} \cap g\overline{M}_{\mathfrak{q}}) \\ &\cong \mathfrak{p}^j(\overline{M}_{\mathfrak{q}}/g\overline{M}_{\mathfrak{q}}). \end{aligned}$$

From the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{p}^{j+1}N \arrow[r] \arrow[d, equal] & \mathfrak{p}^jN \arrow[r] \arrow[d, equal] & \mathfrak{p}^jN/\mathfrak{p}^{j+1}N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{p}^{j+1}(\overline{M}_{\mathfrak{q}}/g\overline{M}_{\mathfrak{q}}) \arrow[r] \arrow[d, "\cong"] & \mathfrak{p}^j(\overline{M}_{\mathfrak{q}}/g\overline{M}_{\mathfrak{q}}) \arrow[r] \arrow[d, "\cong"] & \mathfrak{p}^j(\overline{M}_{\mathfrak{q}}/g\overline{M}_{\mathfrak{q}})/\mathfrak{p}^{j+1}(\overline{M}_{\mathfrak{q}}/g\overline{M}_{\mathfrak{q}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{p}^{j+1}\overline{M}_{\mathfrak{q}}/g\mathfrak{p}^{j+1}\overline{M}_{\mathfrak{q}} \arrow[r] & \mathfrak{p}^j\overline{M}_{\mathfrak{q}}/g\mathfrak{p}^j\overline{M}_{\mathfrak{q}} \arrow[r] & \mathfrak{p}^j\overline{M}_{\mathfrak{q}}/(g\mathfrak{p}^j\overline{M}_{\mathfrak{q}} + \mathfrak{p}^{j+1}\overline{M}_{\mathfrak{q}}) \arrow[r] & 0
\end{tikzcd}$$

we obtain that

$$\begin{aligned} \operatorname{gr}_{\mathfrak{p}}(N) &= \textstyle\bigoplus_{j\in\mathbb{N}} \mathfrak{p}^jN/\mathfrak{p}^{j+1}N \\ &\cong \textstyle\bigoplus_{j\in\mathbb{N}} \mathfrak{p}^j\overline{M}_{\mathfrak{q}}/(g\mathfrak{p}^j\overline{M}_{\mathfrak{q}} + \mathfrak{p}^{j+1}\overline{M}_{\mathfrak{q}}) \\ &\cong \textstyle\bigoplus_{j\in\mathbb{N}} (\mathfrak{p}^j\overline{M}_{\mathfrak{q}}/\mathfrak{p}^{j+1}\overline{M}_{\mathfrak{q}})/g(\mathfrak{p}^j\overline{M}_{\mathfrak{q}}/\mathfrak{p}^{j+1}\overline{M}_{\mathfrak{q}}) \\ &\cong \operatorname{gr}_{\mathfrak{p}}(\overline{M}_{\mathfrak{q}})/g(\operatorname{gr}(\overline{M}_{\mathfrak{q}})). \end{aligned}$$

This proves the claim, and finishes the proof. ☐

Similar to [4, (6.11.8.1)] we have in the graded case:

2.6.2. **Corollary.** *Let A and M be as above and assume that A is excellent. Then for every $\mathfrak{p} \in \operatorname{Spec}(A_0)$ there is an open subset $U^0 \subseteq \operatorname{Spec}(A_0)$ with $\mathfrak{p} \in U^0$, so that for all $\mathfrak{q} \in U^0 \cap V^0(\mathfrak{p})$,*

$$\operatorname{codepth}_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) = \operatorname{codepth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \operatorname{codepth}((A_0)_{\mathfrak{q}}/\mathfrak{p}(A_0)_{\mathfrak{q}})\,.$$

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(A_0)$ and let U_1^0 be as in Proposition 2.6.1, so that $\mathfrak{p} \in U_1^0$, and for all $\mathfrak{q} \in U_1^0 \cap V^0(\mathfrak{p})$,

$$\operatorname{depth}_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) = \operatorname{depth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \operatorname{depth}((A_0)_{\mathfrak{q}}/\mathfrak{p}(A_0)_{\mathfrak{q}})\,.$$

By Proposition 2.3 there is an open subset U_2^0 in $\operatorname{Spec}(A_0)$, so that $\mathfrak{p} \in U_2^0$, and for all $\mathfrak{q} \in U_2^0 \cap V^0(\mathfrak{p})$,

$$\dim_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) = \dim_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim((A_0/\mathfrak{p})_{\mathfrak{q}})\,.$$

Thus with $U^0 = U_1^0 \cap U_2^0$ we have that $\mathfrak{p} \in U^0$, and for all $\mathfrak{q} \in U^0 \cap V^0(\mathfrak{p})$,

$$\operatorname{codepth}_{(A_0)_{\mathfrak{q}}}(M_{\mathfrak{q}}) = \operatorname{codepth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \operatorname{codepth}((A_0)_{\mathfrak{q}}/\mathfrak{p}(A_0)_{\mathfrak{q}})\,. \qquad ☐$$

We are now ready to prove the graded version of [4, (6.11.2)(a)].

2.6.3. **Theorem.** *Let* $A = \bigoplus_{i\in\mathbb{N}} A_i$ *be an excellent graded homogeneous ring and let* $M = \bigoplus_{i\in\mathbb{Z}} M_i$ *be a finitely generated graded A-module. Then for all* $n \in \mathbb{N}$ *the set*

$$U^0_{C_n}(M) = \{\mathfrak{p} \in \operatorname{Spec}(A_0) \mid \operatorname{codepth}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) \le n\}$$

is open in $\operatorname{Spec}(A_0)$.

Proof. According to Nagata's criterion on openness of loci (see [8, Theorem 24.2]) we need to show:

(a) If $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(A_0)$ with $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{p} \in U^0_{C_n}(M)$, then $\mathfrak{q} \in U^0_{C_n}(M)$.

(b) If $\mathfrak{p} \in U^0_{C_n}(M)$, then $U^0_{C_n}(M)$ contains a nonempty open subset of $V(\mathfrak{p})$.

(a) Let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(A_0)$ with $\mathfrak{q} \subseteq \mathfrak{p}$. By Lemma 2.5

$$\operatorname{codepth}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) \ge \operatorname{codepth}_{(A_0)_\mathfrak{q}}(M_\mathfrak{q}),$$

and thus $\mathfrak{p} \in U^0_{C_n}(M)$ implies that $\mathfrak{q} \in U^0_{C_n}(M)$.

(b) Let $\mathfrak{p} \in U^0_{C_n}(M)$. By Corollary 2.6.2 there is an open subset U^0_1 in $\operatorname{Spec}(A_0)$, so that $\mathfrak{p} \in U^0_1$, and for all $\mathfrak{q} \in U^0_1 \cap V^0(\mathfrak{p})$,

$$\operatorname{codepth}_{(A_0)_\mathfrak{q}}(M_\mathfrak{q}) = \operatorname{codepth}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) + \operatorname{codepth}((A_0)_\mathfrak{q}/\mathfrak{p}(A_0)_\mathfrak{q})\,.$$

Since A and A_0 are excellent, there is an open subset U^0_2 in $\operatorname{Spec}(A_0)$, so that $\mathfrak{p} \in U^0_2$, and for all $\mathfrak{q} \in U^0_2 \cap V^0(\mathfrak{p})$, the ring $(A_0/\mathfrak{p})_\mathfrak{q}$ is Cohen-Macaulay. Therefore with $U^0 = U^0_1 \cap U^0_2$ we have that $\mathfrak{p} \in U^0$, and for all $\mathfrak{q} \in U^0 \cap V^0(\mathfrak{p})$,

$$\operatorname{codepth}_{(A_0)_\mathfrak{q}}(M_\mathfrak{q}) = \operatorname{codepth}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p})\,.$$

This implies that $U^0 \cap V^0(\mathfrak{p}) \subseteq U^0_{C_n}(M)$, and the theorem is proved. □

2.6.4. **Corollary.** *Let A and M be as in Theorem* 2.6.3*. Then the Cohen-Macaulay locus of the* A_0*-module M,*

$$U^0_{CM}(M) = U^0_{C_0}(M) = \{\mathfrak{p} \in \operatorname{Spec}(A_0) \mid M_\mathfrak{p} \textit{ is a CM module over } (A_0)_\mathfrak{p}\},$$

is open in $\operatorname{Spec}(A_0)$. □

3. OPENNESS OF THE (S_n)-LOCUS

Throughout this section we assume that $R = A_0$ is the base ring of a graded Noetherian homogeneous ring $A = \bigoplus_{i\ge 0} A_i$ and M is a finitely generated graded A-module. This includes the case of a finitely generated module M over a Noetherian ring R. For those modules we prove that the openness of the C_n-loci of M implies the openness of the (S_k)-loci of M. The argument is due to Grothendieck [4, (5.7.2) and (6.11.2)(b)], but we include it here for the convenience of the reader. The proof also shows that the (S_k)-loci of M only depend on the C_n-loci of M and on the annihilator of M, so that two R-modules M and N with the same annihilators and C_n-loci have identical (S_k)-loci.

Let M be an R-module and suppose that for all $n \in \mathbb{N}_0$, the set

$$U_{C_n}(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{codepth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \le n\}$$

is open in $\operatorname{Spec}(R)$. Define

$$Z_n = V(\mathfrak{b}_n) = \operatorname{Spec}(R) \smallsetminus U_{C_n}(M),$$

where $\mathfrak{b}_n \subseteq R$ is a reduced ideal. Obviously, for all $n \in \mathbb{N}$,

$$U_{C_n}(M) \subseteq U_{C_{n+1}}(M),$$

and therefore

$$Z_{n+1} \subseteq Z_n \quad \text{and} \quad \mathfrak{b}_n \subseteq \mathfrak{b}_{n+1}\,.$$

Since R is Noetherian, there is an $m \in \mathbb{N}$ so that for all $t \in \mathbb{N}$,

$$\mathfrak{b}_m = \mathfrak{b}_{m+t} \quad \text{and} \quad Z_m = Z_{m+t}\,.$$

3.1. **Lemma.** *Let $m \in \mathbb{N}$ be as above. Then $Z_m = \emptyset$.*

Proof. If $\mathfrak{p} \in Z_m$, then $\mathfrak{p} \in Z_{m+t}$ for all $t \in \mathbb{N}$. By definition of Z_{n+t},

$$\operatorname{codepth}_{(R)_\mathfrak{p}}(M_\mathfrak{p}) \geq m + t \quad \text{for all} \quad t \in \mathbb{N}\,.$$

But $\operatorname{codepth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \dim((R)_\mathfrak{p}) \leq \infty$, and therefore $Z_m = \emptyset$. □

Recall that the R-module M satisfies Serre's condition (S_k) if for all $\mathfrak{p} \in \operatorname{Spec}(R)$,

$$(*) \qquad \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq \min(\dim(M_\mathfrak{p}), k).$$

From now on let m denote the minimal $m \in \mathbb{N}$ with $Z_m = \emptyset$.

3.2. **Lemma.** *With the assumptions as above put $\overline{R} = R/\operatorname{ann}_R(M)$ and let $k \in \mathbb{N}$. Then the R-module M satisfies (S_k) if and only if for all $0 \leq n < m$,*

$$\operatorname{ht}(\mathfrak{b}_n\overline{R}) > n + k\,.$$

Proof. Suppose that M satisfies (S_k), and fix an integer n with $0 \leq n < m$. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\mathfrak{b}_n \subseteq \mathfrak{p}$. Then $\mathfrak{p} \in Z_n$, and therefore

$$\operatorname{codepth}_{R_\mathfrak{p}}(M_\mathfrak{p}) > n,$$

or equivalently,

$$\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) - \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) > n\,.$$

Since M satisfies (S_k), we obtain that whenever

$$\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) - \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0,$$

then

$$\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq k\,.$$

Thus, if $\mathfrak{p} \in Z_n$, then

$$\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq n + k,$$

which implies that $\operatorname{ht}(\mathfrak{b}_n\overline{R}) \geq n + k$.

Conversely, fix an integer k and assume that for all $0 \leq n < m$,

$$\operatorname{ht}(\mathfrak{b}_n\overline{R}) > n + k\,.$$

Let $\mathfrak{p} \in \operatorname{Spec}(R)$.

If $M_\mathfrak{p} = 0$, then $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \infty$, and condition $(*)$ is satisfied.

Now assume $M_\mathfrak{p} \neq 0$. If $M_\mathfrak{p}$ is a Cohen-Macaulay R-module, then condition $(*)$ is satisfied. Now assume that

$$\operatorname{codepth}_{R_\mathfrak{p}}(M_\mathfrak{p}) > 0,$$

and let $n \in \mathbb{N}_0$ with

$$\operatorname{codepth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = n + 1\,.$$

Thus $\mathfrak{p} \in Z_n$ and $\mathfrak{b}_n \subseteq \mathfrak{p}$. By assumption,

$$\operatorname{ht}(\mathfrak{b}_n\overline{R}) > n + k \;\Rightarrow\; \operatorname{ht}(\mathfrak{b}_n\overline{R}_\mathfrak{p}) > n + k \;\Rightarrow\; \dim(\overline{R}_\mathfrak{p}) > n + k\,.$$

This implies that

$$\begin{aligned}\operatorname{codepth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) &= n+1\\ &= \dim(\overline{R}_{\mathfrak{p}}) - \operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})\\ &\geq n+1+k-\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}),\end{aligned}$$

and therefore

$$\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq k\,.$$

Thus $M_{\mathfrak{p}}$ satisfies condition $(*)$, and the R-module M satisfies Serre's condition (S_k). $\square$

For all $0 \leq n < m$ consider the closed subset of $\operatorname{Spec}(R)$,

$$Y_{n,k} = \{\mathfrak{q} \in V(\mathfrak{b}_n) \mid \operatorname{ht}(\mathfrak{b}_n \overline{R}_{\mathfrak{q}}) \leq n+k\}\,,$$

and its complement

$$V_{n,k} = \operatorname{Spec}(R) - Y_{n,k}$$

an open subset of $\operatorname{Spec}(R)$. By Lemma 3.2

$$U_{S_k}(M) = \bigcap_{0 \leq n < m} V_{n,k}$$

is an open subset of $\operatorname{Spec}(R)$. We have shown:

3.3. **Theorem.** *Let M be an R-module as above. If for all $n \in \mathbb{N}_0$ the C_n-locus $U_{C_n}(M)$ is open in* $\operatorname{Spec}(R)$*, then for all $k \in \mathbb{N}$, the (S_k)-locus*

$$U_{S_k}(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_{\mathfrak{p}} \textit{ satisfies } (S_k)\}$$

is open in $\operatorname{Spec}(R)$.

In the graded case the theorem states:

3.4. **Corollary.** *Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be an excellent graded homogeneous ring and let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a finitely generated graded A-module. Then for all $k \in \mathbb{N}$, the set*

$$U^0_{S_k}(M) = \{\mathfrak{p} \in \operatorname{Spec}(A_0) \mid \textit{ the } (A_0)_{\mathfrak{p}}\textit{-module } M_{\mathfrak{p}} \textit{ satisfies } (S_k)\}$$

is open in $\operatorname{Spec}(A_0)$.

The proof of the theorem also yields the following corollary:

3.5. **Corollary.** *Suppose that M and N are R-modules as above. Assume that $ann_R(M) = ann_R(N)$ and that for all $n \in \mathbb{N}_0$, the sets $U_{C_n}(M) = U_{C_n}(N)$ are open in* $\operatorname{Spec}(R)$*. Then for all $k \in \mathbb{N}$,*

$$U_{S_k}(M) = U_{S_k}(N),$$

and the (S_k)-loci are open subsets of $\operatorname{Spec}(R)$.

4. Stability on the homogeneous parts

Let $A=\bigoplus_{i\in\mathbb{N}}A_i$ be an excellent graded homogeneous Noetherian ring and let $M=\bigoplus_{i\in\mathbb{Z}}M_i$ be a finitely generated graded A-module. In this section we prove that there is a $k\in\mathbb{N}$, so that for all $n\in\mathbb{N}$ and all $i\geq k$,

$$U^0_{C_n}(M_i)=U^0_{C_n}(M_k)\quad\text{and}\quad U^0_{S_n}(M_i)=U^0_{S_n}(M_k),$$

that is, the codepth and (S_n)-loci of the homogeneous parts of M are eventually stable (considered as an A_0-module). As before we define for all $t\in\mathbb{Z}$

$$N_t=\bigoplus_{i\geq t}M_i,$$

and observe the following simple facts: Let $k_1\in\mathbb{N}$ be an integer so that for all $t\geq k_1$, $\operatorname{ann}_{A_0}(M_t)=\operatorname{ann}_{A_0}(M_{k_1})$. Then for all $t\geq k_1$,

$$U^0_{C_n}(N_t)\supseteq U^0_{C_n}(N_{k_1})\quad\text{and}\quad U^0_{S_n}(N_t)\supseteq U^0_{S_n}(N_{k_1})\,.$$

Since A_0 is Noetherian, there is an integer $k_2\in\mathbb{Z}$, so that $k_2\geq k_1$ and

$$U^0_{C_n}(N_t)=U^0_{C_n}(N_{k_2})\quad\text{and}\quad U^0_{S_n}(N_t)=U^0_{S_n}(N_{k_2})\,.$$

We may also assume for large enough k_2 that

$$N_{k_2}=AM_{k_2},$$

which implies that for all $t\geq k_2$,

$$N_t=AM_t.$$

4.1. **Lemma.** *With the assumptions as above assume additionally that $(A_0,\mathfrak{m}_0)$ is a local ring. Then there is a $k_3\in\mathbb{Z}$, so that for all $t\geq k_3$,*

$$\operatorname{depth}_{A_0}(M_t)=\operatorname{depth}_{A_0}(M_{k_3})=\operatorname{depth}_{A_0}(N_{k_3})\,.$$

Proof. Let k_1 and k_2 be as above and take an integer k with $k>k_2$. Then $\operatorname{codepth}_{A_0}(N_k)=n$ for some $n\in\mathbb{N}$, and therefore

$$\mathfrak{m}_0\in U^0_{C_n}(N_k)\quad\text{and}\quad \mathfrak{m}_0\notin U^0_{C_{n-1}}(N_k)\,.$$

Since $k\geq k_2$, we have for all $t\geq k$

$$\operatorname{codepth}_{A_0}(N_k)=n=\operatorname{codepth}_{A_0}(N_t)\,.$$

For all $t\geq k_1$ we also have that $\operatorname{ann}_{A_0}(N_t)=\operatorname{ann}_{A_0}(N_k)$, and therefore for all $t\geq k$,

$$\operatorname{depth}_{A_0}(N_t)=s=\operatorname{depth}_{A_0}(N_k)\,.$$

Let $r_1,\dots,r_s$ be a maximal regular sequence on N_k and put

$$\overline{N}_k=N_k/(r_1,\dots,r_s)N_k\quad\text{with homogeneous parts}\quad \overline{M}_i=M_i/(r_1,\dots,r_s)M_i$$

for $i\geq k$. Note that the torsion submodule $\Gamma_{A_+}(\overline{N}_k)$ is a finitely generated A-submodule of $\overline{N}_k$. This implies that there is an integer $k_3\geq k$ so that $\Gamma_{A_+}(\overline{N}_k)\cap N_{k_3}=0=\Gamma_{A_+}(\overline{N}_{k_3})$. Thus for k_3 large enough the A-module $\overline{N}_{k_3}$ is A_+-torsion-free. Since by assumption $\operatorname{depth}_{A_0}(N_k)=s=\operatorname{depth}_{A_0}(N_{k_3})$, there is an integer $i\geq k_3$ and an element $\bar{x}\in\overline{M}_i$ so that $\bar{x}\neq 0$ and $\mathfrak{m}_0\bar{x}=0$. Since $\overline{N}_{k_3}$ is A_+-torsion-free, we obtain

$$(A_+)^l\bar{x}\neq 0\quad\text{for all}\quad l\in\mathbb{N}\,.$$

Thus for $k_4 = i > k_3$ we have that $\operatorname{depth}_{A_0}(\overline{M}_{k_4+l}) = 0$ for all $l \in \mathbb{N}_0$, and therefore for all $t \geq k_4$,

$$\operatorname{depth}_{A_0}(M_t) = \operatorname{depth}_{A_0}(M_{k_4}) = s. \qquad \square$$

Choose an integer $k_0 \in \mathbb{Z}$ so that the following conditions are satisfied:

(a) $N_{k_0} = AM_{k_0}$, that is, N_{k_0} is generated in the lowest nonvanishing degree.
(b) For all $t \geq k_0$, $\operatorname{ann}(M_{k_0}) = \operatorname{ann}(M_t)$.
(c) For all $n \in \mathbb{N}_0$ and all $t \geq k_0$,

$$U^0_{C_n}(N_t) = U^0_{C_n}(N_{k_0}) \quad \text{and} \quad U^0_{S_n}(N_t) = U^0_{S_n}(N_{k_0}).$$

As before put

$$Z_n = \operatorname{Spec}(A_0) \smallsetminus U^0_{C_n}(N_{k_0}) = V(\mathfrak{b}_n),$$

where $\mathfrak{b}_n \subseteq A_0$ is a reduced ideal. Then $\mathfrak{b}_n \subseteq \mathfrak{b}_{n+1}$, yielding an increasing sequence of ideals

$$\mathfrak{b}_0 \subseteq \mathfrak{b}_1 \subseteq \ldots \subseteq \mathfrak{b}_{m-1} \subseteq \ldots .$$

We have seen before that the sequence stops with some $\mathfrak{b}_m = A_0$, and let m be minimal with this property, that is, let $\mathfrak{b}_m = A_0$ and $\mathfrak{b}_{m-1} \neq A_0$. For all $0 \leq j \leq m-1$ we consider the set of minimal prime divisors of $\mathfrak{b}_j$:

$$\operatorname{Min}(A_0/\mathfrak{b}_j) = \{\mathfrak{p}_{j1}, \ldots, \mathfrak{p}_{jr_j}\}.$$

By Lemma 4.1, for all $0 \leq j \leq m-1$ and all $r_j \geq h \geq 1$, there is an integer $k_{jh} \in \mathbb{N}$ with $k_{jh} \geq k_0$, so that for all $i \geq k_{jh}$,

$$\operatorname{depth}_{(A_0)_{\mathfrak{p}_{jh}}}((M_i)_{\mathfrak{p}_{jh}}) = \operatorname{depth}_{(A_0)_{\mathfrak{p}_{jh}}}((M_{k_{jh}})_{\mathfrak{p}_{jh}}) = \text{constant}.$$

Let $k = \max\{k_{jh} \mid 0 \leq j \leq m-1; 1 \leq h \leq r_j\}$. Then for all $i \geq k$,

$$\operatorname{depth}_{(A_0)_{\mathfrak{p}_{jh}}}((M_i)_{\mathfrak{p}_{jh}}) = \operatorname{depth}_{(A_0)_{\mathfrak{p}_{jh}}}((M_k)_{\mathfrak{p}_{jh}}) = \operatorname{depth}_{(A_0)_{\mathfrak{p}_{jh}}}((N_k)_{\mathfrak{p}_{jh}}).$$

By assumption on the annihilators we also have for all $i \geq k$

$$\dim_{(A_0)_{\mathfrak{p}_{jh}}}((M_i)_{\mathfrak{p}_{jh}}) = \dim_{(A_0)_{\mathfrak{p}_{jh}}}((M_k)_{\mathfrak{p}_{jh}}) = \dim_{(A_0)_{\mathfrak{p}_{jh}}}((N_k)_{\mathfrak{p}_{jh}}),$$

which implies that for all $i \geq k$ and all primes $\mathfrak{p}_{jh}$,

$$\operatorname{codepth}_{(A_0)_{\mathfrak{p}_{jh}}}((M_i)_{\mathfrak{p}_{jh}}) = \operatorname{codepth}_{(A_0)_{\mathfrak{p}_{jh}}}((M_k)_{\mathfrak{p}_{jh}}) = \operatorname{codepth}_{(A_0)_{\mathfrak{p}_{jh}}}((N_k)_{\mathfrak{p}_{jh}}).$$

We are now ready to prove:

4.2. **Theorem.** *Let k be as above. Then for all $i \geq k$ and all $\mathfrak{p} \in \operatorname{Spec}(A_0)$,*

$$\operatorname{codepth}_{(A_0)_{\mathfrak{p}}}((M_i)_{\mathfrak{p}}) = \operatorname{codepth}_{(A_0)_{\mathfrak{p}}}((M_k)_{\mathfrak{p}}).$$

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(A_0)$. If $\mathfrak{b}_0 \not\subseteq \mathfrak{p}$, then $(N_k)_{\mathfrak{p}}$ is a Cohen-Macaulay module over $(A_0)_{\mathfrak{p}}$. It follows that $(M_i)_{\mathfrak{p}}$ is Cohen-Macaulay for all $i \geq k$.

Assume that $\mathfrak{b}_0 \subseteq \mathfrak{p}$ and let g be minimal so that $\mathfrak{b}_g \subseteq \mathfrak{p}$ and $\mathfrak{b}_{g+1} \not\subseteq \mathfrak{p}$. In this case $\operatorname{codepth}_{(A_0)_{\mathfrak{p}}}((N_k)_{\mathfrak{p}}) = g+1$, and there is an integer $1 \leq j \leq r_j$ so that $\mathfrak{p}_{gj} \subseteq \mathfrak{p}$. By [4, (6.11.5)], the nongraded version of Lemma 2.5, for all $i \geq k$,

$$\operatorname{codepth}_{(A_0)_{\mathfrak{p}}}((M_i)_{\mathfrak{p}}) \geq \operatorname{codepth}_{(A_0)_{p_{gj}}}((M_i)_{p_{gj}}) = \operatorname{codepth}_{(A_0)_{\mathfrak{p}_{gj}}}((N_k)_{\mathfrak{p}_{gj}}) > g.$$

In order to verify the other inequality consider

$$\operatorname{codepth}_{(A_0)_{\mathfrak{p}}}((N_k)_{\mathfrak{p}}) = g+1 = \dim((N_k)_{\mathfrak{p}}) - \operatorname{depth}_{(A_0)_{\mathfrak{p}}}((N_k)_{\mathfrak{p}}),$$

and assume that $\operatorname{depth}_{(A_0)_{\mathfrak{p}}}((N_k)_{\mathfrak{p}}) = s$. Let $x_1, \dots, x_s$ be a regular sequence on $(N_k)_{\mathfrak{p}}$. Then $x_1, \dots, x_s$ is a regular sequence on $(M_i)_{\mathfrak{p}}$ for all $i \geq k$. Since N_k and M_i have the same annihilators, we obtain that

$$\operatorname{codepth}_{(A_0)_{\mathfrak{p}}}((N_k)_{\mathfrak{p}}) = g + 1 \geq \operatorname{codepth}_{(A_0)_{\mathfrak{p}}}((M_i)_{\mathfrak{p}})$$

for all $i \geq k$. This shows that for all $i \geq k$,

$$\operatorname{codepth}_{(A_0)_{\mathfrak{p}}}((M_i)_{\mathfrak{p}}) = g + 1 \,.$$

□

4.3. **Corollary.** *There is an integer $k \in \mathbb{N}$ so that for all $i \geq k$ and all $n \in \mathbb{N}$,*

$$U^0_{C_n}(M_i) = U^0_{C_n}(M_k) = U^0_{C_n}(N_k) \,.$$ □

4.4. **Corollary.** *There is an integer $k \in \mathbb{N}$ so that for all $i \geq k$ and all $n \in \mathbb{N}$,*

$$U^0_{S_n}(M_i) = U^0_{S_n}(M_k) = U^0_{S_n}(N_k) \,.$$

Proof. The second corollary follows from the first by using Corollary 3.5. □

5. Applications

Let A be an excellent ring, let M be a finitely generated A-module, and let $I \subseteq A$ be an ideal of A. By applying the results of the previous section to the Rees algebra/module and to the associated graded ring/module, respectively, we see that there is an integer $k \in \mathbb{N}$, so that for all $i \geq k$ and all $n \in \mathbb{N}$,

$$U_{C_n}(I^iM) = U_{C_n}(I^kM) \quad \text{and} \quad U_{C_n}(I^iM/I^{i+1}M) = U_{C_n}(I^kM/I^{k+1}M),$$
$$U_{S_n}(I^iM) = U_{S_n}(I^kM) \quad \text{and} \quad U_{S_n}(I^iM/I^{i+1}M) = U_{S_n}(I^kM/I^{k+1}M).$$

In the following we want to apply these results to the (S_n)- and codepth-loci of the modules M/I^kM. We want to show that these loci are again eventually stable, provided that M is a Cohen-Macaulay module over A.

5.1. **Lemma.** *Let A be any Noetherian ring, $I \subseteq A$ an ideal, and M a finitely generated A-module. Then for all $k \in \mathbb{N}$,*

$$\operatorname{Supp}(M/I^kM) = \operatorname{Supp}(M/IM) \,.$$

Proof. It suffices to show that for all $k \in \mathbb{N}$,

$$\operatorname{Supp}(M/I^kM) = \operatorname{Supp}(M/I^{k+1}M) \,.$$

Since M/I^kM is a homomorphic image of $M/I^{k+1}M$, we have $\operatorname{Supp}(M/I^kM) \subseteq \operatorname{Supp}(M/I^{k+1}M)$. Consider the exact sequence:

$$0 \to I^kM/I^{k+1}M \to M/I^{k+1}M \to M/I^kM \to 0,$$

and let $\mathfrak{p} \in \operatorname{Spec}(A)$ with $I \subseteq \mathfrak{p}$. The sequence stays exact after localization:

$$0 \to (I^kM/I^{k+1}M)_{\mathfrak{p}} \to (M/I^{k+1}M)_{\mathfrak{p}} \to (M/I^kM)_{\mathfrak{p}} \to 0 \,.$$

If $(M/I^kM)_{\mathfrak{p}} = 0$ with $(M/I^{k+1}M)_{\mathfrak{p}} \neq 0$, then

$$(I^kM/I^{k+1}M)_{\mathfrak{p}} = (M/I^{k+1}M)_{\mathfrak{p}},$$

which implies by Nakayama that $(M/I^{k+1}M)_{\mathfrak{p}} = 0$, a contradiction. □

A more general version of the next result was proved, using different methods, by Kodiyalam [7, Corollary 9].

5.2. **Theorem.** *Suppose that $(A,\mathfrak{m})$ is a local Noetherian ring, let $I \subseteq A$ be an ideal of A, and let M be a finitely generated A-module. Then there is a $k \in \mathbb{N}$, so that for all $i \geq k$,*

$$\operatorname{depth}_A(M/I^iM) = \operatorname{depth}_A(M/I^kM)\,.$$

Proof. Let $\widehat{A}$ be the $\mathfrak{m}$-adic completion of A. Then for any finitely generated A-module T,

$$\operatorname{depth}_A(T) = \operatorname{depth}_{\widehat{A}}(T \otimes_A \widehat{A}),$$

and we may replace A by $\widehat{A}$ and M by $M \otimes_A \widehat{A}$, and assume that A is excellent. By Lemma 4.1 there is a $k_1 \in \mathbb{N}$, so that for all $t \geq k_1$,

$$\operatorname{depth}_A(I^tM/I^{t+1}M) = \operatorname{depth}_A(I^{k_1}M/I^{k_1+1}M) = g.$$

For all $t \geq k_1$ consider the exact sequence

$$0 \to I^tM/I^{t+1}M \to M/I^{t+1}M \to M/I^tM \to 0,$$

which leads to an exact sequence on the cohomology modules:

$$\begin{gathered}\cdots \to H^i_{\mathfrak{m}}(M/I^{t+1}M) \to H^i_{\mathfrak{m}}(M/I^tM) \to 0 \to \cdots \to 0\\ \to \cdots \to H^{g-1}_{\mathfrak{m}}(M/I^{t+1}M) \to H^{g-1}_{\mathfrak{m}}(M/I^tM) \to H^g_{\mathfrak{m}}(I^tM/I^{t+1}M)\\ \to H^g_{\mathfrak{m}}(M/I^{t+1}M) \to H^g_{\mathfrak{m}}(M/I^tM) \to \cdots\,,\end{gathered}$$

where g is minimal with $H^g_{\mathfrak{m}}(I^tM/I^{t+1}M) \neq 0$.

Case 1: There is an $i \leq g-1$ and a $t_0 \geq k_1$, so that $H^i_{\mathfrak{m}}(M/I^{t_0}M) \neq 0$. Then for all $t \geq t_0$, $H^i_{\mathfrak{m}}(M/I^tM) \neq 0$. Let $h \leq g-1$ be the minimal i with this property. Then

$$\operatorname{depth}_A(M/I^tM) = h \quad \text{for all} \quad t \geq t_0\,.$$

Case 2: For all $i \leq g-1$ and all $t \geq k_1$,

$$H^i_{\mathfrak{m}}(M/I^tM) = 0\,.$$

This implies that $\operatorname{depth}_A(M/I^tM) \geq g-1$ for all $t \geq k_1$.

Case 2.1: There are infinitely many $t \geq k_1$, so that

$$H^{g-1}_{\mathfrak{m}}(M/I^tM) \neq 0\,.$$

From the long exact sequence we observe that $H^{g-1}_{\mathfrak{m}}(M/I^tM) \neq 0$ implies that $H^{g-1}_{\mathfrak{m}}(M/I^{t-1}M) \neq 0$ whenever $t-1 \geq k_1$. Thus in this case there is a $t_1 \geq k_1$, so that for all $t \geq t_1$,

$$H^{g-1}_{\mathfrak{m}}(M/I^tM) \neq 0\,,$$

and therefore for all $t \geq t_1$, $\operatorname{depth}_A(M/I^tM) = g-1$.

Case 2.2: There is a $t_2 \geq k_1$, so that for all $t \geq t_2$, $H^{g-1}_{\mathfrak{m}}(M/I^tM) = 0$. Then for all $t \geq t_2$,

$$\operatorname{depth}_A(M/I^tM) = g\,. \qquad \square$$

5.3. **Theorem.** *Let A be an excellent ring and M a finitely generated Cohen-Macaulay A-module. Let $I \subseteq A$ be an ideal of A which is not contained in any minimal prime ideal of M. Then there is an integer $k \in \mathbb{N}$, so that for all $t \geq k$ and all $n \in \mathbb{N}_0$:*

1. $U_{C_n}(M/I^tM) = U_{C_n}(M/I^{k_0}M)$.
2. $U_{S_n}(M/I^tM) = U_{S_n}(M/I^{k_0}M)$.

Proof. (1) Fix $n \in \mathbb{N}$ and let $k \in \mathbb{N}$, so that for all $t \geq k$,

$$U_{C_n}(I^t M) = U_{C_n}(I^k M)\,.$$

We claim that for all $i \geq k$ and all $\mathfrak{p} \in V(I)$,

$$\operatorname{depth}_{A_\mathfrak{p}}((M/I^i M)_\mathfrak{p}) = \operatorname{depth}_{A_\mathfrak{p}}((M/I^k M)_\mathfrak{p})\,.$$

Obviously, for all $i \geq k$, $\dim((I^i M)_\mathfrak{p}) = \dim((I^k M)_\mathfrak{p})$, and thus because of the stability of the codepth-loci, we have for all $\mathfrak{p} \in V(I)$ and all $i \geq k$ that

$$\operatorname{depth}_{A_\mathfrak{p}}((I^i M)_\mathfrak{p}) = \operatorname{depth}_{A_\mathfrak{p}}((I^k M)_\mathfrak{p})\,.$$

Fix an integer $i \geq k$ and a prime ideal $\mathfrak{p} \in V(I)$, and consider the exact sequence

$$0 \to (I^i M)_\mathfrak{p} \to M_\mathfrak{p} \to (M/I^i M)_\mathfrak{p} \to 0\,.$$

With $d = \dim_{A_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p})$ we obtain a long exact sequence of the local cohomology modules

$$\begin{aligned} \cdots \to 0 \to H^{i-1}_\mathfrak{p}((M/I^i M)_\mathfrak{p}) \to H^i_\mathfrak{p}((I^i M)_\mathfrak{p}) \to 0 \to \cdots \to 0 \\ \to H^{d-1}_\mathfrak{p}((M/I^i M)_\mathfrak{p}) \to H^d_\mathfrak{p}((I^i M)_\mathfrak{p}) \to H^d_\mathfrak{p}(M_\mathfrak{p}) \to 0 = H^d_\mathfrak{p}((M/I^i M)_\mathfrak{p}), \end{aligned}$$

where $H^d_\mathfrak{p}((M/I^i M)_\mathfrak{p}) = 0$, since $\dim_{A_\mathfrak{p}}((M/I^i M)_\mathfrak{p}) \leq d-1$. This shows that

$$\operatorname{depth}_{A_\mathfrak{p}}((M/I^i M)_\mathfrak{p}) = \operatorname{depth}_{A_\mathfrak{p}}((I^i M)_\mathfrak{p}) - 1 = \operatorname{depth}_{A_\mathfrak{p}}((I^k M)_\mathfrak{p}) - 1,$$

and the claim is proven. For all $i \geq k$ and all $\mathfrak{p} \in V(I)$ we have

$$\begin{aligned} \operatorname{depth}_{A_\mathfrak{p}}((M/I^i M)_\mathfrak{p}) &= \operatorname{depth}_{A_\mathfrak{p}}((M/I^k M)_\mathfrak{p}), \\ \dim((M/I^i M)_\mathfrak{p}) &= \dim((M/I^k M)_\mathfrak{p})\,. \end{aligned}$$

The last equation is obtained from Lemma 5.1. This yields that for all $n \in \mathbb{N}$ and for all $i \geq k$,

$$U_{C_n}(M/I^i M) = U_{C_n}(M/I^k M)\,.$$

The second assumption follows with Corollary 3.5. □

5.4. **Corollary.** *Let A, M, and I be as in the theorem, and assume that $IM \neq M$. Then there is an element $a \in A$, so that for all $k \in \mathbb{N}$,*

(1) $(M/I^k M)_a \neq 0$.
(2) $(M/I^k M)_a$ *is a Cohen-Macaulay module.* □

5.5. **Corollary.** *Let A be an excellent ring and M a finitely generated A-module. Suppose that the ideal $I \subseteq A$ satisfies the following conditions:*

(i) *I is not contained in a minimal prime of M.*
(ii) *If $\mathfrak{a} \subseteq A$ is the defining ideal of the non-Cohen-Macaulay locus of M, then $\mathfrak{a} \nsubseteq \sqrt{(IM : M)}$.*

Then there is an element $a \in A$, so that for all $k \in \mathbb{N}$,

(1) $(M/I^k M)_a \neq 0$.
(2) $(M/I^k M)_a$ *is a Cohen-Macaulay module.*

Proof. Choose an element $b \in \mathfrak{a} \setminus \sqrt{(IM : M)}$. In order to prove the assertion apply the previous corollary to the Cohen-Macaulay A_b-module M_b. □

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