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JOINTLY HYPONORMAL PAIRS OF COMMUTING SUBNORMAL OPERATORS NEED NOT BE JOINTLY SUBNORMAL

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ABSTRACT. We construct three different families of commuting pairs of subnormal operators, jointly hyponormal but not admitting commuting normal extensions. Each such family can be used to answer in the negative a 1988 conjecture of R. Curto, P. Muhly and J. Xia. We also obtain a sufficient condition under which joint hyponormality does imply joint subnormality.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. For $S,T \in \mathcal{B}(\mathcal{H})$ let $[S,T] := ST - TS$. We say that an n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \cdots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive on the direct sum of n copies of $\mathcal{H}$ (cf. [Ath], [CMX]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. Clearly, normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal. The Bram-Halmos criterion states that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if the k-tuple $(T, T^2, \cdots, T^k)$ is hyponormal for all $k \geq 1$.

For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. The *moments* of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1 & \text{if } k = 0, \\ \alpha_0^2 \cdot \ldots \cdot \alpha_{k-1}^2 & \text{if } k > 0. \end{cases}$$

It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_0 \leq \alpha_1 \leq \cdots$. Similarly, consider double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$, and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert

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space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. (Recall that $\ell^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $\ell^2(\mathbb{Z}_+)\otimes\ell^2(\mathbb{Z}_+)$.) We define the 2-*variable weighted shift* $\mathbf{T}\equiv(T_1,T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1},$$
$$T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1,0)$ and $\varepsilon_2 := (0,1)$. Clearly,

$$T_1T_2 = T_2T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1}\alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}} \ \ (\text{all } \mathbf{k}). \tag{1.1}$$

In an entirely similar way one can define multivariable weighted shifts. Trivially, a pair of unilateral weighted shifts W_α and W_β gives rise to a 2-variable weighted shift $\mathbf{T}\equiv(T_1,T_2)$, if we let $\alpha_{(k_1,k_2)} := \alpha_{k_1}$ and $\beta_{(k_1,k_2)} := \beta_{k_2}$ (all $k_1,k_2\in\mathbb{Z}_+$). In this case, $\mathbf{T}$ is subnormal (resp. hyponormal) if and only if so are T_1 and T_2. In fact, under the canonical identification of $\ell^2(\mathbb{Z}_+^2)$ and $\ell^2(\mathbb{Z}_+)\otimes\ell^2(\mathbb{Z}_+)$, $T_1\cong I\otimes W_\alpha$ and $T_2\cong W_\beta\otimes I$, and $\mathbf{T}$ is also doubly commuting. For this reason, we do not focus our attention on shifts of this type, but use them only when the above-mentioned triviality is desirable or needed.

We now recall a well-known characterization of subnormality for single variable weighted shifts, due to C. Berger (cf. [Con, III.8.16]): W_α is subnormal if and only if there exists a probability measure ξ supported in $[0,\|W_\alpha\|^2]$ (called the *Berger measure* of W_α) such that $\gamma_k(\alpha) := \alpha_0^2\cdot\ldots\cdot\alpha_{k-1}^2 = \int t^k\, d\xi(t)$ $(k\geq 1)$. If W_α is subnormal, and if for $h\geq 1$ we let $\mathcal{M}_h := \bigvee\{e_n : n\geq h\}$ denote the invariant subspace obtained by removing the first h vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_\alpha|_{\mathcal{M}_h}$ is $\frac{1}{\gamma_h}t^h d\xi(t)$.

An important class of subnormal weighted shifts is obtained by considering measures μ with exactly two atoms t_0 and t_1. These shifts arise naturally in the Subnormal Completion Problem [CuFi3] and in the theory of truncated moment problems (cf. [CuFi1], [CuFi4]). For $t_0,t_1\in\mathbb{R}_+$ with $t_0<t_1$, and $\rho_0,\rho_1>0$ with $\rho_0+\rho_1=1$, the moments of the 2-atomic probability measure $\xi := \rho_0\delta_{t_0}+\rho_1\delta_{t_1}$ (here δ_p denotes the point-mass probability measure with support the singleton $\{p\}$) satisfy the 2-step recursive relation $\gamma_{n+2}=\varphi_0\gamma_n+\varphi_1\gamma_{n+1}$ $(n\geq 0)$; at the weight level, this can be written as $\alpha_{n+1}^2=\frac{\varphi_0}{\alpha_n^2}+\varphi_1$ $(n\geq 0)$. More generally, any finitely atomic Berger measure corresponds to a recursively generated weighted shift (i.e., one whose moments satisfy an r-step recursive relation); in fact, $r=$ card supp ξ. In the special case of $r=2$, the theory of recursively generated weighted shifts makes contact with the work of J. Stampfli in [Sta], in which he proved that given three positive numbers $\alpha_0<\alpha_1<\alpha_2$, it is always possible to find a subnormal weighted shift, denoted $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$, whose first three weights are α_0,α_1 and α_2. In this case, the coefficients of recursion (cf. [CuFi2, Example 3.12], [CuFi3, Section 3], [Cu3, Section 1, p. 81]) are given by

$$\varphi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2-\alpha_1^2)}{\alpha_1^2-\alpha_0^2} \text{ and } \varphi_1 = \frac{\alpha_1^2(\alpha_2^2-\alpha_0^2)}{\alpha_1^2-\alpha_0^2}, \tag{1.2}$$

the atoms t_0 and t_1 are the roots of the equation

$$t^2-(\varphi_0+\varphi_1 t)=0, \tag{1.3}$$

and the densities ρ_0 and ρ_1 uniquely solve the 2×2 system of equations

$$\begin{cases} \rho_0+\rho_1 &= 1, \\ \rho_0 t_0+\rho_1 t_1 &= \alpha_0^2. \end{cases} \tag{1.4}$$

We also recall the notion of moment of order $\mathbf{k}$ for a pair (α, β) satisfying (1.1). Given $\mathbf{k} \in \mathbb{Z}_+^2$, the moment of (α, β) of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha, \beta)$$
$$:= \begin{cases} 1 & \text{if } \mathbf{k} = 0, \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0, \\ \beta_{(0,0)}^2 \cdot \ldots \cdot \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1, \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdot \ldots \cdot \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases}$$

We remark that, due to the commutativity condition (1.1), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0,0)$ to (k_1, k_2).

Theorem 1.1 (Berger's Theorem, 2-variable case) ([JeLu])**.** *A 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ admits a commuting normal extension if and only if there is a probability measure μ defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$ ($a_i := \|T_i\|^2$) such that $\gamma_{\mathbf{k}} = \iint_R \mathbf{t}^{\mathbf{k}} d\mu(\mathbf{t}) := \iint_R t_1^{k_1} t_2^{k_2}\, d\mu(t_1, t_2)$ (all $\mathbf{k} \in \mathbb{Z}_+^2$).*

Clearly, each component T_i of a subnormal 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ must be subnormal. For instance, $T_1 \cong \bigoplus_{j=0}^{\infty} W_{\alpha^{(j)}}$, where $\alpha_i^{(j)} := \alpha_{(i,j)}$, so that $W_{\alpha^{(j)}}$ has Berger measure $d\nu_j(t_1) := \frac{1}{\gamma_{(0,j)}} \int_{[0,a_2]} t_2^j d\Phi_{t_1}(t_2)$, where $d\mu(t_1, t_2) \equiv d\Phi_{t_1}(t_2) d\eta(t_1)$ is the canonical disintegration of μ by horizontal slices. On the other hand, if we only know that each of T_1, T_2 is subnormal, and that they commute, the following problem is natural.

Problem 1.2 (Lifting Problem for Commuting Subnormals)**.** Find necessary and sufficient conditions on T_1 and T_2 to guarantee the subnormality of $\mathbf{T} \equiv (T_1, T_2)$.

It is well known that the above-mentioned necessary conditions do not suffice (cf. [Cu1]). In terms of the *marginal* measures, the problem can be phrased as a reconstruction-of-measure problem, that is, under what conditions on the single variable measures $\{\nu_j\}_{j=0}^{\infty}$ and $\{\omega_i\}_{i=0}^{\infty}$ associated with T_1 and T_2, respectively, does there exist a 2-variable measure μ correctly interpolating all the powers $t_1^{k_1} t_2^{k_2}$ $(k_1, k_2 \geq 0)$?

To detect hyponormality for 2-variable weighted shifts, there is a simple criterion involving a base point $\mathbf{k}$ in $\mathbb{Z}_+^2$ and its five neighboring points in $\mathbf{k} + \mathbb{Z}_+^2$ at path distance at most 2 (cf. Figure 1).

Theorem 1.3 (Six-point Test) ([Cu1])**.** *Let $\mathbf{T} \equiv (T_1, T_2)$ be a 2-variable weighted shift, with weight sequences α and β. Then*

$$[\mathbf{T}^*, \mathbf{T}] \geq 0 \Leftrightarrow (([T_j^*, T_i] e_{\mathbf{k}+\varepsilon_j}, e_{\mathbf{k}+\varepsilon_i}))_{i,j=1}^2 \geq 0 \ (\text{all } \mathbf{k} \in \mathbf{Z}_+^2)$$
$$\Leftrightarrow \begin{pmatrix} \alpha_{\mathbf{k}+\varepsilon_1}^2 - \alpha_{\mathbf{k}}^2 & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta_{\mathbf{k}+\varepsilon_2}^2 - \beta_{\mathbf{k}}^2 \end{pmatrix} \geq 0 \ (\text{all } \mathbf{k} \in \mathbf{Z}_+^2).$$

Unlike the single variable case, in which there is a clear separation between hyponormality and subnormality (cf. [CuFi3], [Cu3], [CuLe]), much less is known about the multivariable case. In this paper we will construct three conceptually different families of counterexamples to the following conjecture.

Conjecture 1.4 ([CMX])**.** *Let $\mathbf{T} \equiv (T_1, T_2)$ be a pair of commuting subnormal operators on $\mathcal{H}$. Then $\mathbf{T}$ is subnormal if and only if $\mathbf{T}$ is hyponormal.*

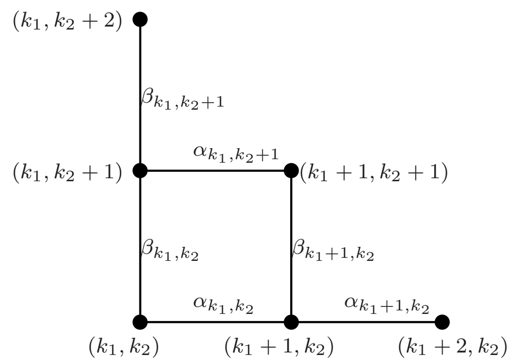

FIGURE 1. Weight diagram used in the Six-point Test

We mention that M. Dritschel and S. McCullough, working independently, have been able to obtain a separate example ([DrMcC]). We shall see in Section 4 that their example is a special case of a general construction that produces nonsubnormal hyponormal pairs with $T_1 \cong T_2$.

We now formulate an improved version of a result due to R. Curto.

Proposition 1.5 (Subnormal backward extension of a 1-variable weighted shift) (cf. [Cu2])**.** *Let T be a weighted shift whose restriction $T_{\mathcal{M}} := T|_{\mathcal{M}}$ to $\mathcal{M} := \bigvee\{e_1, e_2, \cdots\}$ is subnormal, with Berger measure $\mu_{\mathcal{M}}$. Then T is subnormal (with associated measure μ) if and only if*

(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$,

(ii) $\alpha_0^2 \leq (\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})})^{-1}$.

In this case, $d\mu(t) = \frac{\alpha_0^2}{t} d\mu_{\mathcal{M}}(t) + (1 - \alpha_0^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}) d\delta_0(t)$, where δ_0 denotes Dirac measure at 0. In particular, T is never subnormal when $\mu_{\mathcal{M}}(\{0\}) > 0$.

Proof. $\Rightarrow$) We first observe that the moments of T and $T_{\mathcal{M}}$ are related by the equation

$$\gamma_k(T_{\mathcal{M}}) \equiv \alpha_1^2 \cdots \alpha_k^2 = \frac{\gamma_{k+1}(T)}{\alpha_0^2}$$

so that

$$\frac{1}{\alpha_0^2} \int t^{k+1} d\mu(t) = \int t^k d\mu_{\mathcal{M}}(t) \ \ (\text{all } k \geq 0),$$

that is, $t d\mu(t) = \alpha_0^2 d\mu_{\mathcal{M}}(t)$. It follows at once that

$$d\mu(t) = \lambda d\delta_0(t) + \frac{\alpha_0^2}{t} d\mu_{\mathcal{M}}(s),$$

where $\lambda \geq 0$. Since $\int d\mu = 1$, we must have $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$ and $\alpha_0^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} \leq 1$. Finally, it is straightforward to verify that $\lambda = 1 - \alpha_0^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}$.

$\Leftarrow$) Let

$$d\mu(t) := \frac{\alpha_0^2}{t} d\mu_{\mathcal{M}}(t) + (1 - \alpha_0^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}) d\delta_0(t).$$

By hypotheses, μ is a positive Borel measure on $[0, \|T\|^2]$. Moreover,

$$\int d\mu = \alpha_0^2 \int \frac{1}{t} d\mu_{\mathcal{M}} + (1 - \alpha_0^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})}) \int d\delta_0 = 1,$$

and for $k \geq 1$,

$$\begin{aligned} \int t^k d\mu(t) &= \alpha_0^2 \int t^k \frac{1}{t} d\mu_{\mathcal{M}}(t) + (1 - \alpha_0^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})}) \int t^k d\delta_0(t) \\ &= \alpha_0^2 \int t^{k-1} d\mu_{\mathcal{M}}(t) = \alpha_0^2 \gamma_{k-1}(T_{\mathcal{M}}) = \gamma_k(T). \end{aligned}$$

Therefore, T is subnormal, with Berger measure μ. □

Notation 1.6. The maximum possible value for α_0 in Proposition 1.5, namely $(\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})})^{-1}$, will be denoted by

$$\alpha_{ext} \equiv \alpha_{ext}(\mu_{\mathcal{M}}).$$

Observe that $shift(\alpha_{ext}, \alpha_1, \alpha_2, \cdots)$ is subnormal, with Berger measure $d\mu(t) = \frac{\alpha_0^2}{t} d\mu_{\mathcal{M}}(t)$. For example, if B_+ denotes the Bergman shift on $\ell^2(\mathbb{Z}_+)$, then $B_+|_{\mathcal{M}}$ is subnormal, with Berger measure $d\mu(t) := 2tdt$ on $[0,1]$. Then $d\mu_{ext}(t) = dt$, so in this case the extremal measure μ_{ext} is the Berger measure of B_+.

More generally, given a (1-variable) subnormal weighted shift W_η with weight sequence $\eta_1 \leq \eta_2 \leq \cdots$ and Berger measure ν, we let

$$\eta_{ext} := \begin{cases} 0 & \text{if } \frac{1}{t} \notin L^1(\nu), \\ (\left\|\frac{1}{t}\right\|_{L^1(\nu)})^{-1} & \text{if } \frac{1}{t} \in L^1(\nu). \end{cases}$$

Observe that when the weight sequence η is strictly increasing and $\frac{1}{t} \in L^1(\nu)$, we must necessarily have

$$\eta_{ext} < \eta_1, \tag{1.5}$$

by [Sta, Theorem 6]. On occasion, we will write $shift(\alpha_0, \alpha_1, \cdots)$ to denote the weighted shift with weight sequence $\{\alpha_k\}_{k=0}^{\infty}$. We also denote by $U_+ := shift(1, 1, \cdots)$ the (unweighted) unilateral shift, and for $0 < a < 1$ we let $S_a := shift(a, 1, 1, \cdots)$. Observe that the Berger measures of U_+ and S_a are δ_1 and $(1-a^2)\delta_0 + a^2\delta_1$, respectively, where δ_p denotes the point-mass probability measure with support the singleton $\{p\}$. Finally, we let B_+ denote the Bergman shift, whose Berger measure is Lebesgue measure on the interval $[0,1]$; the weights of B_+ are given by the formula $\alpha_n := \sqrt{\frac{n+1}{n+2}}$ $(n \geq 0)$.

We conclude this section with a result that will be needed in Section 3.

Lemma 1.7 (cf. [CuFi3, Theorem 3.10])**.** *For $0 < \alpha_0 < \alpha_1 < \alpha_2$, let $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ be the weighted shift described by* (1.2), (1.3) *and* (1.4). *Now consider $W_\eta := shift(\alpha_1, \alpha_2, \cdots)$, that is, W_η is the restriction of $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ to $\mathcal{M}$. Then $\eta_{ext} = \alpha_0$.*

2. The first family of counterexamples

Recall that a unilateral weighted shift W_α is subnormal if and only if there exists a probability measure $\xi \equiv \xi_\alpha$ supported in $[0, \|W_\alpha\|^2]$ such that $\gamma_k(\alpha) := \alpha_0^2 \cdot ... \cdot \alpha_{k-1}^2 = \int t^k \; d\xi(t) \;\; (k \geq 1)$. For instance, when $\alpha_1 = \alpha_2 = \cdots = 1$ (i.e., $W_\alpha \equiv shift(\alpha_0, 1, 1, \cdots)$), we have $\xi_\alpha = (1 - \alpha_0^2)\delta_0 + \alpha_0^2\delta_1$. The proof of the following lemma is straightforward.

Lemma 2.1. *Given two* 1*-variable weight sequences* α *and* β*, the* 2*-variable weighted shift* $(I \otimes W_\alpha, W_\beta \otimes I)$ *is always subnormal, with Berger measure* $\mu := \xi_\alpha \times \xi_\beta$*.*

Definition 2.2. Let μ and ν be two positive measures on $\mathbb{R}_+$. We say that $\mu \leq \nu$ on $X := \mathbb{R}_+$ if $\mu(E) \leq \nu(E)$ for all Borel subset $E \subseteq \mathbb{R}_+$; equivalently, $\mu \leq \nu$ if and only if $\int f d\mu \leq \int f d\nu$ for all $f \in C(X)$ such that $f \geq 0$ on $\mathbb{R}_+$.

Definition 2.3. Let μ be a probability measure on $X \times Y \equiv \mathbb{R}_+ \times \mathbb{R}_+$, and assume that $\frac{1}{t} \in L^1(\mu)$. The *extremal measure* μ_{ext} (which is also a probability measure) on $X \times Y$ is given by $d\mu_{ext}(s,t) := (1 - \delta_0(t)) \frac{1}{t\left\|\frac{1}{t}\right\|_{L^1(\mu)}} d\mu(s,t)$.

Definition 2.4. Given a measure μ on $X \times Y$, the marginal measure μ^X is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection onto X. Thus, $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$. Observe that if μ is a probability measure, then so is μ^X.

Lemma 2.5. *Let* μ *be the Berger measure of a* 2*-variable weighted shift* $\mathbf{T}$*, and let* ν *be the Berger measure of* $shift(\alpha_{00}, \alpha_{10}, \cdots)$*. Then* $\nu = \mu^X$*. As a consequence,* $\iint f(s) \; d\mu(s,t) = \int f(s) \; d\mu^X(s)$ *for all* $f \in C(X)$*.*

Proof. Observe that $\int s^i \; d\nu(s) = \gamma_{i0} = \iint s^i \; d\mu(s,t)$ for all $i \geq 0$. It follows that $\int f(s) \; d\nu(s) = \iint f(s) \; d\mu(s,t)$ for all $f \in C(X)$. Then, for any Borel set $E \subseteq X$, we have

$$\nu(E) = \int \chi_E \; d\nu = \iint \chi_{E \times Y} \; d\mu = \mu(E \times Y) = \mu^X(E),$$

as desired. The second assertion follows immediately from what we have established. □

Corollary 2.6. *Let* μ *be the Berger measure of a* 2*-variable weighted shift* $\mathbf{T}$*. For* $j \geq 1$*, let* $d\mu_j(s,t) := \frac{1}{\gamma_{0j}} t^j d\mu(s,t)$*. Then the Berger measure of* $shift(\alpha_{0j}, \alpha_{1j}, \cdots)$ *is* $\nu_j \equiv \mu_j^X$*.*

Example 2.7. Let $\mu := \xi \times \eta$ be a probability product measure on $X \times Y$. Then $\mu^X = \xi$.

Lemma 2.8. *Let* μ *and* ω *be two measures on* $X \times Y$*, and assume that* $\mu \leq \omega$*. Then* $\mu^X \leq \omega^X$*.*

Proof. Straightforward from Definition 2.4. □

Proposition 2.9 (Subnormal backward extension of a 2-variable weighted shift)**.** *Consider the following* 2*-variable weighted shift (see Figure* 2*), and let* $\mathcal{M}$ *be the subspace associated to indices* $\mathbf{k}$ *with* $k_2 \geq 1$*. Assume that* $\mathbf{T}_{\mathcal{M}} := \mathbf{T}|_{\mathcal{M}}$ *is subnormal with Berger measure* $\mu_{\mathcal{M}}$ *and that* $W_0 := shift(\alpha_{00}, \alpha_{10}, \cdots)$ *is subnormal*

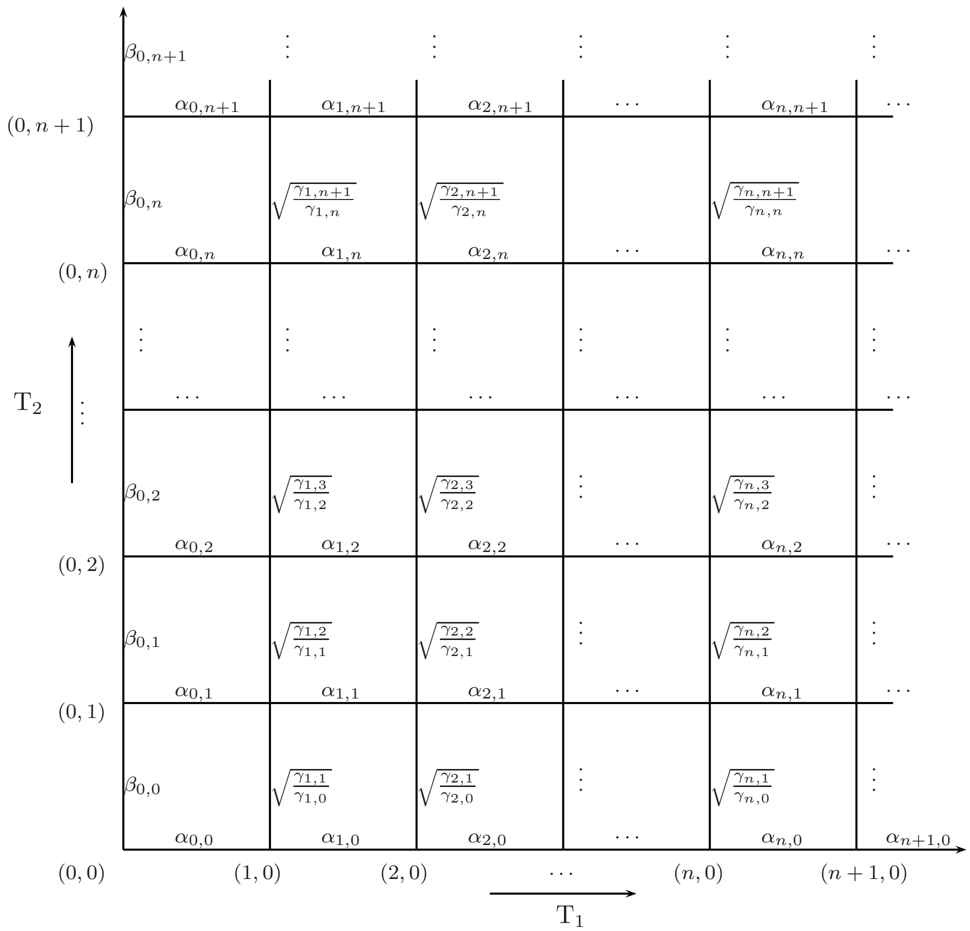

FIGURE 2. Weight diagram of the 2-variable weighted shift in Proposition 2.9

with Berger measure ν. *Then* $\mathbf{T}$ *is subnormal if and only if*

(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;

(ii) $\beta_{00}^2 \le (\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})})^{-1}$;

(iii) $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \le \nu$.

Moreover, if $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} = 1$, *then* $(\mu_{\mathcal{M}})_{ext}^X = \nu$. *In the case when* $\mathbf{T}$ *is subnormal, the Berger measure* μ *of* $\mathbf{T}$ *is given by*

$$d\mu(s,t) = \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}(s,t) + (d\nu(s) - \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X(s))d\delta_0(t).$$

Proof. $(\Rightarrow)$ First, observe that the moments of $\mathbf{T}$ and $\mathbf{T}_{\mathcal{M}}$ are related as follows:

$$\gamma_{\mathbf{k}+\varepsilon_2}(\mathbf{T}) = \beta_{00}^2 \gamma_{\mathbf{k}}(\mathbf{T}_{\mathcal{M}}) \text{ (all } \mathbf{k} \in \mathbb{Z}_+^2), \tag{2.1}$$

so under the assumption that $\mathbf{T}$ is subnormal we must have

$$\begin{aligned}\iint s^i t^j (td\mu)(s,t) &= \iint s^i t^{j+1} d\mu(s,t) = \gamma_{i,j+1}(\mathbf{T}) \\ &= \beta_{00}^2 \gamma_{ij}(\mathbf{T}_{\mathcal{M}}) = \beta_{00}^2 \iint s^i t^j \, d\mu_{\mathcal{M}}(s,t).\end{aligned}$$

Thus $td\mu(s,t) = \beta_{00}^2 d\mu_{\mathcal{M}}(s,t)$ and $\mu_{\mathcal{M}}(E \times \{0\}) = 0$ for all $E \subseteq X$. It follows at once that

$$\begin{aligned}\iint \frac{1}{t} \, d\mu_{\mathcal{M}}(s,t) &= \iint_{(t>0)} \frac{1}{t} \, d\mu_{\mathcal{M}}(s,t) = \frac{1}{\beta_{00}^2} \iint_{(t>0)} \frac{1}{t} \, td\mu(s,t) \\ &= \frac{1}{\beta_{00}^2} \mu((t>0)) \leq \frac{1}{\beta_{00}^2},\end{aligned}$$

which establishes parts (i) and (ii). As for part (iii), let $E \subseteq X$ and $F \subseteq Y$ be two arbitrary Borel sets. Then

$$\begin{aligned}&\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}(E \times F) \\ &= \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} \iint_{E \times F} (1 - \delta_0(t)) \frac{1}{t \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})}} d\mu_{\mathcal{M}}(s,t) \\ &= \iint_{E \times (F \setminus \{0\})} \frac{1}{t} \beta_{00}^2 d\mu_{\mathcal{M}}(s,t) = \mu(E \times (F \setminus \{0\})) \\ &\leq \mu(E \times F),\end{aligned} \tag{2.2}$$

and by Lemmas 2.8 and 2.5, $\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \mu^X = \nu$. Finally, observe that when $\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} = 1$, the inequality in (2.2) becomes an equality, and therefore $(\mu_{\mathcal{M}})_{ext}^X = \nu$.

($\Leftarrow$) Assume that (i), (ii) and (iii) hold, and let

$$\mu := \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext} + [\nu - \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X] \times \delta_0.$$

Of course, if $\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} = 1$, then $\mu := (\mu_{\mathcal{M}})_{ext}$, since the total mass of the second summand is zero. We now compute the moments of μ and verify that they agree with the moments of $\mathbf{T}$. If $j > 0$, then

$$\begin{aligned}\iint s^i t^j \, d\mu(s,t) &= \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} \iint s^i t^j \, d(\mu_{\mathcal{M}})_{ext}(s,t) \\ &= \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} \iint s^i t^j (1 - \delta_0(t)) \frac{1}{t \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})}} d\mu_{\mathcal{M}}(s,t) \\ &= \beta_{00}^2 \iint s^i t^{j-1} \, d\mu_{\mathcal{M}}(s,t) = \beta_{00}^2 \gamma_{(i,j-1)}(\mathbf{T}_{\mathcal{M}}) = \gamma_{(i,j)}(\mathbf{T}),\end{aligned}$$

as desired. When $j = 0$, we have

$$\begin{aligned}\iint s^i \, d\mu(s,t) &= \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} \iint s^i \, d(\mu_{\mathcal{M}})_{ext}(s,t) \\ &\quad + \int s^i \, d(\nu - \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X)(s) \\ &= \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} \int s^i \, d(\mu_{\mathcal{M}})_{ext}^X(s) \\ &\quad + \int s^i \, d\nu(s) - \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} \int s^i \, d(\mu_{\mathcal{M}})_{ext}^X(s) \\ &\quad \text{(using Lemma 2.5 for the first term)} \\ &= \int s^i \, d\nu(s) = \gamma_{(i,0)}(\mathbf{T}),\end{aligned}$$

as desired. It follows that $\mathbf{T}$ is subnormal, with Berger measure μ. □

We are now ready to exhibit our first family of counterexamples to Conjecture 1.4. Consider the 2-variable weighted shift given by Figure 3, where $\max\{x, y\} < 1$ and $a < x$.

Proposition 2.10. *The* 2*-variable weighted shift* $\mathbf{T}$ *given by Figure* 3 *is hyponormal if and only if* $y \leq x\sqrt{\frac{1-x^2}{x^2-2a^2x^2+a^4}}$.

Proof. By the Six-point Test (Theorem 1.3), to show the joint hyponormality of $\mathbf{T}$ it is enough to check that

$$H := \begin{pmatrix} 1-x^2 & \frac{a^2y}{x} - yx \\ \frac{a^2y}{x} - yx & 1-y^2 \end{pmatrix} \geq 0.$$

Since $x < 1$, the positivity of H is equivalent to $\det H \geq 0$, i.e.,

$$(1-x^2)(1-y^2) \geq (\frac{a^2y}{x} - yx)^2,$$

which in turn is equivalent to $y \leq x\sqrt{\frac{1-x^2}{x^2-2a^2x^2+a^4}}$ (observe that $x^2 - 2a^2x^2 + a^4 = x^2(1-x^2) + (x^2-a^2)^2 > 0$). □

Proposition 2.11. *The* 2*-variable weighted shift* $\mathbf{T}$ *given by Figure* 3 *is subnormal if and only if* $y \leq \sqrt{\frac{1-x^2}{1-a^2}}$.

Proof. From Figure 3, it is obvious that $\mathbf{T}_{\mathcal{M}} \cong (I \otimes S_a, U_+ \otimes I)$ (recall that $S_a := shift(a, 1, 1, \cdots)$ and U_+ is the (unweighted) unilateral shift). By Lemma 2.1, $\mathbf{T}_{\mathcal{M}}$

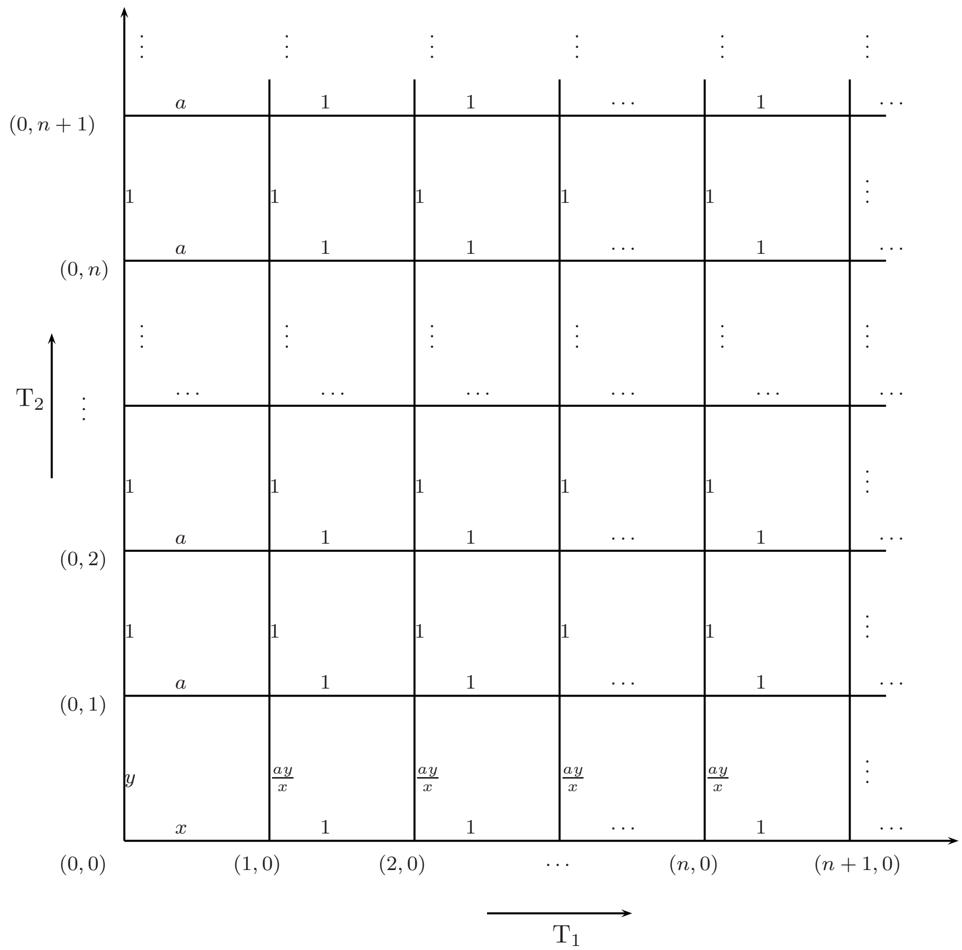

FIGURE 3. Weight diagram of the 2-variable weighted shift in Propositions 2.10 and 2.11

is subnormal, with Berger measure $\mu_{\mathcal{M}} := [(1-a^2)\delta_0 + a^2\delta_1] \times \delta_1$. By Proposition 2.9,

$$\begin{aligned}
\mathbf{T} \text{ is subnormal} \quad &\Leftrightarrow \quad \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \nu \\
&\Leftrightarrow \quad y^2[(1-a^2)\delta_0 + a^2\delta_1] \leq (1-x^2)\delta_0 + x^2\delta_1 \\
&\Leftrightarrow \quad y^2(1-a^2) \leq 1-x^2 \text{ and } ay \leq x \\
&\Leftrightarrow \quad y \leq \min\{\frac{x}{a}, \sqrt{\frac{1-x^2}{1-a^2}}\} \\
&\Leftrightarrow \quad y \leq \sqrt{\frac{1-x^2}{1-a^2}} \text{ (since } \max\{x,y\} < 1 \text{ and } a < x).
\end{aligned}$$

□

We summarize the results in Propositions 2.10 and 2.11 as follows.

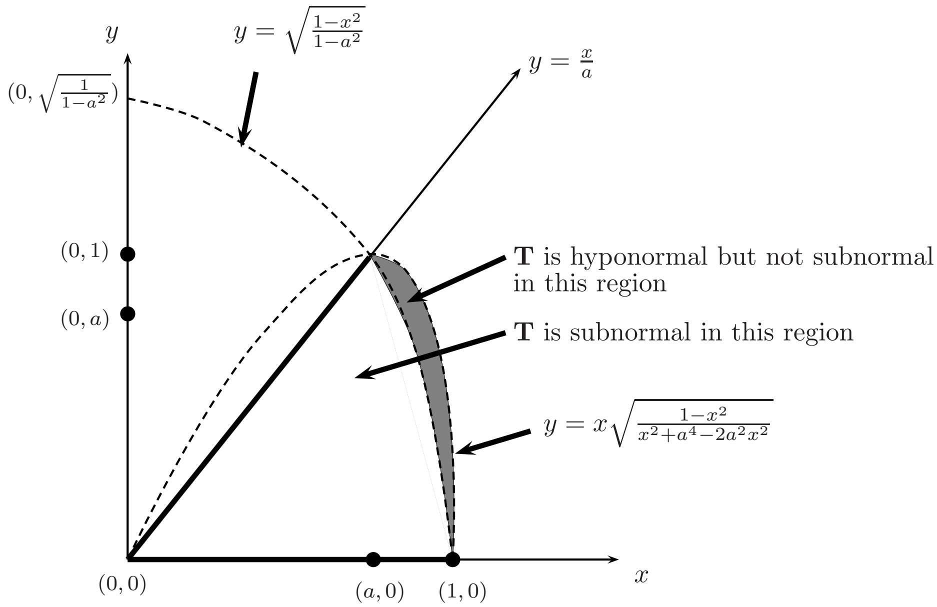

FIGURE 4. Regions of hyponormality and subnormality for the 2-variable weighted shift in Theorem 2.12

Theorem 2.12. *The* 2*-variable weighted shift* $\mathbf{T}$ *given by Figure* 3 *is hyponormal and not subnormal if and only if* $\sqrt{\frac{1-x^2}{1-a^2}} < y \leq x\sqrt{\frac{(1-x^2)}{x^2+a^4-2a^2x^2}}$ *(see Figure* 4*).*

Remark 2.13. As exemplified in Figure 4, observe that for $x > a$, $\sqrt{\frac{1-x^2}{1-a^2}} < x\sqrt{\frac{1-x^2}{x^2+a^4-2a^2x^2}} < \frac{x}{a}$; for, if $a < x$ we have

$$\begin{aligned} a^4 &< a^2x^2 \Rightarrow x^2 + a^4 - 2a^2x^2 < (1-a^2)x^2 \\ &\Rightarrow \frac{1-x^2}{1-a^2} < \frac{x^2(1-x^2)}{x^2+a^4-2a^2x^2} \end{aligned}$$

and

$$\begin{aligned} a^2(1-a^2) &< x^2(1-a^2) \Rightarrow a^2 + a^2x^2 < x^2 + a^4 \\ &\Rightarrow a^2(1-x^2) < x^2 + a^4 - 2a^2x^2 \\ &\Rightarrow \frac{x^2(1-x^2)}{x^2+a^4-2a^2x^2} < \frac{x^2}{a^2}, \end{aligned}$$

as desired.

3. THE SECOND FAMILY OF COUNTEREXAMPLES

Construction of the family. Let $0 < a, b < 1$ and let $\{\xi_k\}_{k=0}^{\infty}$ and $\{\eta_k\}_{k=0}^{\infty}$ be two strictly increasing weight sequences. Consider the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ on $\ell^2(\mathbb{Z}_+^2)$ given by the double-indexed weight sequences

$$\alpha(\mathbf{k}) := \begin{cases} \xi_{k_1} & \text{if } k_1 \geq 1 \text{ or } k_2 \geq 1, \\ a & \text{if } k_1 = 0 \text{ and } k_2 = 0 \end{cases} \tag{3.1}$$

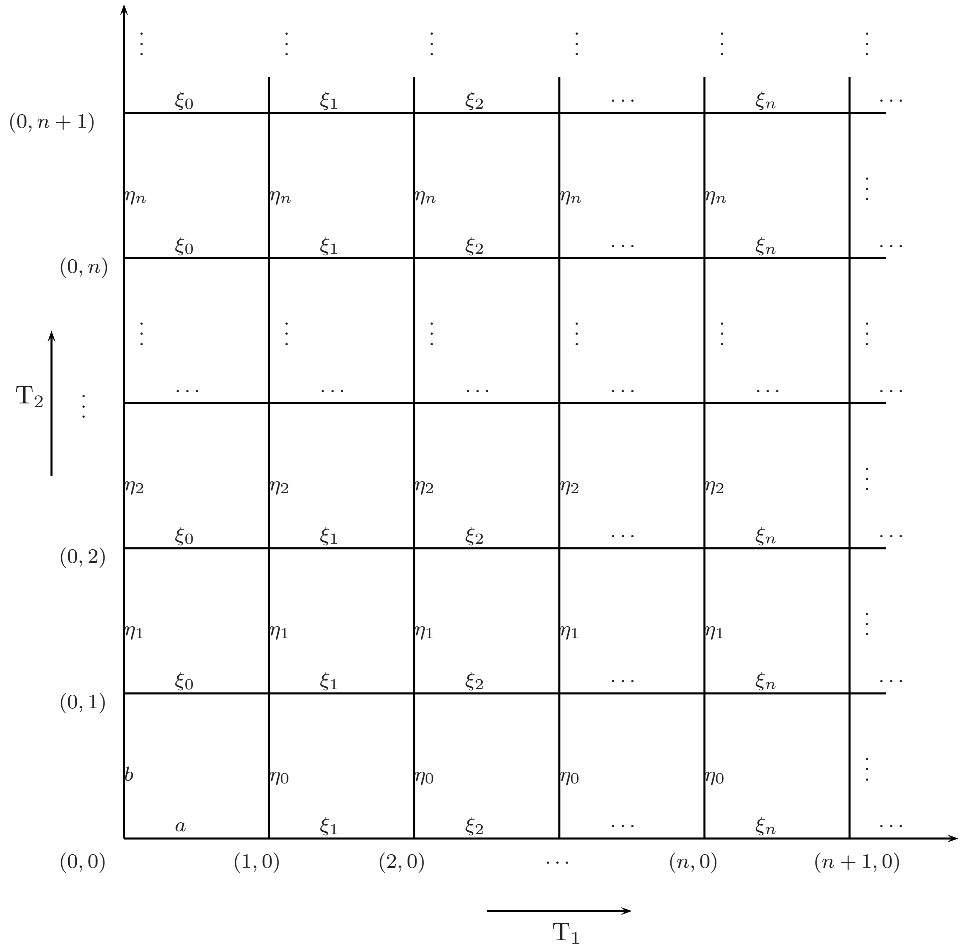

FIGURE 5.

and

$$\beta(\mathbf{k}) := \begin{cases} \eta_{k_2} & \text{if } k_1 \geq 1 \text{ or } k_2 \geq 1, \\ b & \text{if } k_1 = 0 \text{ and } k_2 = 0, \end{cases} \tag{3.2}$$

where W_ξ and W_η are two single-variable subnormal weighted shifts with Berger measures ν and ω, resp., and

$$a\eta_0 = b\xi_0 \tag{3.3}$$

(to guarantee the commutativity of T_1 and T_2; cf. (1.1)). $\mathbf{T}$ can be represented by the following weight diagram (Figure 5). It is then clear that T_1 and T_2 are subnormal provided $a \leq \xi_{ext}(\nu_{\mathcal{M}})$ and $b \leq \eta_{ext}(\omega_{\mathcal{M}})$, where, as usual, $\mathcal{M} := \bigvee\{e_1, e_2, \cdots\}$; in particular, $a < \xi_1$ and $b < \eta_1$.

Proposition 3.1. *The 2-variable weighted shift* $\mathbf{T}$ *defined by* (3.1) *and* (3.2) *is subnormal only if* $a \leq s$, *where* $s := \sqrt{\frac{\xi_0^2\xi_1^2\eta_1^2}{\xi_1^2\eta_0^2+\xi_0^2\eta_1^2-\xi_0^2\eta_0^2}}$.

Proof. Suppose that $\mathbf{T}$ above is subnormal, and let μ be its Berger measure. Then the following partial moment matrix M, corresponding to the moments of μ associated with the monomials $1, s, t$ and ts, must be positive semi-definite:

$$M := \begin{pmatrix} 1 & a^2 & b^2 & a^2\eta_0^2 \\ a^2 & a^2\xi_1^2 & a^2\eta_0^2 & a^2\eta_0^2\xi_1^2 \\ b^2 & a^2\eta_0^2 & b^2\eta_1^2 & a^2\eta_0^2\eta_1^2 \\ a^2\eta_0^2 & a^2\eta_0^2\xi_1^2 & a^2\eta_0^2\eta_1^2 & a^2\eta_0^2\xi_1^2\eta_1^2 \end{pmatrix}.$$

Now, using *Mathematica* we obtain

$$\begin{aligned} \det(M) &\geq 0 \\ &\Leftrightarrow a^6\eta_0^4(\xi_1^2-\xi_0^2)(\eta_1^2-\eta_0^2)(a^2\xi_0^2\eta_0^2 - a^2\xi_1^2\eta_0^2 - a^2\xi_0^2\eta_1^2 + \xi_0^2\xi_1^2\eta_1^2) \geq 0 \\ &\Leftrightarrow a^2\xi_0^2\eta_0^2 - a^2\xi_1^2\eta_0^2 - a^2\xi_0^2\eta_1^2 + \xi_0^2\xi_1^2\eta_1^2 \geq 0 \\ &\Leftrightarrow a \leq \sqrt{\frac{\xi_0^2\xi_1^2\eta_1^2}{\xi_1^2\eta_0^2 + \xi_0^2\eta_1^2 - \xi_0^2\eta_0^2}} = s. \end{aligned}$$

□

Proposition 3.2. *The 2-variable weighted shift* $\mathbf{T}$ *defined by* (3.1) *and* (3.2) *is hyponormal if and only if* $a \leq h$, *where* $h := \xi_0\sqrt{\frac{\xi_1^2\eta_1^2-\xi_0^2\eta_0^2}{\xi_0^2\eta_1^2+\xi_1^2\eta_0^2-2\xi_0^2\eta_0^2}}$.

Proof. From the definition of $\mathbf{T}$ and the Six-point Test (Theorem 1.3), it is clear that all we need is for the following matrix to be positive semi-definite:

$$L := \begin{pmatrix} \xi_1^2 - a^2 & \xi_0\eta_0 - ab \\ \xi_0\eta_0 - ab & \eta_1^2 - b^2 \end{pmatrix}.$$

Observe that

$$\begin{aligned} \det L &\geq 0 \Leftrightarrow \xi_1^2\eta_1^2 - \xi_1^2 b^2 - \xi_0^2\eta_0^2 - a^2\eta_1^2 + 2ab\xi_0\eta_0 \geq 0 \\ &\Leftrightarrow \xi_1^2\eta_1^2 - \xi_1^2\frac{a^2\eta_0^2}{\xi_0^2} - \xi_0^2\eta_0^2 - a^2\eta_1^2 + 2a^2\eta_0^2 \geq 0 \ \text{ (using } b\xi_0 = a\eta_0\text{; cf. (3.3))} \\ &\Leftrightarrow a^2 \leq \frac{\xi_0^2(\xi_1^2\eta_1^2 - \xi_0^2\eta_0^2)}{\xi_0^2\eta_1^2 + \xi_1^2\eta_0^2 - 2\xi_0^2\eta_0^2} = h^2 \end{aligned}$$

(observe that $\xi_0^2\eta_1^2+\xi_1^2\eta_0^2-2\xi_0^2\eta_0^2 = \xi_0^2(\eta_1^2-\eta_0^2)+(\xi_1^2-\xi_0^2)\eta_0^2 > 0$, because the weight sequences are strictly increasing by hypothesis). Thus, $a \leq h$ is clearly a necessary condition for the hyponormality of $\mathbf{T}$. Now, a straightforward calculation shows that $h < \xi_1$; for,

$$\xi_1^2 - h^2 = \frac{\eta_0^2(\xi_1^2-\xi_0^2)^2}{\xi_0^2\eta_1^2 + \xi_1^2\eta_0^2 - 2\xi_0^2\eta_0^2} > 0. \tag{3.4}$$

It follows that $a \leq h$ implies $a < \xi_1$, and therefore $L \geq 0$ by the Nested Determinant Test [Atk]. Thus, the condition $a \leq h$ is also sufficient for the hyponormality of $\mathbf{T}$, and the proof is complete. □

It follows from Propositions 3.1 and 3.2 that to ascertain the existence of a nonsubnormal, hyponormal 2-variable weighted shift $\mathbf{T}$ (with T_1 and T_2 subnormal), it suffices to show that for appropriate choices of ξ_0, ξ_1, η_0 and η_1, it is possible to obtain $s < h$, while keeping $a \leq \xi_{ext}(\nu_{\mathcal{M}})$ and $b \equiv \frac{a\eta_0}{\xi_0} \leq \eta_{ext}(\omega_{\mathcal{M}})$. Now,

$$h^2 - s^2 = \frac{\xi_0^4\eta_0^2(\xi_1^2-\xi_0^2)(\eta_1^2-\eta_0^2)}{(\xi_0^2\eta_1^2+\xi_1^2\eta_0^2-2\xi_0^2\eta_0^2)(\xi_1^2\eta_0^2+\xi_0^2\eta_1^2-\xi_0^2\eta_0^2)} > 0.$$

Therefore, it suffices to prove the existence of strictly increasing weight sequences $\{\xi_i\}$ and $\{\eta_j\}$ such that

(i) $a \leq h$ (hyponormality of $\mathbf{T}$),
(ii) $a > s$ (nonsubnormality of $\mathbf{T}$),
(iii) $a \leq \xi_{ext}(\nu_{\mathcal{M}})$ (subnormality of T_1),
(iv) $a \leq s_2 := \frac{\xi_0}{\eta_0}\eta_{ext}(\omega_{\mathcal{M}})$ (subnormality of T_2).

We now seek to determine the relative positions of h, s, s_2, ξ_0, $\xi_{ext}(\nu_{\mathcal{M}})$ and ξ_1 in the positive real axis.

Claim 1: $\xi_0 \leq \xi_{ext}(\nu_{\mathcal{M}})$. This follows from the fact that $shift(\xi_0, \xi_1, \cdots)$ is subnormal.

Claim 2: $\xi_0 < s$. For,

$$s^2 - \xi_0^2 = \frac{\xi_0^2\xi_1^2\eta_1^2}{\xi_1^2\eta_0^2 + \xi_0^2\eta_1^2 - \xi_0^2\eta_0^2} - \xi_0^2 = \frac{\xi_0^2(\xi_1^2 - \xi_0^2)(\eta_1^2 - \eta_0^2)}{\xi_1^2\eta_0^2 + \xi_0^2\eta_1^2 - \xi_0^2\eta_0^2} > 0.$$

Claim 3: $h < \xi_1$. This was established in the proof of Proposition 3.2; cf. (3.4).

Claim 4: $h \leq s_2$ whenever

$$\eta_0 \leq u := \frac{\xi_1^2(\eta_1^2 - \eta_e^2) + 2\xi_0^2\eta_e^2 - \sqrt{(\eta_1^2 - \eta_e^2)(\xi_1^4(\eta_1^2 - \eta_e^2) + 4\xi_0^2\eta_e^2(\xi_1^2 - \xi_0^2))}}{2\xi_0^2}.$$

Since

$$s_2^2 - h^2 = \frac{\xi_0^2\{\xi_0^2\eta_0^4 - [\xi_1^2(\eta_1^2 - \eta_e^2) + 2\xi_0^2\eta_e^2]\eta_0^2 + \xi_0^2\eta_e^2\eta_1^2\}}{\eta_0^2(\xi_0^2\eta_1^2 + \xi_1^2\eta_0^2 - 2\xi_0^2\eta_0^2)},$$

it follows that $h \leq s_2$ if and only if the quadratic form

$$\begin{aligned} q(t) &\equiv At^2 + Bt + C \\ &:= \xi_0^2 t^2 - [\xi_1^2(\eta_1^2 - \eta_e^2) + 2\xi_0^2\eta_e^2]t + \xi_0^2\eta_e^2\eta_1^2 \end{aligned}$$

is nonnegative. Since A and C are positive, and B is negative, we need to study the discriminant, $\Delta := B^2 - 4AC$. Now,

$$\begin{aligned} \Delta &= (\xi_1^2(\eta_1^2 - \eta_e^2) + 2\xi_0^2\eta_e^2)^2 - 4\xi_0^4\eta_e^2\eta_1^2 \\ &= (\eta_1^2 - \eta_e^2)[\xi_1^4\eta_1^2 - \eta_e^2(2\xi_0^2 - \xi_1^2)^2], \end{aligned}$$

so $\Delta \geq 0 \Leftrightarrow \xi_1^4\eta_1^2 - \eta_e^2(2\xi_0^2 - \xi_1^2)^2 \geq 0$. Since $\xi_1^4\eta_1^2 - \eta_e^2(2\xi_0^2 - \xi_1^2)^2 = \xi_1^4(\eta_1^2 - \eta_e^2)$ $+ 4\xi_0^2\eta_e^2(\xi_1^2 - \xi_0^2)$, we see that Δ is always positive. We conclude that $q \geq 0$ on the interval $[0, t_1]$, where $t_1 := \frac{-B-\sqrt{\Delta}}{2A}$ is the leftmost zero of q. Finally, a straightforward calculation shows that $t_1 = u$.

We now summarize what we have so far. For $\eta_0 \leq u$ we have

$$\begin{cases} \xi_0 < s < h \leq s_2, \\ h < \xi_1, \\ \xi_{ext}(\nu_{\mathcal{M}}) < \xi_1 \text{ (by (1.5))}. \end{cases}$$

Thus, if we can ensure that $h \leq \xi_{ext}(\nu_{\mathcal{M}})$, the construction of the example will be complete by taking a such that $s < a \leq h$. Now, since $h \leq s_2$, an easy way to accomplish this is to build $shift(\xi_0, \xi_1, \cdots)$ in such a way that $\xi_{ext}(\nu_{\mathcal{M}}) = s_2$.

To do this, we appeal to Lemma 1.7, that is, we first build a 2-step recursively generated weighted shift whose first three weights are s_2, ξ_1 and ξ_2, and we then consider the shift $W_{\xi_0(\xi_1,\xi_2,\xi_3)^\wedge}$, where ξ_3 is given by $\xi_3 := \frac{\varphi_0}{\xi_2^2} + \varphi_1$ obtained from the equation $\gamma_4 = \varphi_0\gamma_2 + \varphi_1\gamma_3$. Observe that the extremal value of $W_{(\xi_1,\xi_2,\xi_3)^\wedge}$ is s_2, and that $\xi_0 < s_2$, so the subnormality of $W_{\xi_0(\xi_1,\xi_2,\xi_3)^\wedge}$ is guaranteed. This completes the construction of the example.

Theorem 3.3. *Let* $\mathbf{T} \equiv (T_1, T_2)$ *be the* 2*-variable weighted shift defined by* (3.1) *and* (3.2)*, and let*

$$\begin{cases} h := \xi_0 \sqrt{\frac{\xi_1^2\eta_1^2 - \xi_0^2\eta_0^2}{\xi_0^2\eta_1^2 + \xi_1^2\eta_0^2 - 2\xi_0^2\eta_0^2}}, \\ s := \sqrt{\frac{\xi_0^2\xi_1^2\eta_1^2}{\xi_1^2\eta_0^2 + \xi_0^2\eta_1^2 - \xi_0^2\eta_0^2}}, \\ s_2 := \frac{\xi_0}{\eta_0}\eta_e, \text{ where } \eta_e \equiv \eta_{ext}(\omega_{\mathcal{M}}), \\ u := \frac{\xi_0^2\eta_e^2\eta_1^2}{\xi_1^2(\eta_1^2 - \eta_e^2) + \xi_0^2\eta_e^2}, \text{ and} \\ v := \frac{\xi_1^2(\eta_1^2 - \eta_e^2) + 2\xi_0^2\eta_e^2 - \sqrt{(\eta_1^2 - \eta_e^2)(\xi_1^4(\eta_1^2 - \eta_e^2) + 4\xi_0^2\eta_e^2(\xi_1^2 - \xi_0^2))}}{2\xi_0^2}. \end{cases}$$

Assume further that, as above, $s_2 = \xi_{ext}(\nu_{\mathcal{M}})$ *and* $\eta_0 \le \min\{u, v\}$*. Finally, choose* a *such that* $s < a \le h$*. Then*

(i) $T_1T_2 = T_2T_1$*;*
(ii) T_1 *is subnormal;*
(iii) T_2 *is subnormal*
(iv) $\mathbf{T}$ *is hyponormal; and*
(v) $\mathbf{T}$ *is not subnormal.*

Example 3.4. For a concrete numerical example, let $d\omega_{\mathcal{M}}(t) := 2dt$ on $[\frac{1}{2}, 1]$, so that $\left\|\frac{1}{t}\right\|_{L^1(\omega_{\mathcal{M}})} = 2\ln 2$. It follows that $\eta_e \equiv \eta_{ext}(\omega_{\mathcal{M}}) = \frac{1}{\sqrt{2\ln 2}}$ and $\eta_1 = \frac{\sqrt{3}}{2}$. Now take $\xi_0 := \frac{1}{2}$ and $\xi_1 := 1$. Then $u = \frac{1}{4(2\ln 2 - 1)} \cong 0.647$ and $v = \frac{1}{4}\frac{6\ln 2 - 2 - \sqrt{2}\sqrt{(3\ln 2 - 2)}\sqrt{(6\ln 2 - 1)}}{\ln 2} \cong 0.523$, so we can take $\eta_0 := \frac{1}{2}$. With this choice of η_0 we obtain $s = \frac{\sqrt{2}}{2} \cong 0.707$, $h = \frac{1}{2}\sqrt{\frac{11}{5}} \cong 0.742$ and $s_2 = \eta_e = \frac{1}{\sqrt{2\ln 2}} \cong 0.849$. We can then take $a \in (s, h]$, for instance $a := 0.72$. To build the weighted shift W_ξ we start with s_2, ξ_1 and $\xi_2 := \sqrt{2}$ to obtain $\varphi_0 = \frac{1}{1 - 2\ln 2}$ and $\varphi_1 = \frac{1 - 4\ln 2}{1 - 2\ln 2}$. This gives $\xi_3 = \frac{1}{2}\sqrt{\frac{16\ln 2 - 5}{2\ln 2 - 1}} \cong 1.985$. The 2-atomic measure $\nu_{\mathcal{M}}$ for $W_{(\xi_1,\xi_2,\xi_3)^\wedge}$ has atoms $t_0 \cong 0.659$ and $t_1 \cong 3.93$, and densities $\rho_0 \cong 0.981$ and $\rho_1 \cong 0.019$. With these values we can compute $\left\|\frac{1}{t}\right\|_{L^1(\nu_{\mathcal{M}})} \cong 1.494$; observe that $\xi_0 = \frac{1}{2} \le \xi_{ext}(\nu_{\mathcal{M}}) = \sqrt{\frac{1}{\left\|\frac{1}{t}\right\|_{L^1(\nu_{\mathcal{M}})}}} \cong 0.818$. By Proposition 1.5, the measure associated to $shift(\xi_0, \xi_1, \xi_2, \cdots)$ is $d\nu(t) = \frac{1}{4t}(\rho_0 d\delta_{t_0}(t) + \rho_1 d\delta_{t_1}(t)) + (1 - \frac{1}{4}\left\|\frac{1}{t}\right\|_{L^1(\nu_{\mathcal{M}})})d\delta_0(t)$.

4. The third family of counterexamples

Construction of the family. Let us consider the following 2-variable weighted shift (see Figure 6), where

$$(4.1)\quad \begin{cases} \text{(i)}\ \ 0 < \xi_1 < \xi_2 < \cdots < \xi_n \nearrow 1; \\ \text{(ii)}\ \ W_\xi := shift(\xi_1, \xi_2, \cdots) \text{ is subnormal with Berger measure } \nu; \\ \text{(iii)}\ \ \frac{1}{s^2} \in L^1(\nu) \text{ (this implies that } \frac{1}{s} \in L^1(\nu)\text{, by Jensen's inequality);} \\ \text{(iv)}\ \ \xi_e \equiv \xi_{ext} := (\int \frac{1}{s} d\nu(s))^{-1/2}; \\ \text{(v)}\ \ a \le \frac{1}{\xi_e} (\int \frac{1}{s^2} d\nu(s))^{-1/2}; \\ \text{(vi)}\ \ b \le \xi_e^2 \text{ (this implies the condition } b < \xi_e\text{); and} \\ \text{(vii)}\ \ a^2 \le \frac{b^2 + \xi_e^2}{2}. \end{cases}$$

(Recall that ξ_e is the maximum possible value for ξ_0 in Proposition 1.5.)

Observe that $T_1 \cong T_2$ and that $T_1 T_2 = T_2 T_1$. We claim that T_1 (and therefore T_2) is subnormal. For, the choice of ξ_e immediately implies that $shift(\xi_e, \xi_1, \xi_2, \cdots)$ is subnormal, with Berger measure $d\nu_e(s) := \frac{\xi_e^2}{s} d\nu(s)$ (cf. Proposition 1.5). Another application of Proposition 1.5 shows that $shift(a, \xi_e, \xi_1, \cdots)$ is subnormal if and only if $\frac{1}{s} \in L^1(\nu_e)$ (i.e., $\frac{1}{s^2} \in L^1(\nu)$, which is true by (4.1)(iii)) and $a^2 \xi_e^2 \int \frac{1}{s^2} d\nu(s) \le 1$, which holds by (4.1)(v)). This implies that the restriction of T_1 to $\bigvee\{e_{(i,0)} : i \ge 0\}$ is subnormal. Moreover, the subnormality of T_1 when restricted to $\bigvee\{e_{(i,j)} : i \ge 0\}$ $(j > 0)$ requires that $b \le \xi_e$, which holds by (4.1)(vi).

For a concrete numerical example, consider the probability measure $d\nu(s) := 3s^2 ds$ on the interval $[0, 1]$. The measure ν corresponds to a subnormal weighted shift with weights $\xi_1 = \sqrt{\frac{3}{4}}$, $\xi_2 = \sqrt{\frac{4}{5}}$, $\xi_3 = \sqrt{\frac{5}{6}}$, $\cdots$. Indeed, in this case W_ξ is the restriction of the Bergman shift B_+ to the invariant subspace $\mathcal{M}_2$ obtained by removing the first two basis vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$. Clearly $\frac{1}{s^2} \in L^1(\nu)$ and $\int \frac{1}{s^2} d\nu(s) = 3$; moreover, $\int \frac{1}{s} d\nu(s) = \frac{3}{2}$, so in this case $\xi_e = \sqrt{\frac{2}{3}}$. Choosing $a = \sqrt{\frac{1}{2}}$ and $b = \sqrt{\frac{1}{3}}$ we see that all conditions in (4.1) are satisfied (cf. Corollary 4.4).

Proposition 4.1. *The 2-variable weighted shift* $\mathbf{T}$ *given by Figure* 6 *is hyponormal.*

Proof. Since the restriction of $\mathbf{T}$ to $\bigvee\{e_{(i,j)} : i, j \ge 1\}$ is clearly subnormal (being unitarily equivalent to $(I \otimes W_\xi, W_\xi \otimes I)$, and since the weight diagram of $\mathbf{T}$ is symmetric with respect to the diagonal $j = i$, it suffices to apply the Six-point Test (Theorem 1.3) to $\mathbf{k} = (i, 0)$, with $i \ge 0$.

Case 1: $\mathbf{k} = (0, 0)$. Here we have

$$\begin{pmatrix} \xi_e^2 - a^2 & b^2 - a^2 \\ b^2 - a^2 & \xi_e^2 - a^2 \end{pmatrix} \ge 0 \Leftrightarrow (\xi_e^2 - a^2)^2 \ge (b^2 - a^2)^2$$
$$\Leftrightarrow \ \xi_e^2 - a^2 \ge \left| b^2 - a^2 \right|.$$

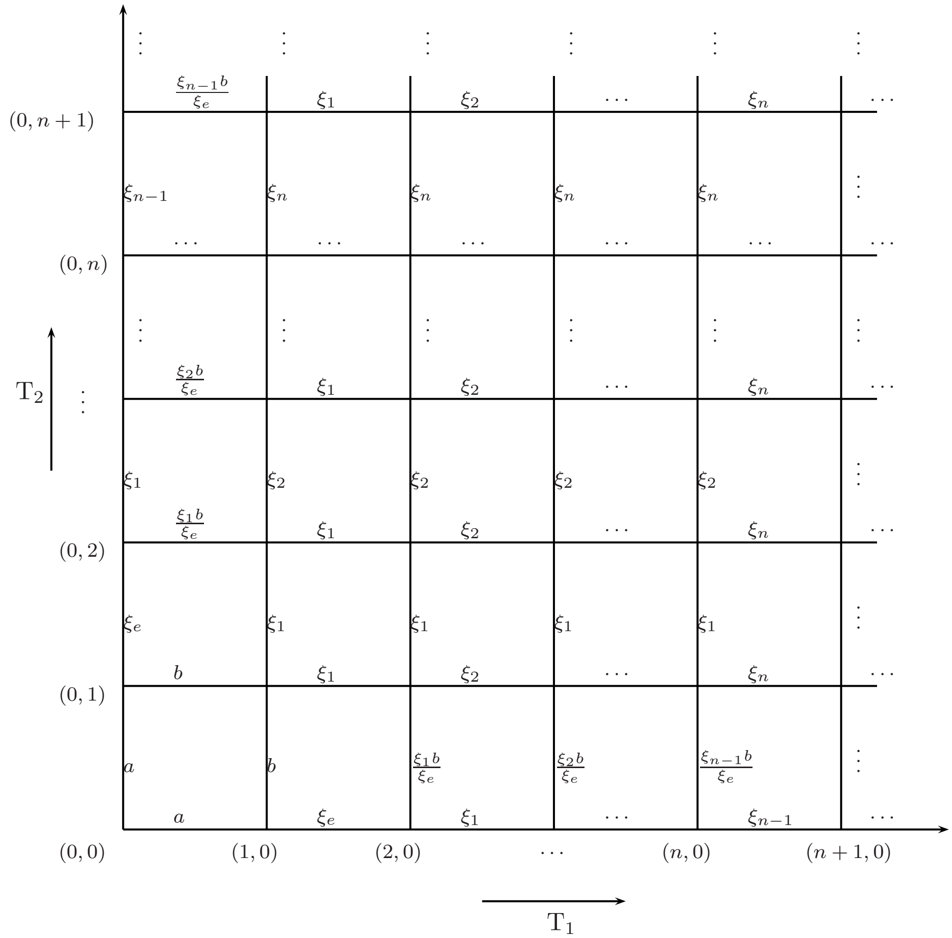

FIGURE 6. Weight diagram of the 2-variable weighted shift in Example 19

When $b \le a$, the last condition is equivalent to $2a^2 \le b^2 + \xi_e^2$, which holds by (4.1)(vii). When $b > a$, the condition is equivalent to $\xi_e \ge b$, which is guaranteed by (4.1)(vi).

Case 2: $\mathbf{k} = (1,0)$. Here

$$\begin{pmatrix} \xi_1^2 - \xi_e^2 & \frac{\xi_1^2 b}{\xi_e} - b\xi_e \\ \frac{\xi_1^2 b}{\xi_e} - b\xi_e & \xi_1^2 - b^2 \end{pmatrix} \ge 0 \Leftrightarrow (\xi_1^2 - \xi_e^2)(\xi_1^2 - b^2) \ge (\frac{\xi_1^2 b}{\xi_e} - b\xi_e)^2$$

$$\Leftrightarrow \xi_1^2 - b^2 \ge (\xi_1^2 - \xi_e^2)\frac{b^2}{\xi_e^2} \Leftrightarrow b \le \xi_e,$$

which again is guaranteed by (4.1)(vi).

Case 3: $k=(n+1,0)$ $(n\geq 1)$. Here

$$\begin{pmatrix} \xi_{n+1}^2-\xi_n^2 & \frac{\xi_{n+1}^2 b}{\xi_e}-\frac{\xi_n^2 b}{\xi_e} \\ \frac{\xi_{n+1}^2 b}{\xi_e}-\frac{\xi_n^2 b}{\xi_e} & \xi_1^2-\frac{\xi_n^2 b^2}{\xi_e^2} \end{pmatrix} \geq 0$$

$$\Leftrightarrow \quad (\xi_{n+1}^2-\xi_n^2)(\xi_1^2-\frac{\xi_n^2 b^2}{\xi_e^2}) \geq (\frac{\xi_{n+1}^2 b}{\xi_e}-\frac{\xi_n^2 b}{\xi_e})^2$$

$$\Leftrightarrow \quad \frac{(\xi_{n+1}^2-\xi_n^2)\left(\xi_1^2\xi_e^2-\xi_{n+1}^2 b^2\right)}{\xi_e^2} \geq 0$$

$$\Leftrightarrow \quad b \leq \frac{\xi_1\xi_e}{\xi_{n+1}} \text{ (all } n\geq 1). \tag{4.2}$$

Since the sequence $\{\xi_n\}$ increases to 1, the last inequality in (4.2) is equivalent to $b\leq \xi_1\xi_e$, which holds by (4.1)(vi).

The proof is now complete. □

Proposition 4.2. *The* 2*-variable weighted shift* $\mathbf{T}$ *given by Figure* 6 *is not subnormal if* $p<0$, *where* $p:=\xi_e^2\xi_1^4+4a^2b^2\xi_1^2-b^2\xi_1^4-a^2b^2\xi_e^2-a^2b^4-2a^2\xi_1^4$.

Proof. Assume that $\mathbf{T}$ is subnormal, and consider the moment matrix associated to the monomials 1, x, y and yx (cf. [CuFi4], [CuFi5]), that is,

$$M:=\begin{pmatrix} 1 & a^2 & a^2 & a^2b^2 \\ a^2 & a^2\xi_e^2 & a^2b^2 & a^2b^2\xi_1^2 \\ a^2 & a^2b^2 & a^2\xi_e^2 & a^2b^2\xi_1^2 \\ a^2b^2 & a^2b^2\xi_1^2 & a^2b^2\xi_1^2 & a^2b^2\xi_1^4 \end{pmatrix}.$$

In the presence of a representing measure, it is well known that M must be positive semi-definite, so in particular $\det M\geq 0$. Now, a straightforward calculation shows that

$$\begin{aligned}\det M &= a^6b^2\left(\xi_e^2-b^2\right)\left(\xi_e^2\xi_1^4-\xi_e^2a^2b^2-2a^2\xi_1^4-b^2\xi_1^4+4a^2b^2\xi_1^2-b^4a^2\right)\\ &= a^6b^2\left(\xi_e^2-b^2\right)p.\end{aligned}$$

It follows that $p\geq 0$. Therefore, $\mathbf{T}$ is not subnormal whenever $p<0$, as desired. □

Theorem 4.3. *Let* $a>0$ *be such that* $\sqrt{\frac{\xi_e^2}{2}}<a\leq\sqrt{\frac{\xi_e^2+\xi_e^4}{2}}$ *and* $a\leq \frac{1}{\xi_e}(\int \frac{1}{s^2}d\nu(s))^{-1/2}$, *and define* $b:=\sqrt{2a^2-\xi_e^2}$. *Then the* 2*-variable weighted shift* $\mathbf{T}\equiv(T_1,T_2)$ *satisfies* (4.1)(i)-(vii), *is hyponormal, and is not subnormal.*

Proof. Observe that the condition $\sqrt{\frac{\xi_e^2}{2}}<a$ guarantees that $2a^2>\xi_e^2$ (so b is well defined) and that the condition $a\leq\sqrt{\frac{\xi_e^2+\xi_e^4}{2}}$ is equivalent to $2a^2-\xi_e^2\leq\xi_e^4$ (so b satisfies (4.1)(vi)). Moreover, $a^2=\frac{b^2+\xi_e^2}{2}$ trivially, so (4.1)(vii) also holds. It follows that $\mathbf{T}$ is hyponormal, by Proposition 4.1. To break subnormality, by Proposition 4.2 it suffices to show that p is negative. Since $b^2=2a^2-\xi_e^2$, we have

$$\begin{aligned} p &= \xi_e^2\xi_1^4-\xi_e^2a^2(2a^2-\xi_e^2)-2a^2\xi_1^4-(2a^2-\xi_e^2)\xi_1^4\\ &\quad +4a^2(2a^2-\xi_e^2)\xi_1^2-(2a^2-\xi_e^2)^2a^2\\ &= -2\left(\xi_1^2-a^2\right)^2\left(2a^2-\xi_e^2\right)<0,\end{aligned}$$

as desired. The proof is now complete. □

Corollary 4.4 ([DrMcC]). *Let $dv(s) := 3s^2 ds$ on $[0,1]$ and choose $a = \sqrt{\frac{1}{2}}$ and $b = \sqrt{\frac{1}{3}}$. Then the 2-variable weighted shift $\mathbf{T}$ given by Figure 6 is commuting, has subnormal components, is hyponormal, but is not subnormal.*

Proof. By Theorem 4.3 and the comments preceding Proposition 4.1, it suffices to verify that $\sqrt{\frac{\xi_e^2}{2}} < a \leq \sqrt{\frac{\xi_e^2+\xi_e^4}{2}}$. Since $\xi_e = \sqrt{\frac{2}{3}}$ and $a = \sqrt{\frac{1}{2}}$, the result follows by a straightforward calculation. $\square$

5. An instance when hyponormality suffices

In this section we will prove that under a suitable condition hyponormality does imply subnormality for commuting pairs of subnormal operators. We begin with an elementary result of independent interest.

Lemma 5.1. *Let ν be a probability measure on $[0,1]$, and let $\gamma_n \equiv \gamma_n(\nu) := \int s^n d\nu(s)$ $(n \geq 0)$ be the moments of ν. The sequence $\{\gamma_n\}_{n=0}^{\infty}$ is bounded below if and only if ν has an atom at $\{1\}$.*

Proof. ($\Leftarrow$) Let $\rho := \nu(\{1\}) > 0$ and write $\nu \equiv (1-\rho)\eta + \rho\delta_1$, where η is a probability measure on $[0,1]$ with $\eta(\{1\}) = 0$. It follows that $\gamma_n(\nu) \geq \rho \int s^n \, d\delta_1(s) = \rho$ (all $n \geq 0$), so $\{\gamma_n\}$ is bounded below by ρ.

($\Rightarrow$) Suppose $\nu(\{1\}) = 0$, let $f_n(s) := s^n$ $(0 \leq s \leq 1,\ n \geq 0)$, and consider the sequence of nonnegative functions $\{f_n\}_{n\geq 0}$. Clearly $f_n \searrow \chi_{\{1\}}$ pointwise, and $|f_n| \leq 1$ (all $n \geq 0$). By the Lebesgue Dominated Convergence Theorem, $\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \int s^n d\nu(s) = \int \chi_{\{1\}} d\nu(s) = \nu(\{1\}) = 0$. Therefore, $\{\gamma_n\}$ is not bounded below. $\square$

We now consider the 2-variable weighted shift $\mathbf{T}$ given by Figure 7, where $W_\xi := shift(\xi_0, \xi_1, \cdots)$ is a subnormal contraction with associated measure ν, and $y \leq 1$.

It is clear that $T_1 T_2 = T_2 T_1$, and that T_1 is subnormal (being the orthogonal direct sum of W_ξ and copies of U_+). To ensure the subnormality of T_2, we must impose the condition $\frac{y}{\sqrt{\gamma_n}} \leq 1$ (all $n \geq 0$), i.e., $y^2 \leq \gamma_n$ (all $n \geq 0$), where $\gamma_n \equiv \gamma_n(\nu)$. Note that this condition also guarantees the boundedness of $\mathbf{T}$.

Theorem 5.2. *Let $\mathbf{T}$ be the 2-variable weighted shift given by Figure 7, and assume that $\mathbf{T}$ is hyponormal. Then $\mathbf{T}$ is subnormal.*

Proof. We apply the Six-point Test (Theorem 1.3) to an arbitrary lattice point of the form $(n,0)$. Since $\mathbf{T}$ is hyponormal by hypothesis, we must have

$$(\xi_{n+1}^2 - \xi_n^2)(1 - \frac{y^2}{\gamma_n}) \geq (\frac{y}{\sqrt{\gamma_{n+1}}} - \frac{y\xi_n}{\sqrt{\gamma_n}})^2,$$

or equivalently $(\xi_{n+1}^2 - \xi_n^2)(1 - \frac{y^2}{\gamma_n}) \geq \frac{y^2}{\gamma_n}(\frac{1}{\xi_n} - \xi_n)^2$, that is, $y^2 \leq (\frac{\xi_{n+1}^2 - \xi_n^2}{\xi_{n+1}^2 + \frac{1}{\xi_n^2} - 2})\gamma_n$. Since $\xi_n^2 + \frac{1}{\xi_n^2} - 2 \geq 0$ and $\frac{\xi_{n+1}^2 - \xi_n^2}{\xi_{n+1}^2 + \frac{1}{\xi_n^2} - 2} = \frac{\xi_{n+1}^2 - \xi_n^2}{(\xi_{n+1}^2 - \xi_n^2) + \xi_n^2 + \frac{1}{\xi_n^2} - 2}$, it follows that $\frac{\xi_{n+1}^2 - \xi_n^2}{\xi_{n+1}^2 + \frac{1}{\xi_n^2} - 2} \leq 1$, so $0 < y^2 \leq \gamma_n$ (all $n \geq 0$). Thus, $\{\gamma_n\}$ is bounded below, and by Lemma 5.1 we can write $\nu = (1-\rho)\eta + \rho\delta_1$, with $\rho := \nu(\{1\})$ and $\eta(\{1\}) = 0$. It follows that $y^2 \leq \rho$. Thus, $y^2\delta_1 \leq \nu$. By Proposition 2.9, $\mathbf{T}$ is subnormal. $\square$

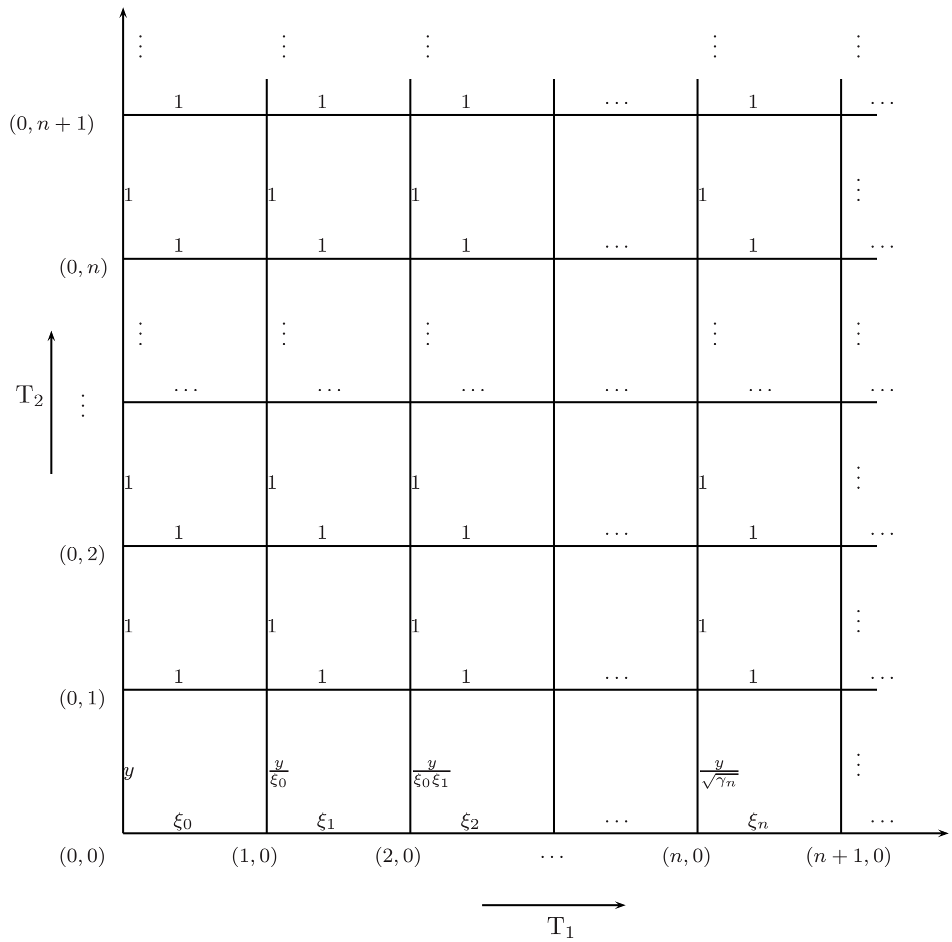

FIGURE 7. Weight diagram of the 2-variable weighted shift in Theorem 5.2

Remark 5.3. Theorem 5.2 (and its proof) reveals that for the 2-variable weighted shift given by Figure 7, the subnormality of T_2 is equivalent to the subnormality of $\mathbf{T}$, which in turn is equivalent to the hyponormality of $\mathbf{T}$.

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