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GROUPOID COHOMOLOGY AND EXTENSIONS

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Abstract. We show that Haefliger's cohomology for étale groupoids, Moore's cohomology for locally compact groups and the Brauer group of a locally compact groupoid are all particular cases of sheaf (or Čech) cohomology for topological simplicial spaces.

1. Introduction

Let G be a locally compact Hausdorff groupoid with Haar system. In [11], the authors studied the group of Morita equivalence classes of actions of G on continuous fields of C^*-algebras over the unit space G_0 such that

- Each fiber is isomorphic to the algebra of compact operators on some Hilbert space (depending on the fiber).
- The bundle satisfies Fell's condition, i.e. each point of G_0 has a neighborhood U such that there exists a section $f(x)$ with $f(x)$ a rank-one projection for all $x \in U$.

They called this group the Brauer group $Br(G)$ of G, and showed that it is naturally isomorphic to the group $\mathrm{Ext}(G, \mathbb{T})$ of Morita equivalence classes of central extensions

$$\mathbb{T} \times G'_0 \to E \to G',$$

where G' is some Morita equivalent groupoid. In the case of discrete groups, it is well known that central extensions of G by $\mathbb{T}$ are classified by $H^2(G, \mathbb{T})$. Actually, given any locally compact group G and any Polish (i.e. metrizable separable complete) G-module A, Moore's cohomology groups $H^2(G, A)$ classify extensions $A \hookrightarrow E \twoheadrightarrow G$ such that the action of G on A by conjugation is exactly the action of G on the G-module A [18, 19]. One of the possible definitions of Moore's cohomology is the following: consider $C^n(G, A)$ the space of all measurable maps $c\colon G^n \to A$ with the differential

$$\begin{aligned}(dc)(g_1, \dots, g_n) &= g_1 c(g_2, \dots, g_n) \\ &\quad + \sum_{k=1}^{n} (-1)^k c(g_1, \dots, g_k g_{k+1}, \dots, g_n) + (-1)^{n+1} c(g_1, \dots, g_n).\end{aligned} \tag{1.1}$$

Then by definition, $H^n(G, A)$ is the n-th cohomology group of the complex $C^*(G, A)$. On the other hand, Haefliger ([6], see also [10]) defined sheaf cohomology groups $H^*(G, \mathcal{A})$ given any étale groupoid G and any Abelian G-sheaf $\mathcal{A}$

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(i.e. an Abelian sheaf on G_0 endowed with a continuous action of G). It was thus natural to expect that a single cohomology theory for groupoids should unify all these: this is the question asked by A. Kumjian in Boulder (1999).

Our approach is to consider the simplicial (topological) space $G_\bullet = (G_n)_{n\in\mathbb{N}}$ associated to the groupoid G and to use sheaf cohomology for simplicial spaces [4] and Čech cohomology (see Section 4). Given a simplicial space $M_\bullet$ and an Abelian sheaf $\mathcal{A}^\bullet$ on $M_\bullet$, let us denote by $H^n(M_\bullet;\mathcal{A}^\bullet)$ and $\check{H}(M_\bullet;\mathcal{A}^\bullet)$ the sheaf and Čech cohomology groups, respectively. We show:

Theorem 1.1. (a) *$H^n(M_\bullet;\mathcal{A}^\bullet) \cong \check{H}(M_\bullet;\mathcal{A}^\bullet)$ if $M_\bullet$ is a simplicial space such that M_n is paracompact for all n;*

(b) *$H^n(G_\bullet;\mathcal{A}^\bullet)$ coincides with Haefliger's cohomology $H^n(G;\mathcal{A}^0)$ if G is an étale groupoid and $\mathcal{A}^\bullet$ is the sheaf on $G_\bullet$ corresponding to an Abelian G-sheaf $\mathcal{A}^0$;*

(c) *$H^n(G_\bullet;\mathcal{A}^\bullet) \cong \check{H}(G_\bullet;\mathcal{A}^\bullet) \cong H^n_{\mathrm{Moore}}(G,A)$ if G is a locally compact group, A is a Polish G-module and $\mathcal{A}^\bullet$ is the sheaf on $G_\bullet$ associated to A;*

(d) *$H^2(G_\bullet;\mathcal{T}) \cong \check{H}(G_\bullet;\mathcal{T})$ is the Brauer group of G if G is a locally compact Hausdorff groupoid with Haar system, where $\mathcal{T}$ denotes the sheaf associated to the G-module $G_0 \times \mathbb{T}$;*

(e) *$H^n(G_\bullet,\mathcal{A}^\bullet)$ and $\check{H}^*(G_\bullet,\mathcal{A}^\bullet)$ are invariant under Morita equivalence of topological groupoids.*

Let us comment on the above theorem in relation to the existing literature.

The construction of Čech cohomology for simplicial spaces as well as part (a) is new. Let us note however that there is another kind of Čech cohomology defined for any topos [1, Exposé V]. It would be interesting to compare it with ours when the topos consists of the category of sheaves on a simplicial space.

Part (b) is already known [15, first part of Theorem 3.1]. We give a different proof in this paper.

Part (c) is easy, and follows from general nonsense analogue to Buchsbaum's criterion (Proposition 6.1). Note also that D. Wigner [25] defined a cohomology theory for topological groups which generalizes Moore's, and which is isomorphic under certain assumptions to sheaf cohomology of the associated simplicial space [25, Theorem 2].

For part (d), it was established by Kumjian [10] that the Brauer group of an *étale* groupoid coincides with the sheaf cohomology group $H^2(G;\mathcal{T})$. However, the authors of [11] were apparently unable to generalize that result to locally compact groupoids [11, p. 909].

Actually we prove more than statement (d): Proposition 5.6 identifies the group of extensions of G by an *(Abelian)* G-module with a 2-cohomology group. In the case of étale groupoids, Moerdijk [17] studies extensions which are even *non-Abelian* by introducing another kind of Čech cohomology. I do not know if his cohomology coincides with the one defined here.

Part (e) is almost clear from the definitions, and is proved in section 8.

In summary, several parts of this paper are known in special cases, or overlap with known results. However, we hope that the present approach, being rather direct and elementary, will still be of interest to the reader.

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2. Simplicial spaces and groupoids

2.1. **Definition of simplicial spaces.** Let us recall some basic facts about simplicial spaces. Let Δ (resp. Δ') be the simplicial (resp. pre-simplicial) category, whose objects are the nonnegative integers, and whose morphisms are the nondecreasing (resp. increasing) maps $[m] \to [n]$ (where $[n]$ denotes the interval $\{0, \dots, n\}$). We denote by $\Delta^{(N)}$ the N-truncated simplicial category, i.e. the full sub-category of Δ whose objects are the integers $\leq N$.

A simplicial (resp. pre-simplicial, N-simplicial) topological space is a contravariant functor from the category Δ (resp. Δ', resp. $\Delta^{(N)}$) to the category of topological spaces. In the same way, one can define the notion of a simplicial (resp. pre-simplicial, N-simplicial) manifold. In this paper we shall work with simplicial topological spaces and will use the terminology "simplicial space", but some of the results can easily be transposed to simplicial manifolds.

In practice, a (pre-)simplicial space is a sequence $M_\bullet = (M_n)_{n\in\mathbb{N}}$ of topological spaces, given with continuous maps $\tilde{f}\colon M_n \to M_k$ for every morphism $f\colon [k] \to [n]$, satisfying the relation $\widetilde{f \circ g} = \tilde{g} \circ \tilde{f}$ for all composable morphisms f and g.

Let $\varepsilon_i^n\colon [n-1] \to [n]$ be the unique increasing map that avoids i, and let $\eta_i^n : [n+1] \to [n]$ be the unique nondecreasing surjective map such that i is reached twice $(0 \leq i \leq n)$. We will usually omit the superscripts for convenience of notation.

If $M_\bullet = (M_n)_{n\in\mathbb{N}}$ is a simplicial space, then the face maps $\tilde{\varepsilon}_i^n : M_n \to M_{n-1}$, $i = 0, \dots, n$, and the degeneracy maps $\tilde{\eta}_i^n : M_n \to M_{n+1}$, $i = 0, \dots, n$, satisfy the following simplicial identities: $\tilde{\varepsilon}_i^{n-1}\tilde{\varepsilon}_j^n = \tilde{\varepsilon}_{j-1}^{n-1}\tilde{\varepsilon}_i^n$ if $i < j$, $\tilde{\eta}_i^{n+1}\tilde{\eta}_j^n = \tilde{\eta}_{j+1}^{n+1}\tilde{\eta}_i^n$ if $i \leq j$, $\tilde{\varepsilon}_i^{n+1}\tilde{\eta}_j^n = \tilde{\eta}_{j-1}^{n-1}\tilde{\varepsilon}_i^n$ if $i < j$, $\tilde{\varepsilon}_i^{n+1}\tilde{\eta}_j^n = \tilde{\eta}_j^{n-1}\tilde{\varepsilon}_{i-1}^n$ if $i > j+1$ and $\tilde{\varepsilon}_j^{n+1}\tilde{\eta}_j^n = \tilde{\varepsilon}_{j+1}^{n+1}\tilde{\eta}_j^n = \mathrm{Id}_{M_n}$.

Conversely, if we are given a sequence $M_\bullet$ of topological spaces and maps satisfying such identities, then there is a unique simplicial structure on $M_\bullet$ such that $\tilde{\varepsilon}_i^n$ are the face maps and $\tilde{\eta}_i^n$ are the degeneracy maps.

2.2. **Groupoids.** In order to fix notations, we first recall some basic facts about groupoids. For more details, see e.g. [22].

A topological groupoid is given by two topological spaces G_0 and G, two maps r and s from G to G_0, called the range and source maps, a unit map $\eta\colon G_0 \to G$, a partially defined multiplication $G_2 = \{(g,h) \in G^2 |\ s(g) = t(h)\} \to G$ denoted by $(g,h) \mapsto gh$, and an inversion map $G \to G$ denoted by $g \mapsto g^{-1}$ such that the following identities hold (for $g, h, k \in G$ and $x \in G_0$):

- $r(gh) = r(g)$, $s(gh) = s(h)$;
- $(gh)k = g(hk)$ whenever $s(g) = r(h)$ and $s(h) = r(k)$;
- $s(\eta(x)) = r(\eta(x)) = x$;
- $g\eta(s(g)) = \eta(r(g))g = g$;
- $r(g^{-1}) = s(g)$, $s(g^{-1}) = r(g)$, $gg^{-1} = \eta(r(g))$, $g^{-1}g = \eta(s(g))$.

We will usually identify the unit space G_0 to a subspace of G by means of the unit map η.

Standard examples are:

- groups, with $G_0 = \mathrm{pt}$;
- spaces M, with $G = G_0 = M$, $r = s = \mathrm{Id}$;

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- the homotopy groupoid of a space M, where $G_0 = M$, G is the set of homotopy classes of paths in M, $s(g)$ is the starting point of the path and $r(g)$ is the endpoint.

Here are a few notations that we will use: $G_x = s^{-1}(x)$, $G^x = r^{-1}(x)$, $G_x^y = G_x \cap G^y$.

A left *action* of a topological groupoid G on a space Z is given by a (continuous) map $p\colon Z \to G_0$ and a map $G \times_{s,p} Z \to Z$, denoted by $(g,z) \mapsto gz$, such that

- $p(gz) = r(g)$;
- $(gh)z = g(hz)$ whenever $(g,h) \in G_2$ and $s(h) = p(z)$;
- $ez = z$ if $e \in G_0 \subseteq G$.

We will say that Z is a (left) G-space. Given a G-space Z, we form the crossed-product groupoid $G \ltimes Z := G \times_{s,p} Z$ with unit space Z, source and range maps $s(g,z) = z$, $r(g,z) = gz$, product $(g,z)(h,z') = (gh,z')$ if $z = hz'$, and inverse $(g,z)^{-1} = (g^{-1}, gz)$.

Any topological groupoid G canonically gives rise to a simplicial space as follows [23]: let

$$G_n = \{(g_1,\dots,g_n) |\ s(g_i) = t(g_{i+1})\ \forall i\}$$

be the set composable n-tuples.

Define the face maps $\tilde{\varepsilon}_i^n : G_n \to G_{n-1}$ for $n > 1$ by

$$\begin{aligned}
&\tilde{\varepsilon}_0^n(g_1,g_2,\dots,g_n) = (g_2,\dots,g_n),\\
&\tilde{\varepsilon}_n^n(g_1,g_2,\dots,g_n) = (g_1,\dots,g_{n-1}),\\
&\tilde{\varepsilon}_i^n(g_1,\dots,g_n) = (g_1,\dots,g_ig_{i+1},\dots,g_n), \quad 1 \le i \le n-1,
\end{aligned}$$

and for $n = 1$ by, $\tilde{\varepsilon}_0^1(g) = s(g)$, $\tilde{\varepsilon}_1^1(g) = r(g)$. Also define the degeneracy maps: $\tilde{\eta}_0^0 : G_0 \to G_1$ is the unit map of the groupoid, and $\tilde{\eta}_i^n : G_n \to G_{n+1}$ by

$$\begin{aligned}
&\tilde{\eta}_0^n(g_1,\dots,g_n) = (r(g_1),g_1,\dots,g_n),\\
&\tilde{\eta}_i^n(g_1,\dots,g_n) = (g_1,\dots,g_i,s(g_i),g_{i+1},\dots,g_n), \quad 1 \le i \le n.
\end{aligned}$$

Another way to view the simplicial structure of $G_\bullet$ is the following: we note that G_n can be identified with the quotient of

$$(EG)_n := \{(\gamma_0,\dots,\gamma_n) \in G^{n+1} |\ r(\gamma_0) = \dots = r(\gamma_n)\}$$

by the left action of G, the correspondence being

$$\begin{aligned}
(g_1,\dots,g_n) &= (\gamma_0^{-1}\gamma_1,\dots,\gamma_{n-1}^{-1}\gamma_n),\\
[\gamma_0,\dots,\gamma_n] &= [r(g_1),g_1,g_1g_2,\dots,g_1\cdots g_n].
\end{aligned}$$

Then, for any morphism $f\colon [k] \to [n]$, $\tilde{f}\colon G_n \to G_k$ is defined by

$$\tilde{f}[\gamma_0,\dots,\gamma_n] = [\gamma_{f(0)},\dots,\gamma_{f(n)}].$$

For instance, in the first picture, if f is injective, then

$$\tilde{f}(g_1,\dots,g_n) = (g_{f(0)+1}\cdots g_{f(1)}, g_{f(1)+1}\cdots g_{f(2)},\dots,g_{f(k-1)+1}\cdots g_{f(k)}). \tag{2.1}$$

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2.3. **Morita equivalence and generalized morphisms.** We recall (see for instance [7, 8, 13, 20, 12, 24]) that a generalized morphism between two (topological, or locally compact, or Lie) groupoids G' and G is given by a topological space (or a locally compact space, or a manifold) Z, two maps $G'_0 \xleftarrow{\rho} Z \xrightarrow{\sigma} G_0$ such that Z admits a left action of G' with respect to ρ, a right action of G with respect to σ, with the property that the two actions commute and $\rho\colon Z \to G'_0$ is a locally trivial G-principal bundle.

Topological (or locally compact...) groupoids and isomorphism classes of generalized morphisms form a category whose isomorphisms are Morita equivalences. Every groupoid morphism naturally defines a generalized morphism.

If $\mathcal{U} = (U_i)_{i\in I}$ is an open cover of G_0, define the cover groupoid

$$G[\mathcal{U}] = \{(i, g, j) \in I \times G \times I |\ r(g) \in U_i, s(g) \in U_j\} \tag{2.2}$$

with unit space $\{(i, x) \in I\times G_0 |\ x \in U_i\}$, source and range maps $s(i, g, j) = (j, s(g))$, $r(i, g, j) = (i, r(g))$ and product $(i, g, j)(j, h, k) = (i, gh, k)$.

Then the canonical morphism $G[\mathcal{U}] \to G$ is a Morita equivalence. Moreover, every generalized morphism $G' \to G$ admits a decomposition $G' \xleftarrow{\sim} G'[\mathcal{U}'] \xrightarrow{f} G$ for some open cover $\mathcal{U}'$ of G'_0 and some *groupoid morphism* f.

Remark 2.1. The simplicial space $G[\mathcal{U}]_\bullet$ is isomorphic to the sub-simplicial space of $(I^{n+1} \times G_n)_{n\in\mathbb{N}}$ such that $G[\mathcal{U}]_n$ consists of $(2n+1)$-tuples $(i_0, \dots, i_n, g_1, \dots, g_n)$ satisfying the condition

$$r(g_1) \in U_{i_0},\ s(g_1) \in U_{i_1}, \dots, s(g_n) \in U_{i_n}. \tag{2.3}$$

Actually, in this paper we will not use the explicit definition of generalized morphisms, but will rather use the characterization of the category of generalized morphisms as the category of fractions obtained by inverting the morphisms $G[\mathcal{U}] \to G$ above [21]. More precisely,

Proposition 2.2. *For any functor F from the category of topological groupoids to any category, the following are equivalent:*

(i) *F is invariant under Morita-equivalence.*
(ii) *F factors through the category whose objects are groupoids and whose morphisms are generalized morphisms.*
(iii) *For any groupoid G and any open cover $\mathcal{U}$ of G_0, the canonical map $G[\mathcal{U}] \to G$ induces an isomorphism $F(G[\mathcal{U}]) \xrightarrow{\sim} F(G)$.*

(I have not been able to locate where this well-known result appears for the first time in the literature. For a proof, see for instance [21] or perhaps [24, Proposition 2.5].)

3. Sheaves on simplicial spaces

3.1. **Basic definitions.** Recall [4] that if $u\colon X \to Y$ is continuous, $\mathcal{A}$ is a sheaf on X and $\mathcal{B}$ is a sheaf on Y, then a u-morphism from $\mathcal{B}$ to $\mathcal{A}$ is by definition an element of $\hom(\mathcal{B}, u_*\mathcal{A}) \cong \hom(u^*\mathcal{B}, \mathcal{A})$. A sheaf on a simplicial (resp. pre-simplicial) space $M_\bullet$ is a sequence $\mathcal{A}^\bullet = (\mathcal{A}^n)_{n\in\mathbb{N}}$ such that $\mathcal{A}^n$ is a sheaf on M_n, and such that for each morphism $f\colon [k] \to [n]$ in the category Δ (resp. Δ') we are given $\tilde{f}$-morphisms

$$\tilde{f}^*\colon \mathcal{A}^k \to \mathcal{A}^n \tag{3.1}$$

such that $\tilde{f}^*\tilde{g}^* = \widetilde{f \circ g}^*$ if $g\colon [\ell] \to [k]$.

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In practice, given open sets $U \subseteq M_n$ and $V \subseteq M_k$ such that $\tilde{f}(U) \subseteq V$, we have a *restriction map* $\tilde{f}^* \colon \mathcal{A}^k(V) \to \mathcal{A}^n(U)$ such that $\tilde{f}^* \circ \tilde{g}^* = \widetilde{f \circ g}^* \colon \mathcal{A}^\ell(W) \to \mathcal{A}^n(U)$ whenever $\tilde{g}(V) \subseteq W$.

A fundamental example is given by G-sheaves. In the definition below, recall that a map $f \colon X \to Y$ is said to be étale if it is a local homeomorphism, i.e. every point $x \in X$ has an open neighborhood U such that $f(U)$ is open and f induces a homeomorphism from U onto $f(U)$. We will also say that X is an étale space over Y. A groupoid is étale if the range (equivalently the source) map is étale. A morphism $\pi_\bullet \colon E_\bullet \to M_\bullet$ is étale if each $\pi_n \colon E_\bullet \to M_\bullet$ is étale. Finally, recall that a sheaf over a space X can be considered as a (not necessarily Hausdorff) étale space over X.

Definition 3.1 ([6])**.** Let G be a topological groupoid. Then a *G-sheaf* is an étale space E_0 over G_0, endowed with a continuous action of G.

Of course, an *Abelian* G-sheaf is a G-sheaf E_0 such that E_0 is an Abelian sheaf on G_0 and such that for each $g \in G$, the action $\alpha_g \colon (E_0)_{s(g)} \to (E_0)_{r(g)}$ is a group morphism.

Example 3.2. If G is a group, then a G-sheaf is just a space endowed with an action of G.

As noted by Moerdijk [15, remark following Theorem 3.1], any G-sheaf determines a sheaf over $G_\bullet$. Let us explain this in detail; we need a few preliminaries.

Definition 3.3. Let $\pi_\bullet \colon E_\bullet \to M_\bullet$ be a morphism of simplicial spaces. We say that $\pi_\bullet$ is *reduced* if for all k, n and all $f \in \hom_\Delta(k, n)$, the map $\tilde{f}$ induces an isomorphism $E_n \cong M_n \times_{\tilde{f}, \pi_k} E_k$. In this case, we will say that $E_\bullet$ is a reduced simplicial space over $M_\bullet$.

Definition 3.4. Let $\mathcal{A}^\bullet$ be a sheaf over the simplicial space $M_\bullet$. We will say that $\mathcal{A}^\bullet$ is *reduced* if for all k, n and all $f \in \hom_\Delta(k, n)$, the morphism $\tilde{f}^* \in \hom(\tilde{f}^* \mathcal{A}^k, \mathcal{A}^n)$ is an isomorphism.

Lemma 3.5. *There is a one-to-one correspondence between reduced sheaves over $M_\bullet$ and reduced étale simplicial spaces over $M_\bullet$.*

Proof. The proof is easy. Let us just explain the construction of the sheaf $\mathcal{A}^\bullet$ out of the reduced simplicial space $E_\bullet$ over $M_\bullet$.

Let $\mathcal{A}^n(U)$ be the space of continuous sections over U of the projection map $\pi_n \colon E_n \to M_n$. If $f \colon [k] \to [n]$ is a morphism in Δ and $\tilde{f}(U) \subseteq V$, then for any section $\sigma \in \mathcal{A}^k(V)$ we define $\tilde{f}^* \sigma \in \mathcal{A}^n(U)$ by $(\tilde{f}^* \sigma)(x) = (x, \sigma(\tilde{f}(x))) \in M_n \times_{\tilde{f}, \pi_k} E_k \cong E_n$. □

Lemma 3.6. *Any reduced simplicial space over $M_\bullet$, étale or not, determines a sheaf over $M_\bullet$.*

Proof. The proof is the same. Note that it is not clear whether all sheaves can be constructed this way. □

Corollary 3.7. *Let G be a topological groupoid. Then any G-space determines a sheaf on $G_\bullet$. If the G-space is étale, then it determines a reduced sheaf on $G_\bullet$.*

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Proof. Suppose Z is a G-space and let π_n be the first projection $(G \ltimes Z)_n = G_n \times_{\tilde{p}_n, p} Z \to G_n$, where $\tilde{p}_n(g_1, \ldots, g_n) = s(g_n)$. Then π_n is clearly a simplicial map $(G \ltimes Z)_\bullet \to G_\bullet$. □

Corollary 3.8. *Every G-sheaf canonically defines a reduced sheaf over the simplicial space $G_\bullet$.*

Another example is given by G-modules.

Definition 3.9. Let G be a topological groupoid. A G-module is a topological groupoid A, with source and range maps equal to a map $p\colon A \to G_0$, such that

- A_x^x is an Abelian group for all x.
- As a space, A is endowed with a G-action $G \times_{s,p} A \to A$.
- For each $g \in G$, the map $\alpha_g\colon A_{s(g)} \to A_{r(g)}$ given by the action is a group morphism.

By Corollary 3.7, any G-module defines a sheaf $\mathcal{A}^\bullet$ which is clearly Abelian.

More explicitly, the simplicial structure on $(G \ltimes A)_\bullet$ is defined as follows: for all $f \in \hom_\Delta(k, n)$,

$$\tilde{f}([\gamma_0, \ldots, \gamma_n], a) = ([\gamma_{f(0)}, \ldots, \gamma_{f(k)}], \gamma_{f(k)}^{-1}\gamma_n a). \tag{3.2}$$

Then $\mathcal{A}^n$ is the sheaf of germs of continuous sections of $(G \ltimes A)_n \to G_n$, i.e. sections are continuous maps $\varphi(g_1, \ldots, g_n) \in A_{s(g_n)}$. However, to recover the usual formulas like (1.1), it is better to work with the maps

$$c(g_1, \ldots, g_n) = g_1 \cdots g_n \varphi(g_1, \ldots, g_n) \in A_{r(g_1)},$$

and this is what we shall usually do.

Note that for all $\vec{g} = (g_1, \ldots, g_n) \in G_n$, the stalk $\mathcal{A}_{\vec{g}}$ maps to $A_{\tilde{p}_n(\vec{g})}$. This map is surjective iff $p\colon A \to G_0$ has enough cross-sections; for injectivity, it is enough that p be an étale map.

If $A = G_0 \times B$ has constant fibers (B being a topological Abelian group with no action of G), then the corresponding sheaf is called the constant sheaf and is again (abusively) denoted by B.

When G is a group, a G-module is just a topological Abelian group A endowed with a continuous action $G \to \mathrm{Aut}(A)$, and the sheaf $\mathcal{A}^n$ is just the constant sheaf A on G_n.

3.2. **G-sheaves and sheaves over simplicial spaces.** Recall (Corollary 3.8) that any G-sheaf defines a sheaf over $G_\bullet$. The converse does not hold: for instance, let G be a nondiscrete Lie group and $\mathcal{A}^n$ the sheaf of germs of C^∞ functions on G^n. Then $\mathcal{A}^\bullet$ is a sheaf on $G_\bullet$ but not a G-sheaf.

However, we have the following remark (this was noted in [15, remark following Theorem 3.1], at least in the case of étale groupoids).

Remark 3.10. Let G be a topological groupoid. There is a one-to-one correspondence between:

(i) G-sheaves;
(ii) reduced sheaves over $G_\bullet$;
(iii) reduced étale spaces over $G_\bullet$.

We omit the easy proof, since this will not be used in the sequel.

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4. Čech cohomology

4.1. Covers of simplicial spaces.

Definition 4.1. An *open cover* of a pre-simplicial space $M_\bullet$ is a sequence $\mathcal{U}_\bullet = (\mathcal{U}_n)_{n\in\mathbb{N}}$ such that $\mathcal{U}_n = (U_i^n)_{i\in I_n}$ is an open cover of the space M_n.

The cover is said to be *pre-simplicial* if $I_\bullet = (I_n)_{n\in\mathbb{N}}$ is a pre-simplicial set such that for all $f \in \hom_{\Delta'}(k,n)$ and for all $i \in I_n$ one has $\tilde{f}(U_i^n) \subseteq U^k_{\tilde{f}(i)}$. In the same way, one defines the notions of simplicial cover and of N-simplicial cover.

The reason why we need to introduce this terminology is that, even when $M_\bullet$ is a simplicial space, there may not exist sufficiently fine simplicial covers. However, given a cover $\mathcal{U}_\bullet$, we can form the pre-simplicial cover $\sigma\mathcal{U}_\bullet$ defined as follows.

Let $\mathcal{P}_n = \bigcup_{k=0}^n \mathcal{P}_n^k$, where $\mathcal{P}_n^k = \hom_{\Delta'}(k,n)$. Note that $\mathcal{P}_n$ can be identified with the set of nonempty subsets of $[n]$.

Let Λ_n (or $\Lambda_n(I)$ if there is a risk of confusion) be the set of maps

$$\lambda\colon \mathcal{P} \to \bigcup_k I_k \text{ such that } \lambda(\mathcal{P}_n^k) \subseteq I_k. \tag{4.1}$$

For all $\lambda \in \Lambda_n$, we let

$$U_\lambda^n = \bigcap_{k\le n} \bigcap_{f\in\mathcal{P}_n^k} \tilde{f}^{-1}(U^k_{\lambda(f)}).$$

It is clear that $(U_\lambda)_{\lambda\in\Lambda_n}$ is an open cover of M_n.

The pre-simplicial structure on $\Lambda_\bullet$ is defined in an obvious way: for all $g \in \hom_{\Delta'}(n,n')$, $\tilde{g}\colon \Lambda_{n'} \to \Lambda_n$ is the map

$$(\tilde{g}\lambda')(f) = \lambda'(g\circ f).$$

It is easily checked that $\tilde{g}(U^{n'}_{\lambda'}) \subseteq U^n_{\tilde{g}(\lambda')}$, thus the cover $\sigma\mathcal{U}_\bullet$ is indeed pre-simplicial.

In the same way, for all integers $n \le N$, let

$$(\sigma_N\mathcal{U})_n = (U_\lambda^n)_{\lambda\in\Lambda_n^N}, \tag{4.2}$$

where Λ_n^N is the set of all maps $\lambda\colon \bigcup_{k\le n} \hom_\Delta(k,n) \to \bigcup_{k\le n} I_k$ which satisfy

$$\lambda(\hom_\Delta(k,n)) \subseteq I_k$$

and

$$U_\lambda^n = \bigcap_{k\le n} \bigcap_{f\in\hom_\Delta(k,n)} \tilde{f}^{-1}(U_\lambda^k(f)).$$

The N-simplicial structure on $\Lambda_\bullet^N$ is defined as follows: for all integers $n, n' \le N$ and all $g \in \hom_\Delta(n,n')$, $\tilde{g}\colon \Lambda_{n'}^N \to \Lambda_n^N$ is the map $(\tilde{g}\lambda')(f) = \lambda'(g\circ f)$.

Then $\sigma_N\mathcal{U}_\bullet = (\sigma_N\mathcal{U}_n)_{n\le N}$ is a N-simplicial cover of the N-skeleton of $M_\bullet$.

Convention 4.2. We will also (abusively) denote by $\sigma_N\mathcal{U}_\bullet$ the open cover which coincides with $\sigma_N\mathcal{U}_\bullet$ for $n \le N$ and with $\mathcal{U}_\bullet$ for $n \ge N+1$.

Example 4.3. Let $M_\bullet = (M)_{n\in\mathbb{N}}$ be the constant simplicial space associated to a topological space M, and suppose $\mathcal{U}_0 = (U_i^0)_{i\in I_0}$ is an open cover of M. Define $I_n = I_0^{n+1}$. Then $I_\bullet = (I_n)_{n\in\mathbb{N}}$ is endowed with a simplicial structure by

$$\tilde{f}(i_0,\dots,i_n) = (i_{f(0)},\dots,i_{f(k)})$$

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for all $f \in \hom_\Delta(k,n)$. Let $U^n_{(i_0,\dots,i_n)} = U^0_{i_0} \cap \dots \cap U^0_{i_n}$, and let $\mathcal{U}_n = (U^n_i)_{i\in I_n}$. Then $\mathcal{U}_\bullet$ is a simplicial cover of $M_\bullet$.

The "set" of covers of a simplicial space $M_\bullet$ is endowed with a partial pre-order. Suppose $\mathcal{U}_\bullet$ and $\mathcal{V}_\bullet$ are open covers of $M_\bullet$, with $\mathcal{U}_n = (U^n_i)_{i\in I_n}$ and $\mathcal{V}_n = (V^n_j)_{j\in J_n}$. We say that $\mathcal{V}$ is *finer* than $\mathcal{U}$ if for all n there exists $\theta_n\colon J_n \to I_n$ such that $\theta_n(V^n_j) \subseteq U^n_{\theta(j)}$ for all j. The map $\theta_\bullet = (\theta_n)_{n\in\mathbb{N}}$ is required to be pre-simplicial (resp. N-simplicial) if $\mathcal{U}$ and $\mathcal{V}$ are pre-simplicial (resp. N-simplicial).

4.2. **Čech cohomology.** Let $\mathcal{U}_\bullet$ be a pre-simplicial open cover of $M_\bullet$ and let $\mathcal{A}^\bullet$ be a pre-simplicial Abelian sheaf. Define a complex

$$C^n_{ss}(\mathcal{U}_\bullet;\mathcal{A}^\bullet) = \prod_{i\in I_n} \mathcal{A}^n(U^n_i),$$

i.e. $C^n_{ss}(\mathcal{U}_\bullet;\mathcal{A}^\bullet)$ is the space of global sections of the pull-back of $\mathcal{A}^n$ on $\coprod_{i\in I_n} U^n_i$.

Define the differential $d\colon C^n_{ss}(\mathcal{U}_\bullet;\mathcal{A}^\bullet) \to C^{n+1}_{ss}(\mathcal{U}_\bullet;\mathcal{A}^\bullet)$ by

$$(dc)_i = \sum_{k=0}^{n+1} (-1)^k \tilde{\varepsilon}^*_k c_{\tilde{\varepsilon}_k(i)},$$

where $\tilde{\varepsilon}^*_k c_{\tilde{\varepsilon}_k(i)}$ is the "restriction" of $c_{\tilde{\varepsilon}_k(i)} \in \mathcal{A}^n(U^n_{\tilde{\varepsilon}_k(i)})$ to a section in $\mathcal{A}^{n+1}(U^{n+1}_i)$.

It is immediate to check that $d^2 = 0$, hence we may define the cohomology groups $H^*_{ss}(\mathcal{U}_\bullet;\mathcal{A}^\bullet)$.

Example 4.4. In Example 4.3, suppose $\mathcal{A}$ is an Abelian sheaf on M and that $\mathcal{A}^n = \mathcal{A}$ for all n. Then $H^*_{ss}(\mathcal{U}_\bullet;\mathcal{A}^\bullet)$ is identical to the usual cohomology group $H^*(\mathcal{U}_0;\mathcal{A})$.

Let $\mathcal{U}_\bullet$ be any open cover of $M_\bullet$. We denote

$$C^n(\mathcal{U}_\bullet;\mathcal{A}^\bullet) = C^n_{ss}(\sigma\mathcal{U}_\bullet;\mathcal{A}^\bullet), \tag{4.3}$$

$$H^n(\mathcal{U}_\bullet;\mathcal{A}^\bullet) = H^n_{ss}(\sigma\mathcal{U}_\bullet;\mathcal{A}^\bullet). \tag{4.4}$$

Now, we want to define Čech cohomology. The idea is to define $\check{H}^n(M_\bullet;\mathcal{A}^\bullet)$ as the inductive limit over $\mathcal{U}_\bullet$ of the groups $H^n(\mathcal{U}_\bullet;\mathcal{A}^\bullet)$. The problem is that if $\theta_\bullet\colon J_\bullet \to I_\bullet$ is a refinement, then $\theta_\bullet$ indeed defines a restriction map

$$\theta^*\colon C^n(\mathcal{U}_\bullet;\mathcal{A}^\bullet) \;\to\; C^n(\mathcal{V}_\bullet;\mathcal{A}^\bullet), \tag{4.5}$$

$$(\theta^*\varphi)_j \;=\; \text{restriction to } V^n_j \text{ of } \varphi_{\theta_n(j)}, \tag{4.6}$$

which commutes with the differentials, and thus θ^* defines a map

$$\theta^*\colon H^n(\mathcal{U}_\bullet;\mathcal{A}^\bullet) \to H^n(\mathcal{V}_\bullet;\mathcal{A}^\bullet).$$

However, that map may depend on the choice of θ. On the other hand we have

Lemma 4.5. *Let $N \in \mathbb{N}$. Suppose that $\mathcal{U}_\bullet$ and $\mathcal{V}_\bullet$ are open covers of $M_\bullet$ such that $\mathcal{V}_\bullet$ admits an N-simplicial structure. Suppose that $\mathcal{V}_\bullet$ is finer than $\mathcal{U}_\bullet$ and that θ_0, $\theta_1\colon \mathcal{U}_\bullet \to \mathcal{V}_\bullet$ are two refinements. Then for all $n \le N$ there exists $H\colon C^n(\mathcal{U}_\bullet;\mathcal{A}^\bullet) \to C^{n-1}(\mathcal{V}_\bullet;\mathcal{A}^\bullet)$ (with the convention $C^{-1} = \{0\}$) such that $\theta_1^* - \theta_0^* = dH + Hd$.*

(In the lemma above, we say that $\mathcal{V}_\bullet$ has an N-simplicial structure if the N-skeleton $(\mathcal{V}_n)_{n\in\mathbb{N}}$ has an N-simplicial structure.)

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Proof. Define for all $\varphi \in C^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ and for all $\lambda \in \Lambda_{n-1}(J)$ (recall notation (4.1)):

$$(H\varphi)_\lambda = \sum_{k=0}^{n-1} (-1)^k \tilde{\eta}_k^* \varphi_{\alpha_k(\lambda)}, \tag{4.7}$$

where as usual $\eta_k \colon [n] \to [n-1]$ is the k-th degeneracy map, and α_k is defined as follows: for all $f \in \hom_{\Delta'}(r, n)$, let

$$\alpha_k(\lambda)(f) = \begin{cases} \theta_0(\lambda(\eta_k \circ f)) & \text{if } \{k, k+1\} \not\subseteq f([r]) \text{ and } f(0) \leq k, \\ \theta_1(\lambda(\eta_k \circ f)) & \text{if } \{k, k+1\} \not\subseteq f([r]) \text{ and } f(0) \geq k+1, \\ \theta_0(\tilde{\eta}_{k'}(\lambda(f'))) & \text{if } \{k, k+1\} \subseteq f([r]), \end{cases}$$

where in the third line k' is the integer such that $f(k') = k$ and f' is the unique morphism in $\hom_{\Delta'}(r-1, n-1)$ such that the following diagram commutes:

$$\begin{tikzcd} {[r]} \arrow[r, "f"] \arrow[d, "\eta_{k'}"] & {[n]} \arrow[d, "\eta_k"] \\ {[r-1]} \arrow[r, "f'"] & {[n-1]} \end{tikzcd} \tag{4.8}$$

i.e. $f'(i) = f(i)$ for $i \leq k'$ and $f'(i) = f(i+1) - 1$ for $i \geq k'+1$.

Let us first check that formula (4.7) makes sense, i.e. that $V_\lambda^{n-1} \subseteq \tilde{\eta}_k^{-1}(U^r_{\alpha_k(\lambda)})$ for all k.

Since $U^n_{\alpha_k(\lambda)} = \bigcap_{r=0}^{n} \bigcap_{f \in \hom_{\Delta'}(r,n)} \tilde{f}^{-1}(U^n_{\alpha_k(\lambda)(f)})$, we need to show that

$$V_\lambda^{n-1} \subseteq \tilde{\eta}_k^{-1} \tilde{f}^{-1}(U^r_{\alpha_k(\lambda)(f)}). \tag{4.9}$$

If $\{k, k+1\} \not\subseteq f([r])$ and $f(0) \leq k$, then

$$\begin{aligned} V_\lambda^{n-1} &\subseteq (\widetilde{\eta_k \circ f})^{-1}(V^r_{\lambda(\eta_k \circ f)}) \quad \text{by definition of } V_\lambda^{n-1} \\ &= \tilde{\eta}_k^{-1} \tilde{f}^{-1}(V^r_{\lambda(\eta_k \circ f)}) \\ &\subseteq \tilde{\eta}_k^{-1} \tilde{f}^{-1}(U^r_{\theta_0(\lambda(\eta_k \circ f))}) \quad \text{since } \theta_0 \colon J_\bullet \to I_\bullet \text{ is a refinement} \\ &= \tilde{\eta}_k^{-1} \tilde{f}^{-1}(U^r_{\alpha_k(\lambda)(f)}). \end{aligned}$$

If $\{k, k+1\} \not\subseteq f([r])$ and $f(0) \geq k+1$ the proof of (4.9) is the same, except that θ_0 is replaced by θ_1.

If $\{k, k+1\} \subseteq f([r])$, then

$$\begin{aligned} V_\lambda^{n-1} &\subseteq (\widetilde{f'})^{-1}(V^{r-1}_{\lambda(f')}) \\ &\subseteq \widetilde{f'}^{-1} \tilde{\eta}_{k'}^{-1}(V^r_{\tilde{\eta}_k(\lambda(f'))}) \quad \text{(recall } \mathcal{V}_\bullet \text{ is } N\text{-simplicial)} \\ &\subseteq \tilde{\eta}_k^{-1} \tilde{f}^{-1}(U^r_{\theta_0(\tilde{\eta}_k(\lambda(f')))}) \quad \text{by (4.8)}, \end{aligned}$$

thus (4.9) is proved.

Let us show that

$$dH + Hd = \theta_1^* - \theta_0^*. \tag{4.10}$$

We have

$$(Hd\varphi)_\lambda = \sum_{\ell=0}^{n} \sum_{k=0}^{n+1} A_{k,\ell} \quad \text{and} \quad (dH\varphi)_\lambda = \sum_{k=0}^{n} \sum_{\ell=0}^{n-1} B_{k,\ell},$$

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where $A_{k,\ell} = (-1)^{k+\ell}\tilde{\eta}_\ell^*\tilde{\varepsilon}_k^*\varphi_{\tilde{\varepsilon}_k(\alpha_\ell(\lambda))}$ and $B_{k,\ell} = (-1)^{k+\ell}\tilde{\varepsilon}_k^*\tilde{\eta}_\ell^*\varphi_{\alpha_\ell(\tilde{\varepsilon}_k(\lambda))}$.

We have $A_{\ell,\ell} = \tilde{\eta}_\ell^*\tilde{\varepsilon}_\ell^*\varphi_{\tilde{\varepsilon}_\ell(\alpha_\ell(\lambda))} = \varphi_{\tilde{\varepsilon}_\ell(\alpha_\ell(\lambda))}$, and for all $f \in \mathcal{P}_n$,

$$\tilde{\varepsilon}_\ell(\alpha_\ell(\lambda))(f) = \alpha_\ell(\lambda)(\varepsilon_\ell \circ f) = \theta_j(\lambda(\eta_\ell \circ \varepsilon_\ell(f))) = \theta_j(\lambda(f)),$$

where $j = 0 \iff \varepsilon_\ell \circ f(0) \le \ell \iff f(0) \le \ell - 1$, and $j = 1$ otherwise.

Let

$$\lambda^{(p)}(f) = \begin{cases} \theta_0(\lambda(f)) & \text{if } f(0) \le p, \\ \theta_1(\lambda(f)) & \text{if } f(0) \ge p+1; \end{cases}$$

then

$$A_{\ell,\ell} = \varphi_{\lambda^{\ell-1}}. \tag{4.11}$$

Similarly, we have $A_{\ell+1,\ell} = -\tilde{\eta}_\ell^*\tilde{\varepsilon}_{\ell+1}^*\varphi_{\tilde{\varepsilon}_{\ell+1}(\alpha_\ell(\lambda))} = -\varphi_{\tilde{\varepsilon}_{\ell+1}(\alpha_\ell(\lambda))}$, and for all $f \in \mathcal{P}_n$,

$$\tilde{\varepsilon}_{\ell+1}(\alpha_\ell(\lambda))(f) = \alpha_\ell(\lambda)(\varepsilon_{\ell+1} \circ f) = \theta_j(\lambda(\eta_\ell \circ \varepsilon_{\ell+1} \circ f)) = \theta_j(\lambda(f)),$$

where $j = 0 \iff \varepsilon_{\ell+1} \circ f(0) \le \ell \iff f(0) \le \ell$, and $j = 1$ otherwise. We thus get

$$A_{\ell+1,\ell} = -\varphi_{\lambda^\ell}. \tag{4.12}$$

From (4.11) and (4.12) we obtain $\sum_{\ell \le k \le \ell+1} A_{k,\ell} = \varphi_{\lambda^{(-1)}} - \varphi_{\lambda^{(n)}} = \theta_1^*\varphi - \theta_0^*\varphi$.

Let us examine the other terms. We have $\sum_{k \le \ell-1} A_{k,\ell} = \sum_{0 \le k \le \ell \le n-1} A_{k,\ell+1}$ and $\sum_{k \ge \ell+2} A_{k,\ell} = \sum_{0 \le \ell < k \le n} A_{k+1,\ell}$. To complete the proof of (4.10) it suffices to show that $A_{k,\ell+1} + B_{k,\ell} = 0$ for $k \le \ell$, and that $A_{k+1,\ell} + B_{k,\ell} = 0$ for $k \ge \ell+1$. Noting that $\eta_{\ell+1}\varepsilon_k = \varepsilon_k\eta_\ell$ for $k \le \ell$ and that $\eta_\ell\varepsilon_{k+1} = \varepsilon_k\eta_\ell$ for $k \ge \ell+1$, it suffices to show that

(a) $\tilde{\varepsilon}_k(\alpha_{\ell+1}(\lambda)) = \alpha_\ell(\tilde{\varepsilon}_k(\lambda))$ for $k \le \ell$,
(b) $\tilde{\varepsilon}_{k+1}(\alpha_\ell(\lambda)) = \alpha_\ell(\tilde{\varepsilon}_k(\lambda))$ for $k \ge \ell+1$.

Let us show (a). Suppose that $f \in \hom_{\Delta'}(r,n)$, and let us first treat the case $\{\ell, \ell+1\} \not\subseteq f([r])$. Then, letting $j = 0$ for $\varepsilon_k \circ f(0) \le \ell+1$ ($\iff f(0) \le \ell$), and $j = 1$ otherwise, we have

$$\begin{aligned} \tilde{\varepsilon}_k(\alpha_{\ell+1}(\lambda))(f) &= \alpha_{\ell+1}(\lambda)(\varepsilon_k \circ f) = \theta_j(\lambda(\eta_{\ell+1} \circ \varepsilon_k \circ f)) \\ &= \theta_j(\lambda(\varepsilon_k \circ \eta_\ell \circ f)) = \alpha_\ell(\tilde{\varepsilon}_k(\lambda))(f). \end{aligned}$$

Let us treat the case $\{\ell, \ell+1\} \subseteq f([r])$. Let ℓ' be such that $f(\ell') = \ell$, and let $f' \colon [r-1] \to [n-1]$ be the increasing map such that the diagram

$$\begin{tikzcd} {[r]} \arrow[r, "f"] \arrow[d, "\eta_{\ell'}"] & {[n]} \arrow[d, "\eta_\ell"] \\ {[r-1]} \arrow[r, "f'"] & {[n-1]} \end{tikzcd} \tag{4.13}$$

commutes. Since $\varepsilon_k \circ \eta_\ell = \eta_{\ell+1} \circ \varepsilon_k \colon [n] \to [n]$, the diagram

$$\begin{tikzcd} {[r]} \arrow[r, "\varepsilon_k \circ f"] \arrow[d, "\eta_{\ell'}"] & {[n+1]} \arrow[d, "\eta_{\ell+1}"] \\ {[r-1]} \arrow[r, "\varepsilon_k \circ f'"] & {[n]} \end{tikzcd} \tag{4.14}$$

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commutes. We thus see that

$$\begin{aligned}\tilde{\varepsilon}_k(\alpha_{\ell+1}(\lambda))(f) &= \alpha_{\ell+1}(\lambda)(\varepsilon_k \circ f)\\ &= \theta_0(\tilde{\eta}_{\ell'}(\lambda(\varepsilon_k \circ f'))) \quad \text{by (4.14)}\\ &= \theta_0(\tilde{\eta}_{\ell'}((\tilde{\varepsilon}_k\lambda)(f')))\\ &= \alpha_\ell(\tilde{\varepsilon}_k\lambda)(f) \quad \text{by (4.13)}.\end{aligned}$$

This completes the proof of (a).

Let us show (b): the method is similar. If $\{\ell, \ell+1\} \not\subseteq f([r])$, then

$$\begin{aligned}\tilde{\varepsilon}_{k+1}(\alpha_\ell(\lambda))(f) &= \alpha_\ell(\lambda)(\varepsilon_{k+1} \circ f) = \theta_j(\lambda(\eta_\ell \circ \varepsilon_{k+1} \circ f))\\ &= \theta_j(\lambda(\varepsilon_k \circ \eta_\ell \circ f)) = \theta_j((\tilde{\varepsilon}_k\lambda)(\eta_\ell \circ f)) = \alpha_\ell(\tilde{\varepsilon}_k(\lambda))(f).\end{aligned}$$

If $\{\ell, \ell+1\} \subseteq f([r])$, let ℓ' be such that $f(\ell') = \ell$ and let f' be such that

$$\begin{tikzcd}[r] \arrow[r, "f"] \arrow[d, "\eta_{\ell'}"] & {[n]} \arrow[d, "\eta_\ell"] \\ {[r-1]} \arrow[r, "f'"] & {[n-1]}\end{tikzcd} \tag{4.15}$$

commutes. Then since $\varepsilon_k \circ \eta_\ell = \eta_\ell \circ \varepsilon_{k+1} \colon [n] \to [n]$, the diagram

$$\begin{tikzcd}[r] \arrow[r, "\varepsilon_{k+1} \circ f"] \arrow[d, "\eta_{\ell'}"] & {[n+1]} \arrow[d, "\eta_\ell"] \\ {[r-1]} \arrow[r, "\varepsilon_k \circ f'"] & {[n]}\end{tikzcd} \tag{4.16}$$

commutes; therefore

$$\begin{aligned}\tilde{\varepsilon}_{k+1}(\alpha_\ell(\lambda))(f) &= \alpha_\ell(\lambda)(\varepsilon_{k+1} \circ f)\\ &= \theta_0(\tilde{\eta}_{\ell'}(\lambda(\varepsilon_k \circ f'))) \quad \text{by (4.16)}\\ &= \theta_0(\tilde{\eta}_{\ell'}((\tilde{\varepsilon}_k\lambda)(f')))\\ &= (\alpha_\ell(\tilde{\varepsilon}_k\lambda))(f) \quad \text{by (4.15)}.\end{aligned}$$

This completes the proof of (b) and hence of (4.10). □

Let us now define

$$\check{H}^n(M_\bullet; \mathcal{A}^\bullet) = \varinjlim H^n(\mathcal{U}_\bullet; A^\bullet), \tag{4.17}$$

where $\mathcal{U}_\bullet$ runs over open covers of $M_\bullet$ whose N-skeleton admits an N-simplicial structure for some $N \geq n+1$. (Recall that $H^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ was defined by (4.4).)

To avoid set-theoretic difficulties (since the collection of open covers is not a set), we can restrict ourselves to open covers indexed by sets of cardinality $\leq \sum_n \#M_n$.

By Lemma 4.5 above, if $\mathcal{V}_\bullet$ is finer than $\mathcal{U}_\bullet$ and if $\mathcal{V}_\bullet$ has an N-simplicial structure, then there is a canonical map $H^*(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \to H^*(\mathcal{V}_\bullet; \mathcal{A}^\bullet)$ defined by (4.6) (θ is *not* required to respect the N-simplicial structures). Since every open cover of $M_\bullet$ admits an N-simplicial refinement (see Convention 4.2), the inductive limit is well defined and is an Abelian group.

Moreover, for every open cover $\mathcal{U}_\bullet$ of $M_\bullet$, N-simplicial or not, there is a canonical map $H^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \to \check{H}^n(M_\bullet; \mathcal{A}^\bullet)$ obtained by mapping $H^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ to $H^n(\mathcal{V}_\bullet; \mathcal{A}^\bullet)$ using (4.6), where $\mathcal{V}_\bullet$ is any refinement admitting an N-simplicial structure for some $N \geq n+1$.

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It is clear that any element $[\varphi]$ of $H^n(\mathcal{U}_\bullet;\mathcal{A}^\bullet)$ maps to 0 in $\check{H}^n(M_\bullet;\mathcal{A}^\bullet)$ if and only if there exists a refinement (N-simplicial or not) such that $[\varphi]$ maps to 0 in $H^n(\mathcal{V}_\bullet;\mathcal{A}^\bullet)$. Thus, in the sense of limit of a functor [14, Chapter 9], $\check{H}^n(M_\bullet,\mathcal{A}^\bullet)$ is the inductive limit of $H^n(\mathcal{U}_\bullet,\mathcal{A}^\bullet)$, where $\mathcal{U}_\bullet$ runs over *all* open covers of $M_\bullet$.

Example 4.6. Consider the case of a discrete group G, and suppose that $\mathcal{A}^\bullet$ is the sheaf associated to a G-module A (Definition 3.9). Then, from (3.2) and below, we see that

$$\begin{aligned}(dc)_\lambda(g_1,\dots,g_{n+1}) &= g_1 c_{\tilde{\varepsilon}_0\lambda}(g_2,\dots,g_{n+1})\\ &\quad+\sum_{k=1}^{n}(-1)^k c_{\tilde{\varepsilon}_k\lambda}(g_1,\dots,g_kg_{k+1},\dots,g_{n+1})\\ &\qquad+(-1)^{n+1}c_{\tilde{\varepsilon}_{n+1}\lambda}(g_1,\dots,g_n).\end{aligned}$$

(Compare with (1.1).) Considering the maximal open cover $(U_x^n)_{x\in G^n}$ where $U_x^n=\{x\}$, one easily sees that Čech cohomology coincides with usual group cohomology.

Remark 4.7. As in [5], $\check{H}^n(M_\bullet;\mathcal{A}^\bullet)$ can be seen as the n-th cohomology group of a canonical Čech complex. Indeed, let $\mathcal{R}(M_\bullet)$ be the set of covers of the form $\mathcal{U}_n=(U_x^n)_{x\in M_n}$. If $\mathcal{U}_\bullet$ and $\mathcal{V}_\bullet$ are in $\mathcal{R}(M_\bullet)$, let us say that $\mathcal{V}_\bullet$ is finer than $\mathcal{U}_\bullet$ if for all n and all $x\in M_n$ we have $V_x^n\subseteq U_x^n$. Given $\mathcal{U}_\bullet$ in $\mathcal{R}(M_\bullet)$, denote by $\sigma_N\mathcal{U}_\bullet$ the associated N-simplicial cover (see Convention 4.2) and let

$$\check{C}_N^n(M_\bullet;\mathcal{A}^\bullet)=\lim_{\mathcal{U}_\bullet}C^n(\sigma_N\mathcal{U}_\bullet;\mathcal{A}^\bullet):=\lim_{\mathcal{U}_\bullet}C_{ss}^n(\sigma\sigma_N\mathcal{U}_\bullet,\mathcal{A}^\bullet),$$

where $\mathcal{U}_\bullet$ runs over open covers in $\mathcal{R}(M_\bullet)$.

Then $\check{H}^n(M_\bullet;\mathcal{A}^\bullet)$ is the cohomology of $\check{C}_N^n(M_\bullet;\mathcal{A}^\bullet)$ whenever $N\geq n+1$.

4.3. **Compatibility with usual Čech cohomology for spaces.** Let M be a space and $\mathcal{A}$ an Abelian sheaf on M. Denote by $M_\bullet$ the constant simplicial space associated to M and by $\mathcal{A}^\bullet$ the sheaf on $M_\bullet$ corresponding to $\mathcal{A}$.

We want to show

Proposition 4.8. *With the above assumptions, the usual Čech cohomology groups $\check{H}^n(M;\mathcal{A})$ are isomorphic to the Čech cohomology groups $\check{H}^n(M_\bullet;\mathcal{A}^\bullet)$.*

Proof. To determine $\check{H}^n(M_\bullet;\mathcal{A}^\bullet)$ we can restrict ourselves to covers $\mathcal{U}_\bullet$ of the form $\mathcal{U}_n=(U_i^n)_{i\in I_n}$, where $I_n=I_0^{n+1}$ and $U_{i_0,\dots,i_n}^n=U_{i_0}^0\cap\dots\cap U_{i_n}^0$. Indeed, choose $N>n$; then the group $\check{H}^n(M_\bullet;\mathcal{A}^\bullet)$ only depends on the N-skeleton ${M_\bullet}^{(N)}$ of $M_\bullet$ (i.e. the N-simplicial space obtained from $M_\bullet$ by truncation). Then, given any open cover $\mathcal{V}$ of ${M_\bullet}^{(N)}$, there exists a cover $(U_i^0)_{i\in I_0}$ of M which is finer than each cover $\mathcal{V}_k$ for $0\leq k\leq N$.

Let us show that $H^*(\mathcal{U}_\bullet;\mathcal{A}^\bullet)\cong H^*(\mathcal{U}_0;\mathcal{A})$. It is not obvious that these two groups are isomorphic, since $C^*(\mathcal{U}_\bullet;\mathcal{A}^\bullet)=C_{ss}^*(\sigma\mathcal{U}_\bullet;\mathcal{A}^\bullet)$ while $C^*(\mathcal{U}_0;\mathcal{A})=C_{ss}^*(\mathcal{U}_\bullet;\mathcal{A}^\bullet)$. However, we show that these two complexes are homotopically equivalent:

First, there is an obvious map

$$q\colon C^*(\mathcal{U}_\bullet;\mathcal{A}^\bullet)\to C^*(\mathcal{U}_0;\mathcal{A})$$

defined by $(q\varphi)_{i_0,\dots,i_n}=\varphi_{\lambda^{(i)}}$, where

$$\lambda^{(i)}(f)=(i_{f(0)},\dots,i_{f(r)})\quad\text{for all } f\in\hom_{\Delta'}(r,n).$$

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In the other direction, define $\iota\colon C^*(\mathcal{U}_0,\mathcal{A}) \to C^*(\mathcal{U}_\bullet;\mathcal{A}^\bullet)$ by

$$(\iota c)_\lambda = c_{\lambda_0,\dots,\lambda_n},$$

where λ_k denotes $\lambda(p_k)$ and $p_k\colon [0] \to [n]$ denotes the map $p_k(0) = k$.

We have $q \circ \iota = \mathrm{Id}$. Indeed, $((q\circ\iota)(c))_{i_0,\dots,i_n} = (\iota c)_{\lambda^{(i)}} = c_{i_0,\dots,i_n}$.

Conversely, we do not have $\iota \circ q = \mathrm{Id}$ since $((\iota\circ q)(\varphi))_\lambda = \varphi_{\lambda'}$, where

$$\lambda'(f) = (\lambda_{f(0)},\dots,\lambda_{f(r)}) \text{ for all } f \in \hom_{\Delta'}(r,n).$$

However, $\iota\circ q$ and Id are homotopic. Indeed, define $H\colon C^n(\mathcal{U}_\bullet;\mathcal{A}^\bullet) \to C^{n-1}(\mathcal{U}_\bullet;\mathcal{A}^\bullet)$ by

$$(H\varphi)_\lambda = \sum_{k=0}^{n-1} (-1)^k \tilde{\eta}_k^* \varphi_{\alpha_k(\lambda)}, \tag{4.18}$$

where

$$\alpha_k(\lambda)(f) = \begin{cases} \lambda(\eta_k \circ f) & \text{if } \{k,k+1\} \not\subseteq f([r]) \text{ and } f(0) \le k, \\ \lambda(\eta_k \circ f) & \text{if } \{k,k+1\} \not\subseteq f([r]) \text{ and } f(0) \ge k+1, \\ \tilde{\eta}_{k'}(\lambda(f')) & \text{if } \{k,k+1\} \subseteq f([r]) \end{cases}$$

($f(k') = k$ and f' is defined as in the proof of Lemma 4.5; also, recall that

$$\tilde{\eta}_{k'}(i_0,\dots,i_{r-1}) = (i_0,\dots,i_{k'},i_{k'},\dots,i_{r-1})).$$

Then the same proof as in Lemma 4.5 shows that $dH + Hd = \iota\circ q - \mathrm{Id}$. We leave out details; anyway we will show later that Čech cohomology coincides with sheaf cohomology for paracompact simplicial spaces, so (at least in the paracompact case) this will provide a second proof that sheaf cohomology for spaces coincides with sheaf cohomology for the associated constant simplicial space. □

Let us introduce some notation.

Notation 4.9. For any simplicial space $M_\bullet$ and any open cover $\mathcal{U}_\bullet$, let us write elements $\lambda \in \Lambda_n$ (see (4.1)) as $(2^{n+1}-1)$-tuples $(\lambda_S)_{\emptyset\neq S\subseteq[n]}$, where subsets S are ordered first by cardinality, then by lexicographic order. For instance, the triple $(\lambda_0,\lambda_1,\lambda_{01})$ represents the element $\lambda \in \Lambda_1$ such that $\lambda(\{0\}) = \lambda_0$, $\lambda(\{1\}) = \lambda_1$, $\lambda(\{0,1\}) = \lambda_{01}$. A cochain in $C^1(\mathcal{U}_\bullet;\mathcal{A}^\bullet)$ is thus a family $(\varphi_{\lambda_0\lambda_1\lambda_{01}})$.

Then, we can write (4.18) more explicitly. For instance, the formulas for $n=1$ and $n=2$ are respectively

$$\begin{aligned} (H\varphi)_{\lambda_0} &= \varphi_{\lambda_0\lambda_0\lambda'_{00}}, \\ (H\varphi)_{\lambda_0\lambda_1\lambda_{01}} &= \varphi_{\lambda_0\lambda_0\lambda_1\lambda'_{00}\lambda_{01}\lambda'_{01}\lambda'_{001}} - \varphi_{\lambda_0\lambda_1\lambda_1\lambda_{01}\lambda_{01}\lambda'_{11}\lambda'_{011}}, \end{aligned}$$

where $\lambda'_{i_0\cdots i_r}$ denotes the $r+1$-tuple $(\lambda_{i_0},\dots,\lambda_{i_r}) \in I_0^{r+1} = I_r$.

4.4. **Long exact sequences in Čech cohomology.** In this section, most proofs are almost identical to [5], thus we will only sketch them.

Proposition 4.10. *If $0 \to \mathcal{A}'^\bullet \to \mathcal{A}^\bullet \to \mathcal{A}''^\bullet \to 0$ is an exact sequence of Abelian presheaves, then the functor $\mathcal{A} \mapsto \check{C}_N^*(M_\bullet;\mathcal{A}^\bullet)$ (see Remark 4.7) maps the above exact sequence to an exact sequence of complexes.*

Proof. $0 \to C_{ss}^*(\sigma\sigma_N\mathcal{U}_\bullet;\mathcal{A}'^\bullet) \to C_{ss}^*(\sigma\sigma_N\mathcal{U}_\bullet;\mathcal{A}^\bullet) \to C_{ss}^*(\sigma\sigma_N\mathcal{U}_\bullet;\mathcal{A}''^\bullet) \to 0$ is exact for every open cover $\mathcal{U}_\bullet$. □

Let us say that a simplicial space $M_\bullet$ is *paracompact* if each M_n is paracompact.

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Proposition 4.11 ([5, Theorem 5.10.2]). *If $\mathcal{A}^\bullet$ is an abelian presheaf on a paracompact simplicial space $M_\bullet$ such that $\mathcal{A}^\bullet$ induces the zero sheaf, then $\check{H}^n(M_\bullet;\mathcal{A}^\bullet)=0$ for all $n\geq 0$.*

Proof. Using paracompactness, every cohomology class is represented by a cocycle in $C^n_N(\mathcal{U}_\bullet;\mathcal{A}^\bullet)$ with $\mathcal{U}_n$ locally finite $\forall n$. Then, using the fact that each $\mathcal{A}^n$ induces the zero sheaf, every *cochain* of that cover becomes zero once restricted to a suitable finer cover. □

Corollary 4.12. *If an abelian presheaf $\mathcal{A}^\bullet$ over a paracompact simplicial space $M_\bullet$ induces the sheaf $\tilde{\mathcal{A}}^\bullet$, then $\check{H}^n(M_\bullet,\mathcal{A}^\bullet)\xrightarrow{\simeq}\check{H}^n(M_\bullet;\tilde{\mathcal{A}}^\bullet)$.*

Proof. There are exact sequences of presheaves

$$0\to\mathcal{N}^\bullet\to\mathcal{A}^\bullet\to\mathcal{J}^\bullet\to 0,$$
$$0\to\mathcal{J}^\bullet\to\tilde{\mathcal{A}}^\bullet\to\mathcal{Q}^\bullet\to 0,$$

where $\mathcal{N}^\bullet$ and $\mathcal{Q}^\bullet$ induce the zero sheaf: $\mathcal{N}^n(U)$ is the set of sections in $\mathcal{A}^n(U)$ whose germ at every point is zero, and $\tilde{\mathcal{A}}^n(U)=\{(\sigma_i)_{i\in I}\}/\sim$, where $\sigma_i\in\mathcal{J}^n(U_i)$ for some open cover $(U_i)_{i\in I}$ of U and the equivalence relation $\sim$ is defined by $(\sigma_i)_{i\in I}\sim(\sigma'_j)_{j\in J}$ iff $\forall i,j$, $\sigma_{i|U_i\cap U'_j}=\sigma'_{j\,|U_i\cap U'_j}$. The conclusion follows from Propositions 4.10 and 4.11 above. □

Corollary 4.13. *If $0\to\mathcal{A}'^\bullet\to\mathcal{A}^\bullet\to\mathcal{A}''^\bullet\to 0$ is an exact sequence of sheaves over a paracompact simplicial space $M_\bullet$, then there is a natural long exact sequence*

$$0\to\check{H}^0(M_\bullet;\mathcal{A}'^\bullet)\to\check{H}^0(M_\bullet;\mathcal{A}^\bullet)\to\check{H}^0(M_\bullet;\mathcal{A}''^\bullet)\xrightarrow{\partial}\check{H}^1(M_\bullet;\mathcal{A}'^\bullet)\to\cdots.$$

Proof. Follows from Proposition 4.10 and Corollary 4.12. □

5. Low-dimensional Čech cohomology

5.1. **The group $\check{H}^0$.** Consider a simplicial space $M_\bullet$. Let $\mathcal{U}_\bullet$ be an open cover of $M_\bullet$. Then, using Notation 4.9, a 0-cocycle is given by a family $(c_{\lambda_0})_{\lambda_0\in I_0}$, with $c_{\lambda_0}\in\mathcal{A}^0(U^0_{\lambda_0})$, and

$$0=(dc)_{\lambda_0\lambda_1\lambda_{01}}=\tilde{\varepsilon}_1^*c_{\lambda_1}-\tilde{\varepsilon}_0^*c_{\lambda_0} \tag{5.1}$$

on $U^1_\lambda=U^1_{\lambda_{01}}\cap\tilde{\varepsilon}_0^{-1}(U^0_{\lambda_0})\cap\tilde{\varepsilon}_1^{-1}(U^0_{\lambda_1})$. Therefore, $\tilde{\varepsilon}_1^*c_{\lambda_1}=\tilde{\varepsilon}_0^*c_{\lambda_0}$ on $\tilde{\varepsilon}_0^{-1}(U^0_{\lambda_0})\cap\tilde{\varepsilon}_1^{-1}(U^0_{\lambda_1})$ for all λ_0, $\lambda_1\in I_0$. Applying $\tilde{\eta}_0^*$ to both sides, we find that $c_{\lambda_0}=c_{\lambda_1}$ on $U^0_{\lambda_0}\cap U^0_{\lambda_1}$. Since $\mathcal{A}^0$ is a sheaf, there exists a global section $\varphi\in\mathcal{A}^0(M_0)$ such that c_{λ_0} is the restriction of φ to $U^0_{\lambda_0}$ for all $\lambda_0\in I_0$. Now, (5.1) is equivalent to $\tilde{\varepsilon}_1^*\varphi=\tilde{\varepsilon}_0^*\varphi$. We have thus proved:

Proposition 5.1. *Let $\mathcal{A}^\bullet$ be an Abelian sheaf on a simplicial space $M_\bullet$ and let $\mathcal{U}_\bullet$ be an open cover of $M_\bullet$. Then*

$$\check{H}^0(M_\bullet;\mathcal{A}^\bullet)=H^0(\mathcal{U}_\bullet;\mathcal{A}^\bullet)=\Gamma_{\mathrm{inv}}(\mathcal{A}^\bullet):=\mathrm{Ker}\,(\mathcal{A}^0(M_0)\rightrightarrows\mathcal{A}^1(M_1)).$$

(Of course, in the case of a groupoid and an Abelian G-sheaf, a section in $\mathcal{A}^0(G_0)$ is in $\Gamma_{\mathrm{inv}}(\mathcal{A}^\bullet)$ if and only if it is an invariant section in the usual sense, i.e. under the action of G.)

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5.2. **The group $\check{H}^1$.** Consider a groupoid G. The cocycle relation in degree 1 is

$$\tilde{\varepsilon}_0^* c_{\lambda_1\lambda_2\lambda_{12}} - \tilde{\varepsilon}_1^* c_{\lambda_0\lambda_2\lambda_{02}} + \tilde{\varepsilon}_2^* c_{\lambda_0\lambda_1\lambda_{01}} = 0$$

on $U^2_{\lambda_0\lambda_1\lambda_2\lambda_{01}\lambda_{02}\lambda_{12}\lambda_{012}}$. Exactly the same method as in the preceding paragraph shows that $c_{\lambda_0\lambda_1\lambda_{01}}$ does not depend on the choice of λ_{01}, hence there exists a section $\varphi_{\lambda_0\lambda_1} \in \mathcal{A}^1(\tilde{\varepsilon}_0^{-1}(U^0_{\lambda_0}) \cap \tilde{\varepsilon}_1^{-1}(U^1_{\lambda_1}))$ such that $c_{\lambda_0\lambda_1\lambda_{01}}$ is the restriction to $U^1_{\lambda_0\lambda_1\lambda_{01}}$ of $\varphi_{\lambda_0\lambda_1}$. The cocycle relation becomes

$$\tilde{\varepsilon}_0^* \varphi_{\lambda_1\lambda_2} - \tilde{\varepsilon}_1^* \varphi_{\lambda_0\lambda_2} + \tilde{\varepsilon}_2^* \varphi_{\lambda_0\lambda_1} = 0. \tag{5.2}$$

Coboundaries are cocycles of the form $\varphi_{\lambda_0\lambda_1} = \tilde{\varepsilon}_0^* \alpha_{\lambda_1} - \tilde{\varepsilon}_1^* \alpha_{\lambda_0}$.

We recall that the group $\check{H}^1$ admits the following interpretation.

Proposition 5.2. *Let G be a topological groupoid and let A be a G-module. Denote by $\mathcal{A}^\bullet$ the associated sheaf on $G_\bullet$. Then $\check{H}^1(G_\bullet; \mathcal{A}^\bullet)$ is the group of G-equivariant locally trivial A-principal bundles over G_0.*

This is well known. We omit the easy and standard proof.

5.3. **The group $\check{H}^2$, extensions and the Brauer group.** Let G be a topological groupoid, and let A be a G-module. Let us denote by $\operatorname{ext}(G, A)$ the set of extensions of the form

$$A \overset{i}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} G$$

such that the unit spaces of the groupoids A, E and G are all equal to G_0, the maps i and π are the identity map on G_0, and such that for all $\gamma \in E$ and all $a \in A_{s(\gamma)}$ we have

$$\gamma a \gamma^{-1} = \pi(\gamma) \cdot a.$$

In $\pi(\gamma) \cdot a$, the dot denotes the action of G on the G-module A.

Two such extensions $A \to E \to G$ and $A \to E' \to G$ are considered equivalent if there is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{Id}"] & E \arrow[r] \arrow[d] & G \arrow[d, "\mathrm{Id}"] \\ A \arrow[r] & E' \arrow[r] & G \end{tikzcd}$$

such that the map $E \to E'$ is a groupoid isomorphism.

There is a canonical extension in $\operatorname{ext}(G, A)$: let $E = A \times_{p,r} G = \{(a, g) \in A \times G |\ p(a) = r(g)\}$. The source and range maps in E are $s(a, g) = s(g)$, $r(a, g) = r(g)$. The product is $(a, g)(b, h) = (a + g \cdot b, gh)$ (defined whenever $s(g) = r(g)$; the product in A is written additively). The inclusion $A \hookrightarrow E$ is $i(a) = (a, p(a))$, and the projection $\pi\colon E \to G$ is $\pi(a, g) = g$.

Let us call this extension the *strictly trivial* extension.

Proposition 5.3. *Let $A \overset{i}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} G$ be an element of* $\operatorname{ext}(G, A)$. *The following are equivalent:*

(i) *the extension is strictly trivial;*
(ii) *there exists a groupoid morphism $\sigma\colon G \to E$ which is a section of π;*
(iii) *there exists $\varphi\colon E \to A$ such that $\varphi(a\gamma) = a\varphi(\gamma)$ for all $(a, \gamma) \in A \times_{p,r} E$ and $\varphi(\gamma_1\gamma_2) = \varphi(\gamma_1) + \pi(\gamma_1) \cdot \varphi(\gamma_2)$ for all composable pairs $(\gamma_1, \gamma_2) \in E^2$.*

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Proof. Let us sketch the easy proof.

(i) $\Longrightarrow$ (ii): take $\sigma(g) = (r(g), g)$.

For (ii) $\Longrightarrow$ (iii), let $\varphi(\gamma) = \gamma[\sigma \circ \pi(\gamma)]^{-1}$.

To prove (iii) $\Longrightarrow$ (i), the map $\gamma \mapsto (\varphi(\gamma), \pi(\gamma))$ is a groupoid isomorphism from E onto $A \times_{p,r} G$. $\square$

The set $\mathrm{ext}(G, A)$ is an Abelian group. The ("Baer") sum $E_1 \oplus E_2$ of two extensions $A \to E_i \to G$ is given by the extension $A \to E \to G$ with

$$E = \{(\gamma_1, \gamma_2) \in E_1 \times E_2 |\ \pi_1(\gamma_1) = \pi_2(\gamma_2)\}/\sim,$$

where $(a\gamma_1, \gamma_2) \sim (\gamma_1, a\gamma_2)$ if $\pi_1(\gamma) = \pi_2(\gamma_2)$ and $p(a) = r(\gamma_1)$. The map $\pi\colon E \to G$ is given by $\pi(\gamma_1, \gamma_2) = \pi_1(\gamma_1) = \pi_2(\gamma_2)$, and the inclusion $i\colon A \to E$ is

$$i(a) = (i_1(a), p(a)) \sim (p(a), i_2(a)).$$

Finally, the inverse of the extension is

$$A \overset{i'}{\hookrightarrow} \bar{E} \overset{\pi}{\twoheadrightarrow} G,$$

where $\bar{E} = E$ as a groupoid, but $i'(a) = i(-a)$. Denoting by $\bar{\gamma}$ the element in $\bar{E}$ which is the same as the element $\gamma \in E$, this means that $\overline{a\gamma} = (-a)\bar{\gamma}$. To check that $E \oplus \bar{E}$ is strictly trivial, just note that for all $g \in G$, the element $(\gamma, \bar{\gamma}) \in E \oplus \bar{E}$ does not depend on the choice of $\gamma \in E$ such that $\pi(\gamma) = g$, since $(a\gamma, \overline{a\gamma}) = (a\gamma, (-a)\bar{\gamma}) \sim (\gamma, \bar{\gamma})$. Therefore, $g \mapsto \sigma(g) = (\gamma, \bar{\gamma})$ defines a cross-section of π.

Example 5.4. When $A = G_0 \times \mathbb{T}$, and G does not act on $\mathbb{T}$, we obtain the group $\mathrm{Tw}(G)$ of twists of G [11].

It is clear that $\mathrm{ext}(G, A)$ is covariant with respect to G-module morphisms. It is also contravariant with respect to groupoid morphisms. Indeed, let $f\colon G' \to G$ be a groupoid morphism, and let $A' = f^*A = \{(x, a) \in G'_0 \times A |\ f(x) = p(a)\}$. Then A' is a G'-module with respect to the action $g' \cdot (x, a) = (r(g'), f(g) \cdot a)$, and there is a "pull-back" morphism

$$f^*\colon \mathrm{ext}(G, A) \to \mathrm{ext}(G', f^*A)$$

defined as follows. Let $A \to E \to G$ be an element of $\mathrm{ext}(G, A)$. Then its pull-back by f is the extension

$$A \overset{i'}{\hookrightarrow} E' \overset{\pi'}{\twoheadrightarrow} G',$$

where $E' = \{(\gamma, g') \in E \times G' |\ \pi(\gamma) = f(g')\}$, $\pi'(\gamma, g') = g'$, $i'(a) = (i(a), p(a))$. The groupoid structure on E' is the one induced from the product groupoid $E \times G'$.

More generally, suppose that $G'_0 \overset{\rho}{\leftarrow} Z \overset{\sigma}{\to} G_0$ is a generalized morphism from G' to G (see Section 2.3). Put $A' = Z \times_G A := \{(z, a) \in Z \times A |\ \sigma(z) = p(a)\}/\sim$, where $(zg, g^{-1}a) \sim (z, a)$ for all triples $(z, a, g) \in Z \times A \times G$ such that $\sigma(a) = p(a) = r(g)$. It is obvious that A' is a G'-module with sum $(z, a) + (z, b) = (z, a + b)$ and left G'-action $g'(z, a) = (g'z, a)$.

The slight defect of the group $\mathrm{ext}(G, A)$ is that it is not invariant by Morita equivalence. To remedy this, let us define

Definition 5.5.

$$\mathrm{Ext}(G, A) = \lim_{\mathcal{U}} \mathrm{ext}(G[\mathcal{U}], A[\mathcal{U}]),$$

where $\mathcal{U}$ runs over open covers of G_0 (see notation (2.2)).

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By construction, the group $\mathrm{Ext}(G, A)$ is invariant under Morita equivalence (see Proposition 2.2).

Let us now come to the relation between 2-cohomology and extensions:

Proposition 5.6. *Let G be a topological groupoid, A a G-module, and $\mathcal{A}^\bullet$ the associated sheaf over $G_\bullet$.*

(a) *For each open cover $\mathcal{U}_\bullet$ of $G_\bullet$, there is a canonical group isomorphism*

$$\mathrm{ext}_{\mathcal{U}}(G[\mathcal{U}_0], A[\mathcal{U}_0]) \cong H^2(\mathcal{U}_\bullet; \mathcal{A}^\bullet), \tag{5.3}$$

where $\mathrm{ext}_{\mathcal{U}}(G[\mathcal{U}_0], A[\mathcal{U}_0])$ denotes the subgroup of elements of $\mathrm{ext}(G[\mathcal{U}_0], A[\mathcal{U}_0])$ consisting of extensions $A[\mathcal{U}_0] \to E \xrightarrow{\pi} G[\mathcal{U}_0]$ such that π admits a continuous lifting over each open set $U^1_\lambda \subseteq G_1$ ($\lambda \in \Lambda_1$).

(b) (5.3) *induces an isomorphism*

$$\mathrm{Ext}(G, A) \cong \check{H}^2(\mathcal{U}_\bullet; \mathcal{A}^\bullet).$$

Proof. As in the previous subsection, one easily sees that a 2-cocycle in $Z^2(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ is given by a family

$$\varphi = (\varphi_{\lambda_0\lambda_1\lambda_2\lambda_{01}\lambda_{02}\lambda_{12}})$$

such that each term is a continuous function $(g, h) \mapsto \varphi_\lambda(g, h) \in A_{r(g)}$, defined on the set of pairs (g, h) where $r(g) \in U^0_{\lambda_0}$, $s(g) \in U^0_{\lambda_1}$, $s(h) \in U^0_{\lambda_2}$, $g \in U^1_{\lambda_{01}}$, $gh \in U^1_{\lambda_{02}}$, and $h \in U^1_{\lambda_{12}}$. The φ's satisfy the cocycle identity

$$\begin{aligned} g\varphi_{\lambda_1\lambda_2\lambda_3\lambda_{12}\lambda_{13}\lambda_{23}}(h, k) - \varphi_{\lambda_0\lambda_2\lambda_3\lambda_{02}\lambda_{03}\lambda_{23}}(gh, k) \\ +\varphi_{\lambda_0\lambda_1\lambda_3\lambda_{01}\lambda_{03}\lambda_{13}}(g, hk) - \varphi_{\lambda_0\lambda_1\lambda_2\lambda_{01}\lambda_{02}\lambda_{12}}(g, h) = 0. \end{aligned} \tag{5.4}$$

Let us consider a cover $\mathcal{V}_\bullet$ of $G[\mathcal{U}]$, $\mathcal{V}_n = (V^n_j)_{j \in J_n}$, such that

- $J_0 = \{\mathrm{pt}\}$ and $\mathcal{V}_0$ is the cover consisting of the unique open set $\coprod_{i \in I_0} U^0_i$;
- $J_1 = I_0 \times I_0 \times I_1$ with $V^1_{ijk} = \{(i, g, j) |\ r(g) \in U^0_i,\ s(g) \in U^0_j,\ g \in U^1_k\}$;
- V^n arbitrary $\forall n \geq 2$.

Consider the group $Z^2(\mathcal{V}_\bullet; \mathcal{A}'^\bullet)$, where $\mathcal{A}'^\bullet$ is the pull-back of the sheaf $\mathcal{A}^\bullet$ by $G[\mathcal{U}] \to G$. As above, it consists of families $\psi_{\mu_{01}\mu_{02}\mu_{12}}$ satisfying the cocycle identity

$$\begin{aligned} g\psi_{\mu_{12}\mu_{13}\mu_{23}}(h, k) - \psi_{\mu_{02}\mu_{03}\mu_{23}}(gh, k) \\ +\psi_{\mu_{01}\mu_{03}\mu_{13}}(g, hk) - \psi_{\mu_{01}\mu_{02}\mu_{12}}(g, h) = 0. \end{aligned} \tag{5.5}$$

We show that $Z^2(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \cong Z^2(\mathcal{V}_\bullet; \mathcal{A}'^\bullet)$, where $\mathcal{A}'^\bullet$ is the pull-back of the sheaf $\mathcal{A}^\bullet$ by $G[\mathcal{U}] \to G$.

In one direction, let $\psi \in Z^2(\mathcal{V}_\bullet; \mathcal{A}'^\bullet)$. For all $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_{01}, \lambda_{02}, \lambda_{12})$ in $I^3_0 \times I^3_1$, define

$$\begin{aligned} \mu_{01} &= (\lambda_0, \lambda_1, \lambda_{01}), \\ \mu_{02} &= (\lambda_0, \lambda_2, \lambda_{02}), \\ \mu_{12} &= (\lambda_1, \lambda_2, \lambda_{12}), \end{aligned} \tag{5.6}$$

and $\varphi_\lambda = \psi_{\mu_{01}\mu_{02}\mu_{12}}$.

In the other direction, if we are given a 2-cocycle $\varphi \in Z^2(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$, we want to define a 2-cocycle $\psi \in Z^2(\mathcal{V}_\bullet; \mathcal{A}'^\bullet)$. Given $\mu = (\mu_{01}, \mu_{02}, \mu_{12}) \in J_1$, write $\mu_{ab} = (i_{ab}, j_{ab}, k_{ab})$. Then $V^1_\mu \neq \emptyset$ implies that $j_{01} = i_{12}$, $i_{01} = i_{02}$, $j_{02} = j_{12}$, hence there exists

$$\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_{01}, \lambda_{02}, \lambda_{12}) \in I^3_0 \times I^3_1$$

such that (5.6) holds. We then define $\psi_\mu = \varphi_\lambda$.

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Comparing (5.4) and (5.5), we see that $Z^2(\mathcal{U}_\bullet;\mathcal{A}^\bullet) \cong Z^2(\mathcal{V}_\bullet;\mathcal{A}'^\bullet)$. Moreover, it is not hard to check that this induces an isomorphism

$$H^2(\mathcal{U}_\bullet;\mathcal{A}^\bullet) \cong H^2(\mathcal{V}_\bullet;\mathcal{A}'^\bullet). \tag{5.7}$$

To prove the first part of the proposition, then, we can (after passing to the groupoid $G[\mathcal{U}_0]$) suppose that $\mathcal{U}_0$ consists of the unique open set G_0.

Consider an extension in $\mathrm{ext}_{\mathcal{U}}(G,A)$:

$$A \hookrightarrow E \twoheadrightarrow G.$$

For each $i \in I_1$, consider a continuous section $\sigma_i\colon U_i^1 \to E$. Define a cochain φ by the equation

$$\sigma_{\lambda_{01}}(g)\sigma_{\lambda_{12}}(h) = \varphi_{\lambda_{01}\lambda_{02}\lambda_{12}}(g,h)\sigma_{\lambda_{02}}(gh). \tag{5.8}$$

To see that it is indeed a cocycle, just write

$$(\sigma_{\lambda_{01}}(g)\sigma_{\lambda_{12}}(h))\sigma_{\lambda_{23}}(k) = \sigma_{\lambda_{01}}(g)(\sigma_{\lambda_{12}}(h)\sigma_{\lambda_{23}}(k))$$

and substitute relations like (5.8) to obtain

$$\begin{aligned} g\varphi_{\lambda_{12}\lambda_{13}\lambda_{23}}(h,k) - \varphi_{\lambda_{02}\lambda_{03}\lambda_{23}}(gh,k) \\ +\varphi_{\lambda_{01}\lambda_{03}\lambda_{13}}(g,hk) - \varphi_{\lambda_{01}\lambda_{02}\lambda_{12}}(g,h) = 0. \end{aligned} \tag{5.9}$$

Suppose that σ_i' is another continuous lifting and let $\alpha_i\colon U_i^1 \to A$ be such that

$$\sigma_i'(g) = \alpha_i(g)\sigma_i(g). \tag{5.10}$$

Define φ' by

$$\sigma'_{\lambda_{01}}(g)\sigma'_{\lambda_{12}}(h) = \varphi'_{\lambda_{01}\lambda_{02}\lambda_{12}}(g,h)\sigma'_{\lambda_{02}}(gh). \tag{5.11}$$

Substituting (5.10) in (5.11) and comparing with (5.8), we find

$$(\varphi' - \varphi)_{\lambda_{01}\lambda_{02}\lambda_{12}}(gh) = g\alpha_{\lambda_{12}}(h) - \alpha_{\lambda_{02}}(gh) + \alpha_{\lambda_{01}}(g),$$

i.e. $\varphi' - \varphi = d\alpha$. This proves that an extension in $\mathrm{ext}_{\mathcal{U}}(G,A)$ determines a unique cohomology class in $H^2(\mathcal{U}_\bullet;\mathcal{A}^\bullet)$.

Conversely, given a cocycle $\varphi_{\lambda_{01}\lambda_{02}\lambda_{12}}$, we want to construct an extension

$$A \to E \to G.$$

The idea is to set

$$E = \coprod_{i\in I_1} \{(a,g,i) |\ a \in A,\ g \in U_i^1,\ p(a) = r(g)\}/\sim \tag{5.12}$$

with the product law

$$[a,g,\lambda_{01}][b,g,\lambda_{12}] = [a + g\cdot b + \varphi_{\lambda_{01}\lambda_{02}\lambda_{12}}(g,h), gh, \lambda_{02}]. \tag{5.13}$$

To determine the correct equivalence relation in (5.12), we note that if $[a,x,i]$ represents a unit element in the groupoid E, then from the product law (5.13) we necessarily have $[a,x,i] = [a,x,i][a,x,i] = [2a + \varphi_{iii}(x,x), x, i]$, thus $[-\varphi_{iii}(x,x), x, i]$ must be the unit element. Using (5.13) again, we necessarily have

$$[-\varphi_{iii}(r(g),r(g)), r(g), i][a,g,k] = [-\varphi_{iii}(r(g),r(g)) + a + \varphi_{ijk}(r(g),g), g, j],$$

thus we necessarily have

$$(a,g,k) \sim (-\varphi_{iii}(r(g),r(g)) + a + \varphi_{ijk}(r(g),g), g, j). \tag{5.14}$$

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Conversely, we want to show that (5.14) defines an equivalence relation. We claim that $\psi_{kj}(g) = -\varphi_{iii}(r(g), r(g)) + \varphi_{ijk}(r(g), g)$ does not depend of the choice of i.

Let us denote $x = r(g)$. Apply (5.9) to (x, x, g) instead of (g, h, k):

$$\varphi_{ijk}(x,g) - \varphi_{\ell mk}(x,g) + \varphi_{nmj}(x,g) - \varphi_{n\ell i}(x,x) = 0. \tag{5.15}$$

Taking $g = x$ and $j = k = \ell = m = n$ we find

$$\varphi_{ijj}(x,x) = \varphi_{jji}(x,x) = \varphi_{iii}(x,x). \tag{5.16}$$

Taking $k = j$ and $n = \ell$ in (5.15) and using (5.16), we get

$$\varphi_{ijj}(x,g) = \varphi_{\ell\ell i}(x,x) = \varphi_{iii}(x,x). \tag{5.17}$$

Then, take $m = j$ and $n = \ell$ in (5.15) and use (5.16) and (5.17):

$$\varphi_{ijk}(x,g) - \varphi_{\ell jk}(x,g) + \varphi_{\ell jj}(x,g) - \varphi_{\ell\ell i}(x,x) = 0,$$
$$\varphi_{ijk}(x,g) - \varphi_{\ell jk}(x,g) + \varphi_{\ell\ell\ell}(x,x) - \varphi_{iii}(x,x) = 0.$$

This proves our claim that ψ_{kj} is well defined. Moreover, taking $n = \ell = i$ in (5.15) we get

$$\psi_{jk}(g) - \psi_{mk}(g) + \psi_{mj}(g) = 0. \tag{5.18}$$

It follows that

$$\begin{aligned} \psi_{jj} &= 0 \quad (\text{use (5.18) for } k = j), \\ \psi_{kj} &= -\psi_{jk} \quad (\text{use (5.18) for } m = k), \\ \psi_{jm} &= \psi_{jk} + \psi_{km}. \end{aligned}$$

Therefore, (5.14) defines an equivalence relation.

It is then elementary to check that (5.13) endows E with a groupoid structure such that the obvious extension

$$A \to E \xrightarrow{\pi} G$$

is an element of $\operatorname{ext}(G, A)$, that π admits a continuous lifting $\sigma_i \colon U_i^1 \to G$ defined by $\sigma_i(g) = [0, g, i]$, and that the associated cocycle is precisely φ. We leave these easy verifications to the reader.

To prove the second part of the proposition, we first pass to the inductive limit over all open covers $\mathcal{U}_1$ of G_1 (leaving $\mathcal{U}_0$ fixed) to find that

$$\operatorname{ext}(G[\mathcal{U}_0], A[\mathcal{U}_0]) = \lim_{\mathcal{U}_1} H^2(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$$

and then take the inductive limit over $\mathcal{U}_0$. □

Remark 5.7. By the same method we used to show (5.7), one can show that for each open cover $\mathcal{U}_\bullet$ of $G_\bullet$, and each sheaf $\mathcal{A}^\bullet$ over $G_\bullet$, the canonical morphism $f \colon G[\mathcal{U}_0] \to G$ induces an isomorphism

$$H^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \cong H^n(\mathcal{V}_\bullet; \mathcal{A}'^\bullet),$$

where $\mathcal{V}_0$ is the cover consisting of the unique open set $\coprod_{i \in I} U_i^0$, and the open cover $\mathcal{V}_n = (V^n_{i_0,\dots,i_n,j})_{i_0,\dots,i_n,j \in I_0^{n+1} \times I_n}$ of $G[\mathcal{U}]_n$ is defined by

$$\begin{aligned} V^n_{i_0,\dots,i_n,j} = \{(i_0, \dots, i_n, g_1, \dots, g_n) \mid\, & r(g_1) \in U^0_{i_0},\ s(g_1) \in U^0_{i_1}, \dots, \\ & s(g_n) \in U^0_{i_n},\ (g_1, \dots, g_n) \in U^n_j\} \end{aligned}$$

(see Remark 2.1).

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Remark 5.8. For completeness, it would remain to examine the relation between the *sheaf* 2-cohomology groups and extensions of nonparacompact groupoids, since it is not obvious whether H^2 is equal to $\check{H}^2$ in this case. However, we will not develop this, due to lack of applications.

In the corollary below, denote by $\mathcal{T}$ the sheaf associated to the G-module $G_0 \times \mathbb{T}$ (for instance, if G is a space X, then $\mathcal{T}$ is the sheaf of germs of continuous maps from X to $\mathbb{T}$).

Corollary 5.9. *If G is a locally compact Hausdorff groupoid with a Haar system, then $\check{H}^2(G_\bullet; \mathcal{T}) \cong \operatorname{Ext}(G, G_0 \times \mathbb{T})$ is the Brauer group of G.*

Proof. Use for instance [24, Proposition 2.13]. □

Remark 5.10. For groups, there is an interpretation of higher degree cohomology in terms of extensions [9], but we will not develop this here.

6. Comparison with Moore's cohomology

Recall [19] that if G is a locally compact group and A is a Polish Abelian group (i.e., as a topological space, A admits a separable complete metric), then A is a G-module if G acts (continuously) by automorphisms on A.

Given a Polish G-module A, let $I(A)$ be the set of μ-measurable functions from G to A (μ being the Haar measure), modulo equality almost everywhere. Then $I(A)$ is again a Polish G-module, with action $(\gamma \cdot f)(x) = \gamma f(\gamma^{-1}x)$ (caution: our definition is different but isomorphic to Moore's definition of $I(A)$).

The G-module A embeds in $I(A)$ via the obvious map

$$i_A \colon A \to I(A), \quad (i_A(a))(x) = a.$$

Let $U(A) = I(A)/A$. Then, using measurable cocycles, Moore defined cohomology groups $H^n(G, A)$ which are are characterized by the proposition below, where $I(A)$ is defined as above and $F(A) = A^G$ (the sub-module of G-fixed points).

Proposition 6.1. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two Abelian categories. Suppose that F is a left-exact functor from $\mathcal{C}_1$ to $\mathcal{C}_2$, that $I \colon \mathcal{C}_1 \to \mathcal{C}_2$ is a functor and $i_A \colon A \hookrightarrow I(A)$ is a natural injection. Then*

(a) *There exists, up to isomorphism, at most one sequence of functors $H^n \colon \mathcal{C}_1 \to \mathcal{C}_2$ such that*
   1) *$H^0 = F$.*
   2) *Any exact sequence $0 \to A' \to A \to A'' \to 0$ induces a natural long exact sequence $0 \to H^0(A') \to H^0(A) \to H^0(A'') \xrightarrow{\partial} H^1(A') \to \dots$.*
   3) *$H^n(I(A)) = 0$ for all A and for all $n \geq 1$.*

(b) *If I is an exact functor and $I(i_A) = i_{I(A)}$ for all A, then 3) may be replaced by 3′) $H^1(I(A)) = 0\ \forall A$.*

(c) *If moreover $F(I(I(A))) \to F(U(I(A)))$ is surjective for all A, where $U(A) = I(A)/A$, then there exists a sequence of functors satisfying 1), 2) and 3).*

Proof. (a) This is essentially [2, Theorem 1] or [19, Theorem 2]: using the long exact sequence associated to $0 \to A \xrightarrow{i_A} I(A) \to U(A) \to 0$, one gets $H^n(A) \cong H^{n-1}(U(A))$ for $n \geq 2$ and $H^1(A) = \operatorname{coker}(F(I(A)) \to F(U(A)))$, thus H^n is uniquely determined by induction on n.

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(b) If I is exact, then

$$0 \to I(A) \overset{I(i_A)}{\to} I(I(A)) \to I(U(A)) \to 0$$

is exact, and

$$0 \to I(A) \overset{i_{I(A)}}{\to} I(I(A)) \to U(I(A)) \to 0$$

is exact by definition. Therefore, the assumption $I(i_A) = i_{I(A)}$ implies $I(U(A)) = U(I(A))$ canonically. Thus, $3')$ implies that for $n \geq 2$,

$$H^n(I(A)) = H^1(U^{n-1}I(A)) = H^1(IU^{n-1}(A)) = 0.$$

(c) Define a resolution $0 \to A \to A_0 \to A_1 \to \cdots$ by $A_0 = I(A)$ and $A_{n+1} = I(A_n/A_{n-1})$. The map $A_n \to A_{n+1}$ is the composition

$$A_n \to A_n/A_{n-1} \overset{i}{\to} I(A_n/A_{n-1}).$$

Define $H^n(A)$ to be the cohomology of the complex $F(A_0) \to F(A_1) \to \cdots$ and let us check that properties 1), 2) and $3')$ hold.

1) $H^0(A) = \operatorname{Ker}(F(A) \to F(I(I(A)/A))) = \operatorname{Ker}(F(I(A)) \to F(I(A)/A))$ since F preserves injectivity of morphisms, and $I(A)/A \to I(I(A)/A)$ is injective. Using left exactness of F, we see that $H^0(A) = F(\operatorname{Ker}(I(A) \to I(A)/A)) = F(A)$.

2) Since I is an exact functor, we have an exact sequence of complexes $0 \to A'_* \to A_* \to A''_* \to 0$, hence the conclusion by the Snake lemma.

$3')$ $H^1(I(A)) = F(U(I(A)))/F(I(I(A))) = 0$. □

For instance, in part (a) of the proposition, if $I(A)$ is an injective object for all A, then H^n are the right derived functors of F.

We are now ready to prove:

Proposition 6.2. *Let G be a locally compact group. Let A be a Polish G-module and let $\mathcal{A}^\bullet$ be the associated sheaf on $G_\bullet$ (see Definition* 3.9 *and below). Then $\check{H}^*(G_\bullet; \mathcal{A}^\bullet) \cong H^*(G, A)$.*

Proof. We just need to check that $\check{H}^*(G_\bullet; \cdot)$ satisfies conditions 1)–3) of Proposition 6.1.

1) $\check{H}^0(G_\bullet; \mathcal{A}^\bullet) = A^G$ was proved in Proposition 5.1 and 2) in Corollary 4.13.

I is an exact functor [19, Proposition 9], and it is obvious that $I(i_A) = i_{I(A)}$. It thus remains to show that $\check{H}^1(G_\bullet; \mathcal{B}^\bullet) = 0$ if $\mathcal{B}^\bullet$ is the sheaf on $G_\bullet$ associated to the G-module $I(A)$. Recall that a 1-cocycle φ is a continuous function $\varphi\colon G \to I(A)$ satisfying

$$g_1\varphi(g_2) - \varphi(g_1g_2) + \varphi(g_1) = 0$$

(see (5.2)), hence φ is a cocycle in the Moore complex. But $H^1_{\text{Moore}}(G, I(A)) = 0$, hence there exists $\psi \in I(A)$ such that $\varphi(g) = g\psi - \psi$ for all $g \in G$. Therefore, $\varphi = 0$ in $H^1(\mathcal{U}_\bullet; \mathcal{B}^\bullet)$. □

Remark 6.3. One can easily construct explicitly the isomorphism $\check{H}^n(G_\bullet; \mathcal{A}^\bullet) \to H^n_{\text{Moore}}(G, A)$. Take $\varphi \in Z^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$, where $\mathcal{U}_\bullet$ is an open cover. Choose measurable maps $\theta_k\colon G^k \to I_k$ such that $x \in U^k_{\theta_k(x)}$ for all $x \in G^k$, and define

$$c(g_1, \ldots, g_n) = \varphi_{\lambda(g_1,\ldots,g_n)}(g_1, \ldots, g_n),$$

where $\lambda(g_1, \ldots, g_n) \in \Lambda_n$ is defined by $\lambda(g_1, \ldots, g_n)(f) = \theta_r(\tilde{f}(g_1, \ldots, g_n)) \in I_k$ for every $f \in \hom_{\Delta'}(k, n)$.

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Remark 6.4. One might wonder if it is possible to define, for every locally compact groupoid and every (say, locally compact) G-module A, an analogue of the Moore complex, using measurable cochains $c(g_1, \dots, g_n) \in A_{r(g_1)}$. In order to get the usual cohomology groups when G is a space, one should probably use the sheaf over G_0 of functions $c(g_1, \dots, g_n) \in A_{r(g_1)}$ which are measurable in the "leaf" direction and continuous in the "transverse" direction. Since this approach does not seem simpler or more useful than Čech or sheaf cohomology of simplicial spaces, we will not develop this further.

7. Comparison with sheaf cohomology and Haefliger's cohomology

Let $M_\bullet$ be a simplicial space and $\mathcal{A}^\bullet$ an Abelian sheaf on $M_\bullet$. By definition [4], the cohomology groups $H^n(M_\bullet; \mathcal{A}^\bullet)$ are the derived functors of the functor $\Gamma_{\mathrm{inv}}(M_\bullet; \mathcal{A}^\bullet) = \mathrm{Ker}\,(\mathcal{A}^0(M_0) \rightrightarrows \mathcal{A}^1(M_1))$.

A practical way of calculating the cohomology groups is to take a resolution $(\mathcal{L}^{\bullet,q})_{q\in\mathbb{N}}$ of $\mathcal{A}^\bullet$ such that $H^n(M_p; \mathcal{L}^{p,q}) = 0$ $\forall n \geq 1$, $\forall p, q \geq 0$ and take the cohomology of the double complex $(\mathcal{L}^{p,q}(M_p))$, where the first differential is $d' = \sum_{k=0}^{p+1} (-1)^k \tilde{\varepsilon}_k^*$ and the second differential d'' is the differential in the resolution

$$\mathcal{A}^p \to \mathcal{L}^{p,0} \to \mathcal{L}^{p,1} \to \cdots .$$

(See [4, §5.2.3] in the general case, [3, §2.7] or [6] in the case of étale groupoids.) Thus, sheaf cohomology of simplicial spaces coincides with Haefliger's cohomology groups in the case of étale groupoids [6].

In this section, we show:

Proposition 7.1. *Let $M_\bullet$ be a paracompact simplicial space, and $\mathcal{A}^\bullet$ an Abelian sheaf on $M_\bullet$. Then $H^*(M_\bullet; \mathcal{A}^\bullet) \cong \check{H}^*(M_\bullet; \mathcal{A}^\bullet)$. In particular, $\check{H}^*(G_\bullet; \mathcal{A}^\bullet)$ are Haefliger's cohomology groups if G is an étale paracompact groupoid and $\mathcal{A}^\bullet$ is an Abelian G-sheaf.*

We will again use Proposition 6.1. Consider $I(\mathcal{A})^\bullet$ the sheaf such that $I(\mathcal{A})^n(U)$ consists of maps f (continuous or not) from $\tilde{\varepsilon}_0^{-1}(U)$ to $\mathcal{A}^{n+1}$ such that $f(x) \in \mathcal{A}_x^{n+1}$ for all $x \in \tilde{\varepsilon}_0^{-1}(U)$. We will need

Lemma 7.2. *For any open cover $\mathcal{U}_\bullet$ and all $n \geq 1$, $H^n(\mathcal{U}_\bullet; I(\mathcal{A})^\bullet) = \{0\}$.*

Proof. Let us first explain the simplicial structure on the sheaf $I(\mathcal{A})^\bullet$. Given $f \in \hom_\Delta(k, n)$, $U \subseteq M_k$, $V \subseteq M_n$ such that $\tilde{f}(V) \subseteq U$ and a section φ of $I(\mathcal{A})^k$ over U, we have to produce a section $\tilde{f}\varphi \in \Gamma(V, I(\mathcal{A})^n)$.

Define $f' \in \hom_\Delta(k+1, n+1)$ such that

$$f'(0) = 0 \text{ and } \varepsilon_0 \circ f = f' \circ \varepsilon_0 \colon [k] \to [n+1]. \tag{7.1}$$

Since $\tilde{f}(V) \subseteq U$ we have $\tilde{f}'(\tilde{\varepsilon}_0^{-1}(V)) \subseteq \tilde{\varepsilon}_0^{-1}(U)$, thus we get a section

$$x \in \tilde{\varepsilon}_0^{-1}(V) \mapsto \varphi(\tilde{f}'(x)) \in \mathcal{A}^{k+1}_{\tilde{f}'(x)} \mapsto (\tilde{f}')^*\varphi(\tilde{f}'(x)) \in \mathcal{A}_x^{n+1}.$$

Now, let us show that the complex $C^*(\mathcal{U}_\bullet; I(\mathcal{A})^\bullet)$ is homotopically trivial. First, for all $n \in \mathbb{N}$ and all $x \in M_n$, let us choose $\theta(x) \in I_n$ such that $x \in U^n_{\theta(x)}$. We define a homotopy

$$H \colon C^n(\mathcal{U}_\bullet; I(\mathcal{A})^\bullet) \to C^{n-1}(\mathcal{U}_\bullet; I(\mathcal{A})^\bullet)$$

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by $(H\varphi)_\lambda(x) = \tilde{\eta}_0^*\varphi_{\lambda'_x}(\tilde{\eta}_0(x))$, $\forall x \in \tilde{\varepsilon}_0^{-1}(U_\lambda) \subseteq M_n$, $\forall \lambda \in \Lambda_{n-1}$, and $\lambda'_x \in \Lambda_n$ is defined as follows: for all $f \in \hom_{\Delta'}(k,n)$, let

$$\lambda'_x(f) = \left\{ \begin{array}{ll} \lambda(\eta_0 \circ f) & \text{if } f(0) \neq 0 \\ \theta(\tilde{f}(x)) & \text{if } f(0) = 0 \end{array} \right\} \in I_k.$$

Let $\varphi \in C^n(\mathcal{U}_\bullet; I(\mathcal{A})^\bullet)$. Let us compute $(dH + Hd)\varphi$ and compare it with φ. We have

$$(d\varphi)_\lambda(x) = \sum_{k=0}^{n+1} (-1)^k \tilde{\varepsilon}_k'^* \varphi_{\tilde{\varepsilon}_k(\lambda)}(\tilde{\varepsilon}'_k(x)) \quad \in \mathcal{A}_x^{n+2}.$$

The meaning of this formula is the following: we take $x \in M_{n+2}$, then its image $\tilde{\varepsilon}'_k(x)$ (see notation (7.1)) is in M_{n+1}. Its image $\varphi_{\tilde{\varepsilon}_k(\lambda)}(\tilde{\varepsilon}'_k(x))$ belongs to $\mathcal{A}^{n+1}_{\tilde{\varepsilon}'_k(x)}$ is restricted (see (3.1)) by $\tilde{\varepsilon}'_k \colon M_{n+2} \to M_{n+1}$ to an element of $\mathcal{A}_x^{n+2}$.

Actually, we have $\varepsilon'_k = \varepsilon_{k+1}$ since $\varepsilon_0 \circ \varepsilon_k = \varepsilon_{k+1} \circ \varepsilon_0 \colon [n] \to [n+2]$, hence

$$\begin{aligned}
(d\varphi)_\lambda(x) &= \sum_{k=0}^{n+1} (-1)^k \tilde{\varepsilon}_{k+1}^* \varphi_{\tilde{\varepsilon}_k(\lambda)}(\tilde{\varepsilon}_{k+1}(x)), \\
(dH\varphi)_\lambda(x) &= \sum_{k=0}^{n} (-1)^k \tilde{\varepsilon}_{k+1}^* (H\varphi)_{\tilde{\varepsilon}_k(\lambda)}(\tilde{\varepsilon}_{k+1}(x)) \\
&= \sum_{k=0}^{n} (-1)^k \tilde{\varepsilon}_{k+1}^* \tilde{\eta}_0^* \varphi_{\tilde{\varepsilon}_k(\lambda)'_{\tilde{\varepsilon}_{k+1}}(x)}(\tilde{\eta}_0 \circ \tilde{\varepsilon}_{k+1}(x)), \\
(Hd\varphi)_\lambda(x) &= \tilde{\eta}_0^*(d\varphi)_{\lambda'_x}(\tilde{\eta}_0(x)) \\
&= \sum_{k=0}^{n+1} (-1)^k \tilde{\eta}_0^* \tilde{\varepsilon}_{k+1}^* \varphi_{\tilde{\varepsilon}_k(\lambda'_x)}(\tilde{\varepsilon}_{k+1}\tilde{\eta}_0(x)).
\end{aligned}$$

In the last sum, for $k = 0$ we get $\varphi_{\tilde{\varepsilon}_0(\lambda'_x)}(x)$. Now, $(\tilde{\varepsilon}_0(\lambda'_x))(f) = \lambda'_x(\varepsilon_0 \circ f) = \lambda(\eta_0 \circ \varepsilon_0 \circ f) = \lambda(f)$, thus the term for $k = 0$ is just $\varphi_\lambda(x)$.

To show that the other terms in the sum $dH + Hd$ cancel out, we just need to check that for all $k \geq 1$,

(a) $\eta_0 \circ \varepsilon_{k+1} = \varepsilon_k \circ \eta_0$, and
(b) $\tilde{\varepsilon}_k(\lambda'_x) = \tilde{\varepsilon}_{k-1}(\lambda)'_{\tilde{\varepsilon}_k(x)}$.

Assertion (a) is straightforward. Let us prove (b).

If $f(0) \neq 0$, then $\tilde{\varepsilon}_k(\lambda'_x)(f) = \lambda'_x(\varepsilon_k \circ f) = \lambda(\eta_0 \circ \varepsilon_k \circ f) = \lambda(\varepsilon_{k-1} \circ \eta_0 \circ f) = (\tilde{\varepsilon}_{k-1}(\lambda))'_{\tilde{\varepsilon}_k(x)}(f)$.

If $f(0) = 0$, then $\tilde{\varepsilon}_k(\lambda'_x)(f) = \lambda'_x(\varepsilon_k \circ f) = \theta(\widetilde{\varepsilon_k \circ f}(x)) = \theta(\tilde{f} \circ \tilde{\varepsilon}_k(x)) = ((\tilde{\varepsilon}_{k-1}\lambda)')_{\tilde{\varepsilon}_k(x)}(f)$. □

Remark 7.3. In the case of an étale groupoid G and cohomology groups with coefficients in G-sheaves, Haefliger [6], following Atiyah and Wall in the case of discrete groups [2], characterized the cohomology groups as the unique sequence of functors H^n such that

(a) $H^0 = \Gamma_{\text{inv}}$,
(b) H^* admits long exact sequences, and
(c) $H^n(G; I(\mathcal{A})) = 0$ for all $n \geq 1$ and for each G-sheaf $I(\mathcal{A})$.

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Let us now prove Proposition 7.1. First, we note that $\mathcal{A}^\bullet \hookrightarrow I(\mathcal{A})^\bullet$ canonically: if $c \in \mathcal{A}^n(U)$, then $\varphi(x) = \tilde{\varepsilon}_0^*[c(\tilde{\varepsilon}_0(x))] \in \mathcal{A}_x^{n+1}$ is a section of $I(\mathcal{A})^n$ over U.

Using the uniqueness part in Proposition 6.1, it suffices to show that

$$\check{H}^n(M_\bullet; I(\mathcal{A}^\bullet)) = H^n(M_\bullet; I(\mathcal{A})^\bullet) = \{0\} \quad \forall n \geq 1.$$

This is true for $\check{H}^n$ thanks to Lemma 7.2.

Inductively define a resolution

$$0 \to I(\mathcal{A})^\bullet \to \mathcal{L}^{\bullet,0} \to \mathcal{L}^{\bullet,1} \to \cdots \tag{7.2}$$

by $\mathcal{L}^{\bullet,0} = I(I(\mathcal{A}))^\bullet$ and $\mathcal{L}^{\bullet,q+1} = I(\mathcal{L}^{\bullet,q}/\mathcal{L}^{\bullet,q-1})$. Since $\mathcal{L}^{p,q}$ is flabby for all $p, q \geq 0$, the double complex $K = (\mathcal{L}^{p,q}(M_p))$ computes $H^*(M_\bullet, I(\mathcal{A})^\bullet)$ (see the introduction of this section).

The E_2-term with respect to the first filtration is $E_2^{p,q} = H^p H^q(K)$. Since

$$0 \to I(\mathcal{A})^p \to \mathcal{L}^{p,0} \to \mathcal{L}^{p,1} \to \cdots$$

is an exact sequence of flabby sheaves,

$$0 \to \Gamma(M_p; I(\mathcal{A})^p) \to \Gamma(M_p; \mathcal{L}^{p,0}) \to \cdots$$

is exact, hence $E_2^{p,q} = 0$ for $q \geq 1$ and $E_2^{p,0} = H^p(\Gamma(M_*; I(\mathcal{A})^*))$. Again using Lemma 7.2 for the cover $\mathcal{U}_n = \{M_n\}$, we get $E_2^{p,0} = 0$ for $p \geq 1$. Finally, $H^n(M_\bullet; I(\mathcal{A})^\bullet) = 0$ for all $n \geq 1$.

8. Invariance by Morita equivalence

Let G be a topological groupoid and let $\mathcal{A}^\bullet$ be an Abelian sheaf on $G_\bullet$. In this (rather easy) section, we show that $H^*(G_\bullet; \mathcal{A}^\bullet)$ and $\check{H}^*(G_\bullet; \mathcal{A}^\bullet)$ are invariant under Morita equivalence (recall Proposition 2.2).

More precisely, if G' is another groupoid and $\mathcal{A}'^\bullet$ is a sheaf on $G'_\bullet$, we say that $(G, \mathcal{A}^\bullet)$ is Morita equivalent to $(G', \mathcal{A}'^\bullet)$ if there exists a groupoid G'', a sheaf $\mathcal{A}''^\bullet$ on $G''^\bullet$ and (continuous) groupoid morphisms

$$G \xleftarrow{f} G'' \xrightarrow{f'} G'$$

such that f and f' are Morita equivalences and $\mathcal{A}''^\bullet \cong f^*\mathcal{A}^\bullet \cong f'^*\mathcal{A}'^\bullet$. Then

Proposition 8.1. *With the above assumptions, f and f' induce isomorphisms in sheaf and Čech cohomology, thus*

$$H^*(G_\bullet; \mathcal{A}^\bullet) \cong H^*(G'_\bullet; \mathcal{A}'^\bullet) \quad \textit{and} \quad \check{H}^*(G_\bullet; \mathcal{A}^\bullet) \cong \check{H}^*(G'_\bullet; \mathcal{A}'^\bullet).$$

Proof. By standard arguments (compare with Proposition 2.2), it suffices to show that for any open cover $\mathcal{U} = (U_i)_{i \in I}$ of G_0, the canonical morphism $f\colon G[\mathcal{U}] \to G$ induces isomorphisms

$$H^*(G_\bullet; \mathcal{A}^\bullet) \cong H^*(G[\mathcal{U}]_\bullet; f^*\mathcal{A}^\bullet) \quad \text{and} \quad \check{H}^*(G_\bullet; \mathcal{A}^\bullet) \cong \check{H}^*(G[\mathcal{U}]_\bullet; f^*\mathcal{A}^\bullet).$$

Below, we will abusively write $H^*(G[\mathcal{U}]_\bullet; \mathcal{A}^\bullet)$ instead of $H^*(G[\mathcal{U}]_\bullet; f^*\mathcal{A}^\bullet)$.

For Čech cohomology, using Remark 5.7, we have

$$\check{H}^n(G_\bullet; \mathcal{A}^\bullet) = \lim_{\mathcal{V}} \lim_{\mathcal{W}_\bullet} H^n(\mathcal{W}_\bullet; \mathcal{A}^\bullet), \tag{8.1}$$

where $\mathcal{V} = (V_j)_{j \in J}$ runs over open covers of G_0 and $\mathcal{W}_\bullet$ runs over open covers of $G[\mathcal{V}]$ such that $\mathcal{W}_0$ consists of the single open set $\coprod V_j$.

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Similarly,

$$\check{H}^n(G[\mathcal{U}]_\bullet;\mathcal{A}^\bullet) = \lim_{\mathcal{V}'}\lim_{\mathcal{W}'_\bullet} H^n(\mathcal{W}'_\bullet;\mathcal{A}^\bullet), \tag{8.2}$$

where $\mathcal{V}' = (V'_j)_{j\in J'}$ runs over open covers of $G[\mathcal{U}]_0$ and $\mathcal{W}'_\bullet$ runs over open covers of $G[\mathcal{U}][\mathcal{V}']$ such that $\mathcal{W}'_0$ consists of the single open set $\coprod V'_j$.

Now, note that if $\mathcal{V}'$ is an open cover of $G[\mathcal{U}]_0$ which is finer than the cover $(\{i\}\times U_i)_{i\in I}$, then there exists an open cover $\mathcal{V}$ of G_0 such that $G[\mathcal{V}] \cong G[\mathcal{U}][\mathcal{V}']$ (the elementary proof is left to the reader). Therefore, on the right-hand sides of (8.1) and (8.2), the terms $\lim_{\mathcal{W}_\bullet} H^n(\mathcal{W}_\bullet;\mathcal{A}^\bullet)$ and $\lim_{\mathcal{W}'_\bullet} H^n(\mathcal{W}'_\bullet;\mathcal{A}^\bullet)$ are identical. It follows that the right-hand sides of (8.1) and (8.2) are equal, hence Čech cohomology is invariant by Morita equivalence.

From the above, we already find that sheaf cohomology is invariant under Morita-equivalence when the groupoid is paracompact. In fact, this holds for a general topological groupoid. Let us sketch the proof for completeness.

Consider the resolution

$$\mathcal{A}^\bullet \to \mathcal{L}^{\bullet,0} \to \mathcal{L}^{\bullet,1} \to \cdots$$

constructed like (7.2). Since $\mathcal{L}^{p,q}$ is flabby for all p, q, the double complex $(\mathcal{L}^{p,q}(G_p))$ computes $H^*(G_\bullet;\mathcal{A}^\bullet)$, and since the lines are exact (7.2), $H^*(G_\bullet;\mathcal{A}^\bullet)$ is the cohomology of the complex $(\Gamma_{\mathrm{inv}}(\mathcal{L}^{\bullet,q}))_{q\in\mathbb{N}}$.

Similarly, $H^*(G[\mathcal{U}]_\bullet;\mathcal{A}^\bullet)$ is the cohomology of the complex $(\Gamma_{\mathrm{inv}}(f^*\mathcal{L}^{\bullet,q}))_{q\in\mathbb{N}}$. Now, it is elementary to check that for every sheaf $\mathcal{B}^\bullet$, $\Gamma_{\mathrm{inv}}(\mathcal{B}^\bullet)$ is isomorphic to $\Gamma_{\mathrm{inv}}(f^*\mathcal{B}^\bullet)$. □

Remark 8.2. It would be interesting to know whether the sheaf cohomology groups of a groupoid G are always the same as the sheaf cohomology groups of some classifying space BG. This is known to be true at least in some cases [25, Theorem 4], [16].

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