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PARTIAL HYPERBOLICITY OR DENSE ELLIPTIC PERIODIC POINTS FOR C^1-GENERIC SYMPLECTIC DIFFEOMORPHISMS

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Abstract. We prove that if a symplectic diffeomorphism is not partially hyperbolic, then with an arbitrarily small C^1 perturbation we can create a totally elliptic periodic point inside any given open set. As a consequence, a C^1-generic symplectic diffeomorphism is either partially hyperbolic or it has dense elliptic periodic points. This extends the similar results of S. Newhouse in dimension 2 and M.-C. Arnaud in dimension 4. Another interesting consequence is that stably ergodic symplectic diffeomorphisms must be partially hyperbolic, a converse to Shub-Pugh's stable ergodicity conjecture for the symplectic case.

1. Introduction and statement of results

We consider M to be a $2n$-dimensional compact connected Riemannian manifold and ω a symplectic form on M, i.e. a non-degenerate closed 2-form. Taking n times the wedge product of ω with itself we obtain a volume form on M. A C^r diffeomorphism f of M, $r \geq 1$, is called *symplectic* if it preserves the symplectic form, $f^*\omega = \omega$. The set of C^r symplectic diffeomorphisms of M will be denoted $\mathbf{Diff}^r_\omega(M)$, and we consider it to have uniform C^r topology. In the 2-dimensional case this is the same as the set of area-preserving C^r diffeomorphisms. In higher dimensions this is just a subset of the set of C^r volume-preserving diffeomorphisms (the volume form corresponding to ω).

A point $p \in M$ is a *periodic point* of period k for f if $f^k(p) = p$. If all the eigenvalues of $Df^k(p)$ have the norm different than 1, then we will say that p is a *hyperbolic periodic point*. In this case we have well-defined stable and unstable manifolds,

$$W^s(p) = \{x \in M, \lim_{l\to\infty} d(f^l(x), f^l(p)) = 0\} \text{ and}$$
$$W^u(p) = \{x \in M, \lim_{l\to\infty} d(f^{-l}(x), f^{-l}(p)) = 0\},$$

where d is the Riemannian metric on the manifold. In the case of symplectic maps the eigenvalues come in pairs, and λ is an eigenvalue of $Df^k(p)$ if and only if λ^{-1} is an eigenvalue, so the dimension of both the stable and unstable manifolds will be n in our case. A point $q \in W^s(p) \cap W^u(p) \setminus \{p\}$ is called a *homoclinic* point of p.

A splitting of the tangent bundle $TM = \mathbf{A} \oplus \mathbf{B}$ is called *invariant* if $\mathbf{A}$ and $\mathbf{B}$ are invariant under Df. An invariant splitting $TM = \mathbf{A} \oplus \mathbf{B}$ is called *dominated* if

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there is an $l > 0$ such that for any $x \in M$ and any two unit vectors $u \in \mathbf{A}_x, v \in \mathbf{B}_x$, we have $\|Df^l(u)\| \geq 2\|Df^l(v)\|$. We will also say that $\mathbf{A}$ *dominates* $\mathbf{B}$. If we want to emphasize the importance of l, we say that $\mathbf{A}$ *l-dominates* $\mathbf{B}$, or the splitting is *l-dominated*.

A map f is called *partially hyperbolic* if there is an invariant splitting of the tangent bundle of M, $TM = \mathbf{A} \oplus \mathbf{B} \oplus \mathbf{C}$, with at least two of them non-trivial, such that

(i) $\mathbf{A}$ is uniformly expanding: there exist $\alpha > 1$ and $C > 0$ such that

$$\|Df^k(u)\| \geq C\alpha^k\|u\|, \forall u \in \mathbf{A}, k \in \mathbb{N},$$

(ii) $\mathbf{C}$ is uniformly contracting: there exist $\beta > 1$ and $D > 0$ such that

$$\|Df^k(v)\| \leq D\beta^{-k}\|v\|, \forall v \in \mathbf{C}, k \in \mathbb{N},$$

(iii) $\mathbf{A}$ dominates $\mathbf{B}$, and $\mathbf{B}$ dominates $\mathbf{C}$.

One can prove that a dominated splitting is continuous, so the angle between the two subbundles is bounded away from 0, and a small C^1-perturbation of a map with a dominated splitting also has a dominated splitting. So the set of partially hyperbolic maps is open in $\mathbf{Diff}^1_\omega(M)$. Also we make the remark that the property of dominance is independent of the Riemannian structure on the manifold, but two different structures can have different constants of dominance l. If $\mathbf{B}$ is trivial, then f is called *uniformly hyperbolic* or *Anosov*. This definitions can be extended to an invariant subset N of M.

A periodic point p of f of period k is called *elliptic* if all the eigenvalues of $Df^k(p)$ are simple, non-real and of norm 1. Obviously the existence of an elliptic periodic point is an obstruction for partial hyperbolicity. We prove here that the converse is also true generically, i.e. if a C^1-generic symplectic diffeomorphism is not partially hyperbolic, then it has an elliptic periodic point (actually it has a dense set of elliptic periodic points). Here we say that a property is *C^r-generic* if it is true for a residual subset of $\mathbf{Diff}^r_\omega(M)$.

Hyperbolicity and ellipticity are responsible for completely different dynamics. A hyperbolic map is on one hand chaotic—it is transitive, it has sensitive dependence on the initial conditions—but on the other hand it has good statistical properties (it is ergodic if it is smooth enough—$C^{1+\alpha}$) and it has a Markov partition, and thus is similar to a subshift of finite type. The Anosov maps are structurally stable and stably ergodic (again, if they are smooth enough). Shub-Pugh's stable ergodicity conjecture states that among the partially hyperbolic diffeomorphisms, the stable ergodic ones form an open set in the C^1 topology, dense in the C^2 topology (see [21]). Here we give an answer to the converse of this for symplectic diffeomorphisms; we show that a form of stable ergodicity implies partial hyperbolicity. The dynamics in a neighborhood of an elliptic periodic point are different. We know from KAM theory that if the map is smooth enough, then close to the elliptic point we have many invariant tori (possibly a positive measure set) on which the map is conjugated to a strongly irrational rotation. Also the elliptic point is accumulated by other quasi-elliptic and hyperbolic periodic points, as well as homoclinic points (see [26]).

If for a periodic point p of period k the tangent map $Df^k(p)$ has exactly $2l$ simple non-real eigenvalues of norm 1 and the other ones have norm different from 1, then we say that p is an *l-elliptic periodic point*. Sometimes it is also called *quasi-elliptic*. S. Newhouse proved that a C^1-generic symplectic diffeomorphism is either Anosov or it has dense 1-elliptic periodic points (see [17]). He concluded from this

that a symplectic diffeomorphism is structurally stable if and only if it is uniformly hyperbolic. In dimension 2 the 1-elliptic periodic points are actually elliptic. M.-C. Arnaud proved (see [1], [2]) that a C^1 generic symplectic diffeomorphism in dimension 4 is either hyperbolic or partially hyperbolic, or it has an elliptic periodic point (in our paper we consider hyperbolicity as a particular case of partial hyperbolicity for simplicity). Our result generalizes this for any dimension.

Theorem 1. *There exists an open dense subset $\mathcal{U}$ of $\mathbf{Diff}^1_\omega(M)$ such that any function in $\mathcal{U}$ is either partially hyperbolic or it has an elliptic periodic point. There exists a residual subset $\mathcal{R}$ of $\mathbf{Diff}^1_\omega(M)$ such that any function in $\mathcal{R}$ is either partially hyperbolic or the set of elliptic periodic points is dense on the manifold.*

The proof is a direct consequence of the following theorem.

Theorem 2. *Suppose $f \in \mathbf{Diff}^1_\omega(M)$ is not partially hyperbolic. Then for any open subset $U \subset M$ there exists an arbitrarily small perturbation g of f such that g has an elliptic periodic point in U.*

Proof of Theorem 1. It is true that the set of partially hyperbolic diffeomorphisms in $\mathbf{Diff}^1_\omega(M)$, denoted by $\mathcal{PH}$, is open, and the set of diffeomorphism with elliptic periodic points, denoted $\mathcal{E}$, is also open. Now if f is not in $\mathcal{PH}$, then we apply Theorem 2 and get that f must be in the closure of $\mathcal{E}$. This proves that $\mathcal{U} = \mathcal{PH} \cup \mathcal{E}$ is open dense. For the second part, if we denote by $\mathcal{E}^\delta$ the set of diffeomorphisms with δ-dense elliptic periodic points, applying Theorem 2 for a finite number of open subsets one can prove that $\mathcal{PH} \cup \mathcal{E}^\delta$ is open dense in $\mathbf{Diff}^1_\omega(M)$, so taking $\mathcal{R}$ as their intersection for $\delta = \frac{1}{k}, k \in \mathbb{N}$, we get the result. $\square$

A map f is called C^1 *robustly transitive* if there is a C^1 neighborhood of f such that every map in this neighborhood is transitive. An example is any Anosov map. It is known that the existence of an elliptic periodic point is an obstruction for C^1 robust transitivity. So as a simple consequence we get the following corollary.

Corollary 3. *If a function $f \in \mathbf{Diff}^1_\omega(M)$ is C^1 robustly transitive, then it must be partially hyperbolic.*

A C^2 map f is called *stable ergodic* if there is a C^1 neighborhood of f such that every C^2 map in this neighborhood is ergodic. Again this is true for Anosov maps. Using the fact that $\mathbf{Diff}^\infty_\omega(M)$ is dense in $\mathbf{Diff}^1_\omega(M)$ in the C^1 topology (see [27]), we can get the following similar result for stable ergodicity.

Corollary 4. *If a function $f \in \mathbf{Diff}^1_\omega(M)$ has a C^1 neighborhood V in $\mathbf{Diff}^1_\omega(M)$ such that every C^∞ diffeomorphism in V is ergodic, then f must be partially hyperbolic.*

We conclude by mentioning some related results. C. Bonatti, L. J. Diaz and E. R. Pujals obtained similar results for the case of general and volume-preserving diffeomorphisms (see [11]). For general diffeomorphisms the elliptic points are substituted by sinks and sources, and the partial hyperbolicity by the existence of a dominated splitting. Also the result is restricted to homoclinic classes of hyperbolic periodic points, because transitivity is needed. For the case of volume-preserving maps the elliptic points are not stable under perturbations, and an arbitrarily small one can make them hyperbolic, so they get a weaker result without the genericity: if a map does not have a dominated splitting, with an arbitrarily small perturbation

one can create periodic points with all the eigenvalues equal to 1. We can get our stronger result for symplectic diffeomorphisms because of three reasons: the elliptic points are stable under perturbations if there are no multiple eigenvalues (which is a generic property)—this is not true for volume-preserving maps, but is true for sinks and sources for general diffeomorphisms; generically the symplectic maps are transitive—this is true in the volume-preserving case but not in the general one; and only in the symplectic case the existence of a dominated splitting implies partial hyperbolicity.

Other related results, obtained by J. Bochi and M. Viana (some of them announced before by R. Mañé), take into consideration the Lyapunov exponents of almost all the points instead of looking at the types of the periodic points (see [6], [7], [8], [15]). They prove that for a generic volume-preserving diffeomorphism for almost all the points either all the Lyapunov exponents are equal to 0 or their Oseledets splitting along the orbit is dominated. Also, a generic symplectic diffeomorphism is either Anosov or almost all the points have 0 as a Lyapunov exponent (with multiplicity at least 2).

We will give the proof of Theorem 2 in the last section. In Section 2 we give a perturbation result from linear algebra. In Section 3 we present some generic properties of symplectic diffeomorphisms. Section 4 is dedicated to Lyapunov exponents and Lyapunov filtrations for homoclinic points. In Section 5 we prove that dominated splitting implies partial hyperbolicity in the symplectic case, and in Section 6 we give the main perturbation proposition which shows how to lower the Lyapunov exponents in the absence of dominance.

2. A lemma from linear algebra

We present here a result we need from linear algebra. We will first give the motivation of the result.

We consider $\mathbb{R}^{2n}$ with the canonical symplectic structure given by the 2-form $\sum_{i=1}^{n} dx_i \wedge dy_i$, where $x_i, y_i, i \in \{1, 2, \ldots, n\}$, are the coordinates on $\mathbb{R}^{2n}$. The norm of the vectors is the euclidean norm in $\mathbb{R}^{2n}$, and the norm of matrices is the norm of the corresponding linear operators on $\mathbb{R}^{2n}$. There exist local coordinates $\phi_j : V_j \to \mathbb{R}^{2n}, j \in J$ finite, $M = \bigcup_{j \in J} V_j$, such that ω has the canonical form: $\omega = (\phi_j)_*(\sum_{i=1}^{n} dx_i \wedge dy_i)$. These will be called *symplectic coordinates*. In these coordinates the tangent map $Df_x : T_xM \to T_{f(x)}M$ can be seen as a symplectic matrix. For each $x \in M$ we fix coordinates ϕ_j with j the smallest number such that $x \in V_j$ and write Df_x using these coordinates at x and $f(x)$. From now on when we talk about l-dominance we use the euclidean norm in these coordinates. We also use them when we talk about the distance between two functions, or the size of a perturbation.

Now the question we ask is the following: suppose we have two unit vectors $u, v \in T_xM$ and $\mathbb{R}u$ does not l-dominate $\mathbb{R}v$, $\|Df_x^l(u)\| \leq \|Df_x^l(v)\|$. Obviously we can perturb the tangent map along the first l iterates of x moving u in the direction of v. Now using Frank's lemma (see [12]) we can realize these purely algebraic perturbations as the tangent map of a perturbation f' of f and get that $Df_x'^{\,l}(u) = cDf_x^l(v)$ for some real constant c. This perturbation is supported in an arbitrarily small neighborhood of the first l iterates of x and leaves these iterates unchanged, and the size of the perturbation is proportional to the size of the algebraic perturbations of the tangent map. The question is, how small will this

perturbation be? We prove that it depends only on l and on the upper bound of the norms of the derivatives K. If l is arbitrarily large, then the perturbation needed can be arbitrarily small.

For the 2-dimensional case or the non-symplectic case the result is known (see [14], [15]).

Lemma 5. *For any $\epsilon > 0$, $K > 0$, there exist an $l \in \mathbb{N}$ such that if $A_0, A_1, \dots, A_l$ are (symplectic, with determinant 1) $2n$-dimensional matrices with $\|A_k\| \le K, k \in \{0, 1, \dots, l\}$, and $\|A_{l-1} \dots A_1 A_0(u)\| \le 2\|A_{l-1} \dots A_1 A_0(v)\|$ for some unit vectors $u, v \in \mathbb{R}^{2n}$, then there exist (symplectic, with determinant 1) matrices $A'_0, A'_1, \dots, A'_l$ with $\|A_k - A'_k\| < \epsilon, k \in \{0, 1, \dots, l\}$, such that*

$$A'_l A'_{l-1} \dots A'_1 A'_0(u) = c A_l A_{l-1} \dots A_1 A_0(v)$$

for some non-zero real number c.

Proof. We will define A'_j as compositions of A_j with symplectic matrices close to the identity, which clearly will work in all the three cases. We give the proof for $n \ge 2$; for the case $n = 1$ the result is known.

We will choose l later in the proof. Denote $A_k A_{k-1} \dots A_0 = B_k$ for all $k \in \{0, 1, \dots, l\}$. For every such k we will consider an orthonormal symplectic basis in $\mathbb{R}^{2n}$, $\{e_1^k, \dots, e_n^k, f_1^k, \dots, f_n^k\}$ such that $\frac{B_k(u)}{\|B_k(u)\|} = e_1^k$ and $\frac{B_k(v)}{\|B_k(v)\|} = a_{k1} e_1^k + a_{k2} e_2^k + b_{k1} f_1^k$ for some $a_{k1}, a_{k2}, b_{k1} \in \mathbb{R}, a_{k1}^2 + a_{k2}^2 + b_{k1}^2 = 1$. We can also make this choice such that a_{k2} and b_{k1} have the same sign. By symplectic basis we mean that $\omega(e_i^k, f_i^k) = -\omega(f_i^k, e_i^k) = 1$ and $\omega(e_i^k, e_j^k) = \omega(f_i^k, f_j^k) = \omega(e_i^k, f_j^k) = \omega(f_i^k, e_j^k) = 0$ for all $i, j \in \{1, 2, \dots, n\}, i \ne j$.

We divide the proof in two steps. In the first one we prove that if the angle between the iterates of u and v is small at some point, then we can use a small rotation moving one into another. In the second step, if the angles between the iterates of u and v are bounded away from 0, then at the first step we make a small rotation of u toward v, then we make small perturbations along the orbit contracting u and expanding v, thus the new orbit of u will move toward v in long enough time, and then we complete with another small rotation in the end to move this new orbit of u to a multiple of $B_l(v)$.

Step 1. If the angle between $B_j(u)$ and $B_j(v)$ is small enough for some $j \in \{0, 1, \dots, l\}$, then we can construct a perturbation moving $B_j(u)$ to the direction of $B_j(v)$ only at the j'th step: if $a_{j1} > 0$ let $A'_j = RA_j$, where R is a symplectic linear map such that

$$\begin{aligned} R(e_1^j) &= a_{j1} e_1^j + a_{j2} e_2^j + b_{j1} f_1^j, \\ R(e_2^j) &= a_{j1}^{-1} e_2^j, \\ R(f_1^j) &= a_{j1}^{-1} f_1^j, \\ R(f_2^j) &= a_{j1} f_2^j - a_{j2} f_1^j, \end{aligned}$$

and R leaves the other vectors of the basis unchanged. This map moves $B_j(u)$ to a multiple of $B_j(v)$. If $a_{j1} < 0$ we can just replace $B_j(v)$ by $-B_j(v)$. Clearly $\|R - Id\|$ tends to 0 as a_{j1} tends to 1, so we can find $\alpha \in (0, 1)$ depending on ϵ and K such that if $|a_{j1}| > \alpha$, then $\|R - Id\| < \frac{\epsilon}{K}$ so $\|A'_j - A_j\| \le \|R - Id\| \|A_j\| < \epsilon$. Taking $A'_i = A_i$ for all $i \in \{0, 1, \dots, l\} \setminus \{j\}$ we get the desired sequence of perturbation.

Step 2. Now we can suppose that $|a_{k1}| \leq \alpha$ for any k. Then we get that $|\frac{a_{k1}a_{k2}}{a_{k2}^2+b_{k1}^2}| \leq \frac{\alpha}{1-\alpha^2}$ and $|\frac{a_{k1}b_{k1}}{a_{k2}^2+b_{k1}^2}| \leq \frac{\alpha}{1-\alpha^2}$ for all k. For a $\sigma > 1$ we can consider the symplectic linear map T_k such that

$$T_k(e_1^k) = \frac{1}{\sigma}e_1^k,$$
$$T_k(e_2^k) = (\sigma - \frac{1}{\sigma})\frac{a_{k1}a_{k2}}{a_{k2}^2+b_{k1}^2}e_1^k + \sigma e_2^k + (\sigma - \frac{1}{\sigma})\frac{a_{k1}b_{k1}}{a_{k2}^2+b_{k1}^2}f_2^k,$$
$$T_k(f_1^k) = (\sigma - \frac{1}{\sigma})\frac{a_{k1}b_{k1}}{a_{k2}^2+b_{k1}^2}e_1^k + \sigma f_1^k - (\sigma - \frac{1}{\sigma})\frac{a_{k1}a_{k2}}{a_{k2}^2+b_{k1}^2}f_2^k,$$
$$T_k(f_2^k) = \frac{1}{\sigma}f_2^k$$

and all the other vectors of the basis are unchanged. This map has the property that $T_kB_k(u) = \frac{1}{\sigma}B_k(u)$ and $T_kB_k(v) = \sigma B_k(v)$. Also $\|T_k - Id\|$ tends to 0 as σ tends to 1 uniformly with respect to $\frac{a_{k1}a_{k2}}{a_{k2}^2+b_{k1}^2}$ and $\frac{a_{k1}b_{k1}}{a_{k2}^2+b_{k1}^2}$ on compact sets, so there exist a $\sigma_0 > 1$ depending on ϵ and K such that if $1 \leq \sigma \leq \sigma_0$, then $\|T_k - Id\| < \frac{\epsilon}{K}$ as long as $|\frac{a_{k1}a_{k2}}{a_{k2}^2+b_{k1}^2}| \leq \frac{\alpha}{1-\alpha^2}$ and $|\frac{a_{k1}b_{k1}}{a_{k2}^2+b_{k1}^2}| \leq \frac{\alpha}{1-\alpha^2}$. From now on we fix $\sigma = \sigma_0$. Let $A'_k = T_kA_k, k \in \{1, 2, \ldots, l-1\}$. For $\delta = 1 - \alpha$ the angle between u and $u + \delta v$ is small enough so we can find a symplectic map R such that $R(u) = u + \delta v$ and $\|R - Id\| < \frac{\epsilon}{K}$ (see the first step of the proof of the lemma). Let $A'_0 = A_0R$.

From the construction of the perturbations we have that

$$A'_{l-1}\ldots A'_1A_0(u) = \sigma^{-(l-1)}A_{l-1}\ldots A_1A_0(u),$$
$$A'_{l-1}\ldots A'_1A_0(v) = \sigma^{l-1}A_{l-1}\ldots A_1A_0(v),$$

and consequently

$$A'_{l-1}\ldots A'_1A'_0(u) = \sigma^{-(l-1)}A_{l-1}\ldots A_1A_0(u) + \delta\sigma^{l-1}A_{l-1}\ldots A_1A_0(v),$$

or $A'_{l-1}\ldots A'_1A'_0(u) = \sigma^{-(l-1)}B_{l-1}(u) + \delta\sigma^{l-1}B_{l-1}(v)$. If we choose l such that $\sigma^{l-1} > \frac{2}{\delta}$ and we use the hypothesis of non-dominance, $\|B_{l-1}(u)\| \leq 2\|B_{l-1}(v)\|$, we get that the angle between $B_{l-1}(v)$ and $A'_{l-1}\ldots A'_1A'_0(u)$ is small enough so there exist a symplectic map R' such that $R'A'_{l-1}\ldots A'_1A'_0(u) = cB_{l-1}(v)$ for some real number c and $\|R' - Id\| < \frac{\epsilon}{K}$. Now if we let $A'_l = A_lR'$, then we get the conclusion of the lemma. □

Remark. The lemma is also true for the odd-dimensional case (not for symplectic matrices). The proof is easier; the perturbations required are restricted to a two-dimensional subspace.

Definition. For a fixed $K > 0$ we define a decreasing function $e : \mathbb{N} \to \mathbb{R}$ as follows: given an $l \in \mathbb{N}$, we define $e(l)$ to be the smallest positive number such that for any sequence of $2n$-dimensional symplectic matrices $A_1, A_2, \ldots, A_l$ with $\|A_k\| \leq K, k \in \{1, 2, \ldots, l\}$, and any two unit vectors $u, v \in \mathbb{R}^{2n}$ such that $\|A_{l-1}\ldots A_2A_1(u)\| \leq 2\|A_{l-1}\ldots A_2A_1(v)\|$, there exist symplectic matrices $A'_1, A'_2, \ldots, A'_l$ with $\|A_k - A'_k\| \leq e(l), k \in \{1, 2, \ldots, l\}$, such that

$$A'_lA'_{l-1}\ldots A'_2A'_1(u) = cA_lA_{l-1}\ldots A_2A_1(v)$$

for some non-zero real number c. The lemma says that $\lim_{l\to\infty} e(l) = 0$.

In the same way for a given symplectic manifold M with fixed symplectic charts and for a fixed $K > 0$ we define a decreasing function $E : \mathbb{N} \to R$ as follows: for

any $l \in \mathbb{N}$ we define $E(l)$ to be the smallest positive number such that for any $f \in \mathbf{Diff}^1_\omega(M)$ with $\|Df_x\| \leq K, \forall x \in M$, in the given charts and any two unit vectors $u, v \in T_xM$ for some $x \in M$ such that $\|Df^l_x(u)\| \leq \|Df^l_x(v)\|$ there exists a perturbation f' of f of size $\frac{E(l)}{2}$ supported on an arbitrarily small neighborhood of $\{x, f(x), \ldots, f^l(x)\}$ such that $f'^k(x) = f^k(x), 1 \leq k \leq l$, and $Df'^l_x(u) = cDf^l_x(v)$ for some real number c. Because of Frank's lemma mentioned above we also have that $\lim_{l\to\infty} E(l) = 0$.

3. Some generic properties of symplectic diffeomorphisms

Here we state some of the known C^1 generic properties of symplectic diffeomorphisms. There exists a residual subset $\mathcal{R}$ of $\mathbf{Diff}^1_\omega(M)$ such that for any $f \in \mathcal{R}$ we have:

1. The periodic points of f are dense in M.
2. Every periodic point of f is either quasi-elliptic or hyperbolic.
3. The hyperbolic periodic points of f are dense in M.
4. The stable and unstable manifolds of hyperbolic periodic points of f intersect transversally.
5. Every hyperbolic periodic point of f has homoclinic orbits.
6. The homoclinic points of f are dense in M.
7. The homoclinic points of a hyperbolic periodic point of f are dense in both the stable and unstable manifolds of the periodic point.
8. The map f is transitive.

Property 1 is a direct consequence of the C^1 closing lemma of Pugh and Robinson (see [20]). Properties 2 and 4 were proved by Robinson (see [22]). Property 3 is a direct consequence of property 1, because we cane make a periodic point hyperbolic with a small perturbation if the period is large enough. An alternate proof can use properties 1 and 2 and the fact that a quasi-elliptic periodic point is generically accumulated by hyperbolic ones from KAM theory (see [17]). Property 5 was proved by Takens (see [23]), which is also a consequence of Hayashi's connecting lemma and the fact that the C^r generically stable (unstable) manifolds accumulate on themselves, so also on the unstable (stable) manifolds - this proof gives also property 7 (see [24], [25]). Property 6 is a consequence of properties 3 and 5. Property 8 is another application of Hayashi's connecting lemma, and it was proved by Bonatti and Crovisier (see [3]).

The next lemma presents another generic property of symplectic (or volume-preserving) diffeomorphisms. The main consequence we use from it is the fact that there is an arbitrarily small C^1 perturbation of f such that the set of homoclinic points of (the continuation of) a hyperbolic periodic point x is dense in M.

Lemma 6. *There exist a residual $\mathcal{R} \in \mathbf{Diff}^1_\omega(M)$ (or in the set of volume-preserving diffeomorphisms) such that for any $f \in \mathcal{R}$ and any hyperbolic periodic point x of f the set of corresponding homoclinic points is dense in M.*

Proof. It is known that for a C^1 generic symplectic (or more general volume-preserving) diffeomorphism for any hyperbolic periodic point y the set of homoclinic points of y is dense in both $W^s(y)$ and $W^u(y)$ (from property 7 above). We need to prove that C^1 generical $W^s(y)$ and $W^u(y)$ are dense in M for all hyperbolic periodic points y, and we are done. We will use the fact that a C^1 generic

symplectic (or volume preserving) diffeomorphism is transitive (property 8) and is the C^1 connection lemma.

Let f be transitive, $U \subset M$ an open set, y a hyperbolic periodic point, $p \in W^s(y)$ and B_k the ball of radius 2^{-k} centered at p. Because f is transitive there is an iterate of U intersecting B_1: $f^{k_1}(U) \cap B_1 \neq \emptyset$. Let U_1 be an open set such that $U_1 \subset \mathrm{cl}(U_1) \subset U \cap f^{-k_1}(B_1)$. Now there is an iterate $f^{k_2}(U_1)$ of U_1 which intersects B_2, so we can choose an open set U_2 such that $U_2 \subset \mathrm{cl}(U_2) \subset U_1 \cap f^{-k_2}(B_2)$, and so on. Then $\mathrm{cl}(U_k)$ is a decreasing sequence of compact sets inside U, so there is a point y_U in their intersection and its orbit accumulates on p. Because p is not periodic, we can use the connection lemma to find an arbitrarily small perturbation of f such that p is a positive iterate of y_U and the positive orbit of p is unchanged, so $y_U \in W^s(y)$. So the stable manifold of y intersects U, and obviously this is also true for small perturbations (replacing y by its continuation).

Now let us denote by $\mathcal{R}(k, U)$ the set of diffeomorphisms with the property that the stable manifold of any hyperbolic periodic point with period less than k intersects U. From what we proved before and from the fact that for an open dense set of diffeomorphisms there are finitely many periodic points of period less than k, we get that $\mathcal{R}(k, U)$ is an open dense subset of $\mathbf{Diff}^1_\omega(M)$ (or the volume preserving diffeomorphisms). But there is a countable base of the topology, so taking the intersection over $k \in \mathbb{N}$ and U in the countable basis we get the residual we are looking for. For $W^u(y)$ the proof is similar. $\square$

4. Lyapunov filtrations for the invariant manifolds of hyperbolic periodic points

For a measure-preserving diffeomorphism almost all the points of the manifold have well-defined Lyapunov exponents and a corresponding Oseledets splitting that give the exponential rate of expansion of the vectors in the tangent bundle under the tangent map. More specific, if $f \in \mathbf{Diff}^1(M)$ preserves a measure μ, then for μ-almost every point $x \in M$ there exist real numbers $\hat{\lambda}_1(x) < \hat{\lambda}_2(x) < \cdots < \hat{\lambda}_{k(x)}(x)$ and an invariant splitting $T_xM = E^1(x) \oplus E^2(x) \oplus \cdots \oplus E^{k(x)}(x)$ with $\dim E^1(x) + \dim E^2(x) + \cdots + \dim E^{k(x)}(x) = \dim M$ such that

$$\lim_{|l|\to\infty} \frac{\log \|Df^l_x(u)\|}{l} = \hat{\lambda}_i(x), \forall u \in E^i(x), i \in \{1, 2, \ldots, k(x)\},$$

and for any two disjoint subsets I and J of $\{1, 2, \ldots, k(x)\}$ we have

$$\lim_{|l|\to\infty} \frac{1}{l} \log \angle\Big(\bigoplus_{i\in I} E^i_{f^l(x)}, \bigoplus_{i\in J} E^i_{f^l(x)}\Big) = 0.$$

Such a point x is called *Lyapunov regular*, the real numbers $\hat{\lambda}_i, i \in \{1, 2, \ldots, k(x)\}$, are called the *Lyapunov exponents* of x and $E^i_x, i \in \{1, 2, \ldots, k(x)\}$, are called *Lyapunov subspaces*. The splitting is also called the *Oseledets splitting*. The dimension of each E^i_x, $m_i(x)$, is called the multiplicity of $\hat{\lambda}_i(x)$.

The Lyapunov regular points $x \in M$ also have invariant forward and backward *Lyapunov filtrations*: $\{0\} = \mathbf{B}^0_x \subset \mathbf{B}^1_x \subset \mathbf{B}^2_x \subset \cdots \subset \mathbf{B}^{k(x)}_x = T_xM$ and $\{0\} = \mathbf{A}^{k(x)+1}_x \subset \mathbf{A}^{k(x)}_x \subset \mathbf{A}^{k(x)-1}_x \subset \cdots \subset \mathbf{A}^1_x = T_xM$ such that for every

$i \in \{1, 2, \ldots, k(x)\}$ we have

$$\lim_{l\to\infty} \frac{\log \|Df_x^{-l}(u)\|}{l} = \hat{\lambda}_i(x), \forall u \in \mathbf{A}_x^i \setminus \mathbf{A}_x^{i+1},$$
$$\lim_{l\to\infty} \frac{\log \|Df_x^{l}(v)\|}{l} = \hat{\lambda}_i(x), \forall v \in \mathbf{B}_x^i \setminus \mathbf{B}_x^{i-1}.$$

In this case $\mathbf{B}_x^i = \bigoplus_{j=1}^i E_x^j$ and $\mathbf{A}_x^i = \bigoplus_{j=i}^{k(x)} E_x^j$. Forward and/or backward Lyapunov filtrations may exist for other points on the manifold which are not Lyapunov regular.

Every periodic point is Lyapunov regular. For example, if x is a periodic point of period p then the Lyapunov exponents are of the form $\lambda = \frac{\log|\gamma|}{p}$, where γ is an eigenvalue of Df_x^p and the Lyapunov subspaces of the Oseledets splitting are given by the direct sum of the corresponding (generalized) eigenspaces. If we have different eigenvalues of the same norm (as is the case of complex eigenvalues), then the number of exponents is smaller than the number of eigenvalues.

We consider $\lambda_1(x) \le \lambda_2(x) \le \cdots \le \lambda_{\dim M}(x)$ to be the Lyapunov exponents $\hat{\lambda}_i(x)$ of a regular point x repeated with multiplicity $m_i(x)$. Also define $\Lambda_i(x) = \lambda_i(x) + \lambda_{i+1}(x) + \cdots + \lambda_{\dim M}(x)$ and the corresponding $\hat{\Lambda}_i(x) = m_i(x)\hat{\lambda}_i(x) + m_{i+1}(x)\hat{\lambda}_{i+1}(x) + \cdots + m_{k(x)}(x)\hat{\lambda}_{k(x)}(x)$. $\Lambda_i(x)$ (or $\hat{\Lambda}_i(x)$) is actually the maximal exponential growth of the $n-i+1$-dimensional (or $m_i(x)+m_{i+1}(x)+\cdots+m_k(x)$-dimensional) volume under Df.

In the symplectic case the Lyapunov exponents come in pairs. If λ is an exponent with multiplicity m, then $-\lambda$ is also an exponent with the same multiplicity m. If the dimension of the manifold M is $2n$, then when we count the eigenvalue with their multiplicity we will denote them $\lambda_{-n} \le \lambda_{-n+1} \le \cdots \le \lambda_{-1} \le 0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, and we have $\lambda_{-i} = -\lambda_i, \forall i \in \{1, 2, \ldots, n\}$.

The following lemma proves that the points on the stable manifold of a hyperbolic periodic point have a forward Lyapunov filtration and the points on the unstable manifold have a backward Lyapunov filtration, with the same exponents as the ones of the periodic point. As a consequence the homoclinic points will have both forward and backward Lyapunov filtrations.

Lemma 7. *Given a hyperbolic periodic point x for the diffeomorphism f of the compact manifold M with Lyapunov exponents $\hat{\lambda}_1 < \hat{\lambda}_2 < \cdots < \hat{\lambda}_k$ and corresponding Lyapunov subspaces $E^1, E^2, \ldots, E^k$, then for any point p on the unstable manifold of x we have a filtration of the tangent space $\{0\} = \mathbf{A}_p^{k+1} \subset \mathbf{A}_p^k \subset \mathbf{A}_p^{k-1} \subset \cdots \subset \mathbf{A}_p^1 = T_pM$ (all the inclusions here are strict) which is invariant under Df and satisfies the following:*

$$\lim_{l\to\infty} \mathbf{A}_{f^{-l}(p)}^j = E^k \oplus E^{k-1} \oplus \cdots \oplus E^j = \mathbf{A}_x^j,$$
$$\lim_{l\to\infty} \frac{-\log \|Df_p^{-l}(u)\|}{l} = \hat{\lambda}_j, \forall u \in \mathbf{A}_p^j \setminus \mathbf{A}_p^{j+1},$$

and for any $j \in \{1, 2, \ldots, k\}$ and any basis $\{u_1, u_2, \ldots, u_{d(j)}\}$ of $\mathbf{A}_p^j$ we have

$$\lim_{l\to\infty} \frac{-\log \| \bigwedge^{d(j)} Df^{-l}(u_1 \wedge u_2 \wedge \cdots \wedge u_{d(j)})\|}{l} = \hat{\Lambda}_j$$

(the $d(j)$-dimensional volume of $\mathbf{A}_p^j$ decreases under backward iterates with an exponential rate of $\hat{\Lambda}_j$).

Similarly for any point q on the stable manifold of x we have a filtration $\{0\} = \mathbf{B}_q^0 \subset \mathbf{B}_q^1 \subset \mathbf{B}_q^2 \subset \cdots \subset \mathbf{B}_q^k = T_pM$, again invariant and satisfying

$$\lim_{l\to\infty} \mathbf{B}^j_{f^l(p)} = E^1 \oplus E^2 \oplus \cdots \oplus E^j = \mathbf{B}^j_x,$$

$$\lim_{l\to\infty} \frac{\log \|Df^l_p(v)\|}{l} = \hat{\lambda}_j, \forall v \in \mathbf{B}^j_p \setminus \mathbf{B}^{j-1}_p,$$

and for any $j \in \{1, 2, \ldots, k\}$ and any basis $\{v_1, v_2, \ldots, v_{d'(j)}\}$ of $\mathbf{B}^j_p$ we have

$$\lim_{l\to\infty} \frac{\log \| \bigwedge^{d'(j)} Df^l(v_1 \wedge v_2 \wedge \cdots \wedge v_{d'(j)})\|}{l} = \hat{\Lambda}_k - \hat{\Lambda}_{j+1}.$$

Proof. First we observe that it is enough to define this filtration on small local stable and unstable manifolds of x and extend them later by invariance. So we suppose p and q are on $W^u_\epsilon(x)$, respectively $W^s_\epsilon(x)$, for some small ϵ and inside a chart around x. If the map is linear or C^1 conjugated to a linear map in a neighborhood of x the proof is trivial. In general we denote $\mathbf{A}^j_x = E^k \oplus E^{k-1} \oplus \cdots \oplus E^j \subset T_xM$, and for any y close to x we denote $\mathbf{A}^j_x(y)$ the translation of $\mathbf{A}^j_x$ to T_yM using the given chart. We define $\mathbf{A}^j_p = \lim_{l\to\infty} Df^l_{f^{-l}(p)} \mathbf{A}^j_x(f^{-l}(p))$. Now one can prove that this is well defined and verifies the required properties using some cone fields in a neighborhood of x.

So let us define the cones

$$\begin{aligned} C^{j,\epsilon}_x &= \mathbb{R}\{a + \epsilon' b, a \in \mathbf{A}^j_x, b \in \mathbf{B}^j_x, \|a\| = \|b\| = 1, \epsilon' < \epsilon\} \\ &= \{a + b, a \in \mathbf{A}^j_x, b \in \mathbf{B}^j_x, \|b\| < \epsilon\|a\|\}. \end{aligned}$$

All these cones are positively strictly invariant, i.e. $\mathrm{cl}(Df(C^{j,\epsilon}_x)) \subset C^{j,\epsilon}_x$. For $\epsilon = 0$, $\delta > 0$ fixed, by eventually replacing f by a power of f we have that any vector in $C^{j,0}_x$ expands under Df by a factor between $e^{\hat{\lambda}_j - \delta}$ and $e^{\hat{\lambda}_n + \delta}$ and the $d(j)$-dimensional volume of the $d(j)$-dimensional subspaces in $C^{j,0}_x$ (which in this case is only $C^{j,0}_x$) expands under $\bigwedge^{d(j)} Df$ by a factor between $e^{\hat{\Lambda}_j - \delta}$ and $e^{\hat{\Lambda}_j + \delta}$. Then we can find $\epsilon > 0$ such that

$$\hat{\lambda}_j - \delta < \log \frac{\|Df(u)\|}{\|u\|} < \hat{\lambda}_n + \delta, \forall u \in C^{j,\epsilon}_x,$$

$$\hat{\Lambda}_j - \delta < \log \frac{\| \bigwedge^{d(j)} Df(u_1 \wedge u_2 \wedge \cdots \wedge u_{d(j)})\|}{\|u_1 \wedge u_2 \wedge \cdots \wedge u_{d(j)}\|} < \hat{\Lambda}_j + \delta,$$

$\forall u_1, u_2, \ldots, u_{d(j)}$ linearly independent, $\mathrm{span}\{u_1, u_2, \ldots, u_{d(j)}\} \subset C^{j,\epsilon}_x$.

Because f is C^1, we can construct a strictly invariant continuous cone field $C^{j,\epsilon}_y$ for y on a small neighborhood of x extending $C^{j,\epsilon}_x$ and having the properties mentioned above. For l large enough $\mathbf{A}^j_x(f^{-l}(p)) \subset C^{j,\epsilon}_{f^{-l}(p)}$, and because the cones are contracting in a neighborhood of x we get that $\mathbf{A}^j_p$ is well defined and obviously $\lim_{l\to\infty} \mathbf{A}^j_{f^{-l}(p)} = \mathbf{A}^j_x$. Also $Df^{-l}\mathbf{A}^j_p = \mathbf{A}^j_{f^{-l}(p)} \subset C^{j,\epsilon}_{f^{-l}(p)}$ for all large enough l so

$$\hat{\lambda}_j - \delta \le \lim_{l\to\infty} \frac{\log \|Df^{-l}(u)\|}{-l} \le \hat{\lambda}_n + \delta, \forall u \in \mathbf{A}^j_p,$$

$$\hat{\Lambda}_j - \delta \le \lim_{l\to\infty} \frac{\log \| \bigwedge^{d(j)} Df^{-l}(u_1 \wedge u_2 \wedge \cdots \wedge u_{d(j)})}{-l} \le \hat{\Lambda}_j + \delta$$

for any basis $u_1, u_2, \ldots, u_{d(j)}$ of $\mathbf{A}^j_p$.

Taking the limit for $\delta \to 0$ we get the desired equality for the $d(j)$-dimensional volume and

$$\hat{\lambda}_j \leq \lim_{l\to\infty} \frac{\log \|Df^{-l}(u)\|}{-l} \leq \hat{\lambda}_n, \forall u \in \mathbf{A}_p^j.$$

We can see from the definition that $\mathbf{A}_p^{j+1} \subset \mathbf{A}_p^j$, so for any vector $u = u_{d(j)} \in \mathbf{A}_p^j \setminus \mathbf{A}_p^{j+1}$ we can choose vectors $u_1, u_2, \ldots, u_{d(j)-1}$ such that $\{u_1, u_2, \ldots, u_{d(j+1)}\}$ is a basis for $\mathbf{A}_p^{j+1}$ and $\{u_1, u_2, \ldots, u_{d(j)}\}$ is a basis for $\mathbf{A}_p^j$. Now using the fact that the exponential volume growth on $\mathbf{A}_p^j$ and $\mathbf{A}_p^{j+1}$ under Df^{-1} is $-\hat{\Lambda}_j$, respectively $-\hat{\Lambda}_{j+1}$, we get that the exponential volume growth under Df^{-1} on the subspace spanned by $\{u_{d(j+1)+1}, \ldots, u_{d(j)}\}$ must be at least $-\hat{\Lambda}_j + \hat{\Lambda}_{j+1} = -m_j\hat{\lambda}_j$. But we know that the exponential growth of the vectors $u_{d(j+1)+1}, \ldots, u_{d(j)}$ under Df^{-1} is at most $-\hat{\lambda}_j$ and $m_j = d(j) - d(j+1)$. From this we can conclude the required relation from the lemma.

For the subspaces $\mathbf{B}_q^j$ for q on the stable manifold of x, the proof is similar. □

Remark. This result can be extended for the partially hyperbolic periodic points—we can allow some 0 Lyapunov exponents and consider the points on the strong stable and strong unstable manifolds.

5. Dominated splittings for symplectic diffeomorphisms

In this section we prove that for a symplectic diffeomorphism the existence of a dominated splitting implies partial hyperbolicity.

Suppose we have a symplectic structure on a $2n$-dimensional vector space V, i.e. there is a non-degenerate skew symmetric bilinear functional on $V \times V$ denoted ω. Two subspaces A and B of V are called *skew orthogonal* if $\omega(a, b) = 0, \forall a \in A, b \in B$. A subspace A is called *symplectic* if ω restricted to A is non-degenerate. The *skew orthogonal complement* of a subspace A is $A^\omega = \{v \in V : \omega(a, v) = 0, \forall a \in A\}$. We have that $(A^\omega)^\omega = A$, and if the dimension of A is d, then the dimension of A^ω is $2n - d$. A subspace is symplectic if and only if $A \cap A^\omega = \emptyset$. If $A = A^\omega$, then we say that A is a *Lagrangian subspace*. The Lagrangian subspaces are the maximal subspaces such that ω restricted to them is trivial (or they are a subset of their skew orthogonal complement).

Lemma 8. *If $f \in \mathbf{Diff}^1_\omega(M)$ has a dominated splitting, then it is partially hyperbolic. More precisely, say that $\mathbf{A} \oplus \mathbf{B}$ is an invariant splitting of TM, $\dim A = i \leq n$ and $\mathbf{A}$ l-dominates $\mathbf{B}$ for some l. Then there is a splitting $\mathbf{C} \oplus \mathbf{D}$ of $\mathbf{B}$ such that $\dim \mathbf{D} = i$, $\dim \mathbf{C} = 2n - 2i$, $\mathbf{C}$ and $\mathbf{A} \oplus \mathbf{D}$ are symplectic and skew orthogonal, $\mathbf{A}$ is uniformly expanding, $\mathbf{D}$ is uniformly contracting, and $\mathbf{A}$ l'-dominates $\mathbf{C}$, $\mathbf{C}$ l'-dominates $\mathbf{D}$ for some $l' \geq l$. In particular if $i = n$, then $\mathbf{C} = 0$ and f is hyperbolic.*

Proof. We define $\mathbf{C} = \mathbf{A}^\omega \cap \mathbf{B}$, $\mathbf{D} = \mathbf{C}^\omega \cap \mathbf{B}$. We remark that $\mathbf{A}$ and $\mathbf{B}$ must be continuous because of the dominance, so $\mathbf{C}$ and $\mathbf{D}$ must also be continuous. There exist $M > 0$ such that for any $x \in M$ and any two vectors $u, v \in T_xM$ we have $|\omega(u, v)| \leq M\|u\|\|v\|$. We divide the proof into two cases.

First case: $i < n$. For any $x \in M$ there are vectors $b_1, b_2 \in \mathbf{B}_x$ with $\omega(b_1, b_2) \neq 0$. Using the continuity of $\mathbf{B}$ and the compactness of M we can find an $m > 0$ such that for any $x \in M$ there are two vectors $b_1^x, b_2^x \in T_xM$ such that $\omega(b_1^x, b_2^x)$

$\geq m\|b_1^x\|\|b_2^x\|$. Then if we take b_1^x and b_2^x to be unit vectors and any other unit vector $a^x \in \mathbf{A}_x$ we get

$$m \leq \omega(b_1^x, b_2^x) = \omega(Df^{kl}(b_1^x), Df^{kl}(b_2^x)) \leq M\|Df^{kl}(b_1^x)\|\|Df^{kl}(b_2^x)\|$$
$$\leq \frac{M}{2^{2k}}\|Df^{kl}(a^x)\|^2 \text{ or } \|Df^{kl}(a^x)\| \geq \sqrt{\frac{2^{2k}m}{M}}, \forall x \in M, a^x \in \mathbf{A}_x.$$

Here we used the dominance hypothesis. Now if we take k large enough so that $\frac{2^{2k}m}{M} > 1$ we get that $\mathbf{A}$ must be uniformly expanding.

For any $x \in M$ and $a_1, a_2 \in \mathbf{A}_x$ we have

$$\omega(a_1, a_2) = \lim_{k\to\infty} \omega(Df^{-k}(a_1), Df^{-k}(a_2)) = 0$$

because $\mathbf{A}$ is expanding, so ω restricted to $\mathbf{A}$ is trivial. Then $\mathbf{A} \subset \mathbf{A}^\omega$ and the dimension of $\mathbf{C} = \mathbf{A}^\omega \cap \mathbf{B}$ must be $2n-2i$. Now $\mathbf{A} \subset (\mathbf{A}\oplus\mathbf{C})^\omega$ and the dimension of $(\mathbf{A}\oplus\mathbf{C})^\omega$ is i, so $\mathbf{A} = (\mathbf{A}\oplus\mathbf{C})^\omega$, and consequently $\mathbf{C}\cap\mathbf{C}^\omega = \emptyset$, so $\mathbf{C}$ is symplectic. Also we know that $\mathbf{A} \subset \mathbf{C}^\omega$ and the dimension of $\mathbf{C}^\omega$ is $2i$, so the dimension of $\mathbf{D} = \mathbf{C}^\omega \cap \mathbf{B}$ must be i. From construction we have that $\mathbf{C}$ and $\mathbf{A}\oplus\mathbf{D}$ are skew orthogonal. Also the fact that $\mathbf{C}$ is symplectic also implies that $\mathbf{A}\oplus\mathbf{D}$ is symplectic and ω restricted to $\mathbf{D}$ is trivial.

What is now left to prove is that $\mathbf{D}$ is uniformly contracting and $\mathbf{C}$ dominates $\mathbf{D}$. As we remarked before $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{D}$ are continuous, M is compact and $\mathbf{C}$, $\mathbf{A}\oplus\mathbf{D}$ are symplectic, *omega* restricted to $\mathbf{A}$ and $\mathbf{D}$ is trivial, so there exists $m > 0$ such that for any any $x \in M, u \in \mathbf{A}_x$ (or $\mathbf{D}_x, \mathbf{C}_x$) there exists $v \in \mathbf{D}_x$ (or $\mathbf{A}_x$ respectively $\mathbf{C}_x$) such that $\omega(u, v) \geq m\|u\|\|v\|$.

Suppose we have $x \in M, d \in \mathbf{D}_x$. As we saw before we can find $a \in \mathbf{A}_x$ such that $\omega(a, d) \geq m\|a\|\|d\|$. Then

$$m\|a\|\|d\| \leq \omega(a, d) = \omega(Df^{-k}(a), Df^{-k}(b)) \leq M\|Df^{-k}(a)\|\|Df^{-k}(d)\|.$$

So $\|Df^{-k}(d)\| \geq \frac{m}{M}\frac{\|a\|}{\|Df^{-k}(a)\|}\|d\|$, and because $\mathbf{A}$ is uniformly expanding, if we take k large enough we get that $\mathbf{D}$ must be uniformly contracting.

We know that $\mathbf{A}$ l-dominates $\mathbf{C}$ because $\mathbf{C} \subset \mathbf{B}$. So let us take $x \in M, d \in \mathbf{D}_x, c \in \mathbf{C}_x, \|c\| = \|d\| = 1$. As before we pick $a \in \mathbf{A}_x, \|a\| = 1, \omega(a, d) \geq m\|a\|\|d\| = m$. Then

$$m \leq \omega(a, d) = \omega(Df^{-kl}(a), Df^{-kl}(d)) \leq M\|Df^{-kl}(a)\|\|Df^{-kl}(d)\|.$$

There exist $c' \in \mathbf{C}_x, \|c'\| = 1$ such that

$$\omega(Df^{-kl}(c), Df^{-kl}(c')) \geq m\|Df^{-kl}(c)\|\|Df^{-kl}(c')\|.$$

From the fact that $\mathbf{A}$ l-dominates $\mathbf{C}$ we get that $\|Df^{-kl}(a)\| \leq 2^{-k}\|Df^{-kl}(c')\|$. Combining these two inequalities we get

$$\|Df^{-kl}(a)\| \leq \frac{2^{-k}}{m\|Df^{-kl}(c)\|}\omega(Df^{-kl}(c), Df^{-kl}(c'))$$
$$= \frac{2^{-k}}{m\|Df^{-kl}(c)\|}\omega(c, c') \leq \frac{2^{-k}}{m\|Df^{-kl}(c)\|}M\|c\|\|c'\| = \frac{2^{-k}M}{m\|Df^{-kl}(c)\|}$$

and furthermore

$$\frac{\|Df^{-kl}(d)\|}{\|Df^{-kl}(c)\|} \geq \frac{2^k m^2}{M^2},$$

which proves that $\mathbf{C}$ also dominates $\mathbf{D}$ if we again take k large enough, and we are done.

Second case: $i = n$. In this case we only have to prove that $\mathbf{A}$ is uniformly expanding and $\mathbf{B}$ is uniformly contracting. For any $x \in M$ we have that either ω restricted to $\mathbf{B}_x$ is trivial or ω restricted to $\mathbf{B}_x$ is not trivial, and as in the proof of the first case we get that $\mathbf{A}_x$ is expanded, so ω restricted to $\mathbf{A}_x$ must be trivial. If we take any $x \in M$ and $a \in \mathbf{A}_x$, because ω restricted to $\mathbf{A}_x$ or to $\mathbf{B}_x$ is trivial, we can find an $b \in \mathbf{B}_x$ such that $\omega(a, b) \neq 0$ and vice-versa. These observations, together with the continuity of $\mathbf{A}$ and $\mathbf{B}$ and the compactness of M, show that there must be again an $m > 0$ such that for any $x \in M$ and for any $a \in \mathbf{A}_x$ ($b \in \mathbf{B}_x$) there exist $b \in \mathbf{B}_x$ ($a \in \mathbf{A}_x$) such that $\omega(a, b) \geq m\|a\|\|b\|$. Now let us suppose that $\mathbf{A}$ is not uniformly expanding, so for any large k there exist $x \in M$ and $a \in \mathbf{A}_x$ such that $\|Df^{kl}(a)\| < 2$. From the dominance condition we get that $\mathbf{B}_x$ must be contracting, $\|Df^{kl}(v)\| < 2^{1-k}, \forall v \in \mathbf{B}_x$. We know that we can find $b \in \mathbf{B}_x$ such that $\omega(a, b) \geq m\|a\|\|b\|$. Then we get

$$\begin{aligned} m\|a\|\|b\| \leq \omega(a, b) = \omega(Df^{kl}(a), Df^{kl}(b)) &\leq M\|Df^{kl}(a)\|\|Df^{kl}(b)\| \\ &< 2^{1-k}M\|Df^{kl}(a)\|, \end{aligned}$$

and from here we find that $\|Df^{kl}(a)\| > 2^{k-1}\frac{m}{M}$. But we can take k arbitrarily large so $2^{k-1}\frac{m}{M}$ becomes larger than 2, and we get a contradiction. The proof that $\mathbf{B}$ is uniformly contracting is similar. In particular in this case we get that $\mathbf{A}$ and $\mathbf{B}$ must be Lagrangian. □

6. The main perturbation result

We say that a splitting $TM = \mathbf{A} \oplus \mathbf{B}$ has *index* k if the dimension of $\mathbf{A}$ is k.

Proposition 9. *Suppose $f \in \mathbf{Diff}^1_\omega(M)$, $x \in M$ is a hyperbolic periodic point for f and $\lambda_{i+1}(f, x) - \lambda_i(f, x) > \delta > 0$. Also suppose that f does not have an l-dominated decomposition of index $n - i$. Then there is a perturbation of f of size less than $E(l)$, say g, and $y \in M$ a hyperbolic periodic point for g arbitrarily close to x such that $\Lambda_{i+1}(g, y) < \Lambda_{i+1}(f, x) - \frac{\delta}{2}$ and $\lambda_n(g, y) \leq \lambda_n(f, x)$.*

Proof. First we remark that if f does not have an l-dominated decomposition of index $n-i$, then the same must be true for any other function in a small C^1 neighborhood of f. Indeed, if this is not true, we can find a sequence of diffeomorphisms f_n converging to f with l-dominated splittings of index $n - i$. For a subsequence the corresponding subbundles will converge to invariant subbundles for f, and by taking the limit we get that this must also be an l-dominated splitting, which is a contradiction.

The strategy is the following: first we make an arbitrarily small perturbation f_1 of f in order to create dense homoclinic points for (the continuation of) x using Lemma 6. We can choose the perturbation small enough so that the continuation of x and its new Lyapunov exponents are arbitrarily close to the old ones and there is no l-dominated splitting of index $n-i$. Let $H(x)$ be the set of homoclinic points of x which is dense in M. Then we use Lemma 7 to define the invariant subbundles $\mathbf{A}$ and $\mathbf{B}$ for the points in $H(x)$ corresponding to the Lyapunov exponents greater than or equal to $\lambda_{i+1}(f_1, x)$ (using the backward iterates), respectively smaller or equal to $\lambda_i(f_1, x)$ (using the forward iterates). If $\mathbf{A}$ l-dominates $\mathbf{B}$, then we can extend these subbundles by continuity to the whole manifold M and get an l-dominated splitting, which is a contradiction.

So we can suppose that $\mathbf{A}$ does not l-dominate $\mathbf{B}$, which means that there exist a homoclinic point p of x and unit vectors $u \in \mathbf{A}_p$, $v \in \mathbf{B}_p$ such that $\|Df_1^l p(u)\| < 2\|Df_1^l p(v)\|$. Now we can use Lemma 5 to create a perturbation f_2 of f_1 of size less than $\frac{E(l)}{2}$ moving the vector u to a multiple of $Df_1^l p(v)$. The perturbation is supported on an arbitrarily small neighborhood of the first l iterates of p, and it does not change the orbit of p, so we can suppose that the orbit and the Lyapunov exponents of x are unchanged and p is still an homoclinic point of x. Also $\mathbf{A}$ is unchanged for the backward iterates of p and $\mathbf{B}$ is unchanged for the forward iterates of $f_1^l(p) = f_2^l(p)$, so we have the vector $u \in \mathbf{A}_p \cap \mathbf{B}_p$ for f_2. Without loss of generality we will denote $f_2 = f$.

In order to finish the proof, we want to close the orbit of p with another arbitrarily small perturbation supported on a small neighborhood of x and thus get the desired hyperbolic periodic orbit with smaller Λ_{i+1}. In the end we have a perturbation of size $\frac{E(l)}{2}$ and finitely many arbitrarily small perturbations, so the total the size of the perturbation will be smaller than $E(l)$.

The next lemma shows how to close the orbit of the homoclinic point p of x.

Lemma 10. *For any p a homoclinic point for the periodic hyperbolic point x of $f \in \mathbf{Diff}^r(M)$ (or $\mathbf{Diff}^r_\omega(M)$, where ω is either a symplectic or a volume form), there exist an arbitrarily small C^r perturbation of f in $\mathbf{Diff}^r(M)$ (or $\mathbf{Diff}^r_\omega(M)$) supported in an arbitrarily small neighborhood of x such that p becomes a periodic point.*

Proof. Let V be a neighborhood of f in $\mathbf{Diff}^r(M)$ (or $\mathbf{Diff}^r_\omega(M)$). We know that there is a small neighborhood U of x, where f^k is C^0-conjugated to the linear map $Df^k(x)$ on a neighborhood of the origin, where k is the period of x. By eventually shrinking the neighborhood we can suppose that the orbit of p intersected with U is inside the ϵ-stable and ϵ-unstable manifolds of x for some small $\epsilon > 0$. Let $y = f^s(p), z = f^{-t}(p), s, t > 0$, be a forward and a backward iterate of p contained in U. Using the perturbation lemma we can find small neighborhoods $U_y \subset \hat{U}_y \subset U$ of y such that $\hat{U}_y \cap f(\hat{U}_y) = \emptyset$, and for any $y' \in U_y$ there exist $f' \in V$ with $f'(y) = f(y')$ and $f' = f$ outside $\hat{U}_y$. In the same way we can find neighborhoods $U_z \subset \hat{U}_z \subset U$ of z using f^{-1}. Now because of the conjugacy to the linear map there is a point $y' \in U_y$ such that a forward iterate of y', say $f^a(y')$, is in U_z. Now let $g \in V$ be a function such that $g(y) = f(y'), g(f^{a-1}(y')) = z$ and $g = f$ outside $\hat{U}_y \cup f^{-1}(\hat{U}_z)$ as before. Then p is a periodic point for g. Obviously U can be taken arbitrarily small. $\square$

We observe that the perturbation we made consists basically of two translations on small neighborhoods of a positive and a negative iterate of p. Also the new orbit of p spends an arbitrarily long time in the neighborhood of x because we can take as many iterates of p (positive and negative) as we want before making the required translations. So the period of p for the perturbation g_N will be $2N + k$, where k is the number of iterates away from x (which is fixed) and $2N$ is the number of iterates close to x (which can be arbitrarily large). By changing the notation we can suppose that $g_N^j(p), -N \le j \le N-1$, are the iterates of p close to x. Furthermore for some $s, t < N$ we have that $g_N^s(p), g_N^{s+1}(p), \ldots, g_N^{2N+k-t}(p)$ is a segment of the orbit of the homoclinic point for f. We remark that p actually depends on N, but $g_N^N(p)$ and $g_N^{-N}(p)$ do not. Because $g_N^j(p), -t \le j \le s$, are in an

arbitrarily small neighborhood and f is C^1, using another small perturbation we can suppose that $Dg_N(g_N^j(p)) = Df(f^{j-s}(g_N^s(p))), 0 \le j \le s$, and $Dg_N(g_N^{-j}(p)) = Df(f^{t-j}(g_N^{-t}(p))), 1 \le j \le t$ (using the coordinate chart around x). In other words, for N positive iterates of p under g_N the derivative is the same as for N iterates of the homoclinic point under f situated on a local unstable manifold of x and similarly for the negative iterates. Because of this we can define $\mathbf{A}_{g_N^j(p)}, 0 \le j \le s$, as the pulled back of $\mathbf{A}_{g_N^s(p)}$ and $\mathbf{B}_{g_N^{-j}(p)}, 0 \le j \le t$, as the pushed forward of $\mathbf{B}_{g_N^{-t}(p)}$ under Dg_N, and we have the relations

$$-\lambda_n(f,x) \le \lim_{N\to\infty} \frac{\log \|Dg^{-N}_{N,g_N^N(p)}(v)\|}{N} \le -\lambda_{i+1}(f,x), \forall v \in \mathbf{A}_{g_N^N(p)},$$
$$-\lambda_n(f,x) \le \lim_{N\to\infty} \frac{\log \|Dg^{N}_{N,g_N^{-N}(p)}(u)\|}{N} \le \lambda_i(f,x), \forall u \in \mathbf{B}_{g_N^{-N}(p)}.$$

Also if $v_n, v_{n-1}, \dots, v_{i+1}$ is an orthonormal basis for $\mathbf{A}_{g_N^N(p)}$ we have

$$\lim_{N\to\infty} \frac{\log \| \bigwedge^{n-i} Dg^{-N}_{N,g_N^N(p)}(v_n \wedge \cdots \wedge v_{i+1})\|}{N} = -\Lambda_{i+1}(f,x).$$

Now we will estimate $\Lambda_{i+1}(g_N,p)$. We know that

$$\Lambda_{i+1}(g_N,p) \le \frac{\log \| \bigwedge^{n-i} Dg^{2N+k}_{N,p}\|}{2N+k}.$$

We want to choose a convenient basis $\{e_{-n}, e_{-n+1}, \dots, e_{-1}, e_1, \dots, e_n\}$ of T_pM such that e_j expands on the first N iterates of Dg_N with an exponential rate close to $\lambda_j(f,x)$, and consequently $e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}}$ expands on the first iterates of Dg_N with an exponential rate not much bigger than $\lambda_{j_1}(f,x)+\lambda_{j_2}(f,x)+\cdots+\lambda_{j_{n-i}}(f,x)$ (the reason we get an estimation only from above for the volume growth is because the angles between vectors may decrease, but that is all that we actually need). For simplicity from now on we denote $\lambda_j = \lambda_j(x,f)$.

We start by supposing that the Lyapunov subspaces at x are orthogonal (this can be done by a change of coordinates; a change of basis does not change the eigenvalues or the Lyapunov exponents). Then we choose an orthonormal basis $\{e'_{-n}, e'_{-n+1}, \dots, e_{-1}, e_1, \dots, e'_n\}$ of $T_{g_N^N(p)}M$ such that it agrees with the backward Lyapunov filtration for f:

$$\lim_{l\to\infty} \frac{\log \|Df^{-l}(e'_j)\|}{-l} = \lambda_j \text{ and}$$
$$\lim_{l\to\infty} \frac{\log \| \bigwedge^{n-j} Df^{-l}(e'_n \wedge e'_{n-1} \wedge \cdots \wedge e'_{j+1})\|}{-l} = \lambda_n + \cdots + \lambda_{j+1} = \Lambda_{j+1}$$

if $\lambda_{j+1} > \lambda_j$ (this can be done using Lemma 7). Also if $e'_j, \dots, e'_k$ are the vectors corresponding to a (possible multiple) exponent, then the pullback under Df of the subspace generated by this vector will converge to the corresponding Lyapunov subspace at x. Now if a Lyapunov exponent λ_j is simple, then we take $e_j = \frac{Dg_N^{-N}(e'_j)}{\|Dg_N^{-N}(e'_j)\|}$. If $\lambda_{j-1} < \lambda_j = \cdots = \lambda_k < \lambda_{k+1}$, then we take $\{e_j, e_{j+1}, \dots, e_k\}$ to be an orthonormal basis of $Dg_N^{-N}(\text{span}\{e'_j, \dots, e'_k\})$. Then this basis of T_pM is 'almost' orthonormal, $\langle e_j, e_k\rangle$ is small for large N, and satisfies our requirements.

Now $\{e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}} \colon -n \leq j_1 < j_2 < \cdots < j_{n-i} \leq n\}$ is a basis for $\bigwedge^{n-i} T_pM$ which is again 'almost' orthonormal (for the dot product induced by the one on T_pM), so it is enough to estimate $\| \bigwedge^{n-i} Dg_N^{2N+k}(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}})\|$. Suppose now that we have $\epsilon > 0$ fixed.

First case: $e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}} \neq e_{i+1} \wedge e_{i+2} \wedge \cdots \wedge e_n$. Then for any ϵ, for large enough N we have

$$\log \| \bigwedge^{n-i} Dg_N^N(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}})\| < N(\lambda_{j_1} + \cdots + \lambda_{j_{n-i}} + \epsilon) \leq N(\Lambda_{i+1} - \delta + \epsilon).$$

So

$$\log \| \bigwedge^{n-i} Dg_N^{N+k}(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}})\| < N(\Lambda_{i+1} - \delta + \epsilon) + K,$$

where K is a constant which does not depend on N.

In order to evaluate the last N iterates we consider an orthonormal basis $\{\bar{e}_{-n}, \bar{e}_{-n+1}, \ldots, \bar{e}_{-1}, \bar{e}_1, \ldots, \bar{e}_n\}$ of $T_{g_N^{-N}(p)}M$ which agrees with the forward Lyapunov filtration for f: $\lim_{l\to\infty} \frac{\log \|Df^l(\bar{e}_j)\|}{l} = \lambda_j$ (this can be done again using Lemma 7). Then we get that

$$\lim_{l\to\infty} \frac{\log \| \bigwedge^{n-i} Df^l(\bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-i}})\|}{l} \leq \lambda_{j_1} + \cdots + \lambda_{j_{n-i}} \leq \Lambda_{i+1}.$$

So for large enough N we have $\| \bigwedge^{n-i} Dg_N^N(\bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-i}})\| < \mathrm{e}^{N(\Lambda_{i+1}+\epsilon)}$ for any $-n \leq j_1 < j_2 < \cdots < j_{n-i} \leq n$. But $\{\bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-i}} \colon -n \leq j_1 < j_2 < \cdots < j_{n-i} \leq n\}$ is again an orthonormal basis of $T_{g_N^{-N}(p)}M$ which has dimension $\binom{n}{n-i}$, so $\| \bigwedge^{n-i} Dg_N^N|_{T_{g_N^{-N}(p)}}\| < \binom{n}{n-i}\mathrm{e}^{N(\Lambda_{i+1}+\epsilon)}$. Taking the log and combining with the inequality for the first $N+k$ iterates we get

$$\log \| \bigwedge^{n-i} Dg_N^{2N+k}(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}})\| < 2N(\Lambda_{i+1} - \frac{\delta}{2} + \epsilon) + K'$$

for all large enough N, where again K' does not depend on N. Dividing by $2N+k$ and taking N sufficiently large we get

$$\frac{\log \| \bigwedge^{n-i} Dg_N^{2N+k}(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}})\|}{2N+k} < \Lambda_{i+1} - \frac{\delta}{2} + \epsilon$$

if $e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}} \neq e_{i+1} \wedge e_{i+2} \wedge \cdots \wedge e_n$.

Second case: the estimation for $e_{i+1} \wedge e_{i+2} \wedge \cdots \wedge e_n$. As in the first case we get for large enough N that

$$\log \| \bigwedge^{n-i} Dg_N^{N+k}(e_{i+1} \wedge e_{i+2} \wedge \cdots \wedge e_n)\| < N(\Lambda_{i+1} + \epsilon) + K.$$

We know that there is a unit vector $u \in \mathrm{span}\{e_{i+1}, e_{i+2}, \ldots, e_n\} = \mathbf{A}_p^{i+1}$ such that

$$Dg_N^{N+k}(u) \in \mathbf{B}_{g_N^{-N}}(p) = \mathrm{span}\{\bar{e}_{-n}, \bar{e}_{-n+1}, \ldots, \bar{e}_i\}.$$

This implies that $Dg_N^{N+k}(e_{i+1} \wedge e_{i+2} \wedge \cdots \wedge e_n)$ is inside the subspace of $\bigwedge^{n-i} T_{g_N^{-N}}(p)M$ generated by all $\bar{e}_{j_1} \wedge \bar{e}_{j_2} \wedge \cdots \wedge \bar{e}_{j_{n-i}}$ different from $\bar{e}_{i+1} \wedge \bar{e}_{i+2} \wedge \cdots \wedge \bar{e}_n$. In the same way as for the first case we get that the norm of $\bigwedge^{n-i} Dg_N^N$

restricted to this subspace is bounded from above by $\binom{n}{n-i}e^{N(\Lambda_{i+1}-\delta+\epsilon)}$ for large N. Again combining and taking N sufficiently large we get

$$\frac{\log \| \bigwedge^{n-i} Dg_N^{2N+k}(e_{i+1} \wedge e_{i+2} \wedge \cdots \wedge e_n)\|}{2N+k} < \Lambda_{i+1} - \frac{\delta}{2} + \epsilon.$$

To conclude, we have the inequality for all the elements of the basis, which is 'almost' orthonormal, so by taking N big enough we get the inequality for any unit vector (in $\bigwedge^{n-i} T_pM$):

$$\frac{\log \| \bigwedge^{n-i} Dg_N^{2N+k}\|}{2N+k} < \Lambda_{i+1} - \frac{\delta}{2} + 2\epsilon$$

which implies that $\Lambda_{i+1}(g_N, p) < \Lambda_{i+1}(f, x) - \frac{\delta}{2} + 2\epsilon$. Now we can do the same thing for a slightly larger δ and arbitrarily small ϵ, so eventually we get $\Lambda_{i+1}(g_N, p) < \Lambda_{i+1}(f, x) - \frac{\delta}{2}$.

The proof of the second inequality of the proposition is similar, but we do not have to use two separate cases. It is enough to remark that the exponential rate of growth of any vector is bounded from above by λ_n around x, so it is enough to take N large, and we get $\lambda_n(g_N, p) < \lambda_n(f, x) + \epsilon$. But again ϵ can be chosen arbitrarily small, so with another small perturbation we get $\lambda_n(g_N, p) \leq \lambda_n(f, x)$.

As a last remark, because the period of p is arbitrarily large we can easily make sure that the point p is actually hyperbolic, so it does not have 0 Lyapunov exponents. □

Corollary 11. *Suppose $f \in \mathbf{Diff}^1_\omega(M)$ does not have a dominated decomposition of index $n-i$. Then there is an arbitrarily small C^1 perturbation g of f such that g has a periodic point x with $\lambda_i = \lambda_{i+1}$.*

Proof. We denote by $\mathrm{Per}(g)$ the set of hyperbolic periodic points of g. Suppose that the result is not true, so there exist a C^1 neighborhood V of f and a $\delta > 0$ such that for any function $g \in V$ and any $x \in \mathrm{Per}(g)$ we have $\lambda_{i+1}(g, x) - \lambda_i(g, x) > \delta$. We also let $\Lambda_{i+1}(g) = \inf_{x \in \mathrm{Per}(g)} \Lambda_{i+1}(g, x)$, $\Lambda = \liminf_{g \to f} \Lambda_{i+1}(g)$. This implies that there exist $g_n \to f$, $x_n \in \mathrm{Per}(g_n)$ such that $\Lambda_{i+1}(g_n, x_n) \to \Lambda$. We can suppose that $g_n \in V$ for all $n > 0$.

We will denote d to be the C^1 distance on $\mathbf{Diff}^1_\omega(M)$ using the fixed symplectic charts. Now for any $l \in \mathbb{N}$ there exist $n_l > l$ such that g_{n_l} does not have an l-dominated splitting of index $n-i$ (otherwise we can pass the decomposition to the limit and get one for f). We also know that $\lambda_{i+1}(g_{n_l}, x_{n_l}) - \lambda_i(g_{n_l}, x_{n_l}) > \delta$, so we can apply the proposition to find $h_l \in \mathbf{Diff}^1_\omega(M)$, $d(h_l, g_{n_l}) < E(l)$, $y_l \in \mathrm{Per}(h_l)$ such that $\Lambda_{i+1}(h_l, y_l) < \Lambda_{i+1}(g_{n_l}, x_{n_l}) - \frac{\delta}{2}$. Because $E(l) \to 0$ as $l \to \infty$ we get that $h_l \to f$. Also

$$\lim_{l\to\infty} \Lambda_{i+1}(h_l) \leq \lim_{l\to\infty} \Lambda_{i+1}(h_l, y_l) \leq \lim_{l\to\infty} \Lambda_{i+1}(g_{n_l}, x_{n_l}) - \frac{\delta}{2} = \Lambda - \frac{\delta}{2},$$

which is a contradiction. □

Remark. One case in which $\lambda_i = \lambda_{i+1}$ is when the corresponding eigenvalues are complex conjugate. Periodic points of this type are used in [11] to construct sinks or sources in the case of general diffeomorphisms or periodic points with all the eigenvalues of modulus 1 for volume-preserving diffeomorphisms in the absence of dominance.

7. Proof of Theorem 2

Now we will give the proof of Theorem 2.

Proof of Theorem 2. We will use the proposition to prove that for any open set U in M we can find an arbitrarily small perturbation of f with an elliptic periodic point in U.

So let us fix an open set U in M. Lemma 8 shows that for symplectic diffeomorphisms the existence of a dominated splitting is equivalent to partial hyperbolicity, so we know that there are no dominated splittings for f. We define the decreasing function $L : \mathbb{R}_+ \to \mathbb{N}$ as follows: for any $\epsilon > 0$ we let $L(\epsilon)$ be the largest integer such that all the perturbations of f of size at most ϵ do not have a $L(\epsilon)$-dominated decomposition. Because for any $l > 0$ there are no sequences of diffeomorphisms with l-dominated splittings converging to f, there is a neighborhood V_l of f such that no function in V_l has an l-dominated splitting. This proves that $\lim_{\epsilon\to 0} L(\epsilon) = \infty$.

For any $g \in \mathbf{Diff}^1_\omega(M)$ we define $\lambda_n(g) = \inf_{x\in \mathrm{Per}(g)\cap U}\lambda_n(g, x)$ and $\lambda = \liminf_{g\to f}\lambda_n(g)$. Because of the C^1 closing lemma λ must be finite. If $\lambda = 0$ we are done. An arbitrarily small perturbation will make all the Lyapunov exponents of a periodic point zero, and consequently we get an elliptic periodic point (making sure there are no multiple eigenvalues). So we can suppose that $\lambda > 0$.

There exist $f_k \to f$ (in the C^1 topology), $x_k \in \mathrm{Per}(f_k) \cap U$, such that $\lambda^k_n := \lambda_n(f_k, x_k) \to \lambda$. Our goal is to use the proposition several times to construct a sequence of perturbations g_k, still converging to f, and having some periodic points $y_k \in \mathrm{Per}(g_k) \cap U$ with $\lim_{k\to\infty} \lambda_n(g_k, y_k) \le (1 - \alpha)\lambda$, which is a contradiction. We will choose $\alpha > 0$ later.

Suppose that $d(f, f_k) = \epsilon_k$ for $\epsilon_k > 0$ small. This means that f_k does not have any $L(\epsilon_k)$-dominated splitting. We denote $f_k = f_{k1}, x_k = x_{k1}$. There exists an $i_1 \in \{-1, 1, 2, \dots, n - 1\}$ such that

$$\lambda_{i_1+1}(f_{k1}, x_{k1}) - \lambda_{i_1}(f_{k1}, x_{k1}) < \frac{\lambda^k_n}{n} = \delta_1.$$

Applying the proposition we can construct $f_{k2} \in \mathbf{Diff}^1_\omega(M)$, $d(f_{k2}, f_{k1}) < EL(\epsilon_k)$, with $x_{k2} \in \mathrm{Per}(f_{k2}) \cap U$ such that

$$\Lambda_{i_1+1}(f_{k2}, x_{k2}) < \Lambda_{i_1+1}(f_{k1}, x_{k1}) - \frac{\delta_1}{2},$$
$$\lambda_n(f_{k2}, x_{k2}) \le \lambda^k_n.$$

Also

$$d(f_{k2}, f) \le d(f_{k2}, f_{k1}) + d(f_k, f) < \epsilon_k + EL(\epsilon_k).$$

We will denote $\phi(\epsilon) = \epsilon + EL(\epsilon)$ and then we can rewrite this as $d(f_{k2}, f) < \phi(\epsilon_k)$.

If $\lambda_n(f_{k2}, x_{k2}) \le (1 - \alpha)\lambda^k_n$, then we stop and take $g_k = f_{k2}$ and $y_k = x_{k2}$. Otherwise there is $j_2 \in \{i_1 + 1, i_1 + 2, \dots, n - 1\}$ such that

$$\lambda_{j_2}(f_{k2}, x_{k2}) < \lambda_{j_2}(f_{k1}, x_{k1}) - \frac{\delta_1}{2n} \le \lambda^k_n - \frac{\delta_1}{2n}.$$

So

$$\lambda^k_n \ge \lambda_n(f_{k2}, x_{k2}) > (1 - \alpha)\lambda^k_n > \lambda^k_n - \frac{\delta_1}{2n} > \lambda_{j_2}(f_{k2}, x_{k2})$$

if $\alpha < \frac{1}{2n^2}$ (remember that $\delta_1 = \frac{\lambda_n^k}{n}$). This implies that there exists $i_2 \in \{j_2, j_2 + 1, \ldots, n-1\}$ such that

$$\lambda_{i_2+1}(f_{k2}, x_{k2}) - \lambda_{i_2}(f_{k2}, x_{k2}) > \frac{\delta_1}{2n^2} - \frac{\alpha\lambda_n^k}{n} = \frac{\lambda_n^k}{2n^3} - \frac{\alpha\lambda_n^k}{n} = \delta_2.$$

Because $d(f_{k2}, f) < \phi(\epsilon_k)$ we get that f_{k2} has no $L(\phi(\epsilon_k))$-dominance. Again applying the proposition we get that there is an $f_{k3} \in \mathbf{Diff}^1_\omega(M)$, with $x_{k3} \in \mathrm{Per}(f_{k3}) \cap U$ such that

$$\Lambda_{i_2+1}(f_{k3}, x_{k3}) < \Lambda_{i_2+1}(f_{k2}, x_{k2}) - \frac{\delta_2}{2},$$
$$\lambda_n(f_{k3}, x_{k3}) \leq \lambda_n(f_{k2}, x_{k2}) \leq \lambda_n^k$$

and $d(f_{k3}, f_{k2}) < EL(\phi(\epsilon_k))$. We observe again that

$$d(f_{k3}, f) \leq d(f_{k3}, f_{k2}) + d(f_{k2}, f) < \phi(\epsilon_k) + EL(\phi(\epsilon_k)) = \phi^2(\epsilon_k).$$

Again if $\lambda_n(f_{k3}, x_{k3}) \leq (1-\alpha)\lambda_n^k$, then we let $g_k = f_{k3}$ and $y_k = x_{k3}$, and we stop. Otherwise, again under the condition that α is sufficiently small, there will be a gap of size at least $\delta_3 = \frac{\delta_2}{2n^2} - \frac{\alpha\lambda_n^k}{n}$ between $\lambda_{i_3+1}(f_{k3}, x_{k3})$ and $\lambda_{i_3}(f_{k3}, x_{k3})$ for some $i_3 > i_2$, and we again apply the proposition to lower Λ_{i_3+1} by at least $\frac{\delta_3}{2}$ for a perturbation f_{k4} and a hyperbolic periodic point x_{k4} in U. The distance from f_{k4} to f will be less than $\phi^3(\epsilon_k)$. We can repeat this argument, and in the end α can be chosen sufficiently small (depends only on n) such that after a finite number of such perturbations (at most n) we actually lower λ_n by $\alpha\lambda_n^k$. So we can indeed find $g_k \in \mathbf{Diff}^1_\omega(M)$, $d(g_k, f) < \phi^n(\epsilon_k)$ where $\epsilon_k = d(f_k, f) \to 0$, and $y_k \in \mathrm{Per}(g_k) \cap U$ such that $\lambda_n(g_k, y_k) < (1-\alpha)\lambda_n^k$. Then

$$\lim_{k\to\infty} \lambda_n(g_k, y_k) \leq \lim_{k\to\infty} (1-\alpha)\lambda_n^k = (1-\alpha)\lambda.$$

We also know that $\lim_{\epsilon\to 0} L(\epsilon) = \infty, \lim_{l\to\infty} E(l) = 0$ so $\lim_{\epsilon\to 0} \phi(\epsilon) = 0$ and furthermore $\lim_{\epsilon\to 0} \phi^n(\epsilon) = 0$ which shows that g_k converges to f in $\mathbf{Diff}^1_\omega(M)$, and we are done because we reached a contradiction. □

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