THE 3-MANIFOLD RECOGNITION PROBLEM

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Abstract. We introduce a natural Relative Simplicial Approximation Property for maps from a 2-cell to a generalized 3-manifold and prove that, modulo the Poincaré Conjecture, 3-manifolds are precisely the generalized 3-manifolds satisfying this approximation property. The central technical result establishes that every generalized 3-manifold with this Relative Simplicial Approximation Property is the cell-like image of some generalized 3-manifold having just a 0-dimensional set of nonmanifold singularities.

1. Introduction

The manifold recognition problem, originally proposed in 1978 by J. W. Cannon [9], asks for a short list of simple topological properties, easy to check, that characterize topological manifolds among topological spaces. Cannon conjectured that n-manifolds might be characterized as those generalized n-manifolds satisfying a minimal amount of general position. To address the latter in dimensions greater than 4 he proposed the following Disjoint Disks Property: any two maps of \( B^2 \) into the space can be approximated by maps with disjoint images.

This paper addresses the 3-manifold recognition problem. For that dimension the fundamental difficulty is to identify an appropriate general position property. The Disjoint Disks Property, possessed by no 3-manifold, is impossibly strong, and the related Disjoint Arcs Property, possessed by all generalized 3-manifolds, is impossibly weak.

A generalized \( n \)-manifold \( X \), abbreviated as \( n\text{-gm} \), is a locally compact, locally contractible, finite dimensional metric space with the relative local homology of \( \mathbb{R}^n \) (i.e., \( H_*(X, X - \{x\}; \mathbb{Z}) \) is isomorphic to \( H_*(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z}) \) for all \( x \in X \)). In such a space \( X \) the manifold set, \( M(X) \), consists of all points of \( X \) having a neighborhood homeomorphic to \( \mathbb{R}^n \), and the singular set, or nonmanifold set, \( S(X) \), is defined as \( S(X) = X - M(X) \). As components of locally compact metric spaces are separable, we simply will view all \( n\text{-gms} \) as separable metric spaces.

Clearly every \( n \)-manifold is an \( n\text{-gm} \), but the converse fails for \( n > 2 \). If \( f: M \to X \) is a proper, cell-like, surjective mapping defined on an \( n \)-manifold, where \( \dim X < \infty \), then \( X \) is an \( n \)-gm, and classical examples like the famous dog-bone space of R. H. Bing [3] demonstrate that \( X \) need not be a genuine manifold. Historically cell-like maps like Bing’s have been used to produce a large class of
examples. To distinguish such images from other possible examples that arise, one calls an $n$-gm $X$ resolvable if there exist a genuine $n$-manifold $M$ and a proper, cell-like, surjective mapping $f: M \to X$. In this case, the pair $(M, f)$ is called a resolution of $X$. Bryant, Ferry, Mio and Weinberger have established the existence of nonresolvable $n$-gms for $n > 5$ [7].

In dimensions greater than 4 the model theorem is provided by the combination of results by Edwards and Quinn. Given a connected $n$-gm $X$, Quinn [19] produced an integer valued obstruction, $i(X)$, which is locally defined, locally constant, and satisfies $i(X \times X') = i(X) \times i(X')$, where $i(X) = 1$ if and only if $X$ is resolvable $(n > 3)$. Edwards [13] (see [11] for details) showed that a resolvable $n$-gm, $n > 4$, is an $n$-manifold if and only if it satisfies the Disjoint Disks Property. Consequently, for $n > 4$ a connected space $X$ is an $n$-manifold if and only if $X$ is an $n$-gm satisfying both the Disjoint Disks Property and $i(X) = 1$.

Daverman and Repovš [12] introduced a kind of general position property—called the spherical simplicial approximation property, abbreviated as SSAP, and defined in Section 4—and showed that every resolvable generalized 3-manifold with the SSAP is a 3-manifold. Here we modify their property, defining a relative simplicial approximation property (RSAP) which is stronger than this SSAP; our main result establishes that, modulo the Poincaré Conjecture, every generalized 3-manifold $X$ satisfying this RSAP is a 3-manifold. Specifically, the fundamental issue is to confirm that $X$ is resolvable, for then [12] applies to give the final 3-manifold recognition step. With no extra hypotheses we produce a cell-like, surjective mapping $\Phi: Y \to X$, where $Y$ is a 3-gm such that $S(Y)$ is 0-dimensional. If the Poincaré Conjecture is true, however, then the Corollary to the Resolution Theorem of [22] (see [23] for corrections) assures that $Y$ has a resolution $\Psi: M \to Y$, and $\Phi \Psi: M \to X$ serves as the desired resolution of $X$.

2. Preliminaries

A subset $C$ of a space $X$ is locally $k$-cocentered, abbreviated as $k$-LCC, if each neighborhood $U$ of an arbitrary point $x \in X$ contains another neighborhood $V$ of $x$ such that every map $\partial I^{k+1} \to V - C$ can be extended to a map $I^{k+1} \to U - C$.

We shall distinguish simplicial complexes from their underlying point sets, called polyhedra. A triangulation of a polyhedron $Q$ is a pair $(T, h)$, where $T$ denotes a simplicial complex and $h$ a homeomorphism of its underlying point set, denoted by $|T|$, onto $Q$. Frequently the polyhedra encountered here will be subsets of a given 3-gm. One should not presume the existence of any compatibility between the (piecewise) linear structure of the simplicial complex associated to a polyhedron $Q$ in a 3-gm $X$ and the possible linear structures arising within $X$. Most of our attention will fall on 2-dimensional polyhedra, called 2-polyhedra for short.

A subpolyhedron $Q'$ of a polyhedron $Q$ is a closed subset of $Q$ such that there exist a triangulation $(T, h)$ of $Q$ and a subcomplex $T'$ of $T$ with $h(|T'|) = Q'$.

Suppose $Q$ is a polyhedron and $z \in Q$. Impose a triangulation $(T, h)$ on $Q$. Suppress $h$, here and throughout the remainder of this paper, and regard $T$ as a simplicial complex whose underlying point set equals $Q$. Subdivide $T$, if necessary, so that $z$ corresponds to a vertex of $T$. For such a $Q$ topologically embedded as a closed subset of a generalized 3-manifold $X$, $X - Q$ is said to have free local fundamental group at $z \in Q$, abbreviated as 1-FLG at $z$, if for each sufficiently small neighborhood $U$ of $z$ there exists another neighborhood $V$ of $z$ with $z \in V \subset U$.
and if \( W \) is any connected open set with \( z \in W \subset V \), then for each nonempty component \( W' \) of \( W - Q \) the (inclusion-induced) image \( \pi_1(W') \to \pi_1(U') \) is a free group on \( m - 1 \) generators, where \( U' \) denotes the component of \( U - Q \) containing \( W' \) and \( m \) is the number of “components” of \( \text{St}(z) - z \) whose images meet \( C(\text{St}(W')) \), where \( \text{St}(z) \) denotes the simplicial star of \( z \) in the complex \( T \). As usual, \( X - Q \) is simply said to be 1-FLG in \( X \) if it is 1-FLG in \( X \) at each point of \( Q \).

For simplicity, we will say that a polyhedron \( Q \) embedded in a generalized 3-manifold \( X \) as a closed subset is tamely embedded in \( X \) if \( X - Q \) is 1-FLG in \( X \). Nicholson [18] has shown that a polyhedron tamely embedded in a genuine 3-manifold \( M \) in this 1-FLG sense is tamely embedded in the geometric sense, where there exists a self-homeomorphism (arbitrarily close to identity) \( M \to M \) of \( M \) that carries \( Q \) onto a subspace underlying a subcomplex of some preassigned triangulation of \( M \), after subdivision.

Given maps \( \phi : Y \to X \) and \( f : Z \to X \), where \( X \) is metrizable, and given \( A \subset Y \), we say that \( f \) approximately lifts to \( A \) (occasionally, for emphasis, under \( \phi \)) if for each metric on \( X \) and each \( \epsilon > 0 \) there exists a map \( \tilde{f} : Z \to A \) such that \( \phi \tilde{f} \) is within \( \epsilon \) (pointwise) of \( f \).

Suppose \( X \) is a connected 3-gm, \( D \) and \( E \) are disjoint, closed subspaces of \( X \), and \( \mu : R \to X - (D \cup E) \) is a map defined on a compact, 2-polyhedron \( R \). We say that \( \mu \) homologically separates \( D \) from \( E \) if there exist \( \alpha \in H_2(R; \mathbb{Z}_2) \) and \( \xi \in H_3(X - E, X - (D \cup E); \mathbb{Z}_2) \) such that, for each \( d \in D \), \( \mu_*(\alpha) = i_*\partial(\xi) \neq 0 \), where \( i \) denotes the inclusion \( X - (D \cup E) \to X - (\{d\} \cup E) \). We say that \( \mu \) strongly separates \( D \) and \( E \) if no component of \( X - \mu(R) \) contains points of both \( D \) and \( E \).

A compact subset \( C \) of any ANR \( Y \) is cell-like if, for each open subset \( U \) of \( Y \) containing \( C \), the inclusion \( C \to U \) is homotopic to a constant. A proper map \( f : Y \to Z \) defined on an ANR \( Y \) is a cell-like map if each \( f^{-1}(z) \), \( z \in Z \), is cell-like. We say that a cell-like map \( f : Y \to Z \) is conservative over \( B \subset Z \) if \( f|f^{-1}(B) \) is 1-1.

Similarly, a compact subset \( C \) of an \( n \)-manifold \( M \) is cellular if \( M \) contains a sequence \( \{D_i\}_{i=1}^\infty \) of \( n \)-cells such that \( \text{Int}(D_i) \supset D_{i+1} \) and \( \bigcap_{i=1}^\infty D_i = C \), and a proper map \( F : M \to Z \) defined on \( M \) is cellular if each \( F^{-1}(z) \) is.

As in [22] a 3-near manifold \( M^* \) is a 3-gm obtained from a 3-manifold \( M \) by identifying a null sequence of pairwise disjoint 3-cells in \( M \) and replacing the interior of each with the interior of a compact, contractible 3-manifold in such a way that \( S(M^*) \) is 0-dimensional and 1-LCC embedded in \( M^* \). A near resolution of a 3-gm \( X \) is a pair \( (M^*, \psi) \), where \( M^* \) is a 3-near manifold and \( \psi : M^* \to X \) is a proper, cell-like surjection. Should the Poincaré Conjecture be false, one could easily produce a 3-near manifold \( M^* \) which is nonresolvable, homotopy equivalent to \( S^3 \), has \( S(M^*) = \text{point} \), and satisfies \( M^* \times \mathbb{R} \cong S^3 \times \mathbb{R} \).

3. Elementary properties of 3-gms and 3-near manifolds

A generalized 3-manifold with boundary \( Z \) is a locally compact, locally contractible, finite dimensional metric space such that, for each \( z \in Z \), either \( H_*(Z, Z - \{z\}; \mathbb{Z}) \cong 0 \) or \( H_*(Z, Z - \{z\}; \mathbb{Z}) \cong H_*(\mathbb{R}^3, \mathbb{R}^3 - \{0\}; \mathbb{Z}) \); the subset consisting of all \( z \in Z \) for which \( H_*(Z, Z - \{z\}; \mathbb{Z}) \cong 0 \) is called the boundary of \( Z \), denoted \( \partial Z \).

Lemma 3.1. If the space \( Z \) is expressed as a union of closed subsets \( Z_1 \) and \( Z_2 \) of \( Z \) which are generalized 3-manifolds with boundary, where \( Z_1 \cap Z_2 = \partial Z_1 = \partial Z_2 \),
then $Z$ is a generalized 3-manifold. Conversely, if $Z$ is a 3-gm and $Z = Z_1 \cup Z_2$, where $Z_1$, $Z_2$ are closed subsets of $Z$ and $Z_0 = Z_1 \cap Z_2$ is a 2-gm, then $Z_1$ and $Z_2$ are generalized 3-manifolds with boundary.

Proof. For the most part—except for ANR properties—this is treated in [20]. When $Z_1$ and $Z_2$ are 3-gms with boundary, work of Mitchell [16] combines with classical results of Wilder [24] to establish that $\partial Z_i (i = 0, 1)$ is a 2-manifold, hence an ANR, and standard results from ANR theory then yield that $Z = Z_1 \cup Z_2$ is an ANR. Similarly, in the converse, $Z_0$ is an ANR, so $Z_1$ and $Z_2$ must be ANRs as well.

Lemma 3.2. Let $\{X_i, p_{i+1,i}\}$ denote a sequence of 3-gms and cell-like maps, with inverse limit $Z$. Then $Z$ is a 3-gm, and the associated projections $q_i : Z \to X_i$ are cell-like maps.

Proof. We provide an argument only for the case in which each $X_i$ is a manifold factor, i.e., $X_i \times \mathbb{R}^k$ is a manifold (one can take $k$ to be any fixed integer greater than 1). It parallels the proof of [17] 3.9(iii)] about the inverse limit of a sequence of ANRs and cell-like maps yielding an ANR. Only this special case matters for our purposes here, because the RSAP implies $X$ contains 2-cells, so $X \times \mathbb{R}^k$ contains codimension one cells, and thus the Quinn obstruction [19] to the existence of a resolution vanishes.

Examine the related sequence $\{X_i \times \mathbb{R}^k, p_{i+1,i} \times \text{Id}\}$ of cell-like maps between manifolds. By [13] or [21] each map $p_{i+1,i} \times \text{Id}$ is a near-homeomorphism, so a result of M. Brown [6] (or see [1]) assures that the induced limiting map $q_i \times \text{Id}: Z \times \mathbb{R}^k \to X_1 \times \mathbb{R}^k$ is a near-homeomorphism. Hence, $Z \times \mathbb{R}^k$ is a $(3+k)$-manifold, and $Z$, being one of its codimension $k$ factors, must be a 3-gm. Furthermore, $q_1 \times \text{Id}$, being a near-homeomorphism, is a cell-like mapping [11] Theorem 17.4]; obviously this means $q_1$ itself is cell-like. $\square$

The next lemma uses the notation of Lemma 3.2, as well as the standard notation for the composite, $p_{2,1} \cdots p_{k-1,k} p_{k+1,k} = p_{k+1,1}$. The map $q_1$ is the inverse limit projection described in Lemma 3.2.

Lemma 3.3. Let $\{X_i, p_{i+1,i}\}$ denote a sequence of 3-gms and cell-like maps such that $p_{k+1,k}$ restricts to a cellular map $p_{k+1,1}(M(X_1)) \to p_{k,1}(M(X_1))$ for each $k > 0$. Then $q_1^{-1}(M(X_1))$ is a 3-manifold.

Proof. Each of the restricted $p_{k+1,k}$ is a near-homeomorphism by Armentrout’s Cellular Approximation Theorem [2], so Brown’s argument [6] applies, just as in 3.2 $\square$

Throughout the remainder of this section $\mathbb{Z}_2$ coefficients will be used for all homology and cohomology computations.

Lemma 3.4. Suppose $E$ is a nonempty, closed subset of the 3-gm $X$ and $d \in X - E$. Then there exist a compact, connected neighborhood $D$ of $d$, a compact 2-polyhedron $R$, and a map $\nu : R \to X$ such that $\nu$ homologically separates $D$ from $E$.

Proof. Note that whenever $X$ has no compact component,

$$\partial : H_3(X, X - \{x\}) \to H_2(X - \{x\})$$

is 1-1. This follows immediately, because, by duality [5], $H_3(X) \cong H_0^0(X) \cong 0$. Fix $0 \neq \xi \in H_3(X - E, X - (E \cup \{d\}))$, and assume $X$ is connected (so $X - E$
Suppose Lemma 3.5.

Let $\xi:X \to \nu$ be a map of a compact 2-polyhedron $R$ which homologically separates $D$ from $E$. Then $\nu$ strongly separates $D$ and $E$.

Proof. If $\nu(R)$ failed to separate $d_0 \in D$ and $e_0 \in E$, then there would be an arc $\gamma \subset X - \nu(R)$ connecting $d_0$ and $e_0$. By hypothesis there exist $\alpha \in H_2(R)$ and $\xi(\neq 0) \in H_3(X - E, X - (E \cup D))$ such that $\nu_*(\alpha) = i_* \partial(\xi)(\neq 0) \in H_2(X - (E \cup \{d_0\}))$. Let $\gamma$ denote the component of $\gamma - E$ containing $d_0$. Certainly here $\eta_*$ would factor through $H_3(X - E, X - (E \cup \{d_0\})) \cong H_0^0(\gamma) \cong H_0^0([0, 1/2]) \cong 0$, a contradiction.

Lemma 3.6. Let $C$ be a closed subset of a 3-manifold $M$, the frontier of which is a surface $S$. Then attachment of an open collar $S \times [0, 1)$ to $C$ along $S = S \times 0$ yields a 3-manifold.

Proof. When $M$ is a 3-sphere and $S$ is a 2-sphere this was proved by Hosay and Lininger [14] (or see [10], [8]). The general case, which localizes to that of a 2-sphere in $S^3$ [2], Theorem 5, follows.

4. A RELATIVE SIMPLICIAL APPROXIMATION PROPERTY

According to [12], a generalized 3-manifold $X$ has the Simplicial Approximation Property (SAP) if for each map $f: I^2 \to X$ and each $\epsilon > 0$, there exist a map $F: I^2 \to X$ and a compact 2-polyhedron $K_F \subset X$ such that (1) dist($F, f) < \epsilon$, (2) $F(I^2) \subset K_F$, and (3) $X - F(I^2)$ is 1-FLG in $X$. Similarly, $X$ has the Spherical Simplicial Approximation Property (SSAP) if the analogous conditions hold for maps $S^2 \to X$ in place of maps $I^2 \to X$. 

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We will say that a map \( f: K \to X \) of a compact 2-dimensional polyhedron \( K \) to a generalized 3-manifold \( X \) is simplicial if \( f(K) \) is a polyhedron whose complement is 1-FLG in \( X \) and \( f: K \to f(K) \) is simplicial with respect to some triangulations of \( K \) and \( f(K) \). Of course, given any map between polyhedra, we can impose triangulations, take fine mesh subdivisions, and then approximate by a simplicial map. In short, the map \( F \) in the SAP (similarly, in the SSAP) can be assumed to be simplicial and onto \( K_F \).

A generalized 3-manifold \( X \) has the Relative Simplicial Approximation Property (RSAP) if for each map \( f: I^2 \to X \), each compact subpolyhedron \( Q \) of \( I^2 \) for which \( f|Q \) is simplicial as above, and each \( \epsilon > 0 \), there exists a simplicial map \( F: I^2 \to X \) such that \( \text{dist}(F, f) < \epsilon \) and \( F|Q = f|Q \).

**Lemma 4.1.** Every 3-gm \( X \) that satisfies the RSAP also satisfies the following stronger property: for each compact 2-polyhedron \( K \), compact subpolyhedron \( L \), map \( g: K \to X \) such that \( g|L \) is simplicial, and \( \epsilon > 0 \), there exists a simplicial map \( G: K \to X \) with \( \text{dist}(G, g) < \epsilon \) and \( G|L = g|L \).

**Proof.** Assume for simplicity that \( X \) is path connected. List the large simplexes \( \Delta_1, \ldots, \Delta_r \) of \( L \)—large in the sense of being proper faces of no other simplexes of \( L \)—and choose any simplex \( \Delta_{r+1} \) of \( K - L \). We show how to approximate \( g \) by a new map \( g_{r+1}: K \to X \) which is simplicial on a complex underlying \( L \cup \Delta_{r+1} \).

Specify a finite collection \( \sigma_1, \ldots, \sigma_r, \sigma_{r+1} \) of pairwise disjoint simplexes in \( \text{Int}(I^2) \) and equip them with simplicial isomorphisms \( e_j: \sigma_j \to \Delta_j \) \((j = 1, \ldots, r+1)\). Define \( \eta = \bigcup e_j: \bigcup \Delta_j \to K \). Think of \( e_{r+1}^{-1}(\Delta_{r+1} \cap L) \) together with all the other \( \sigma_j \) \((j = 1, \ldots, r)\) as \( Q \subset I^2 \). Use the hypothesized path connectedness of \( X \) to extend \( g\eta|Q \) to a map \( f: I^2 \to X \). Apply the RSAP to approximate \( f: I^2 \to X \) by a simplicial map \( F: I^2 \to X \) that agrees with \( g\eta|Q \), and define \( G_{r+1}: Q \cup \Delta_{r+1} \to X \) as \( G_{r+1} = F\eta^{-1} \). Note that \( G_{r+1} \) is a well-defined simplicial map approximating \( g|L \cup \Delta_{r+1} \) and coinciding with \( g \) on \( L \). By a controlled homotopy extension lemma, \( G_{r+1} \) extends to a map \( g_{r+1}: K \to X \) approximating \( g \) and coinciding with \( g \) on \( L \).

A finite number of repetitions of this procedure yields the desired simplicial map \( G: K \to X \).

**Corollary 4.2.** Every generalized 3-manifold \( X \) satisfying the RSAP also satisfies the SSAP.

**Corollary 4.3.** All resolvable generalized 3-manifolds satisfying the RSAP are genuine 3-manifolds.

See Recognition Theorem 3.1 of [12].

**Corollary 4.4.** Suppose \( X \) is a 3-gm satisfying the RSAP, \( L \subset X \) is a tamely embedded 2-polyhedron, and \( \nu: R \to X \) is a map defined on a compact 2-polyhedron. Then for each \( \epsilon > 0 \) there exists a simplicial map \( \mu: R \to X \) with \( \text{dist}(\mu, \nu) < \epsilon \) and \( L \cup \mu(R) \) is a polyhedron tamely embedded in \( X \).

We say that a 2-polyhedron \( P \) is preferred if it contains neither isolated points nor local cut points—equivalently, if in some (hence, each) triangulation of \( P \) the link of every vertex is nonempty and connected. More is said about the role of preferred 2-polyhedra in Section 5. For brevity we call a pair \((K, P)\) of compact, 2-polyhedra in a 3-gm \( X \) a tame-preferred polyhedral pair if \( K \) is tame, \( P \) is preferred and \( P \) is a subpolyhedron of \( K \). Note that if \((K, P)\) is tame-preferred in \( X \), \( P \) is not tame—at least, not \textit{a priori} tame—in \( X \).
Lemma 4.5. Suppose $X$ is a 3-gm satisfying the RSAP and $f : I^2 \to X$ is a map such that $f$ restricts to a simplicial map on $\partial I^2$ with $f|\partial I^2 - I \times 1$ 1-1 and $f(I \times 0) \cap f(I \times 1) = \emptyset$. Then there exists a tame-preferred polyhedral pair $(K, P)$ such that $K \supset P \supset f(I \times 0)$. Furthermore, if $f(\partial I^2 - I \times 1)$ is a subpolyhedron of a compact, tame polyhedron $Q$, then $(K, P)$ can be obtained so $P \cup Q$ is a subpolyhedron of $K$.

Proof. Apply RSAP to obtain an approximation $F : I^2 \to X$ to $f$, with $F|\partial I^2 - I \times 1 = f|\partial I^2 - I \times 1$, and where $F : I^2 \to F(I^2)$ can be regarded as simplicial (also, if need be, where $F(I^2) \cup Q$ is a tame 2-polyhedron). Choose triangulations $T$ of $I^2$ and $T'$ of $F(I^2)$ for which $F$ is simplicial.

Fix a 1-simplex $\tau$ of $T'$, $\tau \subset f(I \times 0)$. We show that some 2-simplex $\sigma \in T'$ contains $\tau$. To see why, consider the unique 2-simplex $\gamma \in T$ containing $f^{-1}(\tau)$. Set $\sigma = F(\gamma)$ if $F(\gamma) \neq \tau$. Otherwise, produce a maximal chain $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_\ell$ of 2-simplices in $T$ such that $\gamma_{j-1} \cap \gamma_j \subset \partial \gamma_j \subset F^{-1}(\tau)$. Since $\partial \gamma_j - \gamma_j = \emptyset \supset \partial I^2$, some other 2-simplex $\xi$ must meet $\gamma_j$ in an edge $e = f^{-1}(\tau)$, and $F(\xi) \in T'$ will be a 2-simplex containing $\tau$.

Let $v$ be a vertex of $f(I \times 0)$ and $w, w'$ the two possible points in the link of $v$ there. Essentially the same argument shows that $w, w'$ belong to a single component of the link of $v$ in $F(I^2)$.

Although $F(I^2)$ itself might not be preferred, we claim that it contains a preferred polyhedron $P \supset f(I \times 0)$. Let $P'$ be $F(I^2)$ after deletion of (the interiors of) all those 1-simplexes $e$ of $T'$ which are edges of no 2-simplex from $T'$. Clearly then $F(I^2) \supset P' \supset f(I \times 0)$. If the vertex $w \in T'$ has disconnected link in $P'$ and $w \notin f(I \times 0)$, delete a small regular neighborhood of $w$ from $P'$; if, however, $w \in f(I \times 0)$, then delete that small neighborhood $N(w)$ but reinsert the closure of the unique component of $N(w) - \{w\}$ containing the intersection of $N(w)$ with $\text{Link}(w, f(I \times 0))$. Repetition of these two operations eliminates or repairs all disconnected links and yields a preferred polyhedron $P \subset P'$ such that $P$ and $P \cup Q$ are subpolyhedra of $K = F(I^2) \cup Q$. \hfill \Box

Lemma 4.6. Suppose $X$ is a 3-gm satisfying the RSAP and $L \subset X$ is a compact 2-polyhedron tamely embedded in $X$ such that each vertex of $L$ belongs to at least two edges. Then there exists a tame-preferred polyhedral pair $(K, P)$ in $X$ such that $L$ is a subpolyhedron of $P$.

Proof. Since components of $L$ can be treated one after another, we will simply assume $L$ is connected.

Assume $\tau$ is a 1-simplex of $L$ which belongs to no 2-simplex. In view of the hypothesis here, there is an embedding $f : \partial I^2 - I \times 1 \to L$ with $f(I \times 0) = \tau$. Since no arc locally separates a 3-gm, obviously $f$ can be extended to a map $f : I^2 \to X$ with $f(I \times 0) \cap f(I \times 1) = \emptyset$, and then Lemma 4.5 assures that $L$ can be expanded by attaching a preferred polyhedron that contains $\tau$. Repeating as often as necessary, we can simply assume each 1-simplex of $L$ is a face of some 2-simplex.

Now assume $v \in L$ is a vertex that has disconnected link in the expanded $L'$. One can build an embedding $f : \partial I^2 - I \times 1 \to L'$ with $v \in f(I \times 0)$, extend $f$ to all of $\partial I^2$, as before, and apply Lemma 4.5 to reduce the number of components of $\text{Link}(v, L')$ in the expanded $L$. This expansion can be localized to affect none of the other vertices of $L$. One can eliminate any 1-simplex contained in no 2-simplex from the expansion and snip at new vertices to prevent disconnected links, just as in
the proof of [15]. Finitely many repetitions yields a preferred polyhedron containing all of \( L \).

Let \( L \) denote a 2-polyhedron. Call \( v \in L \) a negligible vertex if there exists a homeomorphism \( \theta \) from \([0, 1]\) onto a neighborhood of \( v \) such that \( \theta(0) = v \). Note that no point of a preferred 2-polyhedron is a negligible vertex.

Essentially the same argument as in [16] proves the following.

**Lemma 4.7.** Suppose \( X \) is a 3-gm satisfying the RSAP and \( L \subset X \) is a compact 2-polyhedron tamely embedded in \( X \). Let \( L^* \) be a compact, polyhedral subset of \( L \) obtained by deleting a small connected neighborhood about each negligible vertex of \( L \). Then there exists a tame-preferred polyhedral pair \((K, P)\) in \( X \) with \( L^* \) a subpolyhedron of \( P \).

**Theorem 4.8.** Suppose \( X \) is a 3-gm satisfying the RSAP, \( D \) and \( E \) are disjoint closed subsets of \( X \), \( \nu : R \to X \) is a map defined on a compact 2-polyhedron \( R \) such that \( \nu \) homologically separates \( D \) and \( E \), and \( P \) is a preferred 2-polyhedron tamely embedded in \( X \). Then there exists a map \( \mu^* : R \to X \) such that \( \mu^* \) homologically separates \( D \) and \( E \) and there exists a tame-preferred polyhedral pair \((K^*, P^*)\) in \( X \) such that \( P^* \supset P \cup \mu^*(R) \).

**Proof.** First apply Corollary 4.4 to approximate \( \nu \) by a simplicial map \( \mu : R \to X \) so close to \( \nu \) that \( \mu \) homologically separates \( D \) and \( E \) and, in addition, \( P \cup \mu(R) \) is a 2-polyhedron. Then use Lemma 4.7 with \( L = P \cup \mu(R) \) to obtain a tame-preferred polyhedral pair \((K^*, P^*)\) in \( X \), with \( P^* \supset L^* \). Note that any negligible vertex of \( P \cup \mu(R) \) must lie in \( \mu(R) - P \), so \( P^* \supset P \). By construction the map \( \mu \), considered as a map to \( \mu(R) \subset L \), is homotopic in \( \mu(R) \) to a map \( \mu^* \) into \( L^* \). Hence, \( D \) and \( E \) are homologically separated by \( \mu^* \), and \( P \cup \mu^*(R) \subset L^* \subset P^* \).

5. The main result

The aim of this section is to establish the following Near-Resolution Theorem. It immediately yields the promised characterization of 3-manifolds as the generalized 3-manifolds satisfying the RSAP, provided the Poincaré Conjecture holds.

**Theorem 5.1** (Near-Resolution). Every generalized 3-manifold \( X \) satisfying the RSAP has a \( 3 \)-near resolution \((M, \psi)\).

**Corollary 5.2.** Suppose the Poincaré Conjecture is true. Then a generalized 3-manifold \( X \) is a genuine 3-manifold if and only if it satisfies the RSAP.

**Proof.** When \( X \) satisfies the RSAP, Theorem 5.1 certifies the existence of a cell-like, surjective map \( \psi : M \to X \) defined on a \( 3 \)-near manifold \( M \). Under the assumption that the Poincaré Conjecture is true, \( M \) actually is a 3-manifold; in other words, the promised cell-like mapping \( \psi \) itself provides a resolution of \( X \). Corollary 4.3 confirms that \( X \) is a 3-manifold.

The forward implication is trivial. \( \square \)

**Lemma 5.3** (Inflation). Suppose \( X \) is a 3-gm and \( P \subset X \) is a preferred 2-polyhedron. Then there exist a 3-gm \( Y \) and a proper, surjective, cell-like map \( \phi : Y \to X \) satisfying the following conditions:

1. \( \phi \) is conservative over \( X - P \);
2. there is a preferred 2-polyhedron \( \tilde{P} \subset M(\phi^{-1}(P)) \) for which \( \phi : \tilde{P} \to P \) is cell-like;
(3) for each (respectively, preferred) 2-polyhedron \( J \subseteq P \), there is a (respectively, preferred) 2-polyhedron \( J^* \subseteq \phi^{-1}(J) \), for which \( \phi^* : J^* \rightarrow J \) is cell-like;
(4) for each \( x \in X \), \( \phi^{-1}(x) \cap S(Y) \) is finite; and
(5) \( \phi^{-1}(M(X)) \subseteq M(Y) \) and \( \phi : \phi^{-1}(M(X)) \rightarrow M(X) \) is cellular.

Proof. We start by describing a model situation in which \( P \) is a compact, connected 2-manifold separating \( X \) into two components, \( X_+ \) and \( X_- \). Here \( FrX_+ = P = FrX_- \). Let \( Y \) be the space resulting from the disjoint union of \( ClX_- \), \( P \times [-1,1] \) and \( ClX_+ \) after identifying each \( x \in FrX_- \) with \( x \times -1 \in P \times [-1,1] \) and each \( x \in FrX_+ \) with \( x \times 1 \in P \times [-1,1] \). Define \( \phi : Y \rightarrow X \) as the obvious map induced by inclusions on the images of \( ClX_- \), \( ClX_+ \), extended to send all of \( z \times [-1,1], z \in P \), to \( z \in P \subseteq X \). Lemma 3.1 assures that \( Y \) is a generalized 3-manifold. One can check quite easily that \( \phi : Y \rightarrow X \) has all the right features. In particular, the (preferred) 2-polyhedron \( \tilde{P} \) called for in (2) can be spelled out as \( \tilde{P} = P \times \{0\} \cup P \times [-1,1] \), and the polyhedron \( J^* \) called for in (3) can be defined as

\[
J^* = \tilde{P} \cup \phi^{-1}(J - P) \cup [P \cap Cl(J - P)] \times [-1,1].
\]

Conclusion (4) is obvious. Conclusion (5) is assured by Lemma 3.6. Finally, since each point preimage is a cell, cellularity of \( \phi \) over \( M(X) \) is guaranteed here, as well as in subsequent steps, by [15, Cor. 1.4] and [11, Prop. 18.4].

Imposing a triangulation \( T \) on \( P \). Locally the same procedure as in the model case works at interiors of all 2-simplexes \( \sigma \in T \) and leads to a cell-like map \( \phi_2 : Y_2 \rightarrow X \) defined on a 3-gm \( Y_2 \). When replacing \( Int(\sigma) \) by \( Int(\sigma) \times [-1,1] \), \( \sigma \) a 2-simplex of \( T \), the topology of \( Y_2 \) must be regulated so that given any sequence \( \{p_n\} \) in \( Int(\sigma) \) converging to \( p_0 \in \partial \sigma \), then \( p_n \times [-1,1] \rightarrow p_0 \).

The next step is to inflate the 1-skeleton \( T^{(1)} \) of \( T \), treated as a subset of \( Y_2 \), to put it in the manifold set of another 3-gm \( Y_1 \). At each 1-simplex \( \tau \in T \), whereas \( Int(\tau) \) has a neighborhood \( V_\tau \) in \( X \) whose structure is represented schematically in Figure 1(a), the neighborhood \( \phi_2^{-1}(V_\tau) \) in \( Y_2 \) has structure represented in Figure 1(b). This is the spot where the value of preferred 2-polyhedron is exposed. Each \( \tau \in T^{(1)} \) is a face of 2-simplexes \( \sigma_1, \sigma_2, \ldots, \sigma_m, m \geq 1 \), in \( T \); we presume these are arranged in a circular order, in the sense that both \( \sigma_j \) and \( \sigma_{j+1} \) meet the frontier of some component \( W_j \) of \( V_\tau - P \).

With care in the construction of \( V_\tau \), we can assure that \( W_1, W_2, \ldots, W_m \) constitute all the components of \( V_\tau - P \).

The only significant difference between the structures in \( X \) or \( Y_2 \) and the schematics is that \( Int(\tau) \) is an open interval, not just the special point in schematics. The segments emanating from that point in Figure 1 also must be enlarged by taking Cartesian products with that open interval, regarded as \( Int(\tau) \).

In place of each \( Int(\tau) \) we will insert \( Int(\tau) \times B^2 \) into \( Y_2 \) to form a new 3-gm \( Y_1 \) (topologized like \( Y_2 \)) and cell-like map \( \phi_1 : Y_1 \rightarrow Y_2 \), one which is conservative over \( (Y_2 - T^{(1)}) \cup T^{(0)} \). Specifically, replace \( Y_2 - T^{(0)} \) with the space obtained from the disjoint union of \( Int(\tau) \times B^2 \), thickened 2-simplexes \( \sigma_i \times [-1,1] \) and closures \( ClW_j \) of components of the various \( \phi_2^{-1}(V_\tau - P) \) by attaching \( Int(\tau) \times [-1,1] \subset \sigma_i \times [-1,1] \) to an arc of \( Int(\tau) \times \partial B^2 \), as shown in Figure 2, and (localized) by attaching \( ClW_j \) to

\[
Int(\sigma_j) \times [-1,1] \cup Int(\sigma_{j+1}) \times [-1,1] \cup Int(\tau) \times \partial B^2
\]

via the map sending \( z \in ClW_j \cap (\sigma_j - \tau) \) to \( z \times 1 \in Int(\sigma_j) \times [-1,1] \) and sending \( z \in ClW_j \cap (\sigma_{j+1} - \tau) \) to \( z \times -1 \in Int(\sigma_{j+1}) \times [-1,1] \). The cell-like map \( \phi_1 \)
Figure 1.

Figure 2.

amounts to first coordinate projection $\text{Int}(\tau) \times B^2 \rightarrow \text{Int}(\tau)$ wherever that makes sense; elsewhere it is conservative. Let $\tilde{P}_2$ denote the preferred 2-polyhedron of (2) obtained in $Y_2$, and let $P_\tau$ denote the product of $\text{Int}(\tau)$ with the segments in $B^2$. 

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associated with \( \tau \); then the preferred 2-polyhedron \( \widetilde{P}_1 \) of (2), contained, except for its 0-skeleton, in \( M(Y_1) \), is the closure of \( \phi_1^{-1}(P_2 - |T^{(1)}|) \cup (\bigcup \sigma_v(P_v) \setminus P_v) \). The \( J^* \) at this stage are defined similarly. Note that each nontrivial point preimage under \( \phi_1 \) meets \( S(Y_1) \) in a finite set.

When \( y_1 \in \text{Int}(\tau) \cap M(Y_2), \tau \in T^{(1)} \), one can find a small neighborhood \( N \) of \( y_1 \) such that \( \phi_1^{-1}(N) \) can be expressed as \((B^+ \times \mathbb{R}) \cup (\bigcup \text{Cl}(V_i))_{i=1}^s\), where \( \{V_i\} \) is the collection of components of \( N - P_2 \) and \( B^+ \) is a disk \( B \) to which \( s \) open collars on arcs \( \alpha_1, \ldots, \alpha_s \) are attached, where \( \alpha_i \cap \alpha_{i+1} = a_i \) and \( \text{Cl}(V_i) \cap (B \times \mathbb{R}) = a_i \times \mathbb{R} \). Repeated applications of Lemma 3.6 one for each component \( V_i \), yields that \( \phi_1^{-1}(M(Y_2)) \) is a 3-manifold.

Finally we must blow up vertices \( v \in T^{(0)} \) to 3-cells \( B_v \). Each \( v \) has a well-defined link \( L_v \) in \( P \subset X \) and a thickening \( T_v = (\phi_2 \phi_1)^{-1}(L_v) \) of a copy of \( L_v \), namely, \( T_v \cap \widetilde{P}_1 \), to a compact 2-manifold with boundary. We argue that \( T_v \) embeds in a 2-sphere \( S_v \).

**Claim.** The space \( S_v \) obtained by attaching a disk to each component of \( \partial T_v \) is a 2-sphere.

**Proof of the Claim.** There is a closed neighborhood \( C_v \) of \( v \) in \( Y_1 \) that meets \((\phi_2 \phi_1)^{-1}(P) \) in a subset homeomorphic to the cone (from \( v \)) over \( T_v \). Set \( C_v^* = C_v \cap (\phi_2 \phi_1)^{-1}(P) \). Replace the various component closures \( Z_1, \ldots, Z_t \) of \( C_v - C_v^* \) by 3-cells \( B_1, \ldots, B_t \) with

\[
B_j \cap C_v^* = Z_j \cap C_v^* = \text{cone}J \ (= \text{2-cell}),
\]

\( J \) representing a component of \( \partial T_v \), to form \( Q_v = C_v^* \cup \bigcup B_j \). Note that \( Q_v \) can be regarded as the cone from \( v \) over \( S_v \), where \( S_v \) denotes \( \tilde{T}_v \) capped off with 2-cells, one in each \( \partial B_j \setminus \{v\} \). Clearly \( S_v \) is a 2-manifold. Moreover, each 1-cycle \([z]\) in \( S_v - \{v\} \) is homologous to one in \( T_v \subset S_v \). Since loops in \( T_v \) can be deformed in \( X_v \setminus \{v\} \) arbitrarily close to \( v \) and since \( \text{Int}(C_v) \) is a 3-gm, each such loop \( \lambda \) is null homologous in \( C_v - \{v\} \). In view of the fact that the various \( B_j - \{v\} \) are absolute extensors, the inclusion \( C_v - \{v\} \to Q_v - \{v\} \) factors through \( C_v - \{v\} \). It follows that each \( \lambda \) is null-homologous in \( Q_v - \{v\} \simeq S_v \times [0, 1] \). Hence, \( H_1(S_v) \cong 0 \), and \( S_v \) must be a 2-sphere, which completes the proof of the Claim.

Continuing with the proof of 3.6, we regard \( S_v \) as the boundary of a 3-cell \( B_v \), replace each \( v \in T^{(0)} \) with \( B_v \) in a new 3-gm \( Y_0 \), and define \( \phi_0: Y_0 \to Y_1 \) as the map sending each \( B_v \) to the associated vertex \( v \) and being conservative over the complement of the 0-skeleton \( T^{(0)} \). Here the \( S_v \) of the Claim is modified by identifying each of the (abstractly) attached disks to points, which does not change \( S_v \) topologically. It has the benefit of providing a finite set \( F_v \subset S_v \) such that \( S_v - F_v \) has a neighborhood which meets \( \phi_0^{-1}(Y_1 - |T^{(0)}|) \) in a 3-manifold thickening of \( \phi_0^{-1}(\tilde{P}_1 - |T^{(0)}|) \). The topology near \( B_v \) can be arranged so that the closure of \( \phi_0^{-1}(\tilde{P}_1 - |T^{(0)}|) \) meets \( S_v \) in a 1-dimensional polyhedron \( \tilde{K}_v \). Again each \( \phi_0^{-1}(z) \cap S(Y_0) \) is finite, and \( \tilde{K}_v \) is a strong deformation retract of \( S_v - F_v \). The final \( \tilde{P} = \tilde{P}_0 \) of \( \phi_0^{-1}(\tilde{P}_1 - |T^{(0)}|) \cup (\text{cone over } \tilde{K}_v) \); the final \( J^* \) is obtained similarly.

The desired map will be \( \phi = \phi_0 \phi_1 \phi_0^{-1} \). As in Lemma 3.6 it is a near-homeomorphism over \( M(X) \), so its retraction to \( \phi^{-1}(M(X)) \subset M(Y) \) is cellular [11, Prop. 5.1].

The map \( \phi: Y \to X \) in the conclusion of the preceding lemma will be called an inflation of \( X \) at \( K \).
**Lemma 5.4.** Suppose $X$ is a 3-gm satisfying the RSAP. Then there exists a sequence $\{K_i, P_i\}_{i \geq 1}$ of tame-preferred polyhedral pairs in $X$, with $P_i \subset P_{i+1}$ for all $i \geq 1$, and there exists a sequence of maps $\mu_i : R_i \to X$ defined on compact 2-polyhedra $R_i$, with $\mu_i(R_i) \subset P_i$ for all $i \geq 1$, such that corresponding to any two points $x, x' \in X$ is an index $k \in \mathbb{N}$ for which $\mu_k$ homologically separates $x$ from $x'$ in $X$.

**Proof.** Being treatable componentwise as a separable metric space, by an initial assumption, $X$ has a countable basis $\Omega$. Enumerate the countable collection of pairs $\Lambda = (W_j, W'_j)_{j=1}^{\infty} \in \Omega \times \Omega$ for which $\text{Cl}(W_j) \subset W'_j$ and some map $\nu_j : R_j \to X$, defined on a compact 2-polyhedron $R_j$, homologically separates $\text{Cl}(W_j)$ from $X - W'_j$. Lemma 3.3 assures that for any two points $x, x' \in X$ there is a pair $(W_j, W'_j) \in \Lambda$ with $x \in W_j$, $x' \in X - W'_j$.

Since $X$ satisfies RSAP, Theorem 4.8 provides a tame-preferred polyhedron pair $(K_1, P_1)$ in $X$ and a map $\mu_1 : R_1 \to X$ with $\mu_1(R_1) \subset P_1$, such that $\mu_1$ homologically separates $\text{Cl}(W_1)$ and $X - W'_1$.

Assume that we have already produced a finite collection of tame-preferred polyhedral pairs $(K_1, P_1), (K_2, P_2), \ldots, (K_t, P_t)$ in $X$ with $P_1 \subset P_2 \subset \cdots \subset P_t$ and maps $\mu_j : R_j \to X$ with $\mu_j(R_j) \subset P_j$ and with $\mu_j$ strongly separating $\text{Cl}(W_j)$ and $X - W'_j$ ($j = 1, 2, \ldots, t$). Again Theorem 4.8 provides a tame-preferred polyhedron pair $(K_{t+1}, P_{t+1})$ in $X$ with $P_{t+1} \supset P_t$ and a map $\mu_{t+1} : R_{t+1} \to X$ with $\mu_{t+1}(R_{t+1}) \subset P_{t+1}$ such that $\mu_{t+1}$ homologically separates $\text{Cl}(W_{t+1})$ and $X - W'_{t+1}$.

**Lemma 5.5** (Resolution). Suppose the 3-gm $X$ contains a sequence $\{P_i\}_{i=1}^{\infty}$ of compact, preferred 2-polyhedra such that $P_i \subset P_{i+1}$ for all $i \geq 1$. Then there exist a 3-gm $Y$ and a proper, cell-like, surjective map $\Phi : Y \to X$ satisfying the following conditions:

(i) every map $\mu : R \to P_k$, $k \in \mathbb{N}$, defined on a compact 2-polyhedra $R$ has approximate lifts into $M(Y)$, and

(ii) for each $p \in X$, $\dim(\Phi^{-1}(p) \cap S(Y)) \leq 0$.

**Proof.** Set $X_1 = X$ and $\{P^{(1)}_i = P_i\}_{i=1}^{\infty}$. By induction we will construct, for each $n \in \mathbb{N}$, a proper, cell-like map $\phi_{n+1,n} : X_{n+1} \to X_n$ together with a certain sequence, $\{P_{i}^{(n+1)}\}_{i=n+1}^{\infty}$, of compact, preferred 2-polyhedra in $X_{n+1}$. The desired map $\Phi : Y \to X$ will be the inverse limit of the inverse sequence of maps $\{X_n, \phi_{n+1,n}\}$.

Apply Inflation Lemma 5.3 to obtain an inflation $\phi_{2,1} : X_2 \to X_1$ at $P_0^{(1)} = P_1$. Among other features, this provides a 2-polyhedron $\bar{P}_1 \subset M(\phi_{2,1}^{-1}(P_1)) \subset X_2$ where $\phi_{2,1} : \bar{P}_1 \to P_1$ is cell-like. Let $\{P^{(2)}_{i} \}_{i=2}^{\infty}$ be approximate lifts of $P_1$ described in conclusion (3) there. Assuming cell-like maps $\phi_{n+1,n} : X_{n+1} \to X_n$ defined on 3-gms $X_{n+1}$ have been obtained for $n = 1, 2, \ldots, t$, along with approximate lifts $\{P_{j}^{(n+1)}\}_{j=n+1}^{\infty}$ of $\{P_{j}^{(n)}\}_{j=n+1}^{\infty}$, and 2-polyhedra $\bar{P}_n \subset M(\phi_{n+1,n}^{-1}(P_{n}^{(n)})) \subset X_{n+1}$ for which $\phi_{n+1,n} : \bar{P}_n \to P_{n}^{(n)}$ is cell-like, apply Inflation Lemma 5.3 again to obtain an inflation of $X_{n+1}$ at $P_{n+1}^{(n+1)}$, thereby producing the next level of objects for $n = t + 1$.

We conclude immediately from Lemma 3.2 that the inverse sequence $\{X_n, \phi_{n+1,n}\}$ has inverse limit $\Phi : Y \to X_1 = X$, with $Y$ a 3-gm and $\Phi$ a cell-like map.
To verify conclusion (i), note that any map \( \mu: R \to P_k \) can be approximately lifted, successively, to maps \( \mu_i: R \to P_k^{(i)}, i = 1, 2, \ldots, k \), and, finally, to \( \mu_{k+1}: R \to \tilde{P}_k \subset M(\phi_{k+1}^{-1}(P_k^{(k)})) \subset X_{k+1} \). According to Lemma 5.3, \( \Phi_{k+1}^{-1}(M(X_{k+1})) \) is a 3-manifold (where \( \Phi_{k+1} \) satisfies \( \Phi = \phi_{k+1,0} \Phi_{k+1} \)). Hence, \( \mu \) has approximate lifts to \( M(Y) \).

To verify conclusion (ii), let \( A_0 \) denote \( \{p\} \) and recursively let \( A_n \) denote \( \phi_{n,n-1}^{-1}(A_{n-1}) - M(X_n) \) for \( n \in \mathbb{N} \). Each set \( A_n \) is finite, by conclusion (4) of Lemma 5.3. Furthermore, \( \Phi^{-1}(p) \cap S(Y) \subset A_\infty = \varprojlim A_n \). But the inverse limit of finite sets is 0-dimensional.

**Corollary 5.6.** A 3-gm \( X \) has a near-resolution if there exist a sequence \( \{P_i\}_{i=1}^\infty \) of preferred 2-polyhedra in \( X \) and a family of maps \( \{\mu_i: R_i \to X\}_{i=1}^\infty \) satisfying the following conditions:

(i) \( \mu_i(R_i) \subset P_i \) for every \( i \geq 1 \),
(ii) \( P_i \subset P_{i+1} \) for every \( i \geq 1 \), and
(iii) given distinct points \( p, q \in X \) there exists \( k \in \mathbb{N} \) such that \( \Phi_k \) homologically separates \( p \) from \( q \).

**Proof:** Applying Resolution Lemma 5.4 to \( X \) and \( \{P_i\}_{i=1}^\infty \), we obtain \( \Phi: Y \to X \) such that, for all \( x \in X \), \( S(Y) \cap \Phi^{-1}(x) \) is 0-dimensional. We will show that \( \dim S(Y) \leq 0 \), which will imply that \( Y \) has a near resolution \( \psi: M \to Y \). The near-resolution of \( X \) then will be \( \Phi \psi: M \to X \).

To show that \( \dim S(Y) \leq 0 \), we first establish the following

**Claim.** For any two distinct points \( p \) and \( q \) of \( X \) there exists a map \( \kappa: R \to Y \) defined on a compact 2-polyhedron \( R \) such that \( \kappa(R) \) strongly separates \( \Phi^{-1}(p) \) from \( \Phi^{-1}(q) \) and \( \kappa(R) \subset M(Y) \).

**Proof of the Claim.** Choose \( i \in \mathbb{N} \) such that \( \mu_i \) homologically separates \( p \) and \( q \) in \( X \). Endow \( X \) with a metric, and choose \( \epsilon > 0 \) such that any \( \epsilon \)-approximation to \( \mu_i \) is homotopic to \( \mu_i \) in \( X - \{p, q\} \). By Resolution Lemma 5.5, \( \mu_i \) has an \( \epsilon \)-lift \( \kappa \) to \( M(Y) \). Since \( \Phi_k \) is homotopic to \( \mu_i \) in \( X - \{p, q\} \) and \( \Phi \) restricts to a proper homotopy equivalence of the pairs

\[
(Y, Y - \Phi^{-1}(\{p, q\})) \to (X, X - \{p, q\}),
\]

it follows that \( \kappa \) homologically separates \( \Phi^{-1}(p) \) and \( \Phi^{-1}(q) \). By Lemma 5.3, \( \kappa \) strongly separates \( \Phi^{-1}(p) \) and \( \Phi^{-1}(q) \).

Given a component \( C \) of \( S(Y) \), one can immediately produce an \( x_C \in X \) for which \( C \subset \Phi^{-1}(x_C) \), using the Claim. Since \( C \subset S(Y) \cap \Phi^{-1}(x_C) \) and since \( \dim[S(Y) \cap \Phi^{-1}(x_C)] \leq 0 \) by conclusion (ii) of Lemma 5.3, \( C \) must be a singleton. Hence, \( \dim S(Y) \leq 0 \).

**Proof of Theorem 5.1** Apply Lemma 5.4 to obtain a sequence \( \{(K_i, P_i)\}_{i \geq 1} \) of compact, tame-preferred polyhedral pairs such that \( P_i \subset P_{i+1} \) for all \( i \geq 1 \) and any two points of \( X \) are homologically separated by some map \( \mu: R \to P_k \) into one of these \( P_i \). Corollary 5.6 assures that \( X \) has a near-resolution.

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