

THE 3-MANIFOLD RECOGNITION PROBLEM

ROBERT J. DAVERMAN AND THOMAS L. THICKSTUN

ABSTRACT. We introduce a natural Relative Simplicial Approximation Property for maps from a 2-cell to a generalized 3-manifold and prove that, modulo the Poincaré Conjecture, 3-manifolds are precisely the generalized 3-manifolds satisfying this approximation property. The central technical result establishes that every generalized 3-manifold with this Relative Simplicial Approximation Property is the cell-like image of some generalized 3-manifold having just a 0-dimensional set of nonmanifold singularities.

1. INTRODUCTION

The manifold recognition problem, originally proposed in 1978 by J. W. Cannon [9], asks for a short list of simple topological properties, easy to check, that characterize topological manifolds among topological spaces. Cannon conjectured that n -manifolds might be characterized as those generalized n -manifolds satisfying a minimal amount of general position. To address the latter in dimensions greater than 4 he proposed the following Disjoint Disks Property: any two maps of B^2 into the space can be approximated by maps with disjoint images.

This paper addresses the 3-manifold recognition problem. For that dimension the fundamental difficulty is to identify an appropriate general position property. The Disjoint Disks Property, possessed by no 3-manifold, is impossibly strong, and the related Disjoint Arcs Property, possessed by all generalized 3-manifolds, is impossibly weak.

A *generalized n -manifold* X , abbreviated as n -gm, is a locally compact, locally contractible, finite dimensional metric space with the relative local homology of \mathbb{R}^n (i.e., $H_*(X, X - \{x\}; \mathbb{Z})$ is isomorphic to $H_*(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z})$ for all $x \in X$). In such a space X the *manifold set*, $M(X)$, consists of all points of X having a neighborhood homeomorphic to \mathbb{R}^n , and the *singular set*, or *nonmanifold set*, $S(X)$, is defined as $S(X) = X - M(X)$. As components of locally compact metric spaces are separable, we simply will view all n -gms as separable metric spaces.

Clearly every n -manifold is an n -gm, but the converse fails for $n > 2$. If $f: M \rightarrow X$ is a proper, cell-like, surjective mapping defined on an n -manifold, where $\dim X < \infty$, then X is an n -gm, and classical examples like the famous dog-bone space of R. H. Bing [3] demonstrate that X need not be a genuine manifold. Historically cell-like maps like Bing's have been used to produce a large class of

Received by the editors April 21, 2003 and, in revised form, July 21, 2004.

2000 *Mathematics Subject Classification*. Primary 57N10, 57P99; Secondary 57M30, 57N60, 57N75.

Key words and phrases. Generalized 3-manifold, resolvable, simplicial approximation property, relative simplicial approximation, tame embedding, locally 1-coconnected.

©2005 American Mathematical Society
Reverts to public domain 28 years from publication

examples. To distinguish such images from other possible examples that arise, one calls an n -gm X *resolvable* if there exist a genuine n -manifold M and a proper, cell-like, surjective mapping $f: M \rightarrow X$. In this case, the pair (M, f) is called a *resolution* of X . Bryant, Ferry, Mio and Weinberger have established the existence of nonresolvable n -gms for $n > 5$ [7].

In dimensions greater than 4 the model theorem is provided by the combination of results by Edwards and Quinn. Given a connected n -gm X , Quinn [19] produced an integer valued obstruction, $i(X)$, which is locally defined, locally constant, and satisfies $i(X \times X') = i(X) \times i(X')$, where $i(X) = 1$ if and only if X is resolvable ($n > 3$). Edwards [13] (see [11] for details) showed that a resolvable n -gm, $n > 4$, is an n -manifold if and only if it satisfies the Disjoint Disks Property. Consequently, for $n > 4$ a connected space X is an n -manifold if and only if X is an n -gm satisfying both the Disjoint Disks Property and $i(X) = 1$.

Daverman and Repovš [12] introduced a kind of general position property—called the spherical simplicial approximation property, abbreviated as SSAP, and defined in Section 4—and showed that every resolvable generalized 3-manifold with the SSAP is a 3-manifold. Here we modify their property, defining a relative simplicial approximation property (RSAP) which is stronger than this SSAP; our main result establishes that, modulo the Poincaré Conjecture, every generalized 3-manifold X satisfying this RSAP is a 3-manifold. Specifically, the fundamental issue is to confirm that X is resolvable, for then [12] applies to give the final 3-manifold recognition step. With no extra hypotheses we produce a cell-like, surjective mapping $\Phi: Y \rightarrow X$, where Y is a 3-gm such that $S(Y)$ is 0-dimensional. If the Poincaré Conjecture is true, however, then the Corollary to the Resolution Theorem of [22] (see [23] for corrections) assures that Y has a resolution $\Psi: M \rightarrow Y$, and $\Phi\Psi: M \rightarrow X$ serves as the desired resolution of X .

2. PRELIMINARIES

A subset C of a space X is *locally k -coconnected*, abbreviated as k -LCC, if each neighborhood U of an arbitrary point $x \in X$ contains another neighborhood V of x such that every map $\partial I^{k+1} \rightarrow V - C$ can be extended to a map $I^{k+1} \rightarrow U - C$.

We shall distinguish simplicial complexes from their underlying point sets, called *polyhedra*. A *triangulation* of a polyhedron Q is a pair (T, h) , where T denotes a simplicial complex and h a homeomorphism of its underlying point set, denoted by $|T|$, onto Q . Frequently the polyhedra encountered here will be subsets of a given 3-gm. One should not presume the existence of any compatibility between the (piecewise) linear structure of the simplicial complex associated to a polyhedron Q in a 3-gm X and the possible linear structures arising within X . Most of our attention will fall on 2-dimensional polyhedra, called 2-polyhedra for short.

A *subpolyhedron* Q' of a polyhedron Q is a closed subset of Q such that there exist a triangulation (T, h) of Q and a subcomplex T' of T with $h(|T'|) = Q'$.

Suppose Q is a polyhedron and $z \in Q$. Impose a triangulation (T, h) on Q . Suppress h , here and throughout the remainder of this paper, and regard T as a simplicial complex whose underlying point set equals Q . Subdivide T , if necessary, so that z corresponds to a vertex of T . For such a Q topologically embedded as a closed subset of a generalized 3-manifold X , $X - Q$ is said to have *free local fundamental group at $z \in Q$* , abbreviated as 1-FLG at z , if for each sufficiently small neighborhood U of z there exists another neighborhood V of z with $z \in V \subset U$

and if W is any connected open set with $z \in W \subset V$, then for each nonempty component W' of $W - Q$ the (inclusion-induced) image $\pi_1(W') \rightarrow \pi_1(U')$ is a free group on $m - 1$ generators, where U' denotes the component of $U - Q$ containing W' and m is the number of “components” of $St(z) - z$ whose images meet $Cl(W')$, where $St(z)$ denotes the simplicial star of z in the complex T . As usual, $X - Q$ is simply said to be 1-FLG in X if it is 1-FLG in X at each point of Q .

For simplicity, we will say that a polyhedron Q embedded in a generalized 3-manifold X as a closed subset is *tamely embedded in X* if $X - Q$ is 1-FLG in X . Nicholson [18] has shown that a polyhedron tamely embedded in a genuine 3-manifold M in this 1-FLG sense is tamely embedded in the geometric sense, where there exists a self-homeomorphism (arbitrarily close to *Identity*: $M \rightarrow M$) of M that carries Q onto a subspace underlying a subcomplex of some preassigned triangulation of M , after subdivision.

Given maps $\phi: Y \rightarrow X$ and $f: Z \rightarrow X$, where X is metrizable, and given $A \subset Y$, we say that f *approximately lifts to A* (occasionally, for emphasis, *under ϕ*) if for each metric on X and each $\epsilon > 0$ there exists a map $\tilde{f}: Z \rightarrow A$ such that $\phi\tilde{f}$ is within ϵ (pointwise) of f .

Suppose X is a connected 3-gm, D and E are disjoint, closed subspaces of X , and $\mu: R \rightarrow X - (D \cup E)$ is a map defined on a compact, 2-polyhedron R . We say that μ *homologically separates D from E* if there exist $\alpha \in H_2(R; \mathbb{Z}_2)$ and $\xi \in H_3(X - E, X - (D \cup E); \mathbb{Z}_2)$ such that, for each $d \in D$, $\mu_*(\alpha) = i_*\partial(\xi) \not\cong 0$, where i denotes the inclusion $X - (D \cup E) \rightarrow X - (\{d\} \cup E)$. We say that μ *strongly separates D and E* if no component of $X - \mu(R)$ contains points of both D and E .

A compact subset C of any ANR Y is *cell-like* if, for each open subset U of Y containing C , the inclusion $C \rightarrow U$ is homotopic to a constant. A proper map $f: Y \rightarrow Z$ defined on an ANR Y is a *cell-like map* if each $f^{-1}(z)$, $z \in Z$, is cell-like. We say that a cell-like map $f: Y \rightarrow Z$ is *conservative over $B \subset Z$* if $f|f^{-1}(B)$ is 1-1.

Similarly, a compact subset C of an n -manifold M is *cellular* if M contains a sequence $\{D_i\}_{i=1}^\infty$ of n -cells such that $Int(D_i) \supset D_{i+1}$ and $\bigcap_{i=1}^\infty D_i = C$, and a proper map $F: M \rightarrow Z$ defined on M is *cellular* if each $F^{-1}(z)$ is.

As in [22] a 3-*near* manifold M^* is a 3-gm obtained from a 3-manifold M by identifying a null sequence of pairwise disjoint 3-cells in M and replacing the interior of each with the interior of a compact, contractible 3-manifold in such a way that $S(M^*)$ is 0-dimensional and 1-LCC embedded in M^* . A *near resolution* of a 3-gm X is a pair (M^*, ψ) , where M^* is a 3-near manifold and $\psi: M^* \rightarrow X$ is a proper, cell-like surjection. Should the Poincaré Conjecture be false, one could easily produce a 3-near manifold M^* which is nonresolvable, homotopy equivalent to S^3 , has $S(M^*) = \text{point}$, and satisfies $M^* \times \mathbb{R} \cong S^3 \times \mathbb{R}$.

3. ELEMENTARY PROPERTIES OF 3-GMS AND 3-NEAR MANIFOLDS

A *generalized 3-manifold with boundary Z* is a locally compact, locally contractible, finite dimensional metric space such that, for each $z \in Z$, either $H_*(Z, Z - \{z\}; \mathbb{Z}) \cong 0$ or $H_*(Z, Z - \{z\}; \mathbb{Z}) \cong H_*(\mathbb{R}^3, \mathbb{R}^3 - \{0\}; \mathbb{Z})$; the subset consisting of all $z \in Z$ for which $H_*(Z, Z - \{z\}; \mathbb{Z}) \cong 0$ is called the *boundary of Z* , denoted ∂Z .

Lemma 3.1. *If the space Z is expressed as a union of closed subsets Z_1 and Z_2 of Z which are generalized 3-manifolds with boundary, where $Z_1 \cap Z_2 = \partial Z_1 = \partial Z_2$,*

then Z is a generalized 3-manifold. Conversely, if Z is a 3-gm and $Z = Z_1 \cup Z_2$, where Z_1, Z_2 are closed subsets of Z and $Z_0 = Z_1 \cap Z_2$ is a 2-gm, then Z_1 and Z_2 are generalized 3-manifolds with boundary.

Proof. For the most part—except for ANR properties—this is treated in [20]. When Z_1 and Z_2 are 3-gms with boundary, work of Mitchell [16] combines with classical results of Wilder [24] to establish that ∂Z_i ($i = 0, 1$) is a 2-manifold, hence an ANR, and standard results from ANR theory then yield that $Z = Z_1 \cup Z_2$ is an ANR. Similarly, in the converse, Z_0 is an ANR, so Z_1 and Z_2 must be ANRs as well. \square

Lemma 3.2. *Let $\{X_i, p_{i+1,i}\}$ denote a sequence of 3-gms and cell-like maps, with inverse limit Z . Then Z is a 3-gm, and the associated projections $q_i: Z \rightarrow X_i$ are cell-like maps.*

Proof. We provide an argument only for the case in which each X_i is a manifold factor, i.e., $X_i \times \mathbb{R}^k$ is a manifold (one can take k to be any fixed integer greater than 1). It parallels the proof of [17, 3.9(iii)] about the inverse limit of a sequence of ANRs and cell-like maps yielding an ANR. Only this special case matters for our purposes here, because the RSAP implies X contains 2-cells, so $X \times \mathbb{R}^k$ contains codimension one cells, and thus the Quinn obstruction [19] to the existence of a resolution vanishes.

Examine the related sequence $\{X_i \times \mathbb{R}^k, p_{i+1,i} \times Id\}$ of cell-like maps between manifolds. By [13] or [21] each map $p_{i+1,i} \times Id$ is a near-homeomorphism, so a result of M. Brown [6] (or see [1]) assures that the induced limiting map $q_1 \times Id: Z \times \mathbb{R}^k \rightarrow X_1 \times \mathbb{R}^k$ is a near-homeomorphism. Hence, $Z \times \mathbb{R}^k$ is a $(3+k)$ -manifold, and Z , being one of its codimension k factors, must be a 3-gm. Furthermore, $q_1 \times Id$, being a near-homeomorphism, is a cell-like mapping [11, Theorem 17.4]; obviously this means q_1 itself is cell-like. \square

The next lemma uses the notation of Lemma 3.2, as well as the standard notation for the composite, $p_{2,1} \cdots p_{k,k-1} p_{k+1,k} = p_{k+1,1}$. The map q_1 is the inverse limit projection described in Lemma 3.2.

Lemma 3.3. *Let $\{X_i, p_{i+1,i}\}$ denote a sequence of 3-gms and cell-like maps such that $p_{k+1,k}$ restricts to a cellular map $p_{k+1,1}^{-1}(M(X_1)) \rightarrow p_{k,1}^{-1}(M(X_1))$ for each $k > 0$. Then $q_1^{-1}(M(X_1))$ is a 3-manifold.*

Proof. Each of the restricted $p_{k+1,k}$ is a near-homeomorphism by Armentrout's Cellular Approximation Theorem [2], so Brown's argument [6] applies, just as in 3.2. \square

Throughout the remainder of this section \mathbb{Z}_2 coefficients will be used for all homology and cohomology computations.

Lemma 3.4. *Suppose E is a nonempty, closed subset of the 3-gm X and $d \in X - E$. Then there exist a compact, connected neighborhood D of d , a compact 2-polyhedron R , and a map $\nu: R \rightarrow X$ such that ν homologically separates D from E .*

Proof. Note that whenever X has no compact component,

$$\partial: H_3(X, X - \{x\}) \rightarrow H_2(X - \{x\})$$

is 1-1. This follows immediately, because, by duality [5], $H_3(X) \cong H_c^0(X) \cong 0$. Fix $0 \neq \xi \in H_3(X - E, X - (E \cup \{d\}))$, and assume X is connected (so $X - E$

has no compact component). Simply choose R and $\nu: R \rightarrow X$ to be a carrier of $\partial\xi(\neq 0) \in H_2(X - (E \cup \{d\}))$; this choice assures that ν homologically separates $\{d\}$ from E .

Fix a compact connected neighborhood $D \subset X - (E \cup \mu(R))$ of d . Given any $d' \in D$, find an arc $\gamma \subset X - (E \cup \mu(R))$ joining d to d' . The bottom level map in the diagram below is an isomorphism, since all the others are (the vertical ones, by duality in $X - E$):

$$\begin{array}{ccc} H^0(\gamma) & \longrightarrow & H^0(\{d\}) \\ \downarrow & & \downarrow \\ H_3(X - E, X - (E \cup \gamma)) & \longrightarrow & H_3(X - E, X - (E \cup \{d\})) \end{array}$$

Similarly,

$$H_3(X - E, X - (E \cup \gamma)) \rightarrow H_3(X - E, X - (E \cup \{d'\}))$$

is an isomorphism. It follows that ν homologically separates D from E . □

Lemma 3.5. *Suppose D, E are disjoint, closed subsets of the 3-gm X and $\nu: R \rightarrow X$ is a map of a compact 2-polyhedron R which homologically separates D from E . Then ν strongly separates D and E .*

Proof. If $\nu(R)$ failed to separate $d_0 \in D$ and $e_0 \in E$, then there would be an arc $\gamma \subset X - \nu(R)$ connecting d_0 and e_0 . By hypothesis there exist $\alpha \in H_2(R)$ and $\xi(\neq 0) \in H_3(X - E, X - (E \cup D))$ such that $\nu_*(\alpha) = i_*\partial(\xi)(\neq 0) \in H_2(X - (E \cup \{d_0\}))$. Name a compact carrier $C \subset X - E$ for ξ . Then the image of ξ in $H_3(X - E, X - (E \cup \{d_0\}))$ is nonzero and belongs to the inclusion-induced image

$$\eta_*: H_3(C, C \cap (X - (E \cup D))) \rightarrow H_3(X - E, X - (E \cup \{d_0\})).$$

Let $\tilde{\gamma}$ denote the component of $\gamma - E$ containing d_0 . Certainly here η_* would factor through

$$H_3(X - E, X - (E \cup \tilde{\gamma})) \cong H_c^0(\tilde{\gamma}) \cong H_c^0([0, 1/2]) \cong 0,$$

a contradiction. □

Lemma 3.6. *Let C be a closed subset of a 3-manifold M , the frontier of which is a surface S . Then attachment of an open collar $S \times [0, 1)$ to C along $S = S \times 0$ yields a 3-manifold.*

Proof. When M is a 3-sphere and S is a 2-sphere this was proved by Hosay and Lininger [14] (or see [10], [8]). The general case, which localizes to that of a 2-sphere in S^3 [4, Theorem 5], follows. □

4. A RELATIVE SIMPLICIAL APPROXIMATION PROPERTY

According to [12], a generalized 3-manifold X has the *Simplicial Approximation Property* (SAP) if for each map $f: I^2 \rightarrow X$ and each $\epsilon > 0$, there exist a map $F: I^2 \rightarrow X$ and a compact 2-polyhedron $K_F \subset X$ such that (1) $\text{dist}(F, f) < \epsilon$, (2) $F(I^2) \subset K_F$, and (3) $X - F(I^2)$ is 1-FLG in X . Similarly, X has the *Spherical Simplicial Approximation Property* (SSAP) if the analogous conditions hold for maps $S^2 \rightarrow X$ in place of maps $I^2 \rightarrow X$.

We will say that a map $f: K \rightarrow X$ of a compact 2-dimensional polyhedron K to a generalized 3-manifold X is *simplicial* if $f(K)$ is a polyhedron whose complement is 1-FLG in X and $f: K \rightarrow f(K)$ is simplicial with respect to some triangulations of K and $f(K)$. Of course, given any map between polyhedra, we can impose triangulations, take fine mesh subdivisions, and then approximate by a simplicial map. In short, the map F in the SAP (similarly, in the SSAP) can be assumed to be simplicial and onto K_F .

A generalized 3-manifold X has the *Relative Simplicial Approximation Property* (RSAP) if for each map $f: I^2 \rightarrow X$, each compact subpolyhedron Q of I^2 for which $f|_Q$ is simplicial as above, and each $\epsilon > 0$, there exists a simplicial map $F: I^2 \rightarrow X$ such that $\text{dist}(F, f) < \epsilon$ and $F|_Q = f|_Q$.

Lemma 4.1. *Every 3-gm X that satisfies the RSAP also satisfies the following stronger property: for each compact 2-polyhedron K , compact subpolyhedron L , map $g: K \rightarrow X$ such that $g|_L$ is simplicial, and $\epsilon > 0$, there exists a simplicial map $G: K \rightarrow X$ with $\text{dist}(G, g) < \epsilon$ and $G|_L = g|_L$.*

Proof. Assume for simplicity that X is path connected. List the large simplexes $\Delta_1, \dots, \Delta_r$ of L —large in the sense of being proper faces of no other simplexes of L —and choose any simplex Δ_{r+1} of $K - L$. We show how to approximate g by a new map $g_{r+1}: K \rightarrow X$ which is simplicial on a complex underlying $L \cup \Delta_{r+1}$.

Specify a finite collection $\sigma_1, \dots, \sigma_r, \sigma_{r+1}$ of pairwise disjoint simplexes in $\text{Int}(I^2)$ and equip them with simplicial isomorphisms $e_j: \sigma_j \rightarrow \Delta_j$ ($j = 1, \dots, r+1$). Define $\eta = \bigcup e_j: \bigcup \Delta_j \rightarrow K$. Think of $e_{r+1}^{-1}(\Delta_{r+1} \cap L)$ together with all the other σ_j ($j = 1, \dots, r$) as $Q \subset I^2$. Use the hypothesized path connectedness of X to extend $g\eta|_Q$ to a map $f: I^2 \rightarrow X$. Apply the RSAP to approximate $f: I^2 \rightarrow X$ by a simplicial map $F: I^2 \rightarrow X$ that agrees with $g\eta$ on Q , and define $G_{r+1}: Q \cup \Delta_{r+1} \rightarrow X$ as $G_{r+1} = F\eta^{-1}$. Note that G_{r+1} is a well-defined simplicial map approximating $g|_{L \cup \Delta_{r+1}}$ and coinciding with g on L . By a controlled homotopy extension lemma, G_{r+1} extends to a map $g_{r+1}: K \rightarrow X$ approximating g and coinciding with g on L .

A finite number of repetitions of this procedure yields the desired simplicial map $G: K \rightarrow X$. \square

Corollary 4.2. *Every generalized 3-manifold X satisfying the RSAP also satisfies the SSAP.*

Corollary 4.3. *All resolvable generalized 3-manifolds satisfying the RSAP are genuine 3-manifolds.*

See Recognition Theorem 3.1 of [12].

Corollary 4.4. *Suppose X is a 3-gm satisfying the RSAP, $L \subset X$ is a tamely embedded 2-polyhedron, and $\nu: R \rightarrow X$ is a map defined on a compact 2-polyhedron. Then for each $\epsilon > 0$ there exists a simplicial map $\mu: R \rightarrow X$ with $\text{dist}(\mu, \nu) < \epsilon$ and $L \cup \mu(R)$ is a polyhedron tamely embedded in X .*

We say that a 2-polyhedron P is *preferred* if it contains neither isolated points nor local cut points—equivalently, if in some (hence, each) triangulation of P the link of every vertex is nonempty and connected. More is said about the role of preferred 2-polyhedra in Section 5. For brevity we call a pair (K, P) of compact, 2-polyhedra in a 3-gm X a *tame-preferred polyhedral pair* if K is tame, P is preferred and P is a subpolyhedron of K . Note that if (K, P) is tame-preferred in X , P is not tame—at least, not *a priori* tame—in X .

Lemma 4.5. *Suppose X is a 3-gm satisfying the RSAP and $f: I^2 \rightarrow X$ is a map such that f restricts to a simplicial map on ∂I^2 with $f|_{\partial I^2 - I \times 1}$ 1-1 and $f(I \times 0) \cap f(I \times 1) = \emptyset$. Then there exists a tame-preferred polyhedral pair (K, P) such that $K \supset P \supset f(I \times 0)$. Furthermore, if $f(\partial I^2 - I \times 1)$ is a subpolyhedron of a compact, tame polyhedron Q , then (K, P) can be obtained so $P \cup Q$ is a subpolyhedron of K .*

Proof. Apply RSAP to obtain an approximation $F: I^2 \rightarrow X$ to f , with $F|_{\partial I^2 - I \times 1} = f|_{\partial I^2 - I \times 1}$, and where $F: I^2 \rightarrow F(I^2)$ can be regarded as simplicial (also, if need be, where $F(I^2) \cup Q$ is a tame 2-polyhedron). Choose triangulations T of I^2 and T' of $F(I^2)$ for which F is simplicial.

Fix a 1-simplex τ of T' , $\tau \subset f(I \times 0)$. We show that some 2-simplex $\sigma \in T'$ contains τ . To see why, consider the unique 2-simplex $\gamma \in T$ containing $f^{-1}(\tau)$. Set $\sigma = F(\gamma)$ if $F(\gamma) \neq \tau$. Otherwise, produce a maximal chain $\gamma = \gamma_0, \gamma_1, \dots, \gamma_s$ of 2-simplexes in T such that $\gamma_{j-1} \cap \gamma_j = \text{edge}$ and $\gamma_j \subset F^{-1}(\tau)$. Since $\partial \gamma_s - \gamma_{s-1} \not\subset \partial I^2$, some other 2-simplex ξ must meet γ_s in an edge $e = f^{-1}(\tau)$, and $F(\xi) \in T'$ will be a 2-simplex containing τ .

Let v be a vertex of $f(I \times 0)$ and w, w' the two possible points in the link of v there. Essentially the same argument shows that w, w' belong to a single component of the link of v in $F(I^2)$.

Although $F(I^2)$ itself might not be preferred, we claim that it contains a preferred polyhedron $P \supset f(I \times 0)$. Let P' be $F(I^2)$ after deletion of (the interiors of) all those 1-simplexes e of T' which are edges of no 2-simplex from T' . Clearly then $F(I^2) \supset P' \supset f(I \times 0)$. If the vertex $w \in T'$ has disconnected link in P' and $w \notin f(I \times 0)$, delete a small regular neighborhood of w from P' ; if, however, $w \in f(I \times 0)$, then delete that small neighborhood $N(w)$ but reinsert the closure of the unique component of $N(w) - \{w\}$ containing the intersection of $N(w)$ with $\text{Link}(w, f(I \times 0))$. Repetition of these two operations eliminates or repairs all disconnected links and yields a preferred polyhedron $P \subset P'$ such that P and $P \cup Q$ are subpolyhedra of $K = F(I^2) \cup Q$. \square

Lemma 4.6. *Suppose X is a 3-gm satisfying the RSAP and $L \subset X$ is a compact 2-polyhedron tamely embedded in X such that each vertex of L belongs to at least two edges. Then there exists a tame-preferred polyhedral pair (K, P) in X such that L is a subpolyhedron of P .*

Proof. Since components of L can be treated one after another, we will simply assume L is connected.

Assume τ is a 1-simplex of L which belongs to no 2-simplex. In view of the hypothesis here, there is an embedding $f: \partial I^2 - I \times 1 \rightarrow L$ with $f(I \times 0) = \tau$. Since no arc locally separates a 3-gm, obviously f can be extended to a map $f: I^2 \rightarrow X$ with $f(I \times 0) \cap f(I \times 1) = \emptyset$, and then Lemma 4.5 assures that L can be expanded by attaching a preferred polyhedron that contains τ . Repeating as often as necessary, we can simply assume each 1-simplex of L is a face of some 2-simplex.

Now assume $v \in L$ is a vertex that has disconnected link in the expanded L' . One can build an embedding $f: \partial I^2 - I \times 1 \rightarrow L'$ with $v \in f(I \times 0)$, extend f to all of ∂I^2 , as before, and apply Lemma 4.5 to reduce the number of components of $\text{Link}(v, L')$ in the expanded L . This expansion can be localized to affect none of the other vertices of L . One can eliminate any 1-simplex contained in no 2-simplex from the expansion and snip at new vertices to prevent disconnected links, just as in

the proof of 4.5. Finitely many repetitions yields a preferred polyhedron containing all of L . \square

Let L denote a 2-polyhedron. Call $v \in L$ a *negligible vertex* if there exists a homeomorphism θ from $[0, 1)$ onto a neighborhood of v such that $\theta(0) = v$. Note that no point of a preferred 2-polyhedron is a negligible vertex.

Essentially the same argument as in 4.6 proves the following.

Lemma 4.7. *Suppose X is a 3-gm satisfying the RSAP and $L \subset X$ is a compact 2-polyhedron tamely embedded in X . Let L^* be a compact, polyhedral subset of L obtained by deleting a small connected neighborhood about each negligible vertex of L . Then there exists a tame-preferred polyhedral pair (K, P) in X with L^* a subpolyhedron of P .*

Theorem 4.8. *Suppose X is a 3-gm satisfying the RSAP, D and E are disjoint closed subsets of X , $\nu: R \rightarrow X$ is a map defined on a compact 2-polyhedron R such that ν homologically separates D and E , and P is a preferred 2-polyhedron tamely embedded in X . Then there exists a map $\mu^*: R \rightarrow X$ such that μ^* homologically separates D and E and there exists a tame-preferred polyhedral pair (K^*, P^*) in X such that $P^* \supset P \cup \mu^*(R)$.*

Proof. First apply Corollary 4.4 to approximate ν by a simplicial map $\mu: R \rightarrow X$ so close to ν that μ homologically separates D and E and, in addition, $P \cup \mu(R)$ is a 2-polyhedron. Then use Lemma 4.7 with $L = P \cup \mu(R)$ to obtain a tame-preferred polyhedral pair (K^*, P^*) in X , with $P^* \supset L^*$. Note that any negligible vertex of $P \cup \mu(R)$ must lie in $\mu(R) - P$, so $P^* \supset P$. By construction the map μ , considered as a map to $\mu(R) \subset L$, is homotopic in $\mu(R)$ to a map μ^* into L^* . Hence, D and E are homologically separated by μ^* , and $P \cup \mu^*(R) \subset L^* \subset P^*$. \square

5. THE MAIN RESULT

The aim of this section is to establish the following Near-Resolution Theorem. It immediately yields the promised characterization of 3-manifolds as the generalized 3-manifolds satisfying the RSAP, provided the Poincaré Conjecture holds.

Theorem 5.1 (Near-Resolution). *Every generalized 3-manifold X satisfying the RSAP has a 3-near resolution (M, ψ) .*

Corollary 5.2. *Suppose the Poincaré Conjecture is true. Then a generalized 3-manifold X is a genuine 3-manifold if and only if it satisfies the RSAP.*

Proof. When X satisfies the RSAP, Theorem 5.1 certifies the existence of a cell-like, surjective map $\psi: M \rightarrow X$ defined on a 3-near manifold M . Under the assumption that the Poincaré Conjecture is true, M actually is a 3-manifold; in other words, the promised cell-like mapping ψ itself provides a resolution of X . Corollary 4.3 confirms that X is a 3-manifold.

The forward implication is trivial. \square

Lemma 5.3 (Inflation). *Suppose X is a 3-gm and $P \subset X$ is a preferred 2-polyhedron. Then there exist a 3-gm Y and a proper, surjective, cell-like map $\phi: Y \rightarrow X$ satisfying the following conditions:*

- (1) ϕ is conservative over $X - P$;
- (2) there is a preferred 2-polyhedron $\tilde{P} \subset M(\phi^{-1}(P))$ for which $\phi|: \tilde{P} \rightarrow P$ is cell-like;

- (3) for each (respectively, preferred) 2-polyhedron $J \supset P$, there is a (respectively, preferred) 2-polyhedron J^* , $J^* \subset \phi^{-1}(J)$, for which $\phi|: J^* \rightarrow J$ is cell-like;
- (4) for each $x \in X$, $\phi^{-1}(x) \cap S(Y)$ is finite; and
- (5) $\phi^{-1}(M(X)) \subset M(Y)$ and $\phi|: \phi^{-1}(M(X)) \rightarrow M(X)$ is cellular.

Proof. We start by describing a model situation in which P is a compact, connected 2-manifold separating X into two components, X_+ and X_- . Here $FrX_+ = P = FrX_-$. Let Y be the space resulting from the disjoint union of ClX_- , $P \times [-1, 1]$ and ClX_+ after identifying each $x \in FrX_-$ with $x \times -1 \in P \times [-1, 1]$ and each $x \in FrX_+$ with $x \times 1 \in P \times [-1, 1]$. Define $\phi: Y \rightarrow X$ as the obvious map induced by inclusions on the images of ClX_- , ClX_+ , extended to send all of $z \times [-1, 1]$, $z \in P$, to $z \in P \subset X$. Lemma 3.1 assures that Y is a generalized 3-manifold. One can check quite easily that $\phi: Y \rightarrow X$ has all the right features. In particular, the (preferred) 2-polyhedron \tilde{P} called for in (2) can be spelled out as $\tilde{P} = P \times \{0\} \subset P \times [-1, 1]$, and the polyhedron J^* called for in (3) can be defined as

$$J^* = \tilde{P} \cup \phi^{-1}(J - P) \cup [P \cap Cl(J - P)] \times [-1, 1].$$

Conclusion (4) is obvious. Conclusion (5) is assured by Lemma 3.6. Finally, since each point preimage is a cell, cellularity of ϕ over $M(X)$ is guaranteed here, as well as in subsequent steps, by [15, Cor. 1.4] and [11, Prop. 18.4].

Impose a triangulation T on P . Locally the same procedure as in the model case works at interiors of all 2-simplexes $\sigma \in T$ and leads to a cell-like map $\phi_2: Y_2 \rightarrow X$ defined on a 3-gm Y_2 . When replacing $Int(\sigma)$ by $Int(\sigma) \times [-1, 1]$, σ a 2-simplex of T , the topology of Y_2 must be regulated so that given any sequence $\{p_n\}$ in $Int(\sigma)$ converging to $p_0 \in \partial\sigma$, then $p_n \times [-1, 1] \rightarrow p_0$.

The next step is to inflate the 1-skeleton $T^{(1)}$ of T , treated as a subset of Y_2 , to put it in the manifold set of another 3-gm Y_1 . At each 1-simplex $\tau \in T$, whereas $Int(\tau)$ has a neighborhood V_τ in X whose structure is represented schematically in Figure 1(a), the neighborhood $\phi_2^{-1}(V_\tau)$ in Y_2 has structure represented in Figure 1(b). This is the spot where the value of preferred 2-polyhedron is exposed. Each $\tau \in T^{(1)}$ is a face of 2-simplexes $\sigma_1, \sigma_2, \dots, \sigma_m$, $m \geq 1$, in T ; we presume these are arranged in a circular order, in the sense that both σ_j and σ_{j+1} ($j = 1, 2, \dots, m; j+1$ understood to be 1 when $j = m$) meet the frontier of some component W_j of $V_\tau - P$. With care in the construction of V_τ , we can assure that W_1, W_2, \dots, W_m constitute all the components of $V_\tau - P$.

The only significant difference between the structures in X or Y_2 and the schematics is that $Int(\tau)$ is an open interval, not just the special point in schematics. The segments emanating from that point in Figure 1 also must be enlarged by taking Cartesian products with that open interval, regarded as $Int(\tau)$.

In place of each $Int(\tau)$ we will insert $Int(\tau) \times B^2$ into Y_2 to form a new 3-gm Y_1 (topologized like Y_2) and cell-like map $\phi_1: Y_1 \rightarrow Y_2$, one which is conservative over $(Y_2 - |T^{(1)}|) \cup |T^{(0)}|$. Specifically, replace $Y_2 - |T^{(1)}|$ with the space obtained from the disjoint union of $Int(\tau) \times B^2$, thickened 2-simplexes $\sigma_i \times [-1, 1]$ and closures ClW_j of components of the various $\phi_2^{-1}(V_\tau - P)$ by attaching $Int(\tau) \times [-1, 1] \subset \sigma_i \times [-1, 1]$ to an arc of $Int(\tau) \times \partial B^2$, as shown in Figure 2, and (localized) by attaching ClW_j to

$$Int(\sigma_j) \times [-1, 1] \cup Int(\sigma_{j+1}) \times [-1, 1] \cup Int(\tau) \times \partial B^2$$

via the map sending $z \in ClW_j \cap (\sigma_j - \tau)$ to $z \times 1 \subset Int(\sigma_j) \times [-1, 1]$ and sending $z \in ClW_j \cap (\sigma_{j+1} - \tau)$ to $z \times -1 \subset Int(\sigma_{j+1}) \times [-1, 1]$. The cell-like map ϕ_1

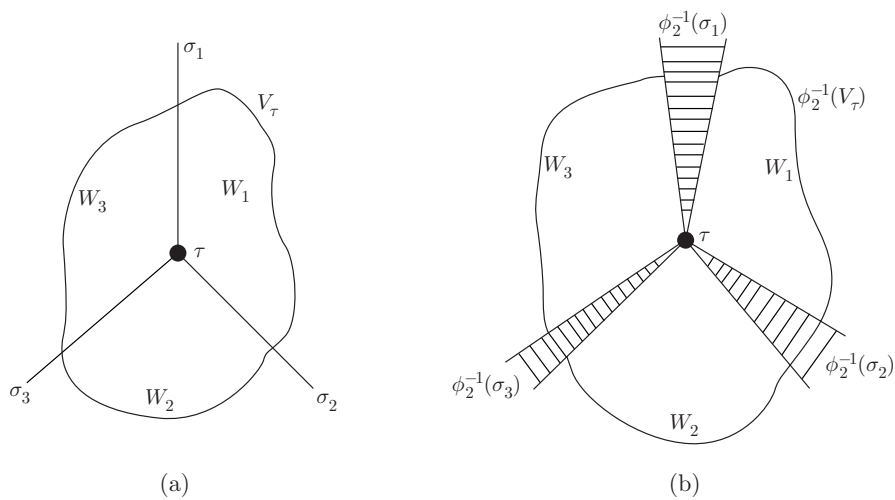


FIGURE 1.

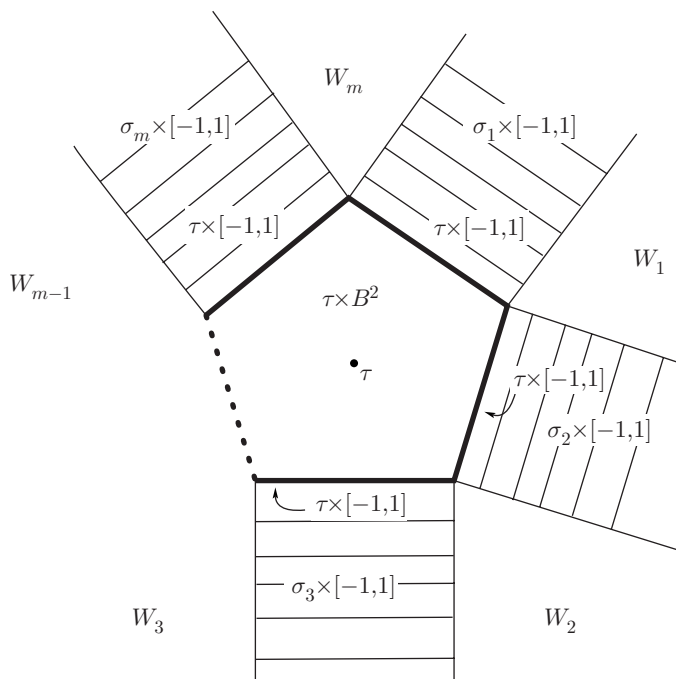


FIGURE 2.

amounts to first coordinate projection $Int(\tau) \times B^2 \rightarrow Int(\tau)$ wherever that makes sense; elsewhere it is conservative. Let \widetilde{P}_2 denote the preferred 2-polyhedron of (2) obtained in Y_2 , and let P_τ denote the product of $Int(\tau)$ with the segments in B^2

associated with τ ; then the preferred 2-polyhedron \widetilde{P}_1 of (2), contained, except for its 0-skeleton, in $M(Y_1)$, is the closure of $\phi_1^{-1}(\widetilde{P}_2 - |T^{(1)}|) \cup (\bigcup_{\tau} P_{\tau})$. The J^* at this stage are defined similarly. Note that each nontrivial point preimage under ϕ_1 meets $S(Y_1)$ in a finite set.

When $y_1 \in \text{Int}(\tau) \cap M(Y_2)$, $\tau \in T^{(1)}$, one can find a small neighborhood N of y_1 such that $\phi_1^{-1}(N)$ can be expressed as $(B^+ \times \mathbb{R}) \cup (\bigcup_i \text{Cl}(V_i))_{i=1}^s$, where $\{V_i\}$ is the collection of components of $N - \widetilde{P}_2$ and B^+ is a disk B to which s open collars on arcs $\alpha_1, \dots, \alpha_s$ are attached, where $\alpha_i \cap \alpha_{i+1} = a_i$ and $\text{Cl}(V_i) \cap (B \times \mathbb{R}) = a_i \times \mathbb{R}$. Repeated applications of Lemma 3.6, one for each component V_i , yields that $\phi_1^{-1}(M(Y_2))$ is a 3-manifold.

Finally we must blow up vertices $v \in T^{(0)}$ to 3-cells B_v . Each v has a well-defined link L_v in $P \subset X$ and a thickening $T_v = (\phi_2\phi_1)^{-1}(L_v)$ of a copy of L_v , namely, $T_v \cap \widetilde{P}_1$, to a compact 2-manifold with boundary. We argue that T_v embeds in a 2-sphere S_v .

Claim. The space S_v obtained by attaching a disk to each component of ∂T_v is a 2-sphere.

Proof of the Claim. There is a closed neighborhood C_v of v in Y_1 that meets $(\phi_2\phi_1)^{-1}(P)$ in a subset homeomorphic to the cone (from v) over T_v . Set $C_v^- = C_v \cap (\phi_2\phi_1)^{-1}(P)$. Replace the various component closures Z_1, \dots, Z_t of $C_v - C_v^-$ by 3-cells B_1, \dots, B_t with

$$B_j \cap C_v^- = Z_j \cap C_v^- = \text{cone}J (= 2\text{-cell}),$$

J representing a component of ∂T_v , to form $Q_v = C_v^- \cup \bigcup_j B_j$. Note that Q_v can be regarded as the cone from v over S_v , where S_v denotes T_v capped off with 2-cells, one in each $\partial B_j - \{v\}$. Clearly S_v is a 2-manifold. Moreover, each 1-cycle $[z]$ in $S_v - \{v\}$ is homologous to one in $T_v \subset S_v$. Since loops in T_v can be deformed in $X_v - \{v\}$ arbitrarily close to v and since $\text{Int}(C_v)$ is a 3-gm, each such loop λ is null homologous in $C_v - \{v\}$. In view of the fact that the various $B_j - \{v\}$ are absolute extensors, the inclusion $C_v^- - \{v\} \rightarrow Q_v - \{v\}$ factors through $C_v - \{v\}$. It follows that each λ is null-homologous in $Q_v - \{v\} \approx S_v \times [0, 1)$. Hence, $H_1(S_v) \cong 0$, and S_v must be a 2-sphere, which completes the proof of the Claim.

Continuing with the proof of 5.3, we regard S_v as the boundary of a 3-cell B_v , replace each $v \in T^{(0)}$ with B_v in a new 3-gm Y_0 , and define $\phi_0: Y_0 \rightarrow Y_1$ as the map sending each B_v to the associated vertex v and being conservative over the complement of the 0-skeleton $T^{(0)}$. Here the S_v of the Claim is modified by identifying each of the (abstractly) attached disks to points, which does not change S_v topologically. It has the benefit of providing a finite set $F_v \subset S_v$ such that $S_v - F_v$ has a neighborhood which meets $\phi_0^{-1}(Y_1 - |T^{(0)}|)$ in a 3-manifold thickening of $\phi_0^{-1}(\widetilde{P}_1 - |T^{(0)}|)$. The topology near B_v can be arranged so that the closure of $\phi_0^{-1}(\widetilde{P}_1 - |T^{(0)}|)$ meets S_v in a 1-dimensional polyhedron K_v . Again each $\phi_0^{-1}(z) \cap S(Y_0)$ is finite, and K_v is a strong deformation retract of $S_v - F_v$. The final $\widetilde{P} = \widetilde{P}_0$ is $\phi_0^{-1}(\widetilde{P}_1 - |T^{(0)}|) \cup (\text{cone over } K_v)$; the final J^* is obtained similarly.

The desired map will be $\phi = \phi_2\phi_1\phi_0$. As in Lemma 3.3, it is a near-homeomorphism over $M(X)$, so its retraction to $\phi^{-1}(M(X)) \subset M(Y)$ is cellular [11, Prop. 5.1]. □

The map $\phi: Y \rightarrow X$ in the conclusion of the preceding lemma will be called an *inflation of X at K* .

Lemma 5.4. *Suppose X is a 3-gm satisfying the RSAP. Then there exists a sequence $\{K_i, P_i\}_{i \geq 1}$ of tame-preferred polyhedral pairs in X , with $P_i \subset P_{i+1}$ for all $i \geq 1$, and there exists a sequence of maps $\mu_i: R_i \rightarrow X$ defined on compact 2-polyhedra R_i , with $\mu_i(R_i) \subset P_i$ for all $i \geq 1$, such that corresponding to any two points $x, x' \in X$ is an index $k \in \mathbb{N}$ for which μ_k homologically separates x from x' in X .*

Proof. Being treatable componentwise as a separable metric space, by an initial assumption, X has a countable basis Ω . Enumerate the countable collection of pairs $\Lambda = (W_j, W'_j)_{j=1}^\infty \in \Omega \times \Omega$ for which $Cl(W_j) \subset W'_j$ and some map $\nu_j: R_j \rightarrow X$, defined on a compact 2-polyhedron R_j , homologically separates $Cl(W_j)$ from $X - W'_j$. Lemma 3.4 assures that for any two points $x, x' \in X$ there is a pair $(W_j, W'_j) \in \Lambda$ with $x \in W_j$, $x' \in X - W'_j$.

Since X satisfies RSAP, Theorem 4.8 provides a tame-preferred polyhedron pair (K_1, P_1) in X and a map $\mu_1: R_1 \rightarrow X$ with $\mu_1(R_1) \subset P_1$, such that μ_1 homologically separates $Cl(W_1)$ and $X - W'_1$.

Assume that we have already produced a finite collection of tame-preferred polyhedral pairs $(K_1, P_1), (K_2, P_2), \dots, (K_t, P_t)$ in X with

$$P_1 \subset P_2 \subset \dots \subset P_t$$

and maps $\mu_j: R_j \rightarrow X$ with $\mu_j(R_j) \subset P_j$ and with μ_j strongly separating $Cl(W_j)$ and $X - W'_j$ ($j = 1, 2, \dots, t$). Again Theorem 4.8 provides a tame-preferred polyhedral pair (K_{t+1}, P_{t+1}) in X with $P_{t+1} \supset P_t$ and a map $\mu_{t+1}: R_{t+1} \rightarrow X$ with $\mu_{t+1}(R_{t+1}) \subset P_{t+1}$ such that μ_{t+1} homologically separates $Cl(W_{t+1})$ and $X - W'_{t+1}$. □

Lemma 5.5 (Resolution). *Suppose the 3-gm X contains a sequence $\{P_i\}_{i=1}^\infty$ of compact, preferred 2-polyhedra such that $P_i \subset P_{i+1}$ for all $i \geq 1$. Then there exist a 3-gm Y and a proper, cell-like, surjective map $\Phi: Y \rightarrow X$ satisfying the following conditions:*

- (i) every map $\mu: R \rightarrow P_k$, $k \in \mathbb{N}$, defined on a compact 2-polyhedra R has approximate lifts into $M(Y)$, and
- (ii) for each $p \in X$, $\dim[\Phi^{-1}(p) \cap S(Y)] \leq 0$.

Proof. Set $X_1 = X$ and $\{P_i^{(1)} = P_i\}_{i=1}^\infty$. By induction we will construct, for each $n \in \mathbb{N}$, a proper, cell-like map $\phi_{n+1,n}: X_{n+1} \rightarrow X_n$ together with a certain sequence, $\{P_i^{n+1}\}_{i=n+1}^\infty$, of compact, preferred 2-polyhedra in X_{n+1} . The desired map $\Phi: Y \rightarrow X$ will be the inverse limit of the inverse sequence of maps $\{X_n, \phi_{n+1,n}\}$.

Apply Inflation Lemma 5.3 to obtain an inflation $\phi_{2,1}: X_2 \rightarrow X_1$ at $P_1^{(1)} = P_1$. Among other features, this provides a 2-polyhedron $\widetilde{P}_1 \subset M(\phi_{2,1}^{-1}(P_1)) \subset X_2$ where $\phi_{2,1}|: \widetilde{P}_1 \rightarrow P_1$ is cell-like. Let $\{P_i^{(2)}\}_{i=2}^\infty$ be approximate lifts of P_i described in conclusion (3) there. Assuming cell-like maps $\phi_{n+1,n}: X_{n+1} \rightarrow X_n$ defined on 3-gms X_{n+1} have been obtained for $n = 1, 2, \dots, t$, along with approximate lifts $\{P_i^{(n+1)}\}_{i=n+1}^\infty$ of $\{P_i^{(n)}\}_{i=n+1}^\infty$, and 2-polyhedra $\widetilde{P}_n \subset M(\phi_{n+1,n}^{-1}(P_n^{(n)})) \subset X_{n+1}$ for which $\phi_{n+1,n}|: \widetilde{P}_n \rightarrow P_n^{(n)}$ is cell-like, apply Inflation Lemma 5.3 again to obtain an inflation of X_{n+1} at $P_{n+1}^{(n+1)}$, thereby producing the next level of objects for $n = t + 1$.

We conclude immediately from Lemma 3.2 that the inverse sequence $\{X_n, \phi_{n+1,n}\}$ has inverse limit $\Phi: Y \rightarrow X_1 = X$, with Y a 3-gm and Φ a cell-like map.

To verify conclusion (i), note that any map $\mu: R \rightarrow P_k$ can be approximately lifted, successively, to maps $\mu_i: R \rightarrow P_k^{(i)}$, $i = 1, 2, \dots, k$, and, finally, to $\mu_{k+1}: R \rightarrow \widetilde{P}_k \subset M(\phi_{k+1,k}^{-1}(P_k^{(k)})) \subset X_{k+1}$. According to Lemma 3.3, $\Phi_{k+1}^{-1}(M(X_{k+1}))$ is a 3-manifold (where Φ_{k+1} satisfies $\Phi = \phi_{k+1,0}\Phi_{k+1}$). Hence, μ has approximate lifts to $M(Y)$.

To verify conclusion (ii), let A_0 denote $\{p\}$ and recursively let A_n denote $\phi_{n,n-1}^{-1}(A_{n-1}) - M(X_n)$ for $n \in \mathbb{N}$. Each set A_n is finite, by conclusion (4) of Lemma 5.3. Furthermore, $\Phi^{-1}(p) \cap S(Y) \subset A_\infty = \varprojlim A_n$. But the inverse limit of finite sets is 0-dimensional. \square

Corollary 5.6. *A 3-gm X has a near-resolution if there exist a sequence $\{P_i\}_{i=1}^\infty$ of preferred 2-polyhedra in X and a family of maps $\{\mu_i: R_i \rightarrow X\}_{i=1}^\infty$ satisfying the following conditions:*

- (i) $\mu_i(R_i) \subset P_i$ for every $i \geq 1$,
- (ii) $P_i \subset P_{i+1}$ for every $i \geq 1$, and
- (iii) given distinct points $p, q \in X$ there exists $k \in \mathbb{N}$ such that μ_k homologically separates p from q .

Proof. Applying Resolution Lemma 5.5 to X and $\{P_i\}_{i=1}^\infty$, we obtain $\Phi: Y \rightarrow X$ such that, for all $x \in X$, $S(Y) \cap \Phi^{-1}(x)$ is 0-dimensional. We will show that $\dim S(Y) \leq 0$, which will imply that Y has a near resolution $\psi: M \rightarrow Y$. The near-resolution of X then will be $\Phi\psi: M \rightarrow X$.

To show that $\dim S(Y) \leq 0$, we first establish the following

Claim. For any two distinct points p and q of X there exists a map $\kappa: R \rightarrow Y$ defined on a compact 2-polyhedron R such that $\kappa(R)$ strongly separates $\Phi^{-1}(p)$ from $\Phi^{-1}(q)$ and $\kappa(R) \subset M(Y)$.

Proof of the Claim. Choose $i \in \mathbb{N}$ such that μ_i homologically separates p and q in X . Endow X with a metric, and choose $\epsilon > 0$ such that any ϵ -approximation to μ_i is homotopic to μ_i in $X - \{p, q\}$. By Resolution Lemma 5.5 μ_i has an ϵ -lift κ to $M(Y)$. Since $\Phi\kappa$ is homotopic to μ_i in $X - \{p, q\}$ and Φ restricts to a proper homotopy equivalence of the pairs

$$(Y, Y - \Phi^{-1}(\{p, q\})) \rightarrow (X, X - \{p, q\}),$$

it follows that κ homologically separates $\Phi^{-1}(p)$ and $\Phi^{-1}(q)$. By Lemma 3.5, κ strongly separates $\Phi^{-1}(p)$ and $\Phi^{-1}(q)$.

Given a component C of $S(Y)$, one can immediately produce an $x_C \in X$ for which $C \subset \Phi^{-1}(x_C)$, using the Claim. Since $C \subset S(Y) \cap \Phi^{-1}(x_C)$ and since $\dim[S(Y) \cap \Phi^{-1}(x_C)] \leq 0$ by conclusion (ii) of Lemma 5.5, C must be a singleton. Hence, $\dim S(Y) \leq 0$. \square

Proof of Theorem 5.1. Apply Lemma 5.4 to obtain a sequence $\{(K_i, P_i)\}_{i \geq 1}$ of compact, tame-preferred polyhedral pairs such that $P_i \subset P_{i+1}$ for all $i \geq 1$ and any two points of X are homologically separated by some map $\mu: R \rightarrow P_k$ into one of these P_i . Corollary 5.6 assures that X has a near-resolution. \square

REFERENCES

1. F. D. Ancel, *An alternative proof of M. Brown's theorem on inverse sequences of near homeomorphisms*, Geometric topology and shape theory (Dubrovnik, 1986), pp. 1-2, Lecture Notes in Math., vol. 1283, Springer, Berlin, New York, 1987. MR89d:54003

2. Steve Armentrout, *Cellular decompositions of 3-manifolds that yield 3-manifolds*, Memoirs Amer. Math. Soc., No. 107, Amer. Math. Soc., Providence, RI, 1971. MR54:1225
3. R. H. Bing, *A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3* , Ann. of Math. (2) **65** (1957), 484–500. MR19:1187g
4. R. H. Bing, *A surface is tame if its complement is 1-ULC*, Trans. Amer. Math. Soc. **101** (1961), 294–305. MR24:A1117
5. G. E. Bredon, *Generalized manifolds, revisited*, 1970 Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, Ga, 1969), pp. 461–469, Markham, Chicago, Ill., 1970. MR43:1193
6. Morton Brown, *Some applications of an approximation theorem for inverse limits*, Proc. Amer. Math. Soc. **11** (1960), 478–483. MR22:5959
7. J. Bryant, S. Ferry, W. Mio and S. Weinberger, *Topology of homology manifolds*, Ann. of Math. (2) **143** (1996), no. 3, 435–467. MR97b:57017
8. J. W. Cannon, *ULC properties in neighbourhoods of embedded surfaces and curves in E^3* , Canad. J. Math. **25** (1973), 31–73. MR47:2589
9. J. W. Cannon, *The recognition problem: what is a topological manifold?* Bull. Amer. Math. Soc. **84** (1978), 832–866. MR58:13043
10. R. J. Daverman, *A new proof for the Hosay-Lininger theorem about crumpled cubes*, Proc. Amer. Math. Soc. **23** (1969), 52–54. MR39:7578
11. R. J. Daverman, *Decompositions of Manifolds*, Pure and Applied Mathematics, 124, Academic Press, Inc., Orlando, Fl., 1986. MR88a:57001
12. R. J. Daverman and D. R. Repovš, *General position properties that characterize 3-manifolds*, Canad. J. Math. **44** (1992), 234–251. MR93d:57038
13. R. D. Edwards, *The topology of manifolds and cell-like maps*, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), pp. 111–127, Acad. Sci. Fennica, Helsinki, 1980. MR81g:57010
14. L. L. Lininger, *Some results on crumpled cubes*, Trans. Amer. Math. Soc. **118** (1965), 534–549. MR31:2717
15. D. R. McMillan, Jr., *A criterion for cellularity in a manifold. II*. Trans. Amer. Math. Soc. **126** (1967), 217–224. MR34:8392
16. W. J. R. Mitchell, *Defining the boundary of a homology manifold*, Proc. Amer. Math. Soc. **110** (1990), 509–513. MR90m:57015
17. W. J. R. Mitchell and D. Repovš, *The topology of cell-like mappings*, Conference on Differential Geometry and Topology (Sardinia, 1988), Rend. Sem. Fac. Sci. Univ. Cagliari **58** (1988), suppl., 265–300. MR92f:54012
18. Victor A. Nicholson, *1-FLG complexes are tame in 3-manifolds*, General Topology and Appl. **2** (1972), 277–285. MR47:5877
19. Frank Quinn, *An obstruction to the resolution of homology manifolds*, Michigan Math. J. **34** (1987), 285–291. MR88j:57016
20. Frank Raymond, *Separation and union theorems for generalized manifolds with boundary*, Michigan Math. J. **7** (1960), 7–21. MR22:11388
21. L. C. Siebenmann, *Approximating cellular maps by homeomorphisms*, Topology **11** (1972), 271–294. MR45:4431
22. T. L. Thickstun, *An extension of the Loop theorem and resolutions of generalized 3-manifolds with 0-dimensional singular set*, Invent. Math. **78** (1984), 161–222. MR86f:57007
23. T. L. Thickstun, *Resolutions of generalized 3-manifolds whose singular sets have general position dimension one*, Topology Appl. **138** (2004), 61–95. MR2005b:57043
24. R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publ., vol. 32, Amer. Math. Soc., Providence, RI, 1979. MR82a:57001

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TENNESSEE AT KNOXVILLE, KNOXVILLE, TENNESSEE 37996-1300

E-mail address: daverman@math.utk.edu

DEPARTMENT OF MATHEMATICS, TEXAS STATE UNIVERSITY, SAN MARCOS, TEXAS 78666

E-mail address: tt04@txstate.edu