

LUSTERNIK-SCHNIRELMANN CATEGORY OF $\mathbf{Spin}(9)$

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ABSTRACT. First we give an upper bound of $\text{cat}(E)$, the L-S category of a principal G -bundle E for a connected compact group G with a characteristic map $\alpha : \Sigma V \rightarrow G$. Assume that there is a cone-decomposition $\{F_i \mid 0 \leq i \leq m\}$ of G in the sense of Ganea that is compatible with multiplication. Then we have $\text{cat}(E) \leq \text{Max}(m+n, m+2)$ for $n \geq 1$, if α is compressible into $F_n \subseteq F_m \simeq G$ with trivial higher Hopf invariant $H_n(\alpha)$. Second, we introduce a new computable lower bound, $\text{Mwgt}(X; \mathbb{F}_2)$ for $\text{cat}(X)$. The two new estimates imply $\text{cat}(\mathbf{Spin}(9)) = \text{Mwgt}(\mathbf{Spin}(9); \mathbb{F}_2) = 8 > 6 = \text{wgt}(\mathbf{Spin}(9); \mathbb{F}_2)$, where $(\text{wgt}-; R)$ is a category weight due to Rudyak and Strom.

INTRODUCTION

In this paper, we work in the category of connected CW-complexes and continuous maps. The Lusternik-Schnirelmann category $\text{cat}(X)$, L-S category for short, is the least integer m such that there is a covering of X by $(m+1)$ open subsets each of which is contractible in X . Ganea introduced a stronger notion of L-S category, $\text{Cat}(X)$ the strong L-S category of X , which is equal to the cone-length by Ganea [4], that is, the least integer m such that there is a set of cofibre sequences $\{A_i \rightarrow X_{i-1} \hookrightarrow X_i\}_{1 \leq i \leq m}$ with $X_0 = \{*\}$ and $X_m \simeq X$. Then by Ganea [4], we have $\text{cat}(X) \leq \text{Cat}(X) \leq \text{cat}(X)+1$. Throughout this paper, we follow the notations in [12], which is based on [9, 10]: For a map $f : S^k \rightarrow X$, a homotopy set of higher Hopf invariants $H_m(f) = \{[H_m^\sigma(f)] \mid \sigma \text{ is a structure map of } \text{cat}(X) \leq m\}$ (or its stabilisation $\mathcal{H}_m(f) = \Sigma_*^\infty H_m(f)$) is referred to simply as a *(stabilized) higher Hopf invariant* of f , which plays a crucial role in this paper.

A computable lower estimate is given by the classical cup-length. Here we give its definition in a slightly general fashion, which is inspired by the proof of $\text{cat}(\mathbf{Sp}(2)) = 3$ given in [14].

Definition 0.1 (I. [12]). (1) Let h be a multiplicative generalized cohomology.

$$(2) \quad \begin{aligned} \text{cup}(X; h) &= \text{Min} \left\{ m \geq 0 \mid \forall \{v_0, \dots, v_m \in \tilde{h}^*(X)\} \ v_0 \cdot v_1 \cdot \dots \cdot v_m = 0 \right\}, \\ \text{cup}(X) &= \text{Max} \left\{ \text{cup}(X; h) \mid h \text{ is a multiplicative generalized} \right. \\ &\quad \left. \text{cohomology} \right\}. \end{aligned}$$

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Then we have $\text{cup}(X; h) \leq \text{cup}(X) \leq \text{cat}(X)$ for any multiplicative generalized cohomology h . When h is the ordinary cohomology with a coefficient ring R , we denote $\text{cup}(X; h)$ by $\text{cup}(X; R)$. This definition immediately implies the following.

Remark 0.2. $\text{cup}(X) = \text{Min} \left\{ m \geq 0 \mid \tilde{\Delta}^{m+1} : X \rightarrow \wedge^{m+1} X \text{ is stably trivial} \right\}$.

Let $\{p_k^{\Omega X} : E^k(\Omega X) \rightarrow P^{k-1}(\Omega X); k \geq 1\}$ be the A_∞ -structure of ΩX in the sense of Stasheff [19] (see also Iwase-Mimura [13] for some more properties). The relation between an A_∞ -structure and an L-S category gives the key observation in [9, 10, 11] producing counter-examples to the Ganea conjecture on L-S category. On the other hand, Rudyak [18] and Strom [20] introduced a homotopy theoretical version of Fadell-Husseini's category weight (see [3]), which can be described as follows, for an element $u \in h^*(X)$ and a generalized cohomology h :

$$\begin{aligned} \text{wgt}(u; h) &= \text{Min} \left\{ m \geq 0 \mid (e_m^X)^*(u) \neq 0 \text{ in } h^*(P^m(\Omega X)) \right\}, \\ \text{wgt}(X; h) &= \text{Min} \left\{ m \geq 0 \mid \begin{array}{l} (e_m^X)^* : h^*(X) \rightarrow h^*(P^m(\Omega X)) \\ \text{is a monomorphism} \end{array} \right\}, \end{aligned}$$

where e_m^X denotes the map $P^m(\Omega X) \hookrightarrow P^\infty(\Omega X) \simeq X$. Then we easily see

$$(0.1) \quad \text{wgt}(X; h) = \text{Max} \{ \text{wgt}(u; h) \mid u \in \tilde{h}^*(X) \}.$$

We remark that $\text{wgt}(u; h) = s$ if and only if u represents a non-zero class in $E_\infty^{s,*}$ of bar spectral sequence $\{(E_r^{*,*}, d_r^{*,*}) \mid r \geq 1\}$ converging to $h^*(X)$ with $E_2^{*,*} \cong \text{Ext}_{h(\Omega X)}^{*,*}(h_*, h_*)$. When h is the ordinary cohomology with a coefficient ring R , we denote $\text{wgt}(X; h)$ by $\text{wgt}(X; R)$. In this paper, we introduce new computable invariants as follows.

Definition 0.3. Let h be a generalized cohomology: A homomorphism $\phi : h^*(X) \rightarrow h^*(Y)$ is called an h -morphism if it preserves the actions of all (unstable) cohomology operations on h^* .

Definition 0.4 (I. [12]). (1) Let h be a generalized cohomology and X a space. A module weight $\text{Mwgt}(X; h)$ of X with respect to h is defined as follows:

$$\text{Mwgt}(X; h) = \text{Min} \left\{ m \geq 0 \mid \begin{array}{l} \text{there is an } h\text{-morphism } \phi : h^*(P^m(\Omega X)) \rightarrow \\ h^*(X), \text{ which is a left homotopy inverse of } (e_m^X)^* \end{array} \right\}.$$

When h is the ordinary cohomology with coefficients in a ring R , we denote $\text{Mwgt}(X; h)$ by $\text{Mwgt}(X; R)$. These invariants satisfy the following inequalities:

$$\text{cup}(X; R) \leq \text{wgt}(X; R) \leq \text{Mwgt}(X; R) \leq \text{cat}(X).$$

Similar to the above definition of $\text{cup}(X)$, we define the following invariants.

Definition 0.5 (I. [12]). (1) $\text{wgt}(X) = \text{Max} \left\{ \text{wgt}(X; h) \mid \begin{array}{l} h \text{ is a generalized} \\ \text{cohomology} \end{array} \right\}$.

(2) $\text{Mwgt}(X) = \text{Max} \{ \text{Mwgt}(X; h) \mid h \text{ is a generalized cohomology} \}$.

Remark 0.6. Let $\text{rcat}(-)$ be Rudyak's stable L-S category, which is denoted as $r(-)$ in [18]. Then we have $\text{cup}(X) \leq \text{wgt}(X) = \text{rcat}(X) \leq \text{Mwgt}(X) \leq \text{cat}(X)$.

Let us denote by $Z^{(k)}$ the k -skeleton of a CW complex Z . To give an upper-bound for the L-S category of the total space of a fibre bundle $F \hookrightarrow E \rightarrow B$, we need a refinement of results of Varadarajan [21] and Hardie [6], and a corresponding result for the strong category of Ganea [4].

Theorem 0.7 ([21, 6, 4]). (1) $\text{cat}(E)+1 \leq (\text{cat}(F)+1) \cdot (\text{cat}(B)+1)$.
 (2) $\text{Cat}(E)+1 \leq (\text{Cat}(F)+1) \cdot (\text{Cat}(B)+1)$.

In [16], Iwase-Mimura-Nishimoto gave such a refinement in the case when the base space B is non-simply connected. But in this paper, we give another refinement in the case when the fibre bundle is a principal bundle over a double suspension space: Let G be a compact Lie group with a cone-decomposition of length m :

$$(m \text{ cofibre sequences}) \quad K_i \xrightarrow{P^i} F_{i-1} \hookrightarrow F_i, \quad i \geq 1,$$

with $F_0 = \{*\}$ and $F_i = F_m \simeq G, i \geq m$. Then we obtain $\sigma_k : F_k \rightarrow P^k \Omega F_k$ for all $k \leq m$ as a right homotopy inverse of $e_k : P^k \Omega F_k \rightarrow P^\infty \Omega F_k \simeq F_k$ by induction on $k \geq 1$. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc} \{*\} & \hookrightarrow & F_1 & \hookrightarrow & \cdots & \hookrightarrow & F_m \\ \sigma_0 \downarrow & & \sigma_1 \downarrow & & & & \sigma_m \downarrow \\ \{*\} & \hookrightarrow & P^1 \Omega F_1 & \hookrightarrow & \cdots & \hookrightarrow & P^m \Omega F_m \\ e_0 \downarrow & & e_1 \downarrow & & & & e_m \downarrow \\ \{*\} & \hookrightarrow & F_1 & \hookrightarrow & \cdots & \hookrightarrow & F_m, \end{array}$$

where $e_k \circ \sigma_k \sim 1_{F_k}$ for all $k \leq m$.

Theorem 0.8. *Let $G \hookrightarrow E \rightarrow \Sigma^2 V$ be a principal bundle with characteristic map $\alpha : A = \Sigma V \rightarrow G$. Then we have $\text{cat}(E) \leq \text{Max}(m+n, m+2)$ for $n \geq 1$, if*

- (1) α is compressible into $F_n \subseteq F_m \simeq G$,
- (2) $H_n^{\sigma_n}(\alpha) = 0$ and
- (3) the restriction of the multiplication $\mu : G \times G \rightarrow G$ to $F_j \times F_n \subseteq F_m \times F_m \simeq G \times G$ is compressible into $F_{j+n} \subseteq F_m \simeq G, j \geq 0$ as $\mu_{j,n} : F_j \times F_n \rightarrow F_{j+n}$ such that $\mu_{j,n}|_{F_{j-1} \times F_n} = \mu_{j-1,n}$.

Remark 0.9. If we choose $n = m+1$, then the assumptions (1) through (3) above are automatically satisfied. Thus we always have $\text{Cat}(E) \leq 2 \text{Cat}(G)+1$ which is a special case of Ganea’s theorem (see Theorem 0.7(2)).

For $\mathbf{Spin}(9)$, we first observe the following result.

Theorem 0.10. $\text{Mwgt}(\mathbf{Spin}(9); \mathbb{F}_2) \geq 8 > 6 = \text{wgt}(\mathbf{Spin}(9); \mathbb{F}_2)$.

These results imply the following result.

Theorem 0.11. $\text{cat}(\mathbf{Spin}(9)) = \text{Mwgt}(\mathbf{Spin}(9); \mathbb{F}_2) = 8$.

1. PROOF OF THEOREM 0.8

From now on, we work in the homotopy category, and so we do not distinguish G and F_m . Let $G \hookrightarrow E \rightarrow \Sigma^2 V$ be a principal bundle with characteristic map $\alpha : A = \Sigma V \rightarrow G$. The assumptions (1) and (3) in Theorem 0.8 allow us to construct a filtration $\{E_k\}_{0 \leq k \leq n+m}$ of E : By using the James-Whitehead decomposition (see Theorem 1.15 of Whitehead [22]), we have

$$E = G \cup_\psi G \times CA, \quad \psi = \mu \circ (1_G \times \alpha) : G \times A \rightarrow G.$$

First, we define two filtrations of E as follows:

$$E_k = \begin{cases} F_k, & k \leq n, \\ F_k \cup_{\psi_{k-n-1}} F_{k-n-1} \times CA, & n < k \leq m+n, \end{cases}$$

$$E'_k = \begin{cases} F_k, & k < n, \\ F_k \cup_{\psi_{k-n}} F_{k-n} \times CA, & n \leq k \leq m+n, \end{cases}$$

where $\psi_j = \mu_{j,n} \circ (\alpha \times 1) : F_j \times A \rightarrow F_{j+n}$ and $E = E'_{m+n}$, which immediately imply that $\text{cat}(E) = \text{cat}(E'_{m+n})$.

By using the assumption (3), we obtain the following cofibre sequences, similar to the arguments given in [16]:

$$K_{k+1} \rightarrow E_k \hookrightarrow E_{k+1}, \text{ for } 0 \leq k < n,$$

$$K_{k+1} \vee (K_{k-n} * A) \rightarrow E_k \hookrightarrow E_{k+1}, \text{ for } n \leq k < m+n,$$

$$K_{k-n} * A \rightarrow E_k \hookrightarrow E'_k.$$

Similar to the arguments given in [15, 16], we obtain

$$(1.1) \quad \text{cat}(E_k) \leq k \text{ and } \text{cat}(E'_k) \leq k+1, \text{ for any } k \geq n,$$

by induction on k . The following lemma can be deduced in a similar but easier manner to the main theorem of [11], using $H_n^{\sigma_n}(\alpha) = 0$, the assumption (2).

Lemma 1.1. $\text{cat}(E'_{j+n}) \leq j+n$ for all $j \geq 0$ and $n \geq 2$.

Lemma 1.1 and (1.1) imply $\text{cat}(E) = \text{cat}(E'_{m+n}) \leq \text{Max}(m+n, m+2)$, and hence we are left to show Lemma 1.1.

Proof of Lemma 1.1. We define a map $\hat{\psi}_j$ as follows:

$$\hat{\psi}_j = \sigma_{j+n} \circ \mu_{j,n} \circ (e_j \times \alpha) : P^j \Omega F_j \times A \rightarrow P^{j+n} \Omega F_{j+n}.$$

Then we have $\hat{\psi}_j \circ (\sigma_j \times 1) = \sigma_{j+n} \circ \mu_{j,n} \circ (e_j \times \alpha) \circ (\sigma_j \times 1) \sim \sigma_{j+n} \circ \mu_{j,n} \circ (1 \times \alpha) = \sigma_{j+n} \circ \psi_j$ and $e_{j+n} \circ \hat{\psi}_j = e_{j+n} \circ \sigma_{j+n} \circ \mu_{j,n} \circ (e_j \times \alpha) \sim \mu_{j,n} \circ (e_j \times \alpha) = \psi_j \circ (e_j \times 1)$. Thus the following diagram is commutative up to homotopy:

$$(1.2) \quad \begin{array}{ccccc} F_j & \xleftarrow{\text{pr}_1} & F_j \times A & \xrightarrow{\psi_j} & F_{j+n} \\ \sigma_j \downarrow & & \sigma_j \times 1 \downarrow & & \downarrow \sigma_{j+n} \\ P^j \Omega F_j & \xleftarrow{\text{pr}_1} & P^j \Omega F_j \times A & \xrightarrow{\hat{\psi}_j} & P^{j+n} \Omega F_{j+n} \\ e_j \downarrow & & e_j \times 1 \downarrow & & \downarrow e_{j+n} \\ F_j & \xleftarrow{\text{pr}_1} & F_j \times A & \xrightarrow{\psi_j} & F_{j+n} \end{array}$$

Therefore, the space $E'_{j+n} = F_{j+n} \cup_{\psi_j} F_j \times CA$ is dominated by $P^{j+n} \Omega F_{j+n} \cup_{\hat{\psi}_j} P^j \Omega F_j \times CA$. Since α satisfies $H_n^{\sigma_n}(\alpha) = 0$, we have the following commutative diagram up to homotopy:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & F_m \\ \Sigma \text{ad } \alpha \downarrow & & \downarrow \sigma_n \\ \Sigma \Omega F_n & \hookrightarrow & P^n \Omega F_n \end{array}$$

where $\text{ad } \alpha : V \rightarrow \Omega F_n$ is the adjoint map of $\alpha : A = \Sigma V \rightarrow F_n$. Thus $\sigma_n \circ \alpha$ is compressible into $\Sigma \Omega F_n$, and hence we have

$$\begin{aligned} \hat{\psi}_j &\sim \sigma_{j+n} \circ \mu_{j,n} \circ (1 \times (e_n \circ \sigma_n)) \circ (e_j \times \alpha) \sim \sigma_{j+n} \circ \mu_{j,n} \circ (e_j \times e_n) \circ (1 \times (\sigma_n \circ \alpha)) \\ &\sim \sigma_{j+n} \circ \mu_{j,n} \circ (e_j \times e_n) \circ (1 \times \Sigma \text{ad } \alpha) = \hat{\psi}'_j|_{P^i \Omega F_j \times \Sigma \Omega F_n} \circ (1 \times \Sigma \text{ad } \alpha), \end{aligned}$$

where $\hat{\psi}'_j = \sigma_{j+n} \circ \mu_{j,n} \circ (e_j \times e_n)$. Since $\text{Cat}(P^i \Omega F_j \times \Sigma \Omega F_n) \leq i+1$, we have that $\hat{\psi}'_j|_{P^i \Omega F_j \times \Sigma \Omega F_n}$ can be compressible into $P^{i+1} \Omega F_{j+n}$ for $i \leq j$. This yields the following cone-decomposition of $P^{j+n} \Omega F_{j+n} \cup_{\hat{\psi}_j} P^j \Omega F_j \times CA$:

$$\begin{aligned} \Omega F_{j+n} &\rightarrow \{*\} \hookrightarrow P^1 \Omega F_{j+n}, \\ E^2 \Omega F_{j+n} \vee A &\rightarrow P^1 \Omega F_{j+n} \hookrightarrow P^2 \Omega F_{j+n} \cup CA, \\ E^{i+2} \Omega F_{j+n} \vee E^i \Omega F_j * A &\rightarrow P^{i+1} \Omega F_{j+n} \cup P^{i-1} \Omega F_j \times CA \\ &\hookrightarrow P^{i+2} \Omega F_{j+n} \cup P^i \Omega F_j \times CA, \quad 0 < i \leq j, \\ E^{j+i} \Omega F_{j+n} &\rightarrow P^{j+i-1} \Omega F_{j+n} \cup P^j \Omega F_j \times CA \\ &\hookrightarrow P^{j+i} \Omega F_{j+n} \cup P^j \Omega F_j \times CA, \quad 2 < i \leq n, \end{aligned}$$

for any $j \geq 0$ and $n \geq 2$. This implies $\text{Cat}(P^{j+n} \Omega F_{j+n} \cup_{\hat{\psi}_j} P^j \Omega F_j \times CA) \leq j+n$ for all $j \geq 0$ and $n \geq 2$, and hence $\text{cat}(E'_{j+n}) = \text{cat}(F_{j+n} \cup F_j \times CA) \leq j+n$ for all $j \geq 0$ and $n \geq 2$. \square

This completes the proof of Theorem 0.8.

2. BAR SPECTRAL SEQUENCE

To calculate our module weight $\text{Mwgt}(X; \mathbb{F}_2)$ together with $\text{wgt}(X; \mathbb{F}_2)$, we need to know the module structure of $H^*(P^m(\mathbf{Spin}(9)); \mathbb{F}_2)$ over the Steenrod algebra modulo 2. By Borel [1], Bott [2], Ishitoya-Kono-Toda [7], Hamanaka-Kono [5] and Kono-Kozima [17], the following are known:

$$\begin{aligned} H^*(\mathbf{Spin}(9); \mathbb{F}_2) &= \mathbb{F}_2[x_3]/(x_3^4) \otimes \bigwedge_{\mathbb{F}_2}(x_5, x_7, x_{15}), \\ Sq^2 x_3 &= x_5, \quad Sq^1 x_5 = x_6, \quad x_i \in H^i(\mathbf{Spin}(9); \mathbb{F}_2), \\ H_*(\Omega \mathbf{Spin}(9); \mathbb{F}_2) &= \bigwedge_{\mathbb{F}_2}(u_2) \otimes \mathbb{F}_2[u_4, u_6, u_{10}, u_{14}], \\ u_4 Sq^2 &= u_2, \quad u_{10} Sq^2 = u_4^2, \quad u_{14} Sq^4 = u_{10}, \quad u_{2i} \in H_{2i}(\Omega \mathbf{Spin}(9); \mathbb{F}_2), \end{aligned}$$

where we denote by $\bigwedge_R(a_{i_1}, \dots, a_{i_t})$ the exterior algebra on a_{i_1}, \dots, a_{i_t} over R . We remark that the cohomology suspension of x_{2i+1} is non-trivially given by u_{2i} for $i = 1, 2, 3$ and 7. To determine $H^*(P^m(\mathbf{Spin}(9)); \mathbb{F}_2)$, we have to study the bar spectral sequence $(E_r^{*,*}, d_r^{*,*})$ converging to $H^*(\mathbf{Spin}(9); \mathbb{F}_2)$:

$$\begin{aligned} E_1^{s,t} &\cong \tilde{H}^{s+t}(P^s(\Omega \mathbf{Spin}(9)), P^{s-1}(\Omega \mathbf{Spin}(9)); \mathbb{F}_2) \cong \tilde{H}^t(\bigwedge^s \Omega \mathbf{Spin}(9); \mathbb{F}_2), \\ D_1^{s,t} &\cong \tilde{H}^{s+t}(P^s(\Omega \mathbf{Spin}(9)); \mathbb{F}_2), \\ E_2^{*,*} &\cong \text{Ext}_{H_*(\Omega \mathbf{Spin}(9); \mathbb{F}_2)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[x_{1,2}] \otimes \bigwedge_{\mathbb{F}_2}(x_{1,4}, x_{1,6}, x_{1,10}, x_{1,14}), \\ E_\infty^{*,*} &\cong \tilde{H}^*(\mathbf{Spin}(9); \mathbb{F}_2) \cong \mathbb{F}_2[x_{1,2}]/(x_{1,2}^4) \otimes \bigwedge_{\mathbb{F}_2}(x_{1,4}, x_{1,6}, x_{1,14}), \end{aligned}$$

where $x_{1,2}, x_{1,4}, x_{1,6}$ and $x_{1,14}$ are permanent cycles by [17]. Therefore, there is only one differential $d_a(x_{1,10})$ ($a \geq 2$) which is possibly non-trivial, and we have $E_a^{*,*} \cong E_2^{*,*}$ and $E_{a+1}^{*,*} \cong E_\infty^{*,*}$. Since x_3 is of height 4 in $H^*(\mathbf{Spin}(9); \mathbb{F}_2)$, we have $d_a(x_{1,10}) = x_{1,2}^4$, and hence $a = 3$. Thus we have the following:

$$d_r = 0 \text{ if } r \neq 3, \quad d_3(x_{1,i}) = 0 \text{ if } i \neq 10, \quad d_3(x_{1,10}) = x_{1,2}^4,$$

$$E_2^{*,*} \cong E_3^{*,*} \cong \mathbb{F}_2[x_{1,2}] \otimes \bigwedge_{\mathbb{F}_2}(x_{1,4}, x_{1,6}, x_{1,10}, x_{1,14}),$$

$$E_4^{*,*} \cong E_\infty^{*,*} \cong \mathbb{F}_2[x_{1,2}]/(x_{1,2}^4) \otimes \bigwedge_{\mathbb{F}_2}(x_{1,4}, x_{1,6}, x_{1,14}).$$

By truncating the above computations with the same differential d_r to the spectral sequence for $P^m(\mathbf{Spin}(9))$ of Stasheff's type (similar to the computation in [8]), we are led to the following proposition, and we leave the details to the reader.

Proposition 2.1. *Let $A = \mathbb{F}_2[x_3]/(x_3^4) \otimes \bigwedge_{\mathbb{F}_2}(x_5, x_7, x_{15})$. Then for $m \geq 0$, we have*

$$H^*(P^m(\mathbf{Spin}(9)); \mathbb{F}_2) \cong \begin{cases} A^{[0]} \cong \mathbb{F}_2, & \text{if } m = 0, \\ A^{[m]} \oplus x_{11} \cdot A^{[m-1]} \oplus S_m, & \text{if } 3 \geq m \geq 1, \\ A^{[m]} \oplus x_{11} \cdot (A^{[m-1]}/A^{[m-4]}) \oplus S_m, & \text{if } m \geq 4, \end{cases}$$

as modules, where $A^{[m]}$ ($m \geq 0$) denotes the quotient module $A/D^{m+1}(A)$ of A by the submodule $D^{m+1}(A) \subseteq A$ generated by all the products of $m+1$ elements in positive dimensions, $x_{11} \cdot (A^{[m-1]}/A^{[m-4]})$ ($m \geq 4$) denotes a submodule corresponding to a submodule in $\mathbb{F}_2[x_3]/(x_3^4) \otimes \bigwedge_{\mathbb{F}_2}(x_5, x_7, x_{11}, x_{15})$ and S_m satisfies $S_m \cdot \tilde{H}^*(P^m(\mathbf{Spin}(9)); \mathbb{F}_2) = 0$ and $S_m|_{P^{m-1}(\mathbf{Spin}(9))} = 0$.

Some more comments might be required to the second direct summand of the above expressions of $H^*(P^m(\mathbf{Spin}(9)); \mathbb{F}_2)$, $m \geq 4$. The multiplication with x_{11} is a fancy notation to describe the module basis and not a usual product. However, we may regard it is a *partial product* in the sense introduced in the next section.

3. PARTIAL PRODUCTS

Since a diagonal map $\Delta_n^{\Omega X} = \Omega(\Delta_n^X) : \Omega X \rightarrow \prod_n \Omega X = \Omega(\prod_n X)$ is a loop map, it induces a map of projective spaces:

$$P^m(\Delta_n^{\Omega X}) : P^m(\Omega X) \rightarrow P^m(\Omega(\prod_n X)),$$

such that $e_m^{\prod_n X} \circ P^m(\Delta_n^{\Omega X}) \sim \Delta_n^X \circ e_m^X$. As is seen in the proof of Theorem 1.1 in [9], there is a natural map

$$\begin{aligned} \varphi_m^X : P^m(\Omega(\prod_n X)) &\rightarrow \bigcup_{i_1+\dots+i_n=m} P^{i_1}(\Omega X) \times \dots \times P^{i_n}(\Omega X) \\ &\subset P^m(\Omega X) \times \dots \times P^m(\Omega X) \end{aligned}$$

such that $(e_m^X \times \dots \times e_m^X) \circ \varphi_m^X = e_m^{\prod_n X}$. Let $\Delta_n^{X,m} = \varphi_m^X \circ P^m(\Delta_n^{\Omega X})$, which we call the n -th partial diagonal of X of height m , or simply a *partial diagonal*

$$\begin{aligned} \Delta_n^{X,m} : P^m(\Omega X) &\rightarrow \bigcup_{i_1+\dots+i_n=m} P^{i_1}(\Omega X) \times \dots \times P^{i_n}(\Omega X) \\ &\subset P^m(\Omega X) \times \dots \times P^m(\Omega X) \end{aligned}$$

such that $(e_m^X \times \cdots \times e_m^X) \circ \Delta_n^{X,m} \sim \Delta_n^X \circ e_m^X$. This partial diagonal also yields the reduced version

$$\begin{aligned} \overline{\Delta}_n^{X,m} : P^m(\Omega X) &\rightarrow \bigcup_{i_1 + \cdots + i_n = m} P^{i_1}(\Omega X) \wedge \cdots \wedge P^{i_n}(\Omega X) \\ &\subset P^{m-n+1}(\Omega X) \wedge \cdots \wedge P^{m-n+1}(\Omega X) \end{aligned}$$

such that $(e_{m-n+1}^X \wedge \cdots \wedge e_{m-n+1}^X) \circ \overline{\Delta}_n^{X,m} \sim \overline{\Delta}_n^X \circ e_m^X$, where $\overline{\Delta}_n^X : X \rightarrow \bigwedge^n X$ denotes the reduced diagonal. Let us call $\overline{\Delta}_n^{X,m}$ the n -th reduced partial diagonal of X of height m , or simply a *reduced partial diagonal*.

As is well known, the product of a multiplicative generalized cohomology h is given by (reduced) diagonal, i.e.,

$$v_1 \cdots v_n = (\overline{\Delta}_n^X)^*(v_1 \otimes \cdots \otimes v_n) \in \bar{h}^*(X), \quad \text{for any } v_1, \dots, v_n \in \bar{h}^*(X),$$

where \bar{h} denotes the reduced cohomology associated with h . So it is natural to define a ‘partial’ product in the following way.

Definition 3.1. For any elements $v_1, \dots, v_n \in \bar{H}^*(\Sigma\Omega X; \mathbb{F}_2)$ which are restrictions of elements in $\bar{H}^*(P^{m-n+1}(\Omega X); \mathbb{F}_2)$, we define a partial product $v_1 \cdots v_n = (\overline{\Delta}_n^{\Omega X, m})^*(v_1 \otimes \cdots \otimes v_n)$ in $\bar{H}^*(P^m(\Omega X); \mathbb{F}_2)$.

Remark 3.2. Since x_{11} can be extended to an element in $\bar{H}^*(P^3(\Omega\mathbf{Spin}(9)); \mathbb{F}_2)$, we have partial products $x_{11} \cdot v_1 \cdots v_{n-1} = (\overline{\Delta}_{n-1}^{\mathbf{Spin}(9), m})^*(x_{11} \otimes v_1 \otimes \cdots \otimes v_{n-1})$ for any elements $v_1, \dots, v_{n-1} \in \bar{H}^*(P^3(\Omega\mathbf{Spin}(9)); \mathbb{F}_2)$, $m-2 \leq n \leq m$. In the direct sum decomposition of $H^*(P^m(\Omega\mathbf{Spin}(9)); \mathbb{F}_2)$ given in Proposition 2.1, the direct summand $x_{11} \cdot (A^{[m-1]}/A^{[m-4]})$ is generated by such partial products.

4. PROOF OF THEOREM 0.10

We know $x_3^3 x_5 x_7 x_{15}$ and $x_{11} \cdot x_3^3 x_5 x_7$ exist non-trivially in $H^*(P^8(\Omega\mathbf{Spin}(9)); \mathbb{F}_2)$ but $x_{11} \cdot x_3^3 x_5 x_7$ does not exist in $H^*(P^9(\Omega\mathbf{Spin}(9)); \mathbb{F}_2)$ by Proposition 2.1. To observe what happens on the element $x_{11} \cdot x_3^3 x_5 x_7$ in $H^*(P^8(\Omega\mathbf{Spin}(9)); \mathbb{F}_2)$, we must recall the bar spectral sequence $(E_r^{*,*}, d_r^{*,*})$:

$$[(p_9^{\Omega\mathbf{Spin}(9)})^*(x_{11} \cdot x_3^3 x_5 x_7)] = d_3(x_{1,2}^3 x_{1,4} x_{1,6} x_{1,10}) = \pm x_{1,2}^7 x_{1,4} x_{1,6} \neq 0 \text{ in } E_3^{*,*},$$

where we denote by $[\beta]$ the corresponding class in $E_3^{*,*}$ to an element $\beta \in E_1^{*,*}$. Thus $(p_9^{\Omega\mathbf{Spin}(9)})^*(x_3^3 x_5 x_7 x_{11}) \neq 0$ in $E_1^{9,*} = \tilde{H}^*(\bigwedge^9 \Omega\mathbf{Spin}(9); \mathbb{F}_2)$, and hence $x_{11} \cdot x_3^3 x_5 x_7$ does not exist in $\tilde{H}^*(P^9(\Omega\mathbf{Spin}(9)); \mathbb{F}_2)$, but does in $\tilde{H}^*(P^8(\Omega\mathbf{Spin}(9)); \mathbb{F}_2)$.

By [17], we know $Sq^4(x_{11}) = x_{15}$ in $H^*(P^1(\mathbf{Spin}(9)); \mathbb{F}_2)$, and hence $Sq^4(x_{11}) = x_{15}$ modulo S_3 in $H^*(P^3(\mathbf{Spin}(9)); \mathbb{F}_2)$ for dimensional reasons. Thus we have

$$(4.1) \quad Sq^4(x_{11} \cdot x_3^3 x_5 x_7) = x_3^3 x_5 x_7 x_{15}, \quad \text{in } H^*(P^7(\Omega\mathbf{Spin}(9)); \mathbb{F}_2),$$

$$(4.2) \quad Sq^4(x_{11} \cdot x_3^3 x_5 x_7) = x_3^3 x_5 x_7 x_{15} + w, \quad w \in S_8 \text{ in } H^*(P^8(\Omega\mathbf{Spin}(9)); \mathbb{F}_2).$$

Equation (4.1) implies that any left inverse epimorphism of $(e_7^{\mathbf{Spin}(9)})^*$,

$$\phi : H^*(P^7(\Omega\mathbf{Spin}(9)); \mathbb{F}_2) \rightarrow H^*(\mathbf{Spin}(9); \mathbb{F}_2)$$

does not preserve the action of the modulo 2 Steenrod operations: If such an epimorphism ϕ did preserve the action of the modulo 2 Steenrod operations, the element $\phi(x_3^3 x_5 x_7 x_{15}) = x_3^3 x_5 x_7 x_{15}$ in $H^*(\mathbf{Spin}(9); \mathbb{F}_2)$ should lie in the image of Sq^4 , since $x_3^3 x_5 x_7 x_{15}$ lies in the image of Sq^4 in $H^*(P^7(\Omega\mathbf{Spin}(9)); \mathbb{F}_2)$ by (4.1). It

contradicts the fact that $H^{32}(\mathbf{Spin}(9); \mathbb{F}_2) = 0$. Thus we have $\text{Mwgt}(\mathbf{Spin}(9); \mathbb{F}_2) \geq 8$.

On the other hand, we can easily see that each generator of $H^*(\mathbf{Spin}(9); \mathbb{F}_2) \cong \mathbb{F}_2[x_3]/(x_3^4) \otimes \bigwedge_{\mathbb{F}_2}(x_5, x_7, x_{15})$ has category weight 1, and hence by (0.1), we have $\text{wgt}(\mathbf{Spin}(9); \mathbb{F}_2) = 6$. This completes the proof of Theorem 0.10.

5. PROOF OF THEOREM 0.11

By [15], we can easily see that $\mathbf{Spin}(7)$ admits a cone-decomposition which satisfies condition (3) in Theorem 0.8. Since $x_{15} \in H^{15}(\mathbf{Spin}(7); \mathbb{F}_2)$ is the modulo 2 reduction of a generator of $H^{15}(\mathbf{Spin}(7); \mathbb{Z}) \cong \mathbb{Z}$, the image of the attaching map α of the 15-cell corresponding to x_{15} must lie in $\mathbf{Spin}(7)^{(13)}$, the 13-skeleton of $\mathbf{Spin}(7)$, where $\mathbf{Spin}(7)^{(13)}$ is contained in $F_3(\mathbf{Spin}(7))$. To observe that the attaching map α satisfies the condition of Theorem 0.8 with $n = 3$, we need to show that $H_3^{\sigma_3}(\alpha) = 0$. Then we obtain $\text{cat}(\mathbf{Spin}(9)) \leq \text{Cat}(\mathbf{Spin}(7)) + n = 5 + 3 = 8$ by Theorem 0.8, while we know $\text{Mwgt}(\mathbf{Spin}(9); \mathbb{F}_2) \geq 8$ by Theorem 0.10, and hence

$$\text{cat}(\mathbf{Spin}(9)) = \text{Mwgt}(\mathbf{Spin}(9); \mathbb{F}_2) = 8.$$

Let $\sigma_3 : F_3(\mathbf{Spin}(7)) \rightarrow P^3(\Omega F_3(\mathbf{Spin}(7)))$ be the canonical structure map of $\text{cat}(F_3(\mathbf{Spin}(7))) = 3$. Then we are left to show that $H_3^{\sigma_3}(\alpha) = 0$. By definition,

$$H_3^{\sigma_3}(\alpha) : S^{14} \rightarrow E^4(\Omega F_3(\mathbf{Spin}(7))),$$

where $F_3(\mathbf{Spin}(7)) = \mathbf{G}_2^{(11)} \cup_{\Sigma \mathbb{C}P^2} \Sigma \mathbb{C}P^3 \cup (\text{higher cells } \geq 8)$. Since $\Omega \mathbf{G}_2^{(11)}$ has the homotopy type of $\mathbb{C}P^2 \cup (\text{higher cells } \geq 6)$, we know $\Omega F_3(\mathbf{Spin}(7))$ has the homotopy type of $\mathbb{C}P^3 \cup (\text{higher cells } \geq 6)$. Thus we observe that $E^4(\Omega F_3(\mathbf{Spin}(7)))$ has the homotopy type of

$$\Sigma^3 \mathbb{C}P^3 \wedge S^2 \wedge S^2 \wedge S^2 \cup \Sigma^3 \mathbb{C}P^2 \wedge \mathbb{C}P^2 \wedge \mathbb{C}P^2 \wedge \mathbb{C}P^2 \cup (\text{higher cells } \geq 15).$$

It is well known that $\Sigma \mathbb{C}P^3 = \Sigma \mathbb{C}P^2 \cup_{\omega_3} e^7$, $\omega_3 : S^6 \rightarrow S^3 \subset \Sigma \mathbb{C}P^3$, and hence we have $\Sigma^3 \mathbb{C}P^3 \wedge S^2 \wedge S^2 \wedge S^2 = \Sigma^3 \mathbb{C}P^2 \wedge S^2 \wedge S^2 \wedge S^2 \cup_{2\nu_{11}} e^{15}$, since $\omega_n = 2\nu_n$ for $n \geq 5$. An easy computation on the cohomology groups shows that $\mathbb{C}P^2 \wedge \mathbb{C}P^2$ has the homotopy type of $(\Sigma^2 \mathbb{C}P^2 \vee S^6) \cup_{\beta} e^8$, $\beta : S^7 \xrightarrow{\mu} S^7 \vee S^7 \xrightarrow{3\nu_4 \vee \eta} S^4 \vee S^6 \subset \Sigma^2 \mathbb{C}P^2 \vee S^6$, where μ denotes the unique co-Hopf structure of S^7 . Then we obtain, up to higher cells in dimension ≥ 10 , that

$$\begin{aligned} [(\Sigma^2 \mathbb{C}P^2 \vee S^6) \cup_{\beta} e^8] \wedge \mathbb{C}P^2 &= (\Sigma^2 \mathbb{C}P^2 \wedge \mathbb{C}P^2 \vee \Sigma^6 \mathbb{C}P^2) \cup_{\Sigma^2 \beta} e^{10} \\ &= (\Sigma^2 \mathbb{C}P^2 \wedge \mathbb{C}P^2 \cup_{3\nu_6} e^{10}) \vee \Sigma^6 \mathbb{C}P^2 \\ &= (\Sigma^4 \mathbb{C}P^2 \cup_{3\nu_6} e^{10} \vee S^8) \cup_{\Sigma^2 \beta} e^{10} \vee \Sigma^6 \mathbb{C}P^2 \\ &= \Sigma^4 \mathbb{C}P^2 \cup_{3\nu_6} e^{10} \vee \Sigma^6 \mathbb{C}P^2 \vee \Sigma^6 \mathbb{C}P^2. \end{aligned}$$

Hence we have, up to higher cells in dimension ≥ 12 , that

$$\begin{aligned} [(\Sigma^2 \mathbb{C}P^2 \vee S^6) \cup_{\beta} e^8] (\wedge \Sigma^2 \mathbb{C}P^2 \vee S^6) \\ = (\Sigma^6 \mathbb{C}P^2 \cup_{3\nu_8} e^{12}) \vee \Sigma^8 \mathbb{C}P^2 \vee \Sigma^8 \mathbb{C}P^2 \vee \Sigma^8 \mathbb{C}P^2. \end{aligned}$$

Thus we obtain that

$$E^4(\Omega F_3(\mathbf{Spin}(7))) = \Sigma^3 \mathbb{C}P^3 \wedge S^2 \wedge S^2 \wedge S^2 \cup \Sigma^3 \mathbb{C}P^2 \wedge \mathbb{C}P^2 \wedge \mathbb{C}P^2 \wedge \mathbb{C}P^2$$

has the homotopy type of

$$\begin{aligned} & \Sigma^3 \mathbb{C}P^3 \wedge S^2 \wedge S^2 \wedge S^2 \cup \Sigma^3 [(\Sigma^2 \mathbb{C}P^2 \vee S^6) \cup_{\beta} e^8] \wedge (\Sigma^2 \mathbb{C}P^2 \vee S^6) \\ & \cup (\Sigma^2 \mathbb{C}P^2 \vee S^6) \wedge [(\Sigma^2 \mathbb{C}P^2 \vee S^6) \cup_{\beta} e^8] \cup (\text{higher cells } \geq 15) \\ & = (\Sigma^9 \mathbb{C}P^2 \cup_{3\nu_{11}} e^{15} \cup_{2\nu_{11}} e^{15}) \\ & \quad \vee \Sigma^{11} \mathbb{C}P^2 \vee \Sigma^{11} \mathbb{C}P^2 \vee \Sigma^{11} \mathbb{C}P^2 \cup (\text{higher cells } \geq 15) \\ & = (\Sigma^9 \mathbb{C}P^2 \cup_{\nu_{11}} e^{15}) \vee \Sigma^{11} \mathbb{C}P^2 \vee \Sigma^{11} \mathbb{C}P^2 \vee \Sigma^{11} \mathbb{C}P^2 \cup (\text{higher cells } \geq 15). \end{aligned}$$

Then an elementary computation shows that $\pi_{14}(E^4(\Omega F_3(\mathbf{Spin}(7)))) = 0$, and hence $H_3^{\sigma^3}(\alpha) = 0$. This completes the proof of Theorem 0.11.

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REFERENCES

- [1] A. Borel, *Sur l'homologie et la cohomologie des groupes de Lie compacts connexes* (French), Amer. J. Math. **76** (1954), 273–342. MR0064056 (16:219b)
- [2] R. Bott, *The space of loops on a Lie group*, Michigan Math. J. **5** (1958), 35–61. MR0102803 (21:1589)
- [3] E. Fadell and S. Husseini, *Category weight and Steenrod operations*, Papers in honor of José Adem (Spanish), Bol. Soc. Mat. Mexicana **37** (1992), 151–161. MR1317569 (95m:55007)
- [4] T. Ganea, *Lusternik-Schnirelmann category and strong category*, Illinois. J. Math. **11** (1967), 417–427. MR0229240 (37:4814)
- [5] H. Hamanaka and A. Kono, *Homotopy-commutativity in spinor groups*, J. Math. Kyoto Univ. **40** (2000), 389–405. MR1787878 (2001k:57044)
- [6] K. A. Hardie, *A note on fibrations and category*, Michigan Math. J. **17** (1970), 351–352. MR0283795 (44:1025)
- [7] K. Ishitoya, A. Kono and H. Toda, *Hopf algebra structure of mod 2 cohomology of simple Lie groups*, Publ. Res. Inst. Math. Sci. **12** (1976/77), 141–167. MR0415651 (54:3731)
- [8] N. Iwase, *On the K-ring structure of X-projective n-space*, Mem. Fac. Sci. Kyushu Univ. Ser. A Math. **38** (1984), 285–297. MR0760190 (86g:55023)
- [9] N. Iwase, *Ganea's conjecture on Lusternik-Schnirelmann category*, Bull. London Math. Soc. **30** (1998), 623–634. MR1642747 (99j:55003)
- [10] N. Iwase, *A_{∞} -method in Lusternik-Schnirelmann category*, Topology **41** (2002), 695–723. MR1905835 (2003h:55004)
- [11] N. Iwase, *Lusternik-Schnirelmann category of a sphere-bundle over a sphere*, Topology **42** (2003), 701–713. MR1953245 (2004c:55030)
- [12] N. Iwase, *The Ganea conjecture and recent developments on the Lusternik-Schnirelmann category* (Japanese), Sūgaku **56** (2004), 281–296. MR2086116 (2005e:55005)
- [13] N. Iwase and M. Mimura, *Higher homotopy associativity*, Algebraic Topology (Arcata, 1986), 193–220, Lecture Notes in Math. 1370, Springer, Berlin, 1989. MR1000378 (90f:55017)
- [14] N. Iwase, M. Mimura, *L-S categories of simply-connected compact simple Lie groups of low rank*, Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001), 199–212, Progr. Math., 215, Birkhäuser, Basel, 2004. MR2039767 (2005a:55001)
- [15] N. Iwase, M. Mimura, T. Nishimoto, *On the cellular decomposition and the Lusternik-Schnirelmann category of $\mathbf{Spin}(7)$* , Topology Appl., **133** (2003), 1–14. MR1996457 (2005b:55002)
- [16] N. Iwase, M. Mimura, T. Nishimoto, *L-S categories of non-simply-connected compact simple Lie groups*, Topology Appl. **150** (2005), 111–123. MR2133672 (2005m:55004)
- [17] A. Kono and K. Kozima, *The adjoint action of a Lie group on the space of loops*, J. Math. Soc. Japan **45** (1993), 495–510. MR1219882 (94h:57053)
- [18] Y. B. Rudyak, *On category weight and its applications*, Topology **38** (1999), 37–55. MR1644063 (99f:55007)

- [19] J. D. Stasheff, *Homotopy associativity of H-spaces, I, II*, Trans. Amer. Math. Soc. **108** (1963), 275–292, 293–312. MR0158400 (28:1623)
- [20] J. Strom, *Essential category weight and phantom maps*, Cohomological methods in homotopy theory (Bellaterra, 1998), 409–415, Progr. Math., 196, Birkhäuser, Basel, 2001. MR1851266 (2002f:55018)
- [21] K. Varadarajan, *On fibrations and category*, Math. Z. **88** (1965), 267–273. MR0180969 (31:5199)
- [22] G. W. Whitehead, “Elements of Homotopy Theory”, Graduate Texts in Mathematics, 61, Springer-Verlag, Berlin, 1978. MR0516508 (80b:55001)

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