\( \alpha \)-CONTINUITY PROPERTIES
OF THE SYMMETRIC \( \alpha \)-STABLE PROCESS

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Abstract. Let \( D \) be a domain of finite Lebesgue measure in \( \mathbb{R}^d \) and let \( X_D^\alpha \) be the symmetric \( \alpha \)-stable process killed upon exiting \( D \). Each element of the set \( \{ \lambda_1^\alpha, \lambda_2^\alpha, \ldots \} \) is right continuous. In addition, if \( D \) is Lipschitz and bounded, then each \( \lambda_i^\alpha \) is continuous in \( \alpha \) and the set of associated eigenfunctions is precompact.

1. Introduction

Let \( X_t \) be a \( d \)-dimensional symmetric \( \alpha \)-stable process of order \( \alpha \in (0, 2) \). The process \( X_t \) has stationary independent increments and its transition density

\[ p^\alpha(t, z, w) = f^\alpha_t(z - w) \]

is determined by its Fourier transform

\[ \exp(-t|z|^\alpha) = \int_{\mathbb{R}^d} e^{iz \cdot w} f^\alpha_t(w) \, dw. \]

These processes have right continuous sample paths and their transition densities satisfy the scaling property

\[ p^\alpha(t, x, y) = t^{-d/\alpha} p^\alpha(1, t^{-1/\alpha} x, t^{-1/\alpha} y). \]

When \( \alpha = 2 \) the process \( X_t \) is a \( d \)-dimensional Brownian motion running at twice the usual speed. The nonlocal operator associated to \( X_t \) is \((-\Delta)^{\alpha/2}\) where \( \Delta \) is the Laplace operator in \( \mathbb{R}^d \).

Let \( D \) be an open set in \( \mathbb{R}^d \) and let \( X_D^\alpha \) be the symmetric \( \alpha \)-stable process killed upon leaving \( D \). We write \( p_{D}^\alpha(t, x, y) \) for the transition density of \( X_D^\alpha \) and \( H_\alpha \) for its associated nonlocal self-adjoint positive operator. It is well known that if \( D \) has finite Lebesgue measure, then the spectrum of \( H_\alpha \) is discrete. Let

\[ 0 < \lambda_1^\alpha(D) \leq \lambda_2^\alpha(D) \leq \lambda_3^\alpha(D) \leq \cdots \]

be the eigenvalues of \( H_\alpha \), and let

\[ \varphi_1^\alpha, \varphi_2^\alpha, \varphi_3^\alpha, \ldots \]

be the corresponding sequence of orthonormal \( L^2(D) \) eigenfunctions. Also, \( \varphi_1^\alpha \) is chosen so as to be positive on \( D \). Note that if \( \alpha < 2 \), then \( \lambda_1^\alpha(D) > \lambda_2^\alpha(D) \), but this need not be true if \( \alpha = 2 \) (unless \( D \) is connected).

Several authors have studied properties of the eigenvalues and eigenfunctions of \( H_\alpha \). One common theme has been to extend results on Brownian motion (\( \alpha = 2 \)) to
analogous results for symmetric $\alpha$-stable processes. For example, R. M. Blumenthal and R. K. Getoor [8] have shown Weyl’s asymptotic law holds: if $D$ is a bounded open set and $N(\lambda)$ is the number of eigenvalues less than or equal to $\lambda$, then there exists a constant $C_{d,\alpha}$, depending only on $d$ and $\alpha$, such that

$$N(\lambda) \approx C_{d,\alpha} \frac{m(D)}{\Gamma(d/\alpha + 1)} \lambda^{d/\alpha}$$

as $\lambda \to \infty$, provided $m(\partial D) = 0$, where $m$ is Lebesgue measure.

If $D \subseteq \mathbb{R}^d$ is a domain, define the inner radius $R_D$ to be the supremum of the radii of all balls contained in $D$. R. Bañuelos et al. [6] and P. Méndez-Hernández [20] have shown that if $D$ is a convex domain with finite inner radius $R_D$ and $I_D$ is the interval $(-R_D, R_D)$, then

$$\lambda_1^\alpha(I_D) \leq \lambda_1^\alpha(D).$$

Moreover, if $D \subseteq \mathbb{R}^d$ has finite volume and $D^*$ is a ball in $\mathbb{R}^d$ with the same volume as $D$, then it was proved in [6] that the Faber-Krahn inequality holds:

$$\lambda_1^\alpha(D^*) \leq \lambda_1^\alpha(D).$$

Another line of inquiry taken by those authors was to consider the eigenvalues as a function of the index $\alpha$. For instance, if $D$ is a convex domain with finite inner radius $R_D$, then

$$\lambda_1^\alpha(D) \leq \left[ \mu_1(D) \right]^{\alpha/2},$$

where $\mu_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on $D$.

For the Cauchy process, i.e. $\alpha = 1$, and bounded Lipschitz domains, R. Bañuelos and T. Kulczycki [5] extended (1.1) to

$$\lambda_1^1(D) \leq \left[ \mu_1(D) \right]^{1/2}, \quad i = 1, 2, \ldots,$$

where

$$0 < \mu_1(D) < \mu_2(D) \leq \cdots$$

are all the Dirichlet eigenvalues of $-\Delta$ on $D$. Their proof of (1.2) is based on a variational formula for $\lambda_1^1(D)$ that they developed from a connection with the Steklov problem for the Laplacian. They also obtained many detailed properties of the eigenfunctions $\varphi_1^i$ for the Cauchy process.

By finding a connection with the symmetric stable process with rational index $\alpha$ and PDEs of order higher than 2, R. D. DeBlassie [14] derived a variational formula for the eigenvalues which led to the following extension of (1.1) and (1.2):

$$\lambda_1^\alpha(D) \leq \left[ \mu_1(D) \right]^{\alpha/2}, \quad i = 1, 2, \ldots,$$

for all rational $\alpha \in (0, 2)$ and certain bounded domains $D \subseteq \mathbb{R}^d$. The class of admissible domains includes convex polyhedra, Lipschitz domains with sufficiently small Lipschitz constant and $C^1$ domains. Please note there is an error in [14] in the derivation of (1.3) above, as pointed out by the referee of the present article. See [15] for a correction. Also, we have learned of a recent preprint of Z.-Q. Chen
and R. Song [11] which extends (1.3) to all indices $\alpha \in (0, 2)$ and domains $D$ of finite Lebesgue measure.

In this article, we study the eigenvalues and eigenfunctions regarded as functions of the index $\alpha$. Our first result concerns continuity of the eigenvalues.

**Theorem 1.1.** Let $D$ be a domain of finite Lebesgue measure. Then, as a function of $\alpha \in (0, 2)$, $\lambda_\alpha^i$ is right continuous for each positive integer $i$.

In order to prove Theorem 1.1 we need the following interesting monotonicity property extending (1.3) above. It is due to Z.-Q. Chen and R. Song [11]; see their Example 5.4.

**Theorem 1.2.** Let $D$ be an open set of finite Lebesgue measure in $\mathbb{R}^d$. If $0 < \alpha < \beta \leq 2$, then for all positive integers $i$,

$$\left[\lambda_\alpha^i(D)\right]^{1/\alpha} \leq \left[\lambda_\beta^i(D)\right]^{1/\beta}.$$

Even though those authors consider bounded open sets $D$, it is clear their argument works for open sets of finite Lebesgue measure.

By requiring more regularity of $\partial D$, we can prove the following extension of Theorem 1.1.

**Theorem 1.3.** Let $D$ be a bounded Lipschitz domain. Then, as a function of $\alpha \in (0, 2)$, $\lambda_\alpha^i$ is continuous for each positive integer $i$.

We will obtain Theorem 1.3 from the following result that we believe is of independent interest.

**Theorem 1.4.** Let $D$ be a bounded Lipschitz domain. If $\alpha_m$ converges to $\alpha \in (0, 2)$, then for each positive integer $i$, $\{\varphi_{\alpha_m}^i : m \geq 1\}$ is precompact in $C(\bar{D})$ equipped with the sup norm. Moreover, if $\lambda_\alpha^i$ converges to $\lambda$, then any limit point of $\{\varphi_{\alpha_m}^i : m \geq 1\}$ is an eigenfunction of $H_\alpha$ and $\lambda$ is the corresponding eigenvalue.

As a corollary of the proof of the last theorem, we obtain continuity of the first eigenfunction as a function of $\alpha$.

**Theorem 1.5.** If $D$ is a bounded Lipschitz domain and $\alpha_m$ converges to $\alpha \in (0, 2)$, then $\varphi_1^{\alpha_m}$ converges uniformly to $\varphi_1^\alpha$ on $D$.

The article is organized as follows. In Section 2 we present some results needed in the proof of Theorem 1.1. In Section 3 we establish Theorem 1.1 by proving upper semicontinuity and right lower semicontinuity of the eigenvalues via Dirichlet forms. Lower semicontinuity of the eigenvalues, for Lipschitz domains, is proved in section 4 using Theorem 1.4. This will yield Theorems 1.3 and 1.5. Section 5 deals with certain weak convergence results needed to prove Theorem 1.4. Finally, in Section 6 we prove Theorem 1.4.

### 2. Preliminary results

Throughout this section we will assume the domain $D$ has finite Lebesgue measure. We denote by $C_c^\infty(D)$ the set of $C^\infty$ functions with compact support in $D$. The inner product and the norm in $L^2(D)$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_2$, respectively.
For any domain $D \subseteq \mathbb{R}^d$, we define $\tau_D$ to be the first exit time of $X_t$ from $D$, i.e.,
$$
\tau_D = \inf\{t > 0 : X_t \notin D\}.
$$

Let
$$
\mathcal{F}_\alpha = \left\{ \varphi \in L^2(\mathbb{R}^d) : \int \int \frac{[\varphi(y) - \varphi(x)]^2}{|y - x|^{d+\alpha}} \, dydx < \infty \right\}.
$$

The Dirichlet form $(\mathcal{E}_\alpha, \mathcal{F}_\alpha)$ associated to $X_t$ is given by
$$
\mathcal{E}_\alpha(\psi, \varphi) = A(d, \alpha) \int \int \frac{[\psi(y) - \psi(x)][\varphi(y) - \varphi(x)]}{|y - x|^{d+\alpha}} \, dydx,
$$
for all $\psi, \varphi \in \mathcal{F}_\alpha$, where
$$
A(d, \alpha) = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^\alpha \pi^{d/2} \Gamma\left(\frac{\alpha}{2}\right)}.
$$

It is well known that the Dirichlet form corresponding to $X^D_t$ is given by $(\mathcal{E}_\alpha, \mathcal{F}_{\alpha,D})$, where
$$\mathcal{F}_{\alpha,D} = \{u \in \mathcal{F}_\alpha : \text{a quasi continuous version of } u \text{ is 0 quasi everywhere in } D^c \}.$$

Recall that for all $\psi, \varphi$ in the domain of $H_\alpha$ we have
$$\mathcal{E}_\alpha(\psi, \varphi) = \langle \psi, H_\alpha \varphi \rangle.
$$

As seen in Theorem 4.4.3 of [17], $\mathcal{F}_{\alpha,D}$ is the closure of $C^\infty_c(D)$ in $\mathcal{F}_\alpha$ with respect to the norm
$$
\|\varphi\|_\alpha = \sqrt{\mathcal{E}_\alpha(\varphi, \varphi)} + \|\varphi\|_2.
$$

**Lemma 2.1.** Let $\varphi, \psi \in C^\infty_c(D)$. Then the function
$$
\mathcal{E}_\alpha(\varphi, \psi) : (0, 2) \to \mathbb{R}
$$
is continuous on $(0, 2)$.

**Proof.** Let $\varphi, \psi \in C^\infty_c(D)$, and let $\beta \in (\alpha - \delta, \alpha + \delta)$, where $\delta = \frac{1}{4} \min \{2 - \alpha, \alpha\}$. Then there exists a constant $C > 0$, depending only on $\varphi$ and $\psi$, such that
$$
\frac{|\psi(y) - \psi(x)|}{|y - x|^{d+\beta}} \leq \frac{C}{|y - x|^{d+\beta - 2}} \leq C \max \left\{ \frac{1}{|y - x|^{d+\alpha+\beta-2}}, \frac{1}{|y - x|^{d+\alpha-\beta-2}} \right\}.
$$

Since $D$ has finite measure, a simple computation using polar coordinates shows that
$$
\max \left\{ \frac{1}{|y - x|^{d+\alpha+\beta-2}}, \frac{1}{|y - x|^{d+\alpha-\beta-2}} \right\}
$$
is integrable in both $(\text{supp}(\varphi) \cup \text{supp}(\psi)) \times D$ and $D \times (\text{supp}(\varphi) \cup \text{supp}(\psi))$. The result immediately follows from the dominated convergence theorem. 

We end this section with some basic estimates on $L^2$ norms to be used in the next section. Suppose $k$ is a positive integer, $0 < \epsilon < 1$, and $\varphi_1, \ldots, \varphi_k \in L^2(D)$ satisfy
$$
|\langle \varphi_i, \varphi_j \rangle| < \frac{\epsilon}{4k^2}, \quad i \neq j,
$$
$$
\left(1 - \frac{\epsilon}{4k^2}\right) < \|\varphi_i\|^2 < \left(1 + \frac{\epsilon}{4k^2}\right),
$$
for all $1 \leq i, j \leq k$. If $\psi = \sum_{i=1}^{k} a_i \varphi_i$ with $\|\psi\|_2 = 1$, then we show that

\begin{equation}
\frac{1}{1 + \epsilon/2} \leq \sum_{i=1}^{k} a_i^2 \leq \frac{1}{1 - \epsilon/2}
\end{equation}

and $\varphi_1, \ldots, \varphi_k$ are linearly independent.

For the proof, note that we have

$$1 = \langle \psi, \psi \rangle = \sum_{i=1}^{k} a_i^2 \|\varphi_i\|_2^2 + 2 \sum_{i=1}^{k} \sum_{j > i} a_i a_j \langle \varphi_i, \varphi_j \rangle$$

$$\geq \sum_{i=1}^{k} a_i^2 \left(1 - \frac{\epsilon}{4k^2}\right) - 2 \sum_{i=1}^{k} \sum_{j > i} |a_i| |a_j| \frac{\epsilon}{4k^2}$$

$$\geq \sum_{i=1}^{k} a_i^2 \left(1 - \frac{\epsilon}{4k^2}\right) - (k^2 - k) \sum_{i=1}^{k} a_i^2 \frac{\epsilon}{4k^2}$$

$$\geq (1 - \epsilon/2) \sum_{i=1}^{k} a_i^2,$$

and we conclude that

\[ \sum_{i=1}^{k} a_i^2 \leq \frac{1}{1 - \epsilon/2}. \]

Similar computations give the remaining assertions.

3. Proof of Theorem 1.1

We will use the following well-known result; see [12].

**Theorem 3.1.** Let $H$ be a nonnegative self-adjoint unbounded operator with discrete spectrum $\{\lambda_i\}_{i=1}^{\infty}$ and domain $\text{Dom}(H)$. Then for $i \geq 1$

\begin{equation}
\lambda_i = \inf \{\lambda(L) : L \subseteq \text{Dom}(H), \dim(L) = i\},
\end{equation}

where

\begin{equation}
\lambda(L) = \sup \{\langle Hf, f \rangle : f \in L, \|f\|_2 = 1\},
\end{equation}

and $L$ is a vector subspace of $\text{Dom}(H)$ of dimension $i$.

We will prove the right continuity of the $k$th eigenvalue in two steps.

**Proposition 3.2.** Let $D$ be a domain of finite Lebesgue measure. Then for all $k \geq 1$

\[ \limsup_{\beta \to \alpha} \lambda_k^\beta(D) \leq \lambda_k^\alpha(D). \]

**Proof.** Let $0 < \epsilon < 1$ and $k \geq 1$. Recall $C_c^\infty(D)$ is dense in $\text{Dom}(H_\alpha)$ under the norm $\| \cdot \|_\alpha$. Then for all $\alpha \in (0, 2)$, there exist $\varphi_1, \ldots, \varphi_k \in C_c^\infty(D)$ such that

\begin{equation}
|\langle \varphi_i^\alpha, \varphi_j^\alpha \rangle - \langle \varphi_i, \varphi_j \rangle| < \frac{\epsilon}{8k^2}
\end{equation}

and

\begin{equation}
|\mathcal{E}_\alpha(\varphi_i^\alpha, \varphi_j^\alpha) - \mathcal{E}_\alpha(\varphi_i, \varphi_j)| < \frac{\epsilon}{8k^2},
\end{equation}

for all $1 \leq i, j \leq k$. 

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Thanks to Lemma 2.1 there exists \( \eta_0 \) such that for all \( \beta \in (\alpha - \eta_0, \alpha + \eta_0) \)

\[
|E_\alpha(\varphi_i, \varphi_j) - E_\beta(\varphi_i, \varphi_j)| < \frac{\epsilon}{8k^2}. \tag{3.5}
\]

Notice that (3.3) implies

\[
|\langle \varphi_i, \varphi_j \rangle| < \frac{\epsilon}{8k^2}, \quad i \neq j,
\]

and

\[
1 - \frac{\epsilon}{8k^2} < \|\varphi_i\|^2 < 1 + \frac{\epsilon}{8k^2},
\]

for all \( 1 \leq i, j \leq k \). Then by the comments at the end of Section 2, we know \( \varphi_1, \ldots, \varphi_k \) are linearly independent.

Theorem 3.1 implies

\[
\lambda_\beta^k(D) \leq \lambda_\beta(L_k),
\]

where \( L_k = \text{span}\{\varphi_1, \ldots, \varphi_k\} \) and

\[
\lambda_\beta(L_k) = \sup \{ \langle H_\beta f, f \rangle : f \in L_k, \|f\|_2 = 1 \}.\]

Take \( \psi = \sum_{i=1}^k a_i \varphi_i \in L_k \) such that

\[
\lambda_\beta(L_k) \leq E_\beta(\psi, \psi) + \epsilon/4 \tag{3.6}
\]

and

\[
\|\psi\|_2 = 1.
\]

Thanks to (2.1), with \( \epsilon \) there replaced by \( \epsilon/2 \), we have

\[
\sum_{i=1}^k a_i^2 \leq 2.
\]

Then since

\[
|E_\beta(\psi, \psi) - E_\alpha(\psi, \psi)| \leq \sum_{i=1}^k \sum_{j=1}^k |a_i a_j| \cdot |E_\beta(\varphi_i, \varphi_j) - E_\alpha(\varphi_i, \varphi_j)|,
\]

(3.5) implies

\[
|E_\beta(\psi, \psi) - E_\alpha(\psi, \psi)| < \frac{\epsilon}{4}. \tag{3.7}
\]

Thus

\[
\lambda_\beta^k(D) \leq E_\alpha(\psi, \psi) + \epsilon/2. \tag{3.8}
\]

Consider \( \psi_0 = \sum_{i=1}^k a_i \varphi_i^2 \). By (2.1) we have

\[
\frac{1}{1 + \epsilon/4} \leq \|\psi_0\|^2 = \sum_{i=1}^k a_i^2 \leq \frac{1}{1 - \epsilon/4}.
\]

Following the argument used to obtain (3.7), one easily proves (3.4) implies

\[
|E_\alpha(\psi_0, \psi_0) - E_\alpha(\psi, \psi)| < \frac{\epsilon}{4}.
\]
Hence

\[
\lambda^\beta_k(D) \leq E_\alpha(\psi, \psi) + \epsilon/2 \\
\leq E_\alpha(\psi_0, \psi_0) + 3\epsilon/4 \\
= \sum_{i=1}^k a_i^2 \lambda^\alpha_i(D) + 3\epsilon/4 \\
\leq \lambda^\alpha_k(D) \sum_{i=1}^k a_i^2 + 3\epsilon/4 \\
\leq \frac{1}{1 - \epsilon/4} \lambda^\alpha_k(D) + 3\epsilon/4,
\]

and the result immediately follows. □

**Proposition 3.3.** Let \( D \) be a domain of finite Lebesgue measure. Then for all \( k \geq 1 \)

\[
\liminf_{\beta \to \alpha^+} \lambda^\beta_k(D) \geq \lambda^\alpha_k(D).
\]

**Proof.** By Theorem 1.2,

\[
\lambda^\alpha_k(D) \leq \left[ \lambda^\alpha_{\beta^\alpha}(D)^{\alpha/(\alpha+\epsilon)} \right].
\]

Now let \( \epsilon \to 0 \) to get the desired \( \liminf \) behavior. □

Combining Propositions 3.2 and 3.3, we get Theorem 1.1.

4. **Proof of Theorems 1.3 and 1.5**

We now show how Theorem 1.4 implies Theorem 1.3. In order to simplify the notation, throughout this section we will write \( \lambda^\alpha_k \) for \( \lambda^\alpha_k(D) \) and \( \mu_k \) for \( \mu_k(D) \).

**Proof of Theorem 1.3.** We proceed by induction on \( i \). For \( i = 1 \), let \( \{\alpha_m\}_{m=1}^\infty \) be a sequence converging to \( \alpha \) in \( (0, 2) \). Consider any subsequence \( \beta_r = \alpha_{m_r} \). Theorem 1.2 implies the sequence \( \{\lambda^\beta_i\}_{m=1}^\infty \) is bounded, and so there is a subsequence \( \gamma_\ell = \beta_\ell \) such that \( \lambda^\gamma_\ell_i \) converges as \( \ell \to \infty \), say, to \( \lambda \). Thanks to Theorem 1.4 we can choose a subsequence \( \eta_p = \gamma_\ell_p \) such that \( \varphi^\eta_p_i \) converges uniformly to \( \varphi \) an eigenfunction of \( H_\alpha \) with eigenvalue \( \lambda \). Since \( \varphi^\eta_p_i \) is nonnegative, so is \( \varphi \). But the only nonnegative eigenfunction of \( H_\alpha \) is \( \varphi^\alpha_1 \). Thus \( \lambda = \lambda^\alpha_1 \) and \( \varphi = \varphi^\alpha_1 \). Hence we have shown any subsequence of \( \lambda^\alpha_m \) contains a further subsequence converging to \( \lambda^\alpha_1 \). We conclude that

\[
\lim_{m \to \infty} \lambda^\alpha_m = \lambda^\alpha_1.
\]

Note this also proves Theorem 1.5.

Next, assume the theorem is true for \( j \leq i \). We verify it is true for \( j = i + 1 \). We will show

\[
\liminf_{\beta \to \alpha} \lambda^\beta_{i+1} \geq \lambda^\alpha_{i+1}.
\]

Combined with the \( \limsup \) behavior from Proposition 3.2, we conclude the desired result

\[
\lim_{\beta \to \alpha} \lambda^\beta_{i+1} = \lambda^\alpha_{i+1}.
\]
To get the lim inf behavior, by way of contradiction, assume \( \lambda = \liminf_{\beta \to \alpha} \lambda_{i+1}^\beta < \lambda_{i+1}^\alpha \). Let \( \{\alpha_m\}_{m=1}^\infty \) be a sequence converging to \( \alpha \) with
\[
\lim_{m \to \infty} \lambda_{i+1}^{\alpha_m} = \lambda.
\]
By the induction hypothesis, \( \lambda_{j^{\alpha}} \) converges to \( \lambda_{j}^\alpha \) for \( j \leq i \). Then Theorem 1.3 implies we can choose a subsequence \( \beta_r = \alpha_m \), such that:
- For each \( j, 1 \leq j \leq i \), \( \lambda_{j}^{\beta_r} \) converges to \( \lambda_{j}^\alpha \), and \( \varphi_{j}^{\beta_r} \) converges uniformly to an eigenfunction \( \varphi_{j} \) of \( H_\alpha \) with corresponding eigenvalue \( \lambda_{j}^\alpha \).
- The limit \( \lambda \) from (4.2) is an eigenvalue of \( H_\alpha \), and \( \varphi_{i+1}^{\beta_r} \) converges uniformly to an eigenfunction \( \varphi_{i+1} \) of \( H_\alpha \) with eigenvalue \( \lambda \).

Since \( \lambda \) is an eigenvalue strictly less than \( \lambda_{i+1}^\alpha \), we can choose positive integers \( \ell \) and \( m \) such that \( \ell \leq m \leq i \), \( \lambda_{m}^\alpha = \lambda \) and
\[
\lambda_{\ell}^\alpha < \lambda_{0}^\alpha = \cdots = \lambda_{m}^\alpha < \lambda_{m+1}^\alpha < \cdots < \lambda_{i+1}^\alpha.
\]
(Here we take \( \lambda_{0}^\alpha := 0 \).) In particular, if \( E \) is the eigenspace corresponding to \( \lambda = \lambda_{m}^\alpha \), then
\[
\dim(E) = m - \ell + 1.
\]
On the other hand, the uniform convergence implies for \( j_1, j_2 \in \{1, \ldots, i+1\} \)
\[
\delta_{j_1j_2} = \int_{D} \varphi_{j_1}^{\beta_r} \varphi_{j_2}^{\beta_r} \, dx \text{ converges to } \int_{D} \varphi_{j_1} \varphi_{j_2} \, dx.
\]
Thus \( \{\varphi_{1}, \ldots, \varphi_{i+1}\} \) is an orthonormal set, and so \( \{\varphi_{i}, \ldots, \varphi_{m}\} \cup \{\varphi_{i+1}\} \) is an orthonormal subset of \( E \). This forces \( \dim(E) \geq m - \ell + 2 \), which contradicts (4.2).

We conclude (4.1) holds.

5. Weak Convergence Results

Let \( D[0, \infty) \) be the space of right continuous functions \( \omega: [0, \infty) \to \mathbb{R}^d \) with left limits. That is, \( \omega(t^+) = \lim_{s \to t^+} \omega(s) = \omega(t) = \lim_{s \to t^-} \omega(s) \) exists. The usual convention is \( \omega(0^-) := \omega(0) \). Let \( X_t(\omega) = \omega(t) \) be the coordinate process and let \( \mathcal{F}_t \) be the \( \sigma \)-algebra generated by the cylindrical sets. We equip \( D[0, \infty) \) with the Skorohod topology. Our main reference is Chapter 3 in Ethier and Kurtz [16]. Let \( P_x^\alpha \) denote the law on \( D[0, \infty) \) of the symmetric \( \alpha \)-stable process started at \( x \); the corresponding expectation will be denoted by \( E_x^\alpha \).

**Lemma 5.1.** If \( (x_n, \alpha_n) \) converges to \( (x, \alpha) \) in \( \mathbb{R}^n \times (0, 2) \), then \( P_{x_n}^{\alpha_n} \) converges weakly to \( P_x^\alpha \) in \( D[0, \infty) \).

**Proof.** Using characteristic functions it is easy to show the corresponding finite dimensional distributions converge. Thus, by Theorem 7.8 on page 131 in [16], it suffices to show \( \{P_{x_n}^\alpha: n \geq 1\} \) is tight on \( D[0, \infty) \).

By Theorem 7.2 and Remark 7.3 of [16] and Theorem 15.2 of [7], it suffices to show that for each \( t_0 \geq 0, \{P_{x_n}^\alpha: n \geq 1\} \) is tight on \( D[0, t_0] \). For this, we proceed as in the proof of Proposition 3.2 in [3], using a theorem of Aldous [2]. For the convenience of the reader, we state the theorem using our notation.

For each \( n \geq 1 \) let \( \tau_n \) be a stopping time with finitely many values, and let \( \delta_n \geq 0 \) converge to 0. Aldous’s Theorem is the following. Suppose for any \( \eta > 0 \),
\[
P_{x_n}^{\alpha_n}(|X(\tau_n + \delta_n) - X(\tau_n)| \geq \eta) \to 0
\]
as \( n \to \infty \). If for each \( t \in [0, t_0] \) the collection
\[
\{ P^\alpha_n \circ X_t : n \geq 1 \}
\]
is tight on \( \mathbb{R}^d \), then \( \{ P^\alpha_n : n \geq 1 \} \) is tight on \( \mathcal{D}[0, t_0] \).

Even though the theorem is stated for dimension one, the argument also works for higher dimensions.

We now verify the conditions of the theorem. First, note that for \( \beta = \alpha \) or \( \alpha_n \) and \( y = x \) or \( x_n \), \( P^\beta_{y} \) solves the martingale problem:

a) \( P^\beta_{y}(X_0 = y) = 1 \),

b) for each \( f \in C^2_b(\mathbb{R}^d) \),
\[
f(X_t) - f(X_0) - \int_0^t \mathcal{L}_\beta f(X_s) ds
\]
is a \( P^\beta_{y} \)-martingale, where \( C^2_b(\mathbb{R}^d) \) is the space of functions with bounded continuous derivatives up to and including order 2 and
\[
\mathcal{L}_\beta f(x) = A(d, \alpha) \int_{\mathbb{R}^d \setminus \{x\}} \frac{f(y) - f(x) - \nabla f(x) \cdot (y - x) I(|y - x| < 1)}{|y - x|^{d+\alpha}} dy;
\]
see Section 2 of [4]. It is easy to show that for any \( f \in C^2_b(\mathbb{R}^d) \) there exists \( C_f > 0 \) independent of \( \alpha_n \) and \( x_n \) such that \( f(X_t) - f(X_0) - C_f t \) is a \( P^\alpha_{x_n} \)-supermartingale.

Then we can argue as in the proof of Proposition 3.2 in [3] to get formula (3.1) from that article: for any bounded stopping time \( \tau_n \),
\[
(5.2) \quad P^\alpha_{x_n} \left( \sup_{\tau_n \leq s \leq \tau_n + \delta} |X_s - X_{\tau_n}| \geq \eta \right) \leq \frac{c \delta}{\eta^2},
\]
where \( c \) is independent of \( n \), \( \delta \) and \( \eta \). Even though the formula from that article is stated for one dimension and \( x_n \equiv x \), the proof works in higher dimensions with \( x_n \) converging to \( x \), since the constant \( C_f \) is independent of \( n \). Note too the requirement there that \( \delta < 1 \) can be dropped.

Replacing \( \delta \) by \( \delta_n \to 0 \) as \( n \to \infty \), upon letting \( n \to \infty \) in (5.2), we obtain condition (5.1). Next we handle tightness of \( \{ P^\alpha_n \circ X_t : n \geq 1 \} \) on \( \mathbb{R}^d \) for \( t \in [0, t_0] \). If \( n \) is large, say \( n \geq N \), then \( |x_n| < |x| + 1 \). Thus, if \( \lambda > |x| + 1 \), then for \( t \in [0, t_0] \), by (5.2) with \( \tau_n \equiv 0 \) and \( \delta = t_0 \)
\[
\sup_{n \geq N} P^\alpha_{x_n} (|X_{t} | \geq \lambda) \leq \sup_{n \geq N} P^\alpha_{x_n} (|X_{t} - x_n| + |x_n| \geq \lambda)
\]

\[
\leq \sup_{n \geq N} P^\alpha_{x_n} (|X_{t} - x_n| \geq \lambda - |x| - 1)
\]

\[
\leq \sup_{n \geq N} P^\alpha_{x_n} (\sup_{s \leq t_0} |X_s - x_n| \geq \lambda - |x| - 1)
\]

\[
\leq \frac{c t_0}{(\lambda - |x| - 1)^2}.
\]

This gives the desired tightness. \( \square \)

The next step is to show for each \( T > 0 \) that the distribution of \( X_{t_0 \wedge \tau_D} \) under \( P^\alpha_{x_n} \) converges to that under \( P^\beta_{x} \) as \( (x_n, \alpha_n) \) converges to \( (x, \alpha) \). To this end, define
\[
\mathcal{A}_D = \{ \omega \in \mathbb{D}[0, \infty) : d(X[0, \tau_D(\omega) - r], D^r) > 0 \text{ for all rational } 0 < r < \tau_D(\omega) \},
\]
and
\[ C_D = \left\{ \omega \in \mathbb{D}(0, \infty) : X(\tau_D(\omega)) \in \mathcal{D} \right\} \cap A_D. \]
Here \( X[0, t] = \{ X_s : 0 \leq s \leq t \} \) and \( d(A, B) \) is the distance between \( A \) and \( B \).

**Lemma 5.2.** For open \( D \subseteq \mathbb{R}^d \), \( \tau_D \) is continuous on \( C_D \).

**Proof.** Let \( \omega \in C_D \) and suppose \( \omega_n \) converges to \( \omega \) in \( \mathbb{D}(0, \infty) \). We will show that \( \tau_D(\omega_n) \) converges to \( \tau_D(\omega) \). Let
\[ \Lambda' = \{ \lambda : [0, \infty) \to [0, \infty) | \lambda \text{ is strictly increasing and surjective} \}. \]
Proposition 5.3 (a) and (c), on page 119 in [16], implies that for each \( T > 0 \) there exist \( \{ \lambda_n \} \subseteq \Lambda' \) such that
\[ \begin{align*}
&\lim_{n \to \infty} \sup_{0 \leq t \leq T} |\lambda_n(t) - t| = 0, \\
&\lim_{n \to \infty} \sup_{0 \leq t \leq T} |\omega_n(t) - \omega(\lambda_n(t))| = 0.
\end{align*} \]
First we show that
\[ \liminf_{n \to \infty} \tau_D(\omega_n) \geq \tau_D(\omega). \]
Let \( \delta \in (0, \tau_D(\omega)/2) \) be rational and set
\[ \varepsilon = d(\omega[0, \tau_D(\omega) - \delta], D^c). \]
Since \( \omega \in C_D \), we have \( \varepsilon > 0 \). Using \( T = \tau_D(\omega) \) in (5.3)–(5.4), there exists \( N \) such that for \( n \geq N \),
\[ \begin{align*}
&\left\{ t - \delta < \lambda_n(t) < t + \delta \text{ for all } t \leq T, \\
&\sup_{0 \leq t \leq T} |\omega_n(t) - \omega(\lambda_n(t))| < \frac{\varepsilon}{2}.
\end{align*} \]
In particular, for all \( t \leq \tau_D(\omega) - 2\delta \) and \( n \geq N \)
\[ \lambda_n(t) < t + \delta \leq \tau_D(\omega) - \delta < T. \]
Thus \( \omega(\lambda_n(t)) \in D \) and \( d(\omega(\lambda_n(t)), D^c) \geq \varepsilon \). Therefore,
\[ \omega_n(t) \in B(\omega(\lambda_n(t)), \epsilon/2) \subseteq D, \]
for all \( t \leq \tau_D(\omega) - 2\delta \) and \( n \geq N \). This implies \( \tau_D(\omega_n) > \tau_D(\omega) - 2\delta \) for \( n \geq N \). Take the \( \lim inf \) as \( n \to \infty \) and then let \( \delta \to 0 \) to get (5.5).

To finish, we show that
\[ \limsup_{n \to \infty} \tau_D(\omega_n) \leq \tau_D(\omega). \]
Given that \( \omega \in C_D \) and \( \omega \) is right continuous, we can choose \( \delta > 0 \) such that
\[ \varepsilon := d(\omega[\tau_D(\omega), \tau_D(\omega) + 2\delta], D) > 0. \]
Using \( T = \tau_D(\omega) + 2\delta \) in (5.3)–(5.4) we can choose \( N \) such that for \( n \geq N \), (5.6) holds for this choice of \( \delta, \varepsilon \) and \( T \). In particular, for \( n \geq N \),
\[ \tau_D(\omega) < \lambda_n(\tau_D(\omega) + \delta) < \tau_D(\omega) + 2\delta \]
and
\[ |\omega_n(\tau_D(\omega) + \delta) - \omega(\lambda_n(\tau_D(\omega) + \delta))| < \frac{\varepsilon}{2}. \]
Together these imply
\[ d(\omega_n(\tau_D(\omega) + \delta), D) > 0, \quad n \geq N, \]
which in turn yields
\[ \tau_D(\omega_n) \leq \tau_D(\omega) + \delta, \quad n \geq N. \]
Taking the lim sup as \( n \to \infty \) and then letting \( \delta \to 0 \) yields (5.7). \( \square \)

**Lemma 5.3.** If \( D \) is a bounded domain that satisfies an exterior cone condition or if \( D \) is a cone, then for all \( x \in D \) and \( 0 < \alpha < 2 \),
\[ P^\alpha_x( C_D \cap \{ X(\tau_D) \in D \} ) = 1. \]

**Proof.** If \( D \) is bounded and satisfies a uniform exterior cone condition, it is known
\[
(5.8) \quad P^\alpha_x( X(\tau_D) \in \partial D ) = 0;
\]
see Lemma 6 in [10]. If \( D \) is a cone, we can apply Lemma 6 in [10] to \( D \cap B_M(0) \)
and letting \( M \to \infty \), we get (5.8).

The proof of Theorem 2 in [18] implies
\[ P^\alpha_x( X(\tau_D) \in \partial D, X(\tau_D) \in E ) = 0, \quad E \subseteq \overline{E} \subseteq \overline{D}^c \]
(see the lines before the footnote on page 89). Combined with (5.8),
\[ P^\alpha_x( X(\tau_D) \in \overline{D}^c, X(\tau_D) \in D ) = 1. \]
Thus to prove the lemma we need to show that
\[ P^\alpha_x( d( X[0, \tau_D - r], D^c ) > 0 \text{ for all rational } r < \tau_D ) = 1. \]
Let
\[ D_n = \{ x \in D : d(x, D^c) > \frac{1}{n} \} \]
and observe \( \tau_{D_n} \leq \tau_D \) increases to some limit \( L \leq \tau_D \). By quasi-left continuity,
\( X(\tau_{D_n}) \to X(L) \) almost surely. One easily sees \( X(L) \notin D \), i.e., \( \tau_D \leq L \). Hence
\( \tau_D = L \), and the increasing limit of \( \tau_{D_n} \) is \( \tau_D \).

If for some rational \( r < \tau_D \) we have
\[ d( X[0, \tau_D - r], D^c ) = 0, \]
then for some sequence \( s_n \leq \tau_D - r \),
\[ d( X_{s_n}, D^c ) \to 0. \]
It is no loss to assume \( s_n \) converges, say to \( s \). Choose \( N \) such that for all \( n \geq N \)
\[ \tau_D - r < \tau_{D_n} \leq \tau_D \]
Given \( n \geq N \), choose \( M_n \) such that for all \( m \geq M_n \),
\[ d( X_{s_m}, D^c ) < \frac{1}{2n} \]
Then for such \( m \), \( X_{s_m} \in D_n^c \), which forces
\[ \tau_{D_n} \leq s_m \leq \tau_D - r. \]
Let \( m \to \infty \) to get \( \tau_{D_n} \leq s \leq \tau_D - r \), then let \( n \to \infty \) to get \( \tau_D = \lim_{n \to \infty} \tau_{D_n} \leq \tau_D - r \);
a contradiction. Thus (5.10) holds. \( \square \)

We will need the following elementary result shortly.

**Lemma 5.4.** Let \( a_n \) and \( b_n \) be nonnegative sequences such that \( a_n \wedge b_n \to 0 \) as \( n \to \infty \). Then for some \( n_k \), either \( a_{n_k} \to 0 \) or \( b_{n_k} \to 0 \) as \( k \to \infty \).
Proof. Suppose \( \lim \inf_{n \to \infty} a_n = a > 0 \). Choose a subsequence \( a_{n_k} \) such that \( a_{n_k} \geq \frac{a}{2} \) for all \( k \). Then

\[
a_{n_k} \wedge b_{n_k} \geq \frac{a}{2} \wedge b_{n_k} \geq 0.
\]

Since \( a_{n_k} \wedge b_{n_k} \to 0 \) as \( k \to \infty \), we must have \( \frac{a}{2} \wedge b_{n_k} \to 0 \), which in turn forces \( b_{n_k} \to 0 \).

**Lemma 5.5.** Let \( D \) be a bounded domain that satisfies an uniform exterior cone condition, and let \( f \) be a bounded continuous function on \( \mathbb{R}^d \). If \( (x_n, \alpha_n) \) converges to \((x, \alpha)\) in \( D \times (0, 2) \), then for each \( T > 0 \),

\[
E_{x_n}^\alpha[f(X_{T \wedge \tau_D})] \to E_x^\alpha[f(X_{T \wedge \tau_D})].
\]

**Proof.** Let \( C_D(T) = C_D \cap \{ X(\tau_D) \in D \} \cap \{ \tau_D \neq T \} \cap \{ \lim_{s \to T} X_s = X_T \} \).

Recall that the symmetric \( \alpha \)-stable process has no fixed discontinuities. Then by the eigenfunction expansion of \( P_x^\alpha(\tau_D > t) \),

\[
P_x^\alpha(\tau_D \neq T, \lim_{s \to T} X_s = X_T) = 1.
\]

Thanks to Lemma 5.3

\[
P_x^\alpha(C_D(T)) = 1.
\]

If we can show that

\[
\omega \in C_D(T) \to \omega(T \wedge \tau_D(\omega))
\]

is continuous, then by an extension of the continuous mapping theorem (Theorem 5.1 in [7]), the desired conclusion will follow.

Let \( \omega \in C_D(T) \) and suppose \( \omega_n \in C_D(T) \) converges to \( \omega \) in \( D[0, \infty) \). Define

\[
t_n = T \wedge \tau_D(\omega_n),
\]

\[
t = T \wedge \tau_D(\omega).
\]

To show that

\[
\lim_{n \to \infty} \omega_n(T \wedge \tau_D(\omega_n)) = \omega(T \wedge \tau_D(\omega)),
\]

we show for every subsequence \( \omega_{n_k} \) there is a further subsequence \( \omega_{n_{k_l}} \) such that

\[
\lim_{l \to \infty} \omega_{n_{k_l}}(T \wedge \tau_D(\omega_{n_{k_l}})) = \omega(T \wedge \tau_D(\omega)).
\]

By Lemma 5.2 \( \lim_{n \to \infty} t_n = t \). Applying Proposition 6.5(a) on page 125 in [10],

\[
\lim_{n \to \infty} |\omega(t_n) - \omega(t)| \wedge |\omega(t_n - \omega(t)| = 0.
\]

Hence by Lemma 5.3 for any subsequence \( \omega_{n_k} \) there is a further subsequence \( \omega_{n_{k_l}} \) such that either

\[
\lim_{l \to \infty} |\omega_{n_{k_l}}(t_{n_{k_l}}) - \omega(t)| = 0
\]

or

\[
(5.11) \quad \lim_{l \to \infty} |\omega_{n_{k_l}}(t_{n_{k_l}}) - \omega(t^-)| = 0.
\]

In the first case, clearly

\[
\lim_{l \to \infty} \omega_{n_{k_l}}(T \wedge \tau_D(\omega_{n_{k_l}})) = \omega(T \wedge \tau_D(\omega)),
\]

as desired.
In the second case (5.11), we distinguish two cases: $T > \tau_D(\omega)$ and $T < \tau_D(\omega)$ (recall $\omega \in C_D(T)$ implies $\tau_D(\omega) \neq T$).

Let us first assume $T > \tau_D(\omega)$. Since $\tau_D(\omega_{n_k})$ converges to $\tau_D(\omega)$ by Lemma 5.2, $t_{n_k} = \tau_D(\omega_{n_k})$ for large $l$. Hence by (5.11)

$$\lim_{l \to \infty} \omega_{n_k}(\tau_D(\omega_{n_k})) = \lim_{l \to \infty} \omega_{n_k}(t_{n_k}) = \omega(t^-) = \omega(\tau_D(\omega^-)).$$

Notice that if $\lim_{l \to \infty} \omega_{n_k}(\tau_D(\omega_{n_k})) = y$ exists, then $y \in D^c$. But then $\omega \in C_D(T)$ implies $y = \omega(\tau_D(\omega^-)) \notin D$; a contradiction. Thus $T > \tau_D(\omega)$ is not possible.

Finally, if $T < \tau_D(\omega)$, then by Lemma 5.2

$$T < \tau_D(\omega_{n_k}),$$

for $l$ large. Since $\omega \in C_D(T)$, (5.11) becomes

$$\omega_{n_k}(T) \to \omega(T^-) = \omega(T).$$

We conclude that

$$\lim_{l \to \infty} \omega_{n_k}(T \land \tau_D(\omega_{n_k})) = \lim_{l \to \infty} \omega_{n_k}(T)$$

$$= \omega(T^-)$$

$$= \omega(T)$$

$$= \omega(T \land \tau_D(\omega)).$$

In any event, we get the desired continuity. $\Box$

6. Proof of Theorem 1.4

We will need the following lemma; it is formula (2.7) in [5]. Although the authors do not mention the statement concerning continuity in $\alpha$, it is possible to trace back through the literature they cite to see the statement holds.

**Lemma 6.1.** If $D \subseteq \mathbb{R}^d$ is a bounded Lipschitz domain, then for some positive continuous functions $C(\alpha)$ and $\beta(\alpha)$,

$$E^\alpha_x(\tau_D) \leq C(\alpha) \delta_D^\beta(x), \quad \text{for all } x \in D.$$  

The next result immediately follows.

**Corollary 6.2.** Given a bounded Lipschitz domain $D$ and compact $K \times [a, b] \subseteq \overline{D} \times (0, 2)$,

$$\sup \{ E^\alpha_x(\tau_D) : (x, \alpha) \in K \times [a, b] \} < \infty.$$  

Corollary 6.2 will allow us to get equicontinuity of the eigenfunctions near $\partial D$. For the interior of $D$ we need the following Krylov–Safanov type of theorem. Let

$$G_0^\alpha g(x) = E^\alpha_x \left[ \int_0^{\tau_D} g(X_t) \, dt \right]$$

be the 0-resolvent of the killed symmetric $\alpha$-stable process in $D$.

**Lemma 6.3.** Suppose $g$ is bounded with support in $\overline{D}$. Then for each $x \in D$ there exist positive continuous functions $C(\alpha)$ and $\beta(\alpha)$, independent of $g$, such that for all $y \in D$

$$| G_0^\alpha g(x) - G_0^\alpha g(y) | \leq C(\alpha) [ \sup |G_0^\alpha g| + \sup |g| ] |x - y|^\beta(\alpha).$$
Proof. This theorem is essentially due to Bass and Levin [4] (see their Proposition 4.2 on page 387). While they consider the 0-resolvent

\[ S_0 g(x) = E_x \left[ \int_0^\infty g(X_t) \, dt \right], \]

their proof also works for the killed resolvent because their crucial formula

\[ S_0 g(y) = E_y \left[ \int_{\tau_{B(x,r)}}^\infty g(X_t) \, dt \right] + E_y \left[ S_0 g(X_{\tau_{B(x,r)}}) \right] \]

holds when \( S_0 \) is replaced by \( G_0^\alpha \) and \( E_y \) is replaced by \( E^\alpha_y \), where \( r > 0 \) is such that \( B(x,r) \subset D \). Since we are restricted to \( D \) instead of \( \mathbb{R}^d \), the numbers \( C(\alpha) \) and \( \beta(\alpha) \) depend on \( x \), in contrast to the case treated by Bass and Levin. Moreover, it is a simple matter to go through their proof and see that the numbers \( C(\alpha) \) and \( \beta(\alpha) \) can be chosen to depend continuously on \( \alpha \). □

Corollary 6.4. Assume \( D \) is bounded and Lipschitz. Then for each \( x \in D \) and \( [a,b] \subseteq (0,2) \), there exist positive \( C \) and \( r \) such that

\[ |G_0^\alpha g(x) - G_0^\alpha g(y)| \leq C |x - y|^r \sup |g| \]

for all \( y \in D \), \( \alpha \in [a,b] \) and bounded \( g \) with support in \( D \).

Proof. By Corollary 6.2

\[ \sup |G^\alpha g| \leq \sup |g| \cdot \sup_x E^\alpha_x (\tau_D) \leq \sup |g| \cdot C \]

where \( C \) is independent of \( \alpha \in [a,b] \) and \( g \). The result follows from this and the continuity of \( C(\alpha) \) and \( \beta(\alpha) \) from Lemma 6.3. □

At last we can prove Theorem 1.4. It is well known that

\[ 0 \leq p^\alpha_D(t,x,y) \leq p^\beta(t,x,y). \]  

Moreover,

\[ p^\alpha(t,x,y) \leq C(\alpha)t^{-d/\alpha} \]

where \( C(\alpha) \) is continuous in \( \alpha \) (see (2.1) in [9]).

Let \( \{\alpha_m\}_{m=1}^\infty \subseteq (0,2) \) be a sequence converging to \( \alpha \in (0,2) \). Recall that for all \( \beta \in (0,2) \), \( \{\varphi_\beta^\alpha\}_m \) is an orthonormal set. Then thanks to the symmetry of the heat kernel and 6.2

\[ \varphi_\beta^\alpha(x) = e^{\lambda_\beta^D t} \int_D p^\beta_D(t,x,y) \varphi_\beta^\alpha(y) \, dy \]

\[ \leq e^{\lambda_\beta^D t} \sqrt{\int_D \left[ p^\beta_D(t,x,y) \right]^2 \, dy} \]

\[ = e^{\lambda_\beta^D t} \sqrt{p^\beta_D(2t,x,x)} \]

\[ \leq e^{\lambda_\beta^D t} \sqrt{\frac{C(\beta)}{(2t)^d/\beta}}. \]
In particular, taking \( t = 1 \) and using Theorem 1.2,

\[
(6.3) \quad \sup_{x \in D, m \geq 1} \varphi_i^{\alpha m}(x) \leq \sup_{m \geq 1} e^{\alpha m} \sqrt{\frac{C(\alpha_m)}{2^{d/\alpha_m}}} < \infty.
\]

Thus for each \( i \geq 1 \), the sequence \( \{\varphi_i^{\alpha m}\}_{m=1}^{\infty} \) is uniformly bounded. Next we show the sequence \( \{\varphi_i^{\alpha m}\}_{m=1}^{\infty} \) is pointwise equicontinuous on \( \overline{D} \). Indeed, since

\[
(6.4) \quad \varphi_i^\beta = \lambda_i^\beta G_0^\beta \varphi_i^1.
\]

Corollary [6.4] implies that for each \( x \in D \) there exist \( C \) and \( r \) such that

\[
|\varphi_i^{\alpha m}(x) - \varphi_i^{\alpha m}(y)| = \lambda_i^{\alpha m} |G_0^{\alpha m} \varphi_i^{\alpha m}(x) - G_0^{\alpha m} \varphi_i^{\alpha m}(y)|
\leq C \left[ \sup_{m \geq 1} \mu_i^{\alpha m/2} \left[ \sup_{u \in D, m \geq 1} |\varphi_i^{\alpha m}(u)| \right] \right] |x - y|^r
\]

for all \( m \geq 1 \) and \( y \in D \). Thanks to (6.3) we get the desired equicontinuity for \( x \in D \).

As for \( x \in \partial D \), first notice (6.4) and Lemma 6.1 imply there are \( r \) and \( C \) independent of \( m \) such that for each \( z \in D \)

\[
|\varphi_i^{\alpha m}(z)| \leq \left[ \sup_{m \geq 1} \mu_i^{\alpha m/2} \left[ \sup_{y \in D, m \geq 1} |\varphi_i^{\alpha m}(y)| \right] \right] E_z^{\alpha m}(\partial D)
\leq C \left[ \delta_D(z) \right]^r.
\]

Thus \( \varphi_i^{\alpha m} \) is continuous on \( \overline{D} \) with boundary value 0. Hence if \( x \in \partial D \), then

\[
|\varphi_i^{\alpha m}(x) - \varphi_i^{\alpha m}(y)| = |\varphi_i^{\alpha m}(y)|
\leq C \left[ \delta_D(y) \right]^r
\leq C|x - y|^r.
\]

By Ascoli’s Theorem, the sequence \( \{\varphi_i^{\alpha m}\}_{m=1}^{\infty} \) is precompact in \( C(\overline{D}) \).

Next assume \( \{\lambda_i^{\alpha m}\}_{m=1}^{\infty} \) converges to \( \lambda \). We show any limit point \( \varphi \) of the sequence \( \{\varphi_i^{\alpha m}\}_{m=1}^{\infty} \) is an eigenfunction of \( H_\alpha \) and the corresponding eigenvalue is \( \lambda \). Choose a subsequence \( \beta_i = \alpha_m \) such that, as \( r \to \infty \), \( \varphi_i^{\beta_i} \) converges uniformly to \( \varphi \) on \( \overline{D} \). Since \( \varphi_i^{\beta_i} \) and \( \varphi \) are 0 on \( \partial D \), we can extend them to all of \( \mathbb{R}^d \) by taking them to be 0 outside \( D \). Then

\[
E_x^{\beta_i} \left[ \varphi_i^{\beta_i}(X_{t \wedge \tau_D}) \right] = E_x^{\beta_i} \left[ \varphi_i^{\beta_i}(X_t) \mathbb{I}_{\tau_D > t} \right],
\]

and \( \varphi_i^{\beta_i} \) converges to \( \varphi \) uniformly on \( \mathbb{R}^d \). Thus we have

\[
e^{-\lambda_i^{\alpha m} t} \varphi_i^{\alpha m}(x) = \int_D \varphi_D(t, x, y) \varphi_i^{\beta_i}(y) dy
\]

\[
(6.5) \quad = E_x^{\beta_i} \left[ \varphi_i^{\beta_i}(X_t) \mathbb{I}_{\tau_D > t} \right]
\]

\[
= E_x^{\beta_i} \left[ \varphi_i^{\beta_i}(X_{t \wedge \tau_D}) \right]
\]

\[
= E_x^{\beta_i} \left[ \varphi_i^{\beta_i}(X_{t \wedge \tau_D}) - \varphi(X_{t \wedge \tau_D}) \right] + E_x^{\beta_i} \left[ \varphi(X_{t \wedge \tau_D}) \right].
\]
Lemma 6.5 and the uniform convergence of $\phi_\alpha^m$ to $\phi$ imply
\[
\lim_{r \to \infty} E_x^\beta \left[ \phi_\alpha^m (X_t \wedge \tau_D) - \phi (X_t \wedge \tau_D) \right] + E_x^\beta \left[ \phi(X_t \wedge \tau_D) \right] = E_x^\alpha [\phi(X_t \wedge \tau_D)].
\]
Since the left-hand side of (6.5) converges to $e^{-\lambda t} \phi(x)$, we conclude that
\[
e^{-\lambda t} \phi(x) = E_x^\alpha \left[ \phi(X_t \wedge \tau_D) \right] = E_x^\alpha \left[ \phi(X_t) I_{\tau_D > \tau} \right] = \int_D p_\alpha^D (t, x, y) \phi(y) dy.
\]
Hence $\phi$ is an eigenfunction of $H_\alpha$, and the corresponding eigenvalue is $\lambda$. \qed

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