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α-CONTINUITY PROPERTIES OF THE SYMMETRIC α-STABLE PROCESS

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Abstract. Let D be a domain of finite Lebesgue measure in $\mathbb{R}^d$ and let X_t^D be the symmetric α-stable process killed upon exiting D. Each element of the set $\{\lambda_i^\alpha\}_{i=1}^\infty$ of eigenvalues associated to X_t^D, regarded as a function of $\alpha \in (0,2)$, is right continuous. In addition, if D is Lipschitz and bounded, then each λ_i^α is continuous in α and the set of associated eigenfunctions is precompact.

1. Introduction

Let X_t be a d-dimensional symmetric α-stable process of order $\alpha \in (0,2]$. The process X_t has stationary independent increments and its transition density $p^\alpha(t,z,w) = f_t^\alpha(z-w)$ is determined by its Fourier transform

$$\exp(-t|z|^\alpha) = \int_{\mathbb{R}^d} e^{iz\cdot w} f_t^\alpha(w)dw.$$

These processes have right continuous sample paths and their transition densities satisfy the scaling property

$$p^\alpha(t,x,y) = t^{-d/\alpha}\, p^\alpha(1, t^{-1/\alpha}x, t^{-1/\alpha}y)\ .$$

When $\alpha = 2$ the process X_t is a d-dimensional Brownian motion running at twice the usual speed. The nonlocal operator associated to X_t is $(-\Delta)^{\alpha/2}$ where Δ is the Laplace operator in $\mathbb{R}^d$.

Let D be an open set in $\mathbb{R}^d$ and let X_t^D be the symmetric α-stable process killed upon leaving D. We write $p_D^\alpha(t,x,y)$ for the transition density of X_t^D and H_α for its associated nonlocal self-adjoint positive operator. It is well known that if D has finite Lebesgue measure, then the spectrum of H_α is discrete. Let

$$0 < \lambda_1^\alpha(D) \le \lambda_2^\alpha(D) \le \lambda_3^\alpha(D) \le \cdots$$

be the eigenvalues of H_α, and let

$$\varphi_1^\alpha,\ \varphi_2^\alpha,\ \varphi_3^\alpha,\ \ldots$$

be the corresponding sequence of orthonormal $L^2(D)$ eigenfunctions. Also, φ_1^α is chosen so as to be positive on D. Note that if $\alpha < 2$, then $\lambda_1^\alpha(D) < \lambda_2^\alpha(D)$, but this need not be true if $\alpha = 2$ (unless D is connected).

Several authors have studied properties of the eigenvalues and eigenfunctions of H_α. One common theme has been to extend results on Brownian motion ($\alpha = 2$) to

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analogous results for symmetric α-stable processes. For example, R. M. Blumenthal and R. K. Getoor [8] have shown Weyl's asymptotic law holds: if D is a bounded open set and $N(\lambda)$ is the number of eigenvalues less than or equal to λ, then there exists a constant $C_{d,\alpha}$, depending only on d and α, such that

$$N(\lambda) \approx C_{d,\alpha} \frac{m(D)}{\Gamma(d/\alpha+1)} \lambda^{d/\alpha} \quad \text{as } \lambda \to \infty,$$

provided $m(\partial D) = 0$, where m is Lebesgue measure.

If $D \subseteq \mathbb{R}^d$ is a domain, define the *inner radius* R_D to be the supremum of the radii of all balls contained in D. R. Bañuelos et al. [6] and P. Méndez-Hernández [20] have shown that if D is a convex domain with finite inner radius R_D and I_D is the interval $(-R_D, R_D)$, then

$$\lambda_1^\alpha(I_D) \le \lambda_1^\alpha(D).$$

Moreover, if $D \subseteq \mathbb{R}^d$ has finite volume and D^* is a ball in $\mathbb{R}^d$ with the same volume as D, then it was proved in [6] that the Faber-Krahn inequality holds:

$$\lambda_1^\alpha(D^*) \le \lambda_1^\alpha(D).$$

Another line of inquiry taken by those authors was to consider the eigenvalues as a function of the index α. For instance, if D is a convex domain with finite inner radius R_D, then

$$\frac{2^\alpha \Gamma(1+\frac{\alpha}{2})\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{1}{2})R_D^\alpha} \le \lambda_1^\alpha(D) \le \lambda_1^\alpha(B_{R_D}),$$

where B_{R_D} is a ball in $\mathbb{R}^d$ of radius R_D. They also proved that if $D \subseteq \mathbb{R}^d$ has finite volume, then

$$\lambda_1^\alpha(D) \le [\mu_1(D)]^{\alpha/2}, \tag{1.1}$$

where $\mu_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D.

For the Cauchy process, i.e. $\alpha = 1$, and bounded Lipshitz domains, R. Bañuelos and T. Kulczycki [5] extended (1.1) to

$$\lambda_i^1(D) \le [\mu_i(D)]^{1/2}, \quad i = 1, 2, \ldots, \tag{1.2}$$

where

$$0 < \mu_1(D) < \mu_2(D) \le \cdots$$

are all the Dirichlet eigenvalues of $-\Delta$ on D. Their proof of (1.2) is based on a variational formula for $\lambda_i^1(D)$ that they developed from a connection with the Steklov problem for the Laplacian. They also obtained many detailed properties of the eigenfunctions φ_i^1 for the Cauchy process.

By finding a connection with the symmetric stable process with rational index α and PDEs of order higher than 2, R. D. DeBlassie [14] derived a variational formula for the eigenvalues which led to the following extension of (1.1) and (1.2):

$$\lambda_i^\alpha(D) \le [\mu_i(D)]^{\alpha/2}, \quad i = 1, 2, \ldots, \tag{1.3}$$

for all rational $\alpha \in (0, 2)$ and certain bounded domains $D \subseteq \mathbb{R}^d$. The class of admissible domains includes convex polyhedra, Lipschitz domains with sufficiently small Lipschitz constant and C^1 domains. Please note there is an error in [14] in the derivation of (1.3) above, as pointed out by the referee of the present article. See [15] for a correction. Also, we have learned of a recent preprint of Z.-Q. Chen

and R. Song [11] which extends (1.3) to all indices $\alpha \in (0,2)$ and domains D of finite Lebesgue measure.

In this article, we study the eigenvalues and eigenfunctions regarded as functions of the index α. Our first result concerns continuity of the eigenvalues.

Theorem 1.1. *Let D be a domain of finite Lebesgue measure. Then, as a function of $\alpha \in (0,2)$, λ_i^α is right continuous for each positive integer i.*

In order to prove Theorem 1.1 we need the following interesting monotonicity property extending (1.3) above. It is due to Z.-Q. Chen and R. Song [11]; see their Example 5.4.

Theorem 1.2. *Let D be an open set of finite Lebesgue measure in $\mathbb{R}^d$. If $0 < \alpha < \beta \leq 2$, then for all positive integers i,*

$$\left[\lambda_i^\alpha(D)\right]^{1/\alpha} \leq \left[\lambda_i^\beta(D)\right]^{1/\beta}.$$

Even though those authors consider bounded open sets D, it is clear their argument works for open sets of finite Lebesgue measure.

By requiring more regularity of ∂D, we can prove the following extension of Theorem 1.1.

Theorem 1.3. *Let D be a bounded Lipschitz domain. Then, as a function of $\alpha \in (0,2)$, λ_i^α is continuous for each positive integer i.*

We will obtain Theorem 1.3 from the following result that we believe is of independent interest.

Theorem 1.4. *Let D be a bounded Lipschitz domain. If α_m converges to $\alpha \in (0,2)$, then for each positive integer i, $\{\varphi_i^{\alpha_m}:\ m \geq 1\}$ is precompact in $C(\overline{D})$ equipped with the sup norm. Moreover, if $\lambda_i^{\alpha_m}$ converges to λ, then any limit point of $\{\varphi_i^{\alpha_m}:\ m \geq 1\}$ is an eigenfunction of H_α and λ is the corresponding eigenvalue.*

As a corollary of the proof of the last theorem, we obtain continuity of the first eigenfunction as a function of α.

Theorem 1.5. *If D is a bounded Lipshitz domain and α_m converges to $\alpha \in (0,2)$, then $\varphi_1^{\alpha_m}$ converges uniformly to φ_1^α on D.*

The article is organized as follows. In Section 2 we present some results needed in the proof of Theorem 1.1. In Section 3 we establish Theorem 1.1 by proving upper semicontinuity and right lower semicontinuity of the eigenvalues via Dirichlet forms. Lower semicontinuity of the eigenvalues, for Lipschitz domains, is proved in section 4 using Theorem 1.4. This will yield Theorems 1.3 and 1.5. Section 5 deals with certain weak convergence results needed to prove Theorem 1.4. Finally, in Section 6 we prove Theorem 1.4.

2. Preliminary results

Throughout this section we will assume the domain D has finite Lebesgue measure. We denote by $C_c^\infty(D)$ the set of C^∞ functions with compact support in D. The inner product and the norm in $L^2(D)$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_2$, respectively.

For any domain $D \subseteq \mathbb{R}^d$, we define τ_D to be the first exit time of X_t from D, i.e.,

$$\tau_D = \inf\{t > 0 : X_t \notin D\}.$$

Let

$$\mathcal{F}_\alpha = \left\{ \varphi \in L^2(\mathbb{R}^d) : \int\int \frac{[\,\varphi(y) - \varphi(x)\,]^2}{|y-x|^{d+\alpha}}\, dydx \,<\, \infty \right\}.$$

The Dirichlet form $(\mathcal{E}_\alpha, \mathcal{F}_\alpha)$ associated to X_t is given by

$$\mathcal{E}_\alpha(\psi, \varphi) = A(d,\alpha) \int\int \frac{[\,\psi(y) - \psi(x)\,]\;[\,\varphi(y) - \varphi(x)\,]}{|y-x|^{d+\alpha}}\, dydx,$$

for all $\psi, \varphi \in \mathcal{F}_\alpha$, where

$$A(d,\alpha) = \frac{\Gamma(\frac{d-\alpha}{2})}{2^\alpha\, \pi^{d/2}\, \Gamma(\frac{\alpha}{2})}.$$

It is well known that the Dirichlet form corresponding to X_t^D is given by $(\mathcal{E}_\alpha, \mathcal{F}_{\alpha,D})$, where

$$\mathcal{F}_{\alpha,D} = \{u \in \mathcal{F}_\alpha : \text{a quasi continuous version of } u \text{ is 0 quasi everywhere in } D^c\}.$$

Recall that for all ψ, φ in the domain of H_α we have

$$\mathcal{E}_\alpha(\psi, \varphi) = \langle \psi, H_\alpha \varphi \rangle.$$

As seen in Theorem 4.4.3 of [17], $\mathcal{F}_{\alpha,D}$ is the closure of $C_c^\infty(D)$ in $\mathcal{F}_\alpha$ with respect to the norm

$$\|\varphi\|_\alpha = \sqrt{\mathcal{E}_\alpha(\varphi,\varphi) + \|\varphi\|_2}\,.$$

Lemma 2.1. *Let $\varphi, \psi \in C_c^\infty(D)$. Then the function*

$$\mathcal{E}_\alpha(\varphi, \psi) : (0,2) \to \mathbb{R}$$

is continuous on $(0,2)$.

Proof. Let $\varphi, \psi \in C_c^\infty(D)$, and let $\beta \in (\alpha - \delta, \alpha + \delta)$, where $\delta = \frac{1}{2}\min\{2-\alpha, \alpha\}$. Then there exists a constant $C > 0$, depending only on φ and ψ, such that

$$\begin{aligned} \frac{[\,\psi(y) - \psi(x)\,]\;[\,\varphi(y) - \varphi(x)\,]}{|y-x|^{d+\beta}} &\leq \frac{C}{|y-x|^{d+\beta-2}} \\ &\leq C \max\left\{ \frac{1}{|y-x|^{d+\alpha-\delta-2}}, \frac{1}{|y-x|^{d+\alpha+\delta-2}} \right\}. \end{aligned}$$

Since D has finite measure, a simple computation using polar coordinates shows that

$$\max\left\{ \frac{1}{|y-x|^{d+\alpha-\delta-2}}, \frac{1}{|y-x|^{d+\alpha+\delta-2}} \right\}$$

is integrable in both $(\mathrm{supp}(\varphi) \cup \mathrm{supp}(\psi)) \times D$ and $D \times (\mathrm{supp}(\varphi) \cup \mathrm{supp}(\psi))$. The result immediately follows from the dominated convergence theorem. □

We end this section with some basic estimates on L^2 norms to be used in the next section. Suppose k is a positive integer, $0 < \epsilon < 1$, and $\varphi_1, \ldots, \varphi_k \in L^2(D)$ satisfy

$$|\langle \varphi_i, \varphi_j \rangle| \,<\, \frac{\epsilon}{4k^2}, \qquad i \neq j,$$

$$\left(1 - \frac{\epsilon}{4k^2}\right) \,<\, \|\varphi_i\|_2^2 < \left(1 + \frac{\epsilon}{4k^2}\right),$$

for all $1 \leq i, j \leq k$. If $\psi = \sum_{i=1}^{k} a_i \varphi_i$ with $\|\psi\|_2 = 1$, then we show that

$$\frac{1}{1+\epsilon/2} \leq \sum_{i=1}^{k} a_i^2 \leq \frac{1}{1-\epsilon/2} \tag{2.1}$$

and $\varphi_1, \ldots \varphi_k$ are linearly independent.

For the proof, note that we have

$$\begin{aligned} 1 = \langle \psi, \psi \rangle &= \sum_{i=1}^{k} a_i^2 \|\varphi_i\|_2^2 + 2 \sum_{i=1}^{k} \sum_{j>i}^{k} a_i a_j \langle \varphi_i, \varphi_j \rangle \\ &\geq \sum_{i=1}^{k} a_i^2 \left(1 - \frac{\epsilon}{4k^2}\right) - 2 \sum_{i=1}^{k} \sum_{j>i}^{k} |a_i| \, |a_j| \, \frac{\epsilon}{4k^2} \\ &\geq \sum_{i=1}^{k} a_i^2 \left(1 - \frac{\epsilon}{4k^2}\right) - (k^2 - k) \sum_{i=1}^{k} a_i^2 \frac{\epsilon}{4k^2} \\ &\geq (1 - \epsilon/2) \sum_{i=1}^{k} a_i^2, \end{aligned}$$

and we conclude that

$$\sum_{i=1}^{k} a_i^2 \leq \frac{1}{1-\epsilon/2}.$$

Similar computations give the remaining assertions.

3. Proof of Theorem 1.1

We will use the following well-known result; see [12].

Theorem 3.1. *Let H be a nonnegative self-adjoint unbounded operator with discrete spectrum $\{\lambda_i\}_{i=1}^{\infty}$ and domain $Dom(H)$. Then for $i \geq 1$*

$$\lambda_i = \inf \{\lambda(L) : L \subseteq Dom(H), dim(L) = i\}, \tag{3.1}$$

where

$$\lambda(L) = \sup \{\langle Hf, f \rangle \, : \, f \in L, \|f\|_2 = 1\}, \tag{3.2}$$

and L is a vector subspace of $Dom(H)$ of dimension i.

We will prove the right continuity of the kth eigenvalue in two steps.

Proposition 3.2. *Let D be a domain of finite Lebesgue measure. Then for all $k \geq 1$*

$$\limsup_{\beta \to \alpha} \lambda_k^{\beta}(D) \leq \lambda_k^{\alpha}(D).$$

Proof. Let $0 < \epsilon < 1$ and $k \geq 1$. Recall $C_c^{\infty}(D)$ is dense in $Dom(H_\alpha)$ under the norm $\| \cdot \|_\alpha$. Then for all $\alpha \in (0, 2)$, there exist $\varphi_1, \ldots, \varphi_k \in C_c^{\infty}(D)$ such that

$$|\langle \varphi_i^{\alpha}, \varphi_j^{\alpha} \rangle - \langle \varphi_i, \varphi_j \rangle| \, < \, \frac{\epsilon}{8k^2} \tag{3.3}$$

and

$$|\mathcal{E}_\alpha(\varphi_i^{\alpha}, \varphi_j^{\alpha}) - \mathcal{E}_\alpha(\varphi_i, \varphi_j)| \, < \, \frac{\epsilon}{8k^2}, \tag{3.4}$$

for all $1 \leq i, j \leq k$.

Thanks to Lemma 2.1 there exists η_0 such that for all $\beta \in (\alpha - \eta_0, \alpha + \eta_0)$

$$|\mathcal{E}_\alpha(\varphi_i, \varphi_j) - \mathcal{E}_\beta(\varphi_i, \varphi_j)| < \frac{\epsilon}{8k^2}. \tag{3.5}$$

Notice that (3.3) implies

$$|\langle \varphi_i, \varphi_j \rangle| < \frac{\epsilon}{8k^2}, \quad i \neq j,$$

and

$$1 - \frac{\epsilon}{8k^2} < \|\varphi_i\|_2^2 < 1 + \frac{\epsilon}{8k^2},$$

for all $1 \leq i, j \leq k$. Then by the comments at the end of Section 2, we know $\varphi_1, \ldots, \varphi_k$ are linearly independent.

Theorem 3.1 implies

$$\lambda_k^\beta(D) \leq \lambda_\beta(L_k),$$

where $L_k = \operatorname{span}\{\varphi_1, \ldots, \varphi_k\}$ and

$$\lambda_\beta(L_k) = \sup\{\, \langle H_\beta f, f \rangle : f \in L_k, \|f\|_2 = 1 \,\}.$$

Take $\psi = \sum_{i=1}^k a_i \varphi_i \in L_k$ such that

$$\lambda_\beta(L_k) \leq \mathcal{E}_\beta(\psi, \psi) + \epsilon/4 \tag{3.6}$$

and

$$\|\psi\|_2 = 1.$$

Thanks to (2.1), with ϵ there replaced by $\epsilon/2$, we have

$$\sum_{i=1}^k a_i^2 \leq 2.$$

Then since

$$|\, \mathcal{E}_\beta(\psi, \psi) - \mathcal{E}_\alpha(\psi, \psi) \,| \leq \sum_{i=1}^k \sum_{j=1}^k |a_i\, a_j| \;\, |\, \mathcal{E}_\beta(\varphi_i, \varphi_j) - \mathcal{E}_\alpha(\varphi_i, \varphi_j) \,|,$$

(3.5) implies

$$|\, \mathcal{E}_\beta(\psi, \psi) - \mathcal{E}_\alpha(\psi, \psi) \,| < \frac{\epsilon}{4}. \tag{3.7}$$

Thus

$$\lambda_k^\beta(D) \leq \mathcal{E}_\alpha(\psi, \psi) + \epsilon/2. \tag{3.8}$$

Consider $\psi_0 = \sum_{i=1}^k a_i \varphi_i^\alpha$. By (2.1) we have

$$\frac{1}{1 + \epsilon/4} \leq \|\psi_0\|_2^2 = \sum_{i=1}^k a_i^2 \leq \frac{1}{1 - \epsilon/4}.$$

Following the argument used to obtain (3.7), one easily proves (3.4) implies

$$|\, \mathcal{E}_\alpha(\psi_0, \psi_0) - \mathcal{E}_\alpha(\psi, \psi) \,| < \frac{\epsilon}{4}.$$

Hence

$$\begin{aligned}
\lambda_k^\beta(D) &\leq \mathcal{E}_\alpha(\psi,\psi)+\epsilon/2\\
&\leq \mathcal{E}_\alpha(\psi_0,\psi_0)+3\epsilon/4\\
&= \sum_{i=1}^k a_i^2\lambda_i^\alpha(D)+3\epsilon/4\\
&\leq \lambda_k^\alpha(D)\sum_{i=1}^k a_i^2+3\epsilon/4\\
&\leq \frac{1}{1-\epsilon/4}\lambda_k^\alpha(D)+3\epsilon/4,
\end{aligned}$$

and the result immediately follows. □

Proposition 3.3. *Let D be a domain of finite Lebesgue measure. Then for all $k \geq 1$*

$$\liminf_{\beta\to\alpha+}\lambda_k^\beta(D)\geq\lambda_k^\alpha(D).$$

Proof. By Theorem 1.2,

$$\lambda_k^\alpha(D)\leq\left[\lambda_k^{\alpha+\epsilon}(D)\right]^{\alpha/(\alpha+\epsilon)}.$$

Now let $\epsilon\to 0$ to get the desired lim inf behavior. □

Combining Propositions 3.2 and 3.3, we get Theorem 1.1.

4. Proof of Theorems 1.3 and 1.5

We now show how Theorem 1.4 implies Theorem 1.3. In order to simplify the notation, throughout this section we will write λ_k^α for $\lambda_k^\alpha(D)$ and μ_k for $\mu_k(D)$.

Proof of Theorem 1.3. We proceed by induction on i. For $i=1$, let $\{\alpha_m\}_{m=1}^\infty$ be a sequence converging to α in $(0,2)$. Consider any subsequence $\beta_r=\alpha_{m_r}$. Theorem 1.2 implies the sequence $\{\lambda_1^{\alpha_m}\}_{m=1}^\infty$ is bounded, and so there is a subsequence $\gamma_\ell=\beta_{r_\ell}$ such that $\lambda_1^{\gamma_\ell}$ converges as $\ell\to\infty$, say to λ. Thanks to Theorem 1.4 we can choose a subsequence $\eta_p=\gamma_{\ell_p}$ such that $\varphi_1^{\eta_p}$ converges uniformly to φ an eigenfunction of H_α with eigenvalue λ. Since $\varphi_1^{\eta_p}$ is nonnegative, so is φ. But the only nonnegative eigenfunction of H_α is φ_1^α. Thus $\lambda=\lambda_1^\alpha$ and $\varphi=\varphi_1^\alpha$. Hence we have shown any subsequence of $\lambda_1^{\alpha_m}$ contains a further subsequence converging to λ_1^α. We conclude that

$$\lim_{m\to\infty}\lambda_1^{\alpha_m}=\lambda_1^\alpha.$$

Note this also proves Theorem 1.5.

Next, assume the theorem is true for $j\leq i$. We verify it is true for $j=i+1$. We will show

$$\liminf_{\beta\to\alpha}\lambda_{i+1}^\beta\geq\lambda_{i+1}^\alpha. \tag{4.1}$$

Combined with the lim sup behavior from Proposition 3.2, we conclude the desired result

$$\lim_{\beta\to\alpha}\lambda_{i+1}^\beta=\lambda_{i+1}^\alpha.$$

To get the lim inf behavior, by way of contradiction, assume $\lambda = \liminf_{\beta\to\alpha} \lambda_{i+1}^{\beta} < \lambda_{i+1}^{\alpha}$. Let $\{\alpha_m\}_{m=1}^{\infty}$ be a sequence converging to α with

$$\lim_{m\to\infty} \lambda_{i+1}^{\alpha_m} = \lambda. \tag{4.2}$$

By the induction hypothesis, $\lambda_j^{\alpha_m}$ converges to λ_j^{α} for $j \le i$. Then Theorem 1.4 implies we can choose a subsequence $\beta_r = \alpha_{m_r}$ such that:

- For each j, $1 \le j \le i$, $\lambda_j^{\beta_r}$ converges to λ_j^{α}, and $\varphi_j^{\beta_r}$ converges uniformly to an eigenfunction φ_j of H_α with corresponding eigenvalue λ_j^{α}.
- The limit λ from (4.2) is an eigenvalue of H_α, and $\varphi_{i+1}^{\beta_r}$ converges uniformly to an eigenfunction φ_{i+1} of H_α with eigenvalue λ.

Since λ is an eigenvalue strictly less than λ_{i+1}^{α}, we can choose positive integers ℓ and m such that $\ell \le m \le i$, $\lambda_m^{\alpha} = \lambda$ and

$$\lambda_{\ell-1}^{\alpha} < \lambda_{\ell}^{\alpha} = \cdots = \lambda_m^{\alpha} < \lambda_{m+1}^{\alpha} \le \cdots \le \lambda_{i+1}^{\alpha}.$$

(Here we take $\lambda_0^{\alpha} := 0$.) In particular, if E is the eigenspace corresponding to $\lambda = \lambda_m^{\alpha}$, then

$$\dim(E) = m - \ell + 1.$$

On the other hand, the uniform convergence implies for $j_1, j_2 \in \{1, \dots, i+1\}$

$$\delta_{j_1 j_2} = \int_D \varphi_{j_1}^{\beta_r} \varphi_{j_2}^{\beta_r}\, dx \text{ converges to } \int_D \varphi_{j_1}\varphi_{j_2}\, dx.$$

Thus $\{\varphi_1, \dots, \varphi_{i+1}\}$ is an orthonormal set, and so $\{\varphi_\ell, \dots, \varphi_m\} \cup \{\varphi_{i+1}\}$ is an orthonormal subset of E. This forces $\dim(E) \ge m - \ell + 2$, which contradicts (4.2). We conclude (4.1) holds. □

5. Weak convergence results

Let $\mathbb{D}[0,\infty)$ be the space of right continuous functions $\omega\colon [0,\infty) \to \mathbb{R}^d$ with left limits. That is, $\omega(t^+) = \lim_{s\to t^+} \omega(s) = \omega(t)$ and $\omega(t^-) = \lim_{s\to t^-} \omega(s)$ exists. The usual convention is $\omega(0^-) := \omega(0)$. Let $X_t(\omega) = \omega(t)$ be the coordinate process and let $\mathcal{F}_t$ be the σ-algebra generated by the cylindrical sets. We equip $\mathbb{D}[0,\infty)$ with the Skorohod topology. Our main reference is Chapter 3 in Ethier and Kurtz [16]. Let P_x^{α} denote the law on $\mathbb{D}[0,\infty)$ of the symmetric α-stable process started at x; the corresponding expectation will be denoted by E_x^{α}.

Lemma 5.1. *If (x_n, α_n) converges to (x, α) in $\mathbb{R}^n \times (0,2)$, then $P_{x_n}^{\alpha_n}$ converges weakly to P_x^{α} in $\mathbb{D}[0,\infty)$.*

Proof. Using characteristic functions it is easy to show the corresponding finite dimensional distributions converge. Thus, by Theorem 7.8 on page 131 in [16], it suffices to show $\{P_{x_n}^{\alpha_n}\colon\ n \ge 1\}$ is tight on $\mathbb{D}[0,\infty)$.

By Theorem 7.2 and Remark 7.3 of [16] and Theorem 15.2 of [7], it suffices to show that for each $t_0 \ge 0$, $\{P_{x_n}^{\alpha_n}\colon\ n \ge 1\}$ is tight on $\mathbb{D}[0,t_0]$. For this, we proceed as in the proof of Proposition 3.2 in [3], using a theorem of Aldous [2]. For the convenience of the reader, we state the theorem using our notation.

For each $n \ge 1$ let τ_n be a stopping time with finitely many values, and let $\delta_n \ge 0$ converge to 0. Aldous's Theorem is the following. Suppose for any $\eta > 0$,

$$P_{x_n}^{\alpha_n}(|X(\tau_n + \delta_n) - X(\tau_n)| \ge \eta) \to 0 \tag{5.1}$$

as $n \to \infty$. If for each $t \in [0, t_0]$ the collection

$$\{P_{x_n}^{\alpha_n} \circ X_t : n \geq 1\}$$

is tight on $\mathbb{R}^d$, then $\{P_{x_n}^{\alpha_n} : \ n \geq 1\}$ is tight on $\mathbb{D}[0, t_0]$.

Even though the theorem is stated for dimension one, the argument also works for higher dimensions.

We now verify the conditions of the theorem. First, note that for $\beta = \alpha$ or α_n and $y = x$ or x_n, P_y^β solves the martingale problem:

a) $P_y^\beta(X_0 = y) = 1$,
b) for each $f \in C_b^2(\mathbb{R}^d)$,

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}_\beta f(X_s) ds$$

is a P_y^β-martingale, where $C_b^2(\mathbb{R}^d)$ is the space of functions with bounded continuous derivatives up to and including order 2 and

$$\mathcal{L}_\beta f(x) = A(d, \alpha) \int\limits_{\mathbb{R}^d \setminus \{x\}} \frac{f(y) - f(x) - \nabla f(x) \cdot (y - x) I(|y - x| < 1)}{|y - x|^{d+\alpha}} dy;$$

see Section 2 of [4]. It is easy to show that for any $f \in C_b^2(\mathbb{R}^d)$ there exists $C_f > 0$ independent of α_n and x_n such that $f(X_t) - f(X_0) - C_f t$ is a $P_{x_n}^{\alpha_n}$-supermartingale. Then we can argue as in the proof of Proposition 3.2 in [3] to get formula (3.1) from that article: for any bounded stopping time τ_n,

$$P_{x_n}^{\alpha_n}\big(\sup_{\tau_n \leq s \leq \tau_n + \delta} |X_s - X_{\tau_n}| \geq \eta \big) \leq \frac{c\delta}{\eta^2}, \tag{5.2}$$

where c is independent of n, δ and η. Even though the formula from that article is stated for one dimension and $x_n \equiv x$, the proof works in higher dimensions with x_n converging to x, since the constant C_f is independent of n. Note too the requirement there that $\delta < 1$ can be dropped.

Replacing δ by $\delta_n \to 0$ as $n \to \infty$, upon letting $n \to \infty$ in (5.2), we obtain condition (5.1). Next we handle tightness of $\{P_{x_n}^{\alpha_n} \circ X_t : n \geq 1\}$ on $\mathbb{R}^d$ for $t \in [0, t_0]$. If n is large, say $n \geq N$, then $|x_n| < |x| + 1$. Thus, if $\lambda > |x| + 1$, then for $t \in [0, t_0]$, by (5.2) with $\tau_n \equiv 0$ and $\delta = t_0$

$$\begin{aligned}
\sup_{n \geq N} P_{x_n}^{\alpha_n}(|X_t| \geq \lambda) &\leq \sup_{n \geq N} P_{x_n}^{\alpha_n}(|X_t - x_n| + |x_n| \geq \lambda) \\
&\leq \sup_{n \geq N} P_{x_n}^{\alpha_n}(|X_t - x_n| \geq \lambda - |x| - 1) \\
&\leq \sup_{n \geq N} P_{x_n}^{\alpha_n}(\sup_{s \leq t_0} |X_s - x_n| \geq \lambda - |x| - 1) \\
&\leq \frac{ct_0}{(\lambda - |x| - 1)^2}.
\end{aligned}$$

This gives the desired tightness. □

The next step is to show for each $T > 0$ that the distribution of $X_{T \wedge \tau_D}$ under $P_{x_n}^{\alpha_n}$ converges to that under P_x^α as (x_n, α_n) converges to (x, α). To this end, define

$$\mathcal{A}_D = \{\, \omega \in \mathbb{D}[0, \infty) : \ d(X[0, \tau_D(\omega) - r], D^c) > 0 \text{ for all rational } 0 < r < \tau_D(\omega) \,\},$$

and

$$\mathcal{C}_D = \left\{ \omega \in \mathbb{D}[0,\infty) \colon \ X(\tau_D(\omega)) \in \overline{D}^c \right\} \cap \mathcal{A}_D.$$

Here $X[0,t] = \{X_s \colon \ 0 \le s \le t\}$ and $d(A,B)$ is the distance between A and B.

Lemma 5.2. *For open $D \subseteq \mathbb{R}^d$, τ_D is continuous on $\mathcal{C}_D$.*

Proof. Let $\omega \in \mathcal{C}_D$ and suppose ω_n converges to ω in $\mathbb{D}[0,\infty)$. We will show that $\tau_D(\omega_n)$ converges to $\tau_D(\omega)$. Let

$$\Lambda' = \{\, \lambda \colon \ [0,\infty) \to [0,\infty) \mid \lambda \text{ is strictly increasing and surjective} \,\}.$$

Proposition 5.3 (a) and (c), on page 119 in [16], implies that for each $T > 0$ there exist $\{\lambda_n\} \subseteq \Lambda'$ such that

$$\lim_{n\to\infty} \sup_{0\le t\le T} |\lambda_n(t) - t| = 0, \tag{5.3}$$

$$\lim_{n\to\infty} \sup_{0\le t\le T} |\, \omega_n(t) - \omega(\, \lambda_n(t)\,)\,| = 0. \tag{5.4}$$

First we show that

$$\liminf_{n\to\infty} \tau_D(\omega_n) \ge \tau_D(\omega). \tag{5.5}$$

Let $\delta \in (0, \tau_D(\omega)/2)$ be rational and set

$$\varepsilon = d(\, \omega[0, \tau_D(\omega) - \delta], D^c\,).$$

Since $\omega \in \mathcal{C}_D$, we have $\varepsilon > 0$. Using $T = \tau_D(\omega)$ in (5.3)–(5.4), there exists N such that for $n \ge N$,

$$\begin{cases} t - \delta < \lambda_n(t) < t + \delta \quad \text{for all} \quad t \le T, \\ \displaystyle\sup_{0\le t\le T} |\, \omega_n(t) - \omega(\lambda_n(t))\,| < \frac{\varepsilon}{2}. \end{cases} \tag{5.6}$$

In particular, for all $t \le \tau_D(\omega) - 2\delta$ and $n \ge N$

$$\lambda_n(t) < t + \delta \le \tau_D(\omega) - \delta < T.$$

Thus $\omega(\lambda_n(t)) \in D$ and $d(\, \omega(\lambda_n(t))\,, D^c\,) \ge \varepsilon$. Therefore,

$$\omega_n(t) \in B(\, \omega(\lambda_n(t))\,, \epsilon/2\,) \subseteq D,$$

for all $t \le \tau_D(\omega) - 2\delta$ and $n \ge N$. This implies $\tau_D(\omega_n) > \tau_D(\omega) - 2\delta$ for $n \ge N$. Take the lim inf as $n \to \infty$ and then let $\delta \to 0$ to get (5.5).

To finish, we show that

$$\limsup_{n\to\infty} \tau_D(\omega_n) \le \tau_D(\omega). \tag{5.7}$$

Given that $\omega \in \mathcal{C}_D$ and ω is right continuous, we can choose $\delta > 0$ such that

$$\varepsilon := d(\, \omega[\, \tau_D(\omega), \tau_D(\omega) + 2\delta\,], D\,) > 0.$$

Using $T = \tau_D(\omega) + 2\delta$ in (5.3)–(5.4) we can choose N such that for $n \ge N$, (5.6) holds for this choice of δ, ε and T. In particular, for $n \ge N$,

$$\tau_D(\omega) < \lambda_n(\tau_D(\omega) + \delta) < \tau_D(\omega) + 2\delta$$

and

$$|\, \omega_n(\, \tau_D(\omega) + \delta\,) \, - \, \omega(\, \lambda_n(\tau_D(\omega) + \delta)\,)\,| < \frac{\varepsilon}{2}.$$

Together these imply

$$d(\omega_n(\tau_D(\omega) + \delta), D) > 0, \quad n \ge N,$$

which in turn yields

$$\tau_D(\omega_n) \le \tau_D(\omega) + \delta, \quad n \ge N.$$

Taking the lim sup as $n \to \infty$ and then letting $\delta \to 0$ yields (5.7). □

Lemma 5.3. *If D is a bounded domain that satisfies an exterior cone condition or if D is a cone, then for all $x \in D$ and $0 < \alpha < 2$,*

$$P_x^\alpha(\, \mathcal{C}_D \cap \{X(\tau_D^-) \in D\}\,) = 1.$$

Proof. If D is bounded and satisfies a uniform exterior cone condition, it is known that

$$P_x^\alpha(\, X(\tau_D) \in \partial D\,) = 0; \tag{5.8}$$

see Lemma 6 in [10]. If D is a cone, we can apply Lemma 6 in [10] to $D \cap B_M(0)$ and letting $M \to \infty$, we get (5.8).

The proof of Theorem 2 in [18] implies

$$P_x^\alpha(\, X(\tau_D^-) \in \partial D, X(\tau_D) \in E\,) = 0, \qquad E \subseteq \overline{E} \subseteq \overline{D}^c \tag{5.9}$$

(see the lines before the footnote on page 89). Combined with (5.8),

$$P_x^\alpha(\, X(\tau_D) \in \overline{D}^c, X(\tau_D^-) \in D\,) = 1.$$

Thus to prove the lemma we need to show that

$$P_x^\alpha(\, d(X[0, \tau_D - r], D^c\,) > 0 \text{ for all rational } r < \tau_D\,) = 1. \tag{5.10}$$

Let

$$D_n = \left\{ x \in D\colon\ d(x, D^c) > \frac{1}{n} \right\}$$

and observe $\tau_{D_n} \le \tau_D$ increases to some limit $L \le \tau_D$. By quasi-left continuity, $X(\tau_{D_n}) \to X(L)$ almost surely. One easily sees $X(L) \notin D$, i.e., $\tau_D \le L$. Hence $\tau_D = L$, and the increasing limit of τ_{D_n} is τ_D.

If for some rational $r < \tau_D$ we have

$$d(X[0, \tau_D - r], D^c) = 0,$$

then for some sequence $s_n \le \tau_D - r$,

$$d(\, X_{s_n}, D^c\,) \to 0.$$

It is no loss to assume s_n converges, say to s. Choose N such that for all $n \ge N$

$$\tau_D - r < \tau_{D_n} \le \tau_D.$$

Given $n \ge N$, choose M_n such that for all $m \ge M_n$,

$$d(\, X_{s_m}, D^c\,) < \frac{1}{2n}.$$

Then for such m, $X_{s_m} \in D_n^c$, which forces

$$\tau_{D_n} \le s_m \le \tau_D - r.$$

Let $m \to \infty$ to get $\tau_{D_n} \le s \le \tau_D - r$, then let $n \to \infty$ to get $\tau_D = \lim_{n\to\infty} \tau_{D_n} \le \tau_D - r$; a contradiction. Thus (5.10) holds. □

We will need the following elementary result shortly.

Lemma 5.4. *Let a_n and b_n be nonnegative sequences such that $a_n \wedge b_n \to 0$ as $n \to \infty$. Then for some n_k, either $a_{n_k} \to 0$ or $b_{n_k} \to 0$ as $k \to \infty$.*

Proof. Suppose $\liminf_{n\to\infty} a_n = a > 0$. Choose a subsequence a_{n_k} such that $a_{n_k} \geq \frac{a}{2}$ for all k. Then

$$a_{n_k} \wedge b_{n_k} \geq \frac{a}{2} \wedge b_{n_k} \geq 0.$$

Since $a_{n_k} \wedge b_{n_k} \to 0$ as $k \to \infty$, we must have $\frac{a}{2} \wedge b_{n_k} \to 0$, which in turn forces $b_{n_k} \to 0$. □

Lemma 5.5. *Let D be a bounded domain that satisfies an uniform exterior cone condition, and let f be a bounded continuous function on $\mathbb{R}^d$. If (x_n, α_n) converges to (x, α) in $D \times (0, 2)$, then for each $T > 0$,*

$$E^{\alpha_n}_{x_n}[\, f(X_{T\wedge\tau_D})\,] \to E^{\alpha}_{x}[\, f(X_{T\wedge\tau_D})\,].$$

Proof. Let

$$\mathcal{C}_D(T) = \mathcal{C}_D \cap \{X(\tau_D^-) \in D\} \cap \{\tau_D \neq T\} \cap \{\lim_{s\to T} X_s = X_T\}.$$

Recall that the symmetric α-stable process has no fixed discontinuities. Then by the eigenfunction expansion of $P^\alpha_x(\tau_D > t)$,

$$P^\alpha_x(\tau_D \neq T, \lim_{s\to T} X_s = X_T) = 1.$$

Thanks to Lemma 5.3,

$$P^\alpha_x(\mathcal{C}_D(T)) = 1.$$

If we can show that

$$\omega \in \mathcal{C}_D(T) \to \omega(T \wedge \tau_D(\omega))$$

is continuous, then by an extension of the continuous mapping theorem (Theorem 5.1 in [7]), the desired conclusion will follow.

Let $\omega \in \mathcal{C}_D(T)$ and suppose $\omega_n \in \mathcal{C}_D(T)$ converges to ω in $\mathbb{D}[0, \infty)$. Define

$$t_n = T \wedge \tau_D(\omega_n),$$
$$t = T \wedge \tau_D(\omega).$$

To show that

$$\lim_{n\to\infty} \omega_n(T \wedge \tau_D(\omega_n)) = \omega(T \wedge \tau_D(\omega)),$$

we show for every subsequence ω_{n_k} there is a further subsequence $\omega_{n_{k_l}}$ such that

$$\lim_{l\to\infty} \omega_{n_{k_l}}(T \wedge \tau_D(\omega_{n_{k_l}})) = \omega(T \wedge \tau_D(\omega)).$$

By Lemma 5.2, $\lim_{n\to\infty} t_n = t$. Applying Proposition 6.5(a) on page 125 in [16],

$$\lim_{n\to\infty} |\omega_n(t_n) - \omega(t)| \wedge |\omega_n(t_n) - \omega(t^-)| \;=\; 0.$$

Hence by Lemma 5.4, for any subsequence ω_{n_k} there is a further subsequence $\omega_{n_{k_l}}$ such that either

$$\lim_{l\to\infty} |\omega_{n_{k_l}}(t_{n_{k_l}}) - \omega(t)| = 0$$

or

$$\lim_{l\to\infty} |\omega_{n_{k_l}}(t_{n_{k_l}}) - \omega(t^-)| = 0. \tag{5.11}$$

In the first case, clearly

$$\lim_{l\to\infty} \omega_{n_{k_l}}(T \wedge \tau_D(\omega_{n_{k_l}})) = \omega(T \wedge \tau_D(\omega)),$$

as desired.

In the second case (5.11), we distinguish two cases: $T > \tau_D(\omega)$ and $T < \tau_D(\omega)$ (recall $\omega \in \mathcal{C}_D(T)$ implies $\tau_D(\omega) \neq T$).

Let us first assume $T > \tau_D(\omega)$. Since $\tau_D(\omega_{n_{k_l}})$ converges to $\tau_D(\omega)$ by Lemma 5.2, $t_{n_{k_l}} = \tau_D(\omega_{n_{k_l}})$ for large l. Hence by (5.11)

$$\lim_{l\to\infty} \omega_{n_{k_l}}(\tau_D(\omega_{n_{k_l}})) = \lim_{l\to\infty} \omega_{n_{k_l}}(t_{n_{k_l}}) = \omega(t^-) = \omega(\tau_D(\omega)^-).$$

Notice that if $\lim_{l\to\infty} \omega_{n_{k_l}}(\tau_D(\omega_{n_{k_l}})) = y$ exists, then $y \in D^c$. But then $\omega \in \mathcal{C}_D(T)$ implies $y = \omega(\tau_D(\omega)^-) \in D$; a contradiction. Thus $T > \tau_D(\omega)$ is not possible.

Finally, if $T < \tau_D(\omega)$, then by Lemma 5.2

$$T < \tau_D(\omega_{n_{k_l}}),$$

for l large. Since $\omega \in \mathcal{C}_D(T)$, (5.11) becomes

$$\omega_{n_{k_l}}(T) \to \omega(T^-) = \omega(T).$$

We conclude that

$$\begin{aligned}\lim_{l\to\infty} \omega_{n_{k_l}}(T \wedge \tau_D(\omega_{n_{k_l}})) &= \lim_{l\to\infty} \omega_{n_{k_l}}(T)\\ &= \omega(T^-)\\ &= \omega(T)\\ &= \omega(T \wedge \tau_D(\omega)).\end{aligned}$$

In any event, we get the desired continuity. □

6. Proof of Theorem 1.4

We will need the following lemma; it is formula (2.7) in [5]. Although the authors do not mention the statement concerning continuity in α, it is possible to trace back through the literature they cite to see the statement holds.

Lemma 6.1. *If $D \subseteq \mathbb{R}^d$ is a bounded Lipschitz domain, then for some positive continuous functions $C(\alpha)$ and $\beta(\alpha)$,*

$$E_x^\alpha(\tau_D) \le C(\alpha)\, \delta_D^{\beta(\alpha)}(x), \qquad \text{for all } x \in D.$$

The next result immediately follows.

Corollary 6.2. *Given a bounded Lipschitz domain D and compact $K \times [a,b] \subseteq \overline{D} \times (0,2)$,*

$$\sup\{\, E_x^\alpha[\tau_D] \colon\ (x,\alpha) \in K \times [a,b] \,\} < \infty.$$

Corollary 6.2 will allow us to get equicontinuity of the eigenfunctions near ∂D. For the interior of D we need the following Krylov–Safanov type of theorem. Let

$$G_0^\alpha g(x) = E_x^\alpha \left[\int_0^{\tau_D} g(X_t)dt \right]$$

be the 0-resolvent of the killed symmetric α-stable process in D.

Lemma 6.3. *Suppose g is bounded with support in $\overline{D}$. Then for each $x \in D$ there exist positive continuous functions $C(\alpha)$ and $\beta(\alpha)$, independent of g, such that for all $y \in D$*

$$|\, G_0^\alpha g(x) - G_0^\alpha g(y) \,| \le C(\alpha)\ [\sup |G_0^\alpha g| + \sup |g|\,]\ |x-y|^{\beta(\alpha)}.$$

Proof. This theorem is essentially due to Bass and Levin [4] (see their Proposition 4.2 on page 387). While they consider the 0-resolvent

$$S_0 g(x) = E_x \left[\int_0^\infty g(X_t) dt \right],$$

their proof also works for the killed resolvent because their crucial formula

$$S_0 g(y) = E_y \left[\int_0^{\tau_{B(x,r)}} g(X_t) dt \right] + E_y \left[S_0 g(X_{\tau_{B(x,r)}}) \right]$$

holds when S_0 is replaced by G_0^α and E_y is replaced by E_y^α, where $r > 0$ is such that $B(x,r) \subset D$. Since we are restricted to D instead of $\mathbb{R}^d$, the numbers $C(\alpha)$ and $\beta(\alpha)$ depend on x, in contrast to the case treated by Bass and Levin. Moreover, it is a simple matter to go through their proof and see that the numbers $C(\alpha)$ and $\beta(\alpha)$ can be chosen to depend continuously on α. □

Corollary 6.4. *Assume D is bounded and Lipschitz. Then for each $x \in D$ and $[a,b] \subseteq (0,2)$, there exist positive C and r such that*

$$| G_0^\alpha g(x) - G_0^\alpha g(y) | \leq C |x-y|^r \sup |g|$$

for all $y \in D$, $\alpha \in [a,b]$ and bounded g with support in $\overline{D}$.

Proof. By Corollary 6.2

$$\begin{aligned} \sup |G^\alpha g| &\leq \sup |g| \cdot \sup_x E_x^\alpha(\tau_D) \\ &\leq \sup |g| \cdot C \end{aligned}$$

where C is independent of $\alpha \in [a,b]$ and g. The result follows from this and the continuity of $C(\alpha)$ and $\beta(\alpha)$ from Lemma 6.3. □

At last we can prove Theorem 1.4. It is well known that

$$0 \leq p_D^\alpha(t,x,y) \leq p^\alpha(t,x,y). \tag{6.1}$$

Moreover,

$$p^\alpha(t,x,y) \leq C(\alpha) t^{-d/\alpha} \tag{6.2}$$

where $C(\alpha)$ is continuous in α (see (2.1) in [9]).

Let $\{\alpha_m\}_{m=1}^\infty \subseteq (0,2)$ be a sequence converging to $\alpha \in (0,2)$. Recall that for all $\beta \in (0,2)$, $\{\varphi_m^\beta\}_{m=1}^\infty$ is an orthonormal set. Then thanks to the symmetry of the heat kernel and (6.2)

$$\begin{aligned} \varphi_i^\beta(x) &= e^{\lambda_i^\beta t} \int_D p_D^\beta(t,x,y) \varphi_i^\beta(y) dy \\ &\leq e^{\lambda_i^\beta t} \sqrt{\int_D \left[p_D^\beta(t,x,y) \right]^2 dy} \\ &= e^{\lambda_i^\beta t} \sqrt{p_D^\beta(2t,x,x)} \\ &\leq e^{\lambda_i^\beta t} \sqrt{\frac{C(\beta)}{(2t)^{d/\beta}}}. \end{aligned}$$

In particular, taking $t = 1$ and using Theorem 1.2,

$$\sup_{x\in D, m\geq 1} \varphi_i^{\alpha_m}(x) \leq \sup_{m\geq 1} e^{\lambda_i^{\alpha_m}} \sqrt{\frac{C(\alpha_m)}{2^{d/\alpha_m}}} < \infty. \tag{6.3}$$

Thus for each $i \geq 1$, the sequence $\{\varphi_i^{\alpha_m}\}_{m=1}^{\infty}$ is uniformly bounded. Next we show the sequence $\{\varphi_i^{\alpha_m}\}_{m=1}^{\infty}$ is pointwise equicontinuous on $\overline{D}$. Indeed, since

$$\varphi_i^{\beta} = \lambda_i^{\beta} G_0^{\beta} \varphi_i^{\beta}, \tag{6.4}$$

Corollary 6.4 implies that for each $x \in D$ there exist C and r such that

$$\begin{aligned} |\varphi_i^{\alpha_m}(x) - \varphi_i^{\alpha_m}(y)| &= \lambda_i^{\alpha_m} \, |G_0^{\alpha_m}\varphi_i^{\alpha_m}(x) - G_0^{\alpha_m}\varphi_i^{\alpha_m}(y)| \\ &\leq C \left[\sup_{m\geq 1} [\,\mu_i\,]^{\alpha_m/2} \right] \left[\sup_{u\in D, m\geq 1} |\varphi_i^{\alpha_m}(u)| \right] |x-y|^r \end{aligned}$$

for all $m \geq 1$ and $y \in D$. Thanks to (6.3) we get the desired equicontinuity for $x \in D$.

As for $x \in \partial D$, first notice (6.4) and Lemma 6.1 imply there are r and C independent of m such that for each $z \in D$

$$\begin{aligned} |\varphi_i^{\alpha_m}(z)| &\leq \left[\sup_{m\geq 1} [\,\mu_i\,]^{\alpha_m/2} \right] \left[\sup_{y\in D, m\geq 1} |\varphi_i^{\alpha_m}(y)| \right] E_z^{\alpha_m}(\,\tau_D\,) \\ &\leq C\,[\,\delta_D(z)\,]^r\,. \end{aligned}$$

Thus $\varphi_i^{\alpha_m}$ is continuous on $\overline{D}$ with boundary value 0. Hence if $x \in \partial D$, then

$$\begin{aligned} |\varphi_i^{\alpha_m}(x) - \varphi_i^{\alpha_m}(y)| &= |\varphi_i^{\alpha_m}(y)| \\ &\leq C\,[\,\delta_D(y)\,]^r \\ &\leq C|x-y|^r. \end{aligned}$$

By Ascoli's Theorem, the sequence $\{\varphi_i^{\alpha_m}\}_{m=1}^{\infty}$ is precompact in $C(\overline{D})$.

Next assume $\{\lambda_i^{\alpha_m}\}_{m=1}^{\infty}$ converges to λ. We show any limit point φ of the sequence $\{\varphi_i^{\alpha_m}\}_{m=1}^{\infty}$ is an eigenfunction of H_α and the corresponding eigenvalue is λ. Choose a subsequence $\beta_r = \alpha_{m_r}$ such that, as $r \to \infty$, $\varphi_i^{\beta_r}$ converges uniformly to φ on $\overline{D}$. Since $\varphi_i^{\beta_r}$ and φ are 0 on ∂D, we can extend them to all of $\mathbb{R}^d$ by taking them to be 0 outside D. Then

$$\begin{aligned} E_x^{\beta_r}\left[\varphi_i^{\beta_r}(X_{t\wedge\tau_D})\right] &= E_x^{\beta_r}\left[\varphi_i^{\beta_r}(X_t)\, I_{\tau_D>t}\right], \\ E_x^{\beta_r}\left[\,\varphi(X_{t\wedge\tau_D})\,\right] &= E_x^{\beta_r}\left[\,\varphi(X_t)\, I_{\tau_D>t}\,\right], \end{aligned}$$

and $\varphi_i^{\beta_r}$ converges to φ uniformly on $\mathbb{R}^d$. Thus we have

$$\begin{aligned} e^{-\lambda_i^{\beta_r} t}\varphi_i^{\beta_r}(x) &= \int_D p_D^{\beta_r}(t,x,y)\varphi_i^{\beta_r}(y)dy \\ &= E_x^{\beta_r}\left[\varphi_i^{\beta_r}(X_t)\, I_{\tau_D>t}\right] \\ &= E_x^{\beta_r}\left[\varphi_i^{\beta_r}(X_{t\wedge\tau_D})\right] \\ &= E_x^{\beta_r}\left[\varphi_i^{\beta_r}(X_{t\wedge\tau_D}) - \varphi(X_{t\wedge\tau_D})\right] + E_x^{\beta_r}\left[\,\varphi(X_{t\wedge\tau_D})\right]. \end{aligned} \tag{6.5}$$

Lemma 5.5 and the uniform convergence of $\varphi_i^{\alpha_m}$ to φ imply

$$\lim_{r\to\infty} E_x^{\beta_r}\left[\varphi_i^{\beta_r}(X_{t\wedge\tau_D}) - \varphi(X_{t\wedge\tau_D})\right] + E_x^{\beta_r}\left[\varphi(X_{t\wedge\tau_D})\right] = E_x^{\alpha}\left[\varphi(X_{t\wedge\tau_D})\right].$$

Since the left-hand side of (6.5) converges to $e^{-\lambda t}\varphi(x)$, we conclude that

$$\begin{aligned} e^{-\lambda t}\varphi(x) &= E_x^{\alpha}\left[\varphi(X_{t\wedge\tau_D})\right] \\ &= E_x^{\alpha}\left[\varphi(X_t)\, I_{\tau_D>t}\right] \\ &= \int_D p_D^{\alpha}(t,x,y)\varphi(y)dy. \end{aligned}$$

Hence φ is an eigenfunction of H_α, and the corresponding eigenvalue is λ. □

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References

[1] R.A. Adams, L.I. Hedberg (1984). Inclusion relations among fine topologies in nonlinear potential theory, Indiana Univ. Math. J. **33** 117–126. MR0726108 (85c:31011)

[2] D. Aldous (1978). Stopping times and tightness, Annals of Probability **6** 335–340. MR0474446 (57:14086)

[3] R.F. Bass (1988). Uniqueness in law for pure jump processes, Probability Theory and Related Fields **79** 271–287. MR0958291 (89h:60118)

[4] R.F. Bass and D.A. Levin (2002). Harnack inequalities for jump processes, Potential Analysis **17** 375–388. MR1918242 (2003e:60194)

[5] R. Bañuelos and T. Kulczycki (2004). The Cauchy process and the Steklov problem, J. Funct. Anal. **211** 355-423. MR2056835 (2005b:60124)

[6] R. Bañuelos, R. Latała, and P. Méndez–Hernández (2001). A Brascamp–Lieb–Luttinger-type inequality and applications to symmetric stable processes, Proc. Amer. Math. Soc. **129** 2997–3008. MR1840105 (2002c:60125)

[7] P. Billingsley (1968). *Convergence of Probability Measures*, Wiley, New York. MR0233396 (38:1718)

[8] R.M. Blumenthal and R.K. Getoor (1959). The asymptotic distribution of the eigenvalues for a class of Markov operators, Pacific J. Math. **9** 399–408. MR0107298 (21:6023)

[9] R.M. Blumenthal and R.K. Getoor (1960). Some theorems on stable processes, Trans. Amer. Math. Soc. **95** 263–273. MR0119247 (22:10013)

[10] K. Bogdan (1997). The boundary Harnack principle for the fractional Laplacian, Studia Mathematica **123** 43–80. MR1438304 (98g:31005)

[11] Z.-Q. Chen and R. Song (2005). Two-sided eigenvalue estimates for subordinate processes in domains, J. Funct. Anal. **226** (2005), 90–113. MR2158176 (2006d:60116)

[12] E.B. Davies (1995). *Spectral Theory and Differential Operators*, Cambridge Univ. Press, Cambridge. MR1349825 (96h:47056)

[13] R.D. DeBlassie (1993). The first exit time of a two-dimensional symmetric stable process from a wedge, Annals of Probability **18** 134–170. MR1062058 (91j:60078)

[14] R.D. DeBlassie (2004). Higher order PDE's and symmetric stable processes, Probability Theory and Related Fields **129** 495–536. MR2078980 (2005d:60079)

[15] R.D. DeBlassie (2005). Correction to "Higher order PDE's and symmetric stable processes," Probab. Theory Related Fields **133** 141–143. MR2197141

[16] S.N. Ethier and T.G. Kurtz (1986). *Markov Processes*, Wiley, New York. MR0838085 (88a:60130)

[17] M. Fukushima, Y. Oshima, and M. Takeda (1994). *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter Studies in Mathematics **19** Walter de Gruyter, Berlin. MR1303354 (96f:60126)

[18] N. Ikeda and S. Watanabe (1962). On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes, J. Math. Kyoto Univ. **2** 79–95. MR0142153 (25:5546)

[19] P.J. Méndez–Hernández (2002). Exit times from cones in $\mathbb{R}^n$ of symmetric stable processes, Illinois Journal of Mathematics **46** 155–163. MR1936081 (2003i:60070)

[20] P.J. Méndez–Hernández (2002). Brascamp-Lieb-Luttinger inequalities for convex domains of finite inradius, Duke Math. J. **13** 93–131. MR1905393 (2003b:31005)

[21] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambribge University Press, Cambridge, 1990. MR1739520 (2003b:60064)

[22] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970. MR0290095 (44:7280)

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