Thin Stationary Sets and Disjoint Club Sequences

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Abstract. We describe two opposing combinatorial properties related to adding clubs to \( \omega_2 \): the existence of a thin stationary subset of \( P_{\omega_1}(\omega_2) \) and the existence of a disjoint club sequence on \( \omega_2 \). A special Aronszajn tree on \( \omega_2 \) implies there exists a thin stationary set. If there exists a disjoint club sequence, then there is no thin stationary set, and moreover there is a fat stationary subset of \( \omega_2 \) which cannot acquire a club subset by any forcing poset which preserves \( \omega_1 \) and \( \omega_2 \). We prove that the existence of a disjoint club sequence follows from Martin’s Maximum and is equiconsistent with a Mahlo cardinal.

Suppose that \( S \) is a fat stationary subset of \( \omega_2 \), that is, for every club set \( C \subseteq \omega_2 \), \( S \cap C \) contains a closed subset with order type \( \omega_1 + 1 \). A number of forcing posets have been defined which add a club subset to \( S \) and preserve cardinals under various assumptions. Abraham and Shelah proved that, assuming CH, the poset consisting of closed bounded subsets of \( S \) ordered by end-extension adds a club subset to \( S \) and is \( \omega_1 \)-distributive. S. Friedman discovered a different poset for adding a club subset to a fat set \( S \subseteq \omega_2 \) with finite conditions. This finite club poset preserves all cardinals provided that there exists a thin stationary subset of \( P_{\omega_1}(\omega_2) \), that is, a stationary set \( T \subseteq P_{\omega_1}(\omega_2) \) such that for all \( \beta < \omega_2 \), \( |\{a \cap \beta : a \in T\}| \leq \omega_1 \). This notion of stationarity appears in and was discovered independently by Friedman. The question remained whether it is always possible to add a club subset to a given fat set and preserve cardinals, without any assumptions.

J. Krueger introduced a combinatorial principle on \( \omega_2 \) which asserts the existence of a disjoint club sequence, which is a pairwise disjoint sequence \( \langle C_\alpha : \alpha \in A \rangle \) indexed by a stationary subset of \( \omega_2 \cap \text{cof}(\omega_1) \), where each \( C_\alpha \) is club in \( P_{\omega_1}(\alpha) \). Krueger proved that the existence of such a sequence implies there is a fat stationary set \( S \subseteq \omega_2 \) which cannot acquire a club subset by any forcing poset which preserves \( \omega_1 \) and \( \omega_2 \).

We prove that a special Aronszajn tree on \( \omega_2 \) implies there exists a thin stationary subset of \( P_{\omega_1}(\omega_2) \). On the other hand assuming Martin’s Maximum there exists a disjoint club sequence on \( \omega_2 \). Moreover, we have the following equiconsistency result.

Theorem 0.1. Each of the following statements is equiconsistent with a Mahlo cardinal: (1) There does not exist a thin stationary subset of \( P_{\omega_1}(\omega_2) \). (2) There exists a disjoint club sequence on \( \omega_2 \). (3) There exists a fat stationary set \( S \subseteq \omega_2 \)
such that any forcing poset which preserves $\omega_1$ and $\omega_2$ does not add a club subset to $S$.

Our proof of this theorem gives a totally different construction of the following result of Mitchell [5]: If $\kappa$ is Mahlo in $L$, then there is a generic extension of $L$ in which $\kappa = \omega_2$ and there is no special Aronszajn tree on $\omega_2$. The consistency of Theorem 0.1(3) provides a negative solution to the following problem of Abraham and Shelah [1]: If $S \subseteq \omega_2$ is fat, does there exist an $\omega_1$-distributive forcing poset which adds a club subset to $S$?

Section 1 outlines notation and background material. In Section 2 we discuss thin stationarity and prove that a special Aronszajn tree implies the existence of a thin stationary set. In Section 3 we introduce disjoint club sequences and prove that a special Aronszajn tree implies the existence of a club sequence. In Section 4 we prove that Martin’s Maximum implies there exists a disjoint club sequence. In Section 5 we construct a model in which there is a disjoint club using an RCS iteration up to a Mahlo cardinal.

Sections 3 and 4 are due for the most part to J. Krueger. We would like to thank Boban Velicković and Mirna Džamonja for pointing out Theorem 2.3 to the authors.

1. Preliminaries

For a set $X$ which contains $\omega_1$, $P_{\omega_1}(X)$ denotes the collection of countable subsets of $X$. A set $C \subseteq P_{\omega_1}(X)$ is club if it is closed under unions of countable increasing sequences and is cofinal. A set $S \subseteq P_{\omega_1}(X)$ is stationary if it meets every club. If $C \subseteq P_{\omega_1}(X)$ is club, then there exists a function $F : X^{<\omega} \to X$ such that every $a$ in $P_{\omega_1}(X)$ closed under $F$ is in $C$. If $F : X^{<\omega} \to P_{\omega_1}(X)$ is a function and $Y \subseteq X$, we say that $Y$ is closed under $F$ if for all $\vec{y}$ from $Y^{<\omega}$, $F(\vec{y}) \subseteq Y$. A partial function $H : P_{\omega_1}(X) \to X$ is regressive if for all $a$ in the domain of $H$, $H(a)$ is a member of $a$. Fodor’s Lemma asserts that whenever $S \subseteq P_{\omega_1}(X)$ is stationary and $H : S \to X$ is a total regressive function, there is a stationary set $S^* \subseteq S$ and a set $x$ in $X$ such that for all $a$ in $S^*$, $H(a) = x$.

If $\kappa$ is a regular cardinal let $\text{cof}(\kappa)$ (respectively, $\text{cof}(<\kappa)$) denote the class of ordinals with cofinality $\kappa$ (respectively, cofinality less than $\kappa$). If $A$ is a cofinal subset of a cardinal $\lambda$ and $\kappa < \lambda$, we write for example $A \cup \text{cof}(\kappa)$ to abbreviate $A \cup (\lambda \cap \text{cof}(\kappa))$.

A stationary set $S \subseteq \kappa$ is fat if for every club $C \subseteq \kappa$, $S \cap C$ contains closed subsets with arbitrarily large order types less than $\kappa$. If $\kappa$ is the successor of a regular uncountable cardinal $\mu$, this is equivalent to the statement that for every club $C \subseteq \kappa$, $S \cap C$ contains a closed subset with order type $\mu + 1$. In particular, if $A \subseteq \kappa^+ \cap \text{cof}(\mu)$ is stationary, then $A \cup \text{cof}(<\mu)$ is fat.

We write $\theta \gg \kappa$ to indicate $\theta$ is larger than $2^{2|H(\kappa)|}$.

A tree $T$ is a special Aronszajn tree on $\omega_2$ if:

1. $T$ has height $\omega_2$ and each level has size less than $\omega_2$,
2. each node in $T$ is an injective function $f : \alpha \to \omega_1$ for some $\alpha < \omega_2$,
3. the ordering on $T$ is by extension of functions, and if $f$ is in $T$, then $f \restriction \beta$ is in $T$ for all $\beta < \text{dom}(f)$.

By [8] if there does not exist a special Aronszajn tree on $\omega_2$, then $\omega_2$ is a Mahlo cardinal in $L$. 

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If $V$ is a transitive model of $\mathsf{ZFC}$, we say that $W$ is an outer model of $V$ if $W$ is a transitive model of $\mathsf{ZFC}$ such that $V \subseteq W$ and $W$ has the same ordinals as $V$.

A forcing poset $\mathbb{P}$ is $\kappa$-distributive if forcing with $\mathbb{P}$ does not add any new sets of ordinals with size $\kappa$.

If $\mathbb{P}$ is a forcing poset, $\dot{a}$ is a $\mathbb{P}$-name, and $G$ is a generic filter for $\mathbb{P}$, we write $a$ for the set $\check{a}$.

Martin's Maximum is the statement that whenever $\mathbb{P}$ is a forcing poset which preserves stationary subsets of $\omega_1$, then for any collection $D$ of dense subsets of $\mathbb{P}$ with $|D| \leq \omega_1$, there is a filter $G \subseteq \mathbb{P}$ which intersects each dense set in $D$.

A forcing poset $\mathbb{P}$ is proper if for all sufficiently large regular cardinals $\theta > 2^{|\mathbb{P}|}$, there is a club of countable elementary substructures $N$ of $(H(\theta), \in)$ such that for all $p$ in $N \cap \mathbb{P}$, there is $q \leq p$ which is generic for $N$, i.e. $q$ forces $N[\dot{G}] \cap \text{On} = N \cap \text{On}$. If $\mathbb{P}$ is proper, then $\mathbb{P}$ preserves $\omega_1$ and preserves stationary subsets of $P_{\omega_1}(\lambda)$ for all $\lambda \geq \omega_1$. A forcing poset $\mathbb{P}$ is semiproper if the same statement holds as above except the requirement that $q$ is generic is replaced by $q$ being semigeneric, i.e. $q$ forces $N[\dot{G}] \cap \omega_1 = N \cap \omega_1$. If $\mathbb{P}$ is semigeneric, then $\mathbb{P}$ preserves $\omega_1$ and preserves stationary subsets of $\omega_1$.

If $\mathbb{P}$ is $\omega_1$-c.c. and $N$ is a countable elementary substructure of $H(\theta)$, then $\mathbb{P}$ forces $N[\dot{G}] \cap \text{On} = N \cap \text{On}$; so every condition in $\mathbb{P}$ is generic for $N$.

We let $<^\omega \text{On}$ denote the class of finite strictly increasing sequences of ordinals. If $\eta$ and $\nu$ are in $<^\omega \text{On}$, write $\eta \leq \nu$ if $\eta$ is an initial segment of $\nu$, and write $\eta < \nu$ if $\eta \leq \nu$ and $\eta \neq \nu$. Let $l(\eta)$ denote the length of $\eta$. A set $T \subseteq <^\omega \text{On}$ is a tree if for all $\eta$ in $T$ and $k < l(\eta)$, $\eta \upharpoonright k$ is in $T$. A cofinal branch of $T$ is a function $b : \omega \rightarrow \kappa$ such that for all $n < \omega$, $b \upharpoonright n$ is in $T$.

Suppose $I$ is an ideal on a set $X$. Then $I^+$ is the collection of subsets of $X$ which are not in $I$. If $S$ is in $I^+$ let $I \upharpoonright S$ denote the ideal $I \cap \mathcal{P}(S)$. For example if $I = NS_\kappa$, the ideal of non-stationary subsets of $\kappa$, a set $S$ is in $I^+$ iff $S$ is stationary. In this case $NS_\kappa \upharpoonright S$ is the ideal of non-stationary subsets of $S$, and $(NS_\kappa \upharpoonright S)^+$ is the collection of stationary subsets of $S$.

If $\kappa$ is regular and $\lambda \geq \kappa$ is a cardinal, then $\text{Coll}(\kappa, \lambda)$ is a forcing poset for collapsing $\lambda$ to have cardinality $\kappa$: Conditions are partial functions $p : \kappa \rightarrow \lambda$ with size less than $\kappa$, ordered by an extension of functions.

**2. Thin stationary sets**

Let $T$ be a cofinal subset of $P_{\omega_1}(\omega_2)$. We say that $T$ is thin if for all $\beta < \omega_2$ the set $\{ a \cap \beta : a \in T \}$ has size less than $\omega_2$. Note that if $\text{CH}$ holds, then $P_{\omega_1}(\omega_2)$ itself is thin. A set $S \subseteq P_{\omega_1}(\omega_2)$ is closed under initial segments if for all $a$ in $S$ and $\beta < \omega_2$, $a \cap \beta$ is in $S$.

**Lemma 2.1.** If $S \subseteq P_{\omega_1}(\omega_2)$ is stationary and closed under initial segments, then for all uncountable $\beta < \omega_2$, the set $S \cap P_{\omega_1}(\beta)$ is stationary in $P_{\omega_1}(\beta)$.

**Proof.** Consider $\beta < \omega_2$ and let $C \subseteq P_{\omega_1}(\beta)$ be a club set. Then the set $D = \{ a \in P_{\omega_1}(\omega_2) : a \cap \beta \in C \}$ is a club subset of $P_{\omega_1}(\omega_2)$. Fix $a$ in $S \cap D$. Since $S$ is closed under initial segments, $a \cap \beta$ is in $S \cap C$. \qed

**Lemma 2.2.** If there exists a thin stationary subset of $P_{\omega_1}(\omega_2)$, then there is a thin stationary set $S$ such that for all uncountable $\beta < \omega_2$, $S \cap P_{\omega_1}(\beta)$ is stationary in $P_{\omega_1}(\beta)$. 
Proof. Let $T$ be a thin stationary set. Define $S = \{ \alpha \cap \beta : \alpha \in T, \beta < \omega_2 \}$. Then $S$ is thin stationary and closed under initial segments.

A set $S \subseteq P_{\omega_1}(\omega_2)$ is a local club if there is a club set $C \subseteq \omega_2$ such that for all uncountable $\alpha$ in $C$, $S \cap P_{\omega_1}(\alpha)$ contains a club in $P_{\omega_1}(\alpha)$ (see [3]). Note that local clubs are stationary.

Theorem 2.3. If there is a special Aronszajn tree on $\omega_2$, then there is a thin local club subset of $P_{\omega_1}(\omega_2)$.

Proof. Let $T$ be a special Aronszajn tree on $\omega_2$. For each $f$ in $T$ with $\text{dom}(f) \geq \omega_1$, define $S_f = \{ \alpha \in \text{dom}(f) : f(\alpha) < i \} : i < \omega_1 \}$. Note that $S_f$ is a club subset of $P_{\omega_1}(\text{dom}(f))$. For each uncountable $\beta < \omega_2$ define $S_\beta = \bigcup \{ S_f : f \in T, \text{dom}(f) = \beta \}$. Then $S_\beta$ has size $\omega_1$. Define $S = \bigcup \{ S_\beta : \omega_1 \leq \beta < \omega_2 \}$. Clearly $S$ is a local club. To show $S$ is thin, it suffices to prove that whenever $\beta < \gamma$ are uncountable and $a$ is in $S_\gamma$, then $a \cap \beta$ is in $S_\beta$. Fix $f$ in $T$ and $i < \omega_1$ such that $a = f^{-1}(i)$. Then $f \upharpoonright \beta$ is in $T$, so $(f \upharpoonright \beta)^{-1}(i) \in S_\beta$. But $(f \upharpoonright \beta)^{-1}(i) \cap \beta = a \cap \beta$. □

In later sections of the paper we will construct models in which there does not exist a thin stationary subset of $P_{\omega_1}(\omega_2)$. Theorem 2.3 shows that in such a model there cannot exist a special Aronszajn tree on $\omega_2$, so by [3] $\omega_2$ is Mahlo in L. Mitchell [3] constructed a model in which there is no special Aronszajn tree on $\omega_2$ by collapsing a Mahlo cardinal in L to become $\omega_2$ with a proper forcing poset. However, in Mitchell’s model the set $(P_{\omega_1}(\kappa))^L$ is a thin stationary subset of $P_{\omega_1}(\omega_2)$.

Lemma 2.4. Suppose $S \subseteq P_{\omega_1}(\omega_2)$ is a local club. Then $S$ is a local club in any outer model $W$ with the same $\omega_1$ and $\omega_2$.

Proof. Let $C$ be a club subset of $\omega_2$ such that for every uncountable $\alpha$ in $C$, $S \cap P_{\omega_1}(\alpha)$ contains a club in $P_{\omega_1}(\alpha)$. Then $C$ remains club in $W$. For each uncountable $\alpha$ in $C$, fix a bijection $g_\alpha : \omega_1 \rightarrow \alpha$. Then $\{ g_\alpha^{-1}(i) : i < \omega_1 \}$ is a club subset of $P_{\omega_1}(\alpha)$. By intersecting this club with $S$, we get a club subset of $S \cap P_{\omega_1}(\alpha)$ of the form $\{ \alpha_i : i < \omega_1 \}$ which is increasing and continuous. Clearly this set remains a club subset of $P_{\omega_1}(\alpha)$ in $W$. □

Proposition 2.5. (1) Suppose there exists a thin local club in $P_{\omega_1}(\omega_2)$. Then there exists a thin local club in any outer model with the same $\omega_1$ and $\omega_2$. (2) Suppose $\kappa$ is a cardinal such that for all $\mu < \kappa$, $\mu^\omega < \kappa$, and assume $P$ is a proper forcing poset which collapses $\kappa$ to become $\omega_2$. Then $P$ forces that there is a thin stationary subset of $P_{\omega_1}(\omega_2)$.

Proof. (1) is immediate from Lemma 2.4 and the absoluteness of thinness. (2) Let $G$ be generic for $P$ over $V$ and work in $V[G]$. Since $P$ is proper, $\omega_1$ is preserved and the set $S = (P_{\omega_1}(\kappa))^V$ is stationary in $P_{\omega_1}(\omega_2)$. We claim that $S$ is thin. If $\beta < \omega_2$, then $\{ a \cap \beta : a \in S \} = (P_{\omega_1}(\beta))^V$, and $|(P_{\omega_1}(\beta))^V| \leq |(P_{\omega_1}(\beta))^V |^V = |(\beta^\omega)^V |^V < \kappa$. □

As we mentioned above, if CH holds, then the set $P_{\omega_1}(\omega_2)$ itself is thin. We show on the other hand that if CH fails, then no club subset of $P_{\omega_1}(\omega_2)$ is thin. The proof is due to Baumgartner and Taylor [2] who showed that for any club set $C \subseteq P_{\omega_1}(\omega_2)$, there is a countable set $A \subseteq \omega_2$ such that $C \cap P(A)$ has size at least $2^\omega$. Their method of proof is described in the next lemma; we include the proof since we will use similar arguments later in the paper.
Lemma 2.6. Suppose $Z$ is a stationary subset of $\omega_2 \cap \text{cof}(\omega)$ and for each $\alpha$ in $Z$, $M_\alpha$ is a countable cofinal subset of $\alpha$. Then there is a sequence $(Z_s, \xi_s : s \in \omega^2)$ satisfying:

1. each $Z_s$ is a stationary subset of $Z$,
2. if $s \subseteq t$, then $Z_t \subseteq Z_s$,
3. if $\alpha$ is in $Z_s$, then $\xi_s$ is in $M_\alpha$,
4. if $\alpha$ is in $Z_s \setminus 0$ and $\beta$ is in $Z_s \setminus 1$, then $\xi_s \setminus 0$ is not in $M_\beta$ and $\xi_s \setminus 1$ is not in $M_\alpha$.

Proof. Let $Z_0 = Z$ and let $\xi_0 = 0$. Suppose $Z_s$ is given. Define $X_s$ as the set of $\xi$ in $\omega_2$ such that the set $\{\alpha \in Z_s : \xi \in M_\alpha\}$ is stationary. A straightforward argument using Fodor’s Lemma shows that $X_s$ is unbounded in $\omega_2$. For each $\alpha$ in $Z_s$ such that $X_s \cap \alpha$ has size $\omega_1$, there exists $\xi < \alpha$ in $X_s$ such that $\xi$ is not in $M_\alpha$.

By Fodor’s Lemma there is a stationary set $Z'_{s \setminus 1} \subseteq Z_s$ and $\xi_{s \setminus 0}$ in $X_s$ such that for all $\alpha$ in $Z'_{s \setminus 1}$, $\xi_{s \setminus 0}$ is not in $M_\alpha$. Let $Z'_{s \setminus 0}$ denote the set of $\alpha$ in $Z_s$ such that $\xi_{s \setminus 0}$ is in $M_\alpha$, which is stationary since $\xi_{s \setminus 0}$ is in $X_s$. Now define $Y_s$ as the set of $\xi$ in $\omega_2$ such that $\{\alpha \in Z'_{s \setminus 1} : \xi \in M_\alpha\}$ is stationary. Then $Y_s$ is unbounded in $\omega_2$. So for each $\alpha$ in $Z'_{s \setminus 0}$ such that $Y_s \cap \alpha$ has size $\omega_1$, there is $\xi < \alpha$ in $Y_s$ which is not in $M_\alpha$. By Fodor’s Lemma there is a stationary set $\xi_{s \setminus 1}$ in $Y_s$ and $Z_{s \setminus 0} \subseteq Z'_{s \setminus 0}$ stationary such that for all $\alpha$ in $Z_{s \setminus 0}$, $\xi_{s \setminus 1}$ is not in $M_\alpha$. Now define $Z_{s \setminus 1}$ as the set of $\alpha$ in $Z_{s \setminus 1}$ such that $\xi_{s \setminus 1}$ is in $M_s$.

Theorem 2.7 (Baumgartner and Taylor). If $C \subseteq P_{\omega_1}(\omega_2)$ is club, then there is a countable set $A \subseteq \omega_2$ such that $C \cap P(A)$ has size at least $2^\omega$. Hence if CH fails, then there does not exist a thin club subset of $P_{\omega_1}(\omega_2)$.

Proof. Let $F : \omega_2 \rightarrow \omega_2$ be a function such that any $a$ in $P_{\omega_1}(\omega_2)$ closed under $F$ is in $C$. Let $Z$ be the stationary set of $\alpha$ in $\omega_2 \cap \text{cof}(\omega)$ closed under $F$. For each $\alpha$ in $Z$ fix a countable set $M_\alpha \subseteq \alpha$ such that $\sup(M_\alpha) = \alpha$ and $M_\alpha$ is closed under $F$. Fix a sequence $(Z_s, \xi_s : s \in \omega^2)$ as described in Lemma 2.6.

For each function $f : \omega \rightarrow 2$, define $bf = cl_F(\{\xi_{f(n)} : n < \omega\})$. Then $bf$ is in $C$. Note that if $n < \omega$ and $\alpha$ is in $Z_{f|n}$, then $cl_F(\{\xi_{f|m} : m \leq n\}) \subseteq M_\alpha$. Fix $bf$ in $C$. For by Lemma 2.6(2), for $m \leq n$, $Z_{f|m} \subseteq Z_{f|n}$. So $\alpha$ is in $Z_{f|m}$, and hence $\xi_{f|m}$ is in $M_\alpha$ by (3). But $M_\alpha$ is closed under $F$.

Let $A = cl_F(\{\xi_s : s \in \omega^2\})$. Since $\omega^2$ has size $\omega$, $A$ is countable, and clearly each $bf$ is a subset of $A$. We claim that for distinct $f$ and $g$, $bf \neq bg$. Let $n < \omega$ be least such that $f(n) \neq g(n)$. If $bf = bg$, then there is $k > n$ such that $\xi_{g|(n+1)}$ is in $cl_F(\{\xi_{f|m} : m \leq k\})$. Fix $\alpha$ in $Z_{f|k}$. By the last paragraph, $\xi_{g|(n+1)}$ is in $M_\alpha$. But $\alpha$ is in $Z_{f|(n+1)}$ by (2), which contradicts (4).

Let $\kappa$ be an uncountable cardinal. The Weak Reflection Principle at $\kappa$ is the statement that whenever $S$ is a stationary subset of $P_{\kappa}(\kappa)$, there is a set $Y$ in $P_{\omega_2}(\kappa)$ such that $\omega_1 \subseteq Y$ and $S \cap P_{\omega_1}(Y)$ is stationary in $P_{\omega_1}(Y)$. Martin’s Maximum implies the Weak Reflection Principle holds for all uncountable cardinals $\kappa$ [3]. The Weak Reflection Principle at $\omega_2$ is equivalent to the statement that for every stationary set $S \subseteq P_{\omega_1}(\omega_2)$, there is a stationary set of uncountable $\beta < \omega_2$ such that $S \cap P_{\omega_1}(\beta)$ is stationary in $P_{\omega_1}(\beta)$. This is equivalent to the statement that every local club subset of $P_{\omega_1}(\omega_2)$ contains a club. The Weak Reflection Principle at $\omega_2$ is equiconsistent with a weakly compact cardinal [3].
Corollary 2.8. Suppose CH fails and there is a special Aronszajn tree on $\omega_2$. Then the Weak Reflection Principle at $\omega_2$ fails.

Proof. By Theorems 2.3 and 2.7, there is a thin local club subset of $P_{\omega_1}(\omega_2)$ which is not club. Hence the Weak Reflection Principle at $\omega_2$ fails. \hfill $\square$

In Sections 4 and 5 we describe models in which there is no thin stationary subset of $P_{\omega_1}(\omega_2)$. On the other hand S. Friedman proved there always exists a thin cofinal set.

Theorem 2.9 (Friedman). There exists a thin cofinal subset of $P_{\omega_1}(\omega_2)$.

Proof. We construct by induction a sequence $(S_\alpha : \omega_1 \leq \alpha < \omega_2)$ satisfying the properties: (1) each $S_\alpha$ is a cofinal subset of $P_{\omega_1}(\alpha)$ with size $\omega_1$, (2) for uncountable $\beta < \gamma$, if $a$ is in $S_\gamma$, then $a \cap \beta$ is in $\bigcup \{ S_\alpha : \omega_1 \leq \alpha \leq \beta \}$, and (3) if $\beta < \gamma < \omega_2$, $a$ is in $P_{\omega_1}(\gamma)$, and $a \cap \beta$ is in $S_\beta$, then there is $b$ in $S_\gamma$ such that $a \subseteq b$ and $a \cap \beta = b \cap \beta$.

Let $S_{\omega_1} = \omega_1$. Given $S_\alpha$, let $S_{\alpha+1}$ be the collection $\{ b \cup \{ \alpha \} : b \in S_\alpha \}$. Conditions (1), (2), and (3) follow by induction. Suppose $\gamma < \omega_2$ is an uncountable limit ordinal and $S_\alpha$ is defined for all uncountable $\alpha < \gamma$. If $c(\gamma) = \omega_1$, then let $S_\gamma = \bigcup \{ S_\alpha : \omega_1 \leq \alpha < \gamma \}$. The required conditions follow by induction.

Assume $c(\gamma) = \omega$. Fix an increasing sequence of uncountable ordinals $(\gamma_n : n < \omega)$ unbounded in $\gamma$. Let $T_\gamma$ be some cofinal subset of $P_{\omega_1}(\gamma)$ with size $\omega_1$. Fix $n < \omega$. For each $x$ in $T_\gamma$ and $a$ in $S_\gamma$, define a set $b(a,x,n)$ in $P_{\omega_1}(\gamma)$ inductively as follows. Let $b(a,x,n) \cap \gamma_m = a$. Given $b(a,x,n) \cap \gamma_m$ in $S_{\gamma_m}$ for some $m \geq n$, apply condition (3) to $\gamma_m, \gamma_{m+1}$, and the set

$$(b(a,x,n) \cap \gamma_m) \cup (x \cap [\gamma_m, \gamma_{m+1}])$$

to find $y$ in $S_{\gamma_{m+1}}$ such that $y \cap \gamma_m = b(a,x,n) \cap \gamma_m$ and $x \cap [\gamma_m, \gamma_{m+1}] \subseteq y$. Let $b(a,x,n) \cap \gamma_{m+1} = y$. This completes the definition of $b(a,x,n)$. Clearly $b(a,x,n) \cap \gamma_m = a$, $x \cap \gamma_m \subseteq b(a,x,n)$, and for all $k \geq n$, $b(a,x,n) \cap \gamma_k$ is in $S_{\gamma_k}$.

Now define $S_\gamma = \{ b(a,x,n) : n < \omega, a \in S_\gamma, x \in T_\gamma \}$. We verify conditions (1), (2), and (3). Clearly $S_\gamma$ has size $\omega_1$. Let $\beta < \gamma$ and consider $b(a,x,n)$ in $S_\gamma$. Fix $k > n$ such that $\beta < \gamma_k$. Then $b(a,x,n) \cap \gamma_k$ is in $S_{\gamma_k}$. So by induction $b(a,x,n) \cap \beta$ is in $\bigcup \{ S_\alpha : \omega_1 \leq \alpha \leq \beta \}$. Now assume $a$ is in $P_{\omega_1}(\gamma)$, $\beta < \gamma$, and $a \cap \beta$ is in $S_\beta$. Choose $x$ in $T_\gamma$ such that $a \subseteq x$. Fix $k$ such that $\beta < \gamma_k$. By the induction hypothesis there is $a'$ in $S_{\gamma_k}$ such that $a \cap \gamma_k \subseteq a'$ and $a' \cap \beta = a \cap \beta$. Let $c = b(a',x,k)$. Then $c$ is in $S_\gamma$, $c \cap \beta = (c \cap \gamma_k) \cap \beta = a' \cap \beta = a \cap \beta$, and $a \subseteq c$.

To prove $S_\gamma$ is cofinal consider $a$ in $P_{\omega_1}(\gamma)$. Fix $x$ in $T_\gamma$ such that $a \subseteq x$. By induction $S_{\gamma_0}$ is cofinal in $P_{\omega_1}(\gamma_0)$. So let $y$ be in $S_{\gamma_0}$ such that $x \cap \gamma_0 \subseteq y$. Then $a$ is a subset of $b(y,x,0)$.

Now define $S = \bigcup \{ S_\beta : \omega_1 \leq \beta < \omega_2 \}$. Conditions (1) and (2) imply that $S$ is thin and cofinal in $P_{\omega_1}(\omega_2)$. \hfill $\square$

3. Disjoint Club Sequences

We introduce a combinatorial property of $\omega_2$ which implies there does not exist a thin stationary subset of $P_{\omega_1}(\omega_2)$. This property follows from Martin’s Maximum and is equiconsistent with a Mahlo cardinal. It implies there exists a fat stationary subset of $\omega_2$ which cannot acquire a club subset by any forcing poset which preserves $\omega_1$ and $\omega_2$. 


Definition 3.1. A disjoint club sequence on $\omega_2$ is a sequence $\langle C_\alpha : \alpha \in A \rangle$ such that $A$ is a stationary subset of $\omega_2 \cap \text{cof}(\omega_1)$, each $C_\alpha$ is a club subset of $P_{\omega_1}(\alpha)$, and $C_\alpha \cap C_\beta$ is empty for all $\alpha < \beta$ in $A$.

Proposition 3.2. Suppose there is a disjoint club sequence on $\omega_2$. Then there does not exist a thin stationary subset of $P_{\omega_1}(\omega_2)$.

Proof. Let $\langle C_\alpha : \alpha \in A \rangle$ be a disjoint club sequence. Suppose for a contradiction there exists a thin stationary set. By Lemma 2.2 fix a thin stationary set $T \subseteq P_{\omega_1}(\omega_2)$ such that for all uncountable $\beta < \omega_2$, $T \cap P_{\omega_1}(\beta)$ is stationary in $P_{\omega_1}(\beta)$. Then for each $\beta$ in $A$ we can choose a set $a_\beta$ in $C_\beta \cap T$. Since $\text{cf}(\beta) = \omega_1$, $\sup(a_\beta) < \beta$. By Fodor’s Lemma there is a stationary set $B \subseteq A$ and a fixed $\gamma < \omega_2$ such that for all $\beta$ in $B$, $\sup(a_\beta) = \gamma$. If $\alpha < \beta$ are in $B$, then $a_\alpha \neq a_\beta$ since $C_\alpha \cap C_\beta$ is empty. So the set $\{a_\beta : \beta \in B\}$ witnesses that $T$ is not thin, which is a contradiction. \qed

Lemma 3.3. Suppose there is a disjoint club sequence $\langle C_\alpha : \alpha \in A \rangle$ on $\omega_2$. Let $W$ be an outer model with the same $\omega_1$ and $\omega_2$ in which $A$ is still stationary. Then there is a disjoint club sequence $\langle D_\alpha : \alpha \in A \rangle$ in $W$.

Proof. By the proof of Lemma 2.4, each $C_\alpha$ contains a club set $D_\alpha$ in $W$. Since $\omega_1$ is preserved, each $\alpha$ in $A$ still has cofinality $\omega_1$. \qed

Theorem 3.4. Suppose $\langle C_\alpha : \alpha \in A \rangle$ is a disjoint club sequence on $\omega_2$. Then $A \cup \text{cof}(\omega)$ does not contain a club.

Proof. Suppose for a contradiction that $A \cup \text{cof}(\omega)$ contains a club. Without loss of generality $2^{<\omega_1} = \omega_2$. Otherwise work in a generic extension $W$ by $\text{Coll}(\omega_2, 2^{<\omega_1})$: In $W$ the set $A \cup \text{cof}(\omega)$ contains a club, and by Lemma 3.3 there is a disjoint club sequence $\langle D_\alpha : \alpha \in A \rangle$.

Since $2^{<\omega_1} = \omega_2$, $H(\omega_2)$ has size $\omega_2$. Fix a bijection $h : H(\omega_2) \to \omega_2$. Let $A$ denote the structure $(H(\omega_2), \in, h)$. Define $B$ as the set of $\alpha$ in $\omega_2 \cap \text{cof}(\omega_1)$ such that there exists an increasing and continuous sequence $\langle N_i : i < \omega_1 \rangle$ of countable elementary substructures of $A$ such that:

1. for $i < \omega_1$, $N_i$ is in $\text{cof}(\omega_1)$,
2. the set $\{N_i \cap \omega_2 : i < \omega_1\}$ is club in $P_{\omega_1}(\alpha)$.

We claim that $B$ is stationary in $\omega_2$. To prove this let $C \subseteq \omega_2$ be club. Let $B$ be the expansion of $A$ by the function $\alpha \mapsto \text{min}(C \setminus \alpha)$. Define by induction an increasing and continuous sequence $\langle N_i : i < \omega_1 \rangle$ of elementary substructures of $B$ such that for all $i < \omega_1$, $N_i$ is in $N_{i+1}$. Let $N = \bigcup \{N_i : i < \omega_1\}$. Then $\omega_1 \subseteq N$ so $N \cap \omega_2$ is an ordinal. Write $\alpha = N \cap \omega_2$. Then $\alpha$ is in $C$ and $\{N_i \cap \omega_2 : i < \omega_1\}$ is club in $P_{\omega_1}(\alpha)$. So $\alpha$ is in $B \cap C$.

Since $A \cup \text{cof}(\omega)$ contains a club, $A \cap B$ is stationary. For each $\alpha$ in $A \cap B$ fix a sequence $\langle N_\alpha^i : i < \omega_1 \rangle$ as described in the definition of $B$. Then $\{N_\alpha^i \cap \omega_2 : i < \omega_1\}$ is in $C_\alpha$ in $\omega_2$ such that $\{N_\alpha^i \cap \omega_2 : i \in c_\alpha\}$ is club and $\alpha$ is a subset of $C_\alpha$. Write $i_\alpha = \text{min}(c_\alpha)$ and let $d_\alpha = c_\alpha \setminus \{i_\alpha\}$.

Define $S = \{N_\alpha^i : \alpha \in A \cap B, i \in d_\alpha\}$. If $N$ is in $S$, then there is a unique pair $\alpha$ in $A \cap B$ and $i$ in $d_\alpha$ such that $N = N_\alpha^i$. For if $N = N_\alpha^i = N_\beta^j$, then $N \cap \omega_2$ is in $C_\alpha \cap C_\beta$, so $\alpha = \beta$. Clearly then $i = j$. Also note that if $N_\alpha$ is in $S$, then $N_\alpha^i$ is in $N_\alpha^i$. So the function $H : S \to H(\omega_2)$ defined by $H(N_\alpha^i) = N_\alpha^i$ is well defined and regressive.
Corollary 3.5. Let $\omega_2$ be a function. Define $G : \omega_2^{<\omega} \rightarrow \omega_2$ by letting $G(\alpha_0, \ldots, \alpha_n)$ be equal to $h(F(h^{-1}(\alpha_0), \ldots, h^{-1}(\alpha_n)))$. Let $E$ be the club set of $\alpha$ in $\omega_2$ closed under $G$. Fix $\alpha$ in $E \cap A \cap B$. Then there is $i$ in $d_\alpha$ such that $N^\alpha_i \cap \omega_2$ is closed under $G$. We claim that $N^\alpha_i$ is closed under $F$. Given $\alpha_0, \ldots, \alpha_n$ in $N^\alpha_i$, the ordinals $h(\alpha_0), \ldots, h(\alpha_n)$ are in $N^\alpha_i \cap \omega_2$. So $\gamma = G(h(\alpha_0), \ldots, h(\alpha_n)) = h(F(\alpha_0, \ldots, \alpha_n))$ is in $N^\alpha_i \cap \omega_2$. Therefore $h^{-1}(\gamma) = F(\alpha_0, \ldots, \alpha_n)$ is in $N^\alpha_i$.

Since $S$ is stationary and $H : S \rightarrow H(\omega_2)$ is regressive, there is a stationary set $S^* \subseteq S$ and a fixed $N$ such that for all $N^\alpha_i$ in $S^*$, $H(N^\alpha_i) = N$. The set $S^*$, being stationary, must have size $\omega_2$. So there are distinct $\alpha$ and $\beta$ such that for some $i$ in $d_\alpha$ and $j$ in $d_\beta$, $N^\alpha_i$ and $N^\beta_j$ are in $S^*$. Then $N = N^\alpha_i = N^\beta_j$. So $N \cap \omega_2$ is in $C_\alpha \cap C_\beta$, which is a contradiction.

Abraham and Shelah [1] asked the following question: Assume that $A$ is a stationary subset of $\omega_2 \cap \text{cof}(\omega_1)$. Does there exist an $\omega_1$-distributive forcing poset which adds a club subset to $A \cup \text{cof}(\omega)$? We answer this question in the negative.

**Corollary 3.5.** Assume that $(C_\alpha : \alpha \in A)$ is a disjoint club sequence. Let $W$ be an outer model of $V$ with the same $\omega_1$ and $\omega_2$. Then in $W$, $A \cup \text{cof}(\omega)$ does not contain a club subset.

**Proof.** If $A$ remains stationary in $W$, then by Lemma 3.3 there is a disjoint club sequence $(D_\alpha : \alpha \in A)$ in $W$. By Theorem 3.4 $A \cup \text{cof}(\omega)$ does not contain a club in $W$. \hfill $\square$

### 4. Martin’s Maximum

In this section we prove that Martin’s Maximum implies there exists a disjoint club sequence on $\omega_2$. We apply MM to the poset for adding a Cohen real and then forcing a continuous $\omega_1$-chain through $P_{\omega_1}(\omega_2) \setminus V$.

**Theorem 4.1** (Krueger). Martin’s Maximum implies there exists a disjoint club sequence on $\omega_2$.

We will use the following theorem from [1].

**Theorem 4.2.** Suppose $P$ is $\omega_1$-c.c. and adds a real. Then $P$ forces that $(P_{\omega_1}(\omega_2) \setminus V)$ is stationary in $P_{\omega_1}(\omega_2)$.

Note: Gitik [3] proved that the conclusion of Theorem 4.2 holds for any outer model of $V$ which contains a new real and computes the same $\omega_1$.

Suppose that $S$ is a stationary subset of $P_{\omega_1}(\omega_2)$. Following [1] we define a forcing poset $P(S)$ which adds a continuous $\omega_1$-chain through $S$. A condition in $P(S)$ is a countable increasing and continuous sequence $\langle a_i : i < \beta \rangle$ of elements from $S$, where for each $i < \beta$, $a_i \cap \omega_1 < a_{i+1} \cap \omega_1$. The ordering on $P(S)$ is by extension of sequences.

**Proposition 4.3.** If $S \subseteq P_{\omega_1}(\omega_2)$ is stationary, then $P(S)$ is $\omega$-distributive.

**Proof.** Suppose $p$ forces $\dot{f} : \omega \rightarrow \text{On}$. Let $\theta \gg \omega_2$ be a regular cardinal such that $\dot{f}$ is in $H(\theta)$. Since $S$ is stationary, we can fix a countable elementary substructure $N$ of the model $\langle H(\theta), \in, S, P(S), p, \dot{f} \rangle$.
such that \( N \cap \omega_2 \) is in \( S \). Let \( \langle D_n : n < \omega \rangle \) be an enumeration of all the dense subsets of \( \mathcal{P}(S) \) in \( N \). Inductively define a decreasing sequence \( \langle p_n : n < \omega \rangle \) of elements of \( N \cap \mathcal{P} \) such that \( p_0 = p \) and \( p_{n+1} \) is a refinement of \( p_n \) in \( D_n \cap N \). Write \( \bigcup \{ p_n : n < \omega \} = \{ b_i : i < \gamma \} \). Clearly \( \bigcup \{ b_i : i < \gamma \} = N \cap \omega_2 \). Since \( N \cap \omega_2 \) is in \( S \), the sequence \( \langle b_i : i < \gamma \rangle \cup \{ \langle \gamma, N \cap \omega_2 \rangle \} \) is a condition below \( p \) which decides \( f(n) \) for all \( n < \omega \).

\[ \square \]

**Theorem 4.4.** Suppose \( \mathbb{P} \) is an \( \omega_1 \)-c.c. forcing poset which adds a real. Let \( \dot{S} \) be a name such that \( \mathbb{P} \) forces \( \dot{S} = (P_{\omega_1}(\omega_2) \setminus V) \). Then \( \mathbb{P} \ast \mathbb{P}(\dot{S}) \) preserves stationary subsets of \( \omega_1 \).

**Proof.** By Theorem 4.2 and Proposition 4.3, the poset \( \mathbb{P} \ast \mathbb{P}(\dot{S}) \) preserves \( \omega_1 \). Let \( A \) be a stationary subset of \( \omega_1 \) in \( V \). Suppose \( p \ast \dot{q} \) is a condition in \( \mathbb{P} \ast \mathbb{P}(\dot{S}) \) which forces that \( \check{C} \) is a club subset of \( \omega_1 \).

Let \( G \) be a generic filter for \( \mathbb{P} \) over \( V \) which contains \( p \). In \( V[G] \) fix a regular cardinal \( \theta \gg \omega_2 \) and let

\[ A = \langle H(\theta), \in, A, S, q, \check{C} \rangle. \]

Fix a Skolem function \( F : H(\theta) \to H(\theta) \) for \( A \). Define \( F^* : \omega_2^{<\omega} \to P_{\omega_1}(\omega_2) \) by letting

\[ F^*(\alpha_0, \ldots, \alpha_n) = \text{cl}_F(\{\alpha_0, \ldots, \alpha_n\}) \cap \omega_2. \]

Since \( \mathbb{P} \) is \( \omega_1 \)-c.c. there is a function \( H : \omega_2^{<\omega} \to P_{\omega_1}(\omega_2) \) in \( V \) such that for all \( \alpha \in \omega_2^{<\omega} \), \( F^*(\alpha) \subseteq H(\alpha) \). Let \( Z^* \) be the stationary set of \( \alpha \in \omega_2 \cap \text{cof}(\omega) \) closed under \( H \).

Working in \( V \), since \( A \) is stationary we can fix for each \( \alpha \) in \( Z^* \) a countable cofinal set \( M_\alpha \subseteq \alpha \) closed under \( H \) with \( M_\alpha \cap \omega_1 \in A \). By Fodor’s Lemma there is \( Z \subseteq Z^* \) stationary and \( \delta \in A \) such that for all \( \alpha \in Z, M_\alpha \cap \omega_1 = \delta \). Fix a sequence \( \langle \xi_s, Z_s : s \in \omega^+ \rangle \) satisfying conditions (1)–(4) of Lemma 2.6.

Let \( f : \omega \to 2 \) be a function in \( V[G] \setminus V \). For each \( n < \omega \) let \( M_n \) denote \( \text{cl}_H(\delta \cup \{\xi_s(m) : m \leq n\}) \). Define \( M = \bigcup \{ M_n : n < \omega \} \). Note that \( M \) is closed under \( H \) and hence it is closed under \( F^* \). Therefore \( N = \text{cl}_F(M) \) is an elementary substructure of \( A \) such that \( N \cap \omega_2 = M \).

As in the proof of Theorem 2.7, for all \( n < \omega \), if \( \alpha \) is in \( Z_{f[n]} \) then \( M_n \subseteq M_\alpha \). Note that \( M \cap \omega_1 = \delta \). For if \( \gamma \) is in \( M \cap \omega_1 \), there is \( n < \omega \) such that \( \gamma \in M_n \). Fix \( \alpha \) in \( Z_{f[n]} \). Then \( \gamma \) is in \( M_\alpha \cap \omega_1 = \delta \).

We prove that \( M \) is not in \( V \) by showing how to compute \( f \) by induction from \( M \). Suppose \( f \downarrow n \) is known. Fix \( j < k \) such that \( f(n) \neq j \). We claim that \( \xi_{(f[n])^{-j}} \) is not in \( M \). Otherwise there is \( k > n \) such that \( \xi_{(f[n])^{-j}} \) is in \( M_k \). Fix \( \alpha \) in \( Z_{f[k]} \). Then \( \xi_{(f[n])^{-j}} \) is in \( M_\alpha \). But \( \alpha \) is in \( Z_{f(n+1)} \), contradicting Lemma 2.6(4). So \( f(n) \) is the unique \( i < k \) such that \( \xi_{(f[n])^{-i}} \) is in \( M \). This completes the definition of \( f \) from \( M \). Since \( f \) is not in \( V \), neither is \( M \).

Let \( \langle D_n : n < \omega \rangle \) enumerate the dense subsets of \( \mathcal{P}(S) \) lying in \( N \). Inductively define a decreasing sequence \( \langle q_n : n < \omega \rangle \) in \( N \cap \mathcal{P}(S) \) such that \( q_0 = q \) and \( q_{n+1} \) is in \( D_n \cap N \). Write \( \bigcup \{ q_n : n < \omega \} = \{ b_i : i < \gamma \} \). Clearly \( \bigcup \{ b_i : i < \gamma \} = N \cap \omega_2 = M \), and since \( N \) is not in \( V \), \( r = \{ b_i : i < \gamma \} \cup \{ \langle \gamma, M \rangle \} \) is a condition in \( \mathbb{P}(S) \). By an easy density argument, \( r \) forces that \( N \cap \omega_1 = \delta \) is a limit point of \( \check{C} \), and hence is in \( \check{C} \). Let \( \check{r} \) be a name for \( r \). Then \( p \ast \check{r} \leq p \ast \dot{q} \) and \( p \ast \check{r} \) forces that \( \delta \) is in \( A \cap \check{C} \).
Now we are ready to prove that MM implies there exists a disjoint club sequence on \( \omega_2 \).

**Proof of Theorem 4.1.** Assume Martin’s Maximum. Inductively define \( A \) and \( \langle C_\alpha : \alpha \in A \rangle \) as follows. Suppose \( \alpha \) is in \( \omega_2 \cap \text{cof}(\omega_1) \) and \( A \cap \alpha \) and \( \langle C_\beta : \beta \in A \cap \alpha \rangle \) are defined. Let \( \alpha \) be in \( A \) if the set \( \bigcup \{ C_\beta : \beta \in A \cap \alpha \} \) is non-stationary in \( P_{\omega_1}(\alpha) \). If \( \alpha \) is in \( A \), then choose a club set \( C_\alpha \subseteq P_{\omega_1}(\alpha) \) with size \( \omega_1 \) which is disjoint from this union.

This completes the definition. We prove that \( A \) is stationary. Then clearly \( \langle C_\alpha : \alpha \in A \rangle \) is a disjoint club sequence. Fix a club set \( C \subseteq \omega_2 \).

Let \( \text{ADD} \) denote the forcing poset for adding a single Cohen real with finite conditions and let \( \dot{S} \) be an \( \text{ADD} \)-name for the set \( (P_{\omega_1}(\omega_2) \setminus V) \). By Theorem 4.4 the poset \( \text{ADD} \star \mathbb{P}(\dot{S}) \) preserves stationary subsets of \( \omega_1 \). We will apply Martin’s Maximum to this poset after choosing a suitable collection of dense sets.

For each \( \alpha < \omega_2 \) fix a surjection \( f_\alpha : \omega_1 \to \alpha \). If \( \beta \) is in \( A \) enumerate \( C_\beta \) as \( \langle a_\beta^i : i < \omega_1 \rangle \). For every quadruple \( i, j, k, l \) of countable ordinals let \( D(i, j, k, l) \) denote the set of conditions \( p \) in \( \text{dom}(\dot{q}) \) such that:

1. \( p \) forces that \( i \) and \( j \) are in the domain of \( \dot{q} \), and for some \( \beta_i \) and \( \beta_j \), \( p \) forces \( \beta_i = \sup(\dot{q}(i)) \) and \( \beta_j = \sup(\dot{q}(j)) \).

2. \( \exists \zeta < \omega_1 \) such that \( p \) forces \( \zeta \) is the least element in \( \text{dom}(\dot{q}) \) such that \( f_{\beta_i}(j) \in \dot{q}(\zeta) \).

3. \( \exists \xi \in C \) larger than \( \beta_i \) and \( \beta_j \) such that \( p \) forces \( \xi \) is the supremum of the maximal set in \( \dot{q} \).

4. If \( f_{\beta_i}(k) = \gamma \) is in \( A \), then there is \( z \) such that \( p \) forces \( z \) is in the symmetric difference \( \dot{q}(i) \triangle a_\beta^j \).

It is routine to check that \( D(i, j, k, l) \) is dense.

Let \( G \star H \) be a filter on \( \text{ADD} \star \mathbb{P}(\dot{S}) \) intersecting each \( D(i, j, k, l) \). For \( i < \omega_1 \) define \( a_i \) as the set of \( \beta \) for which there exists some \( p \star \dot{q} \) in \( G \star H \) such that \( p \) forces \( i \in \text{dom}(\dot{q}) \) and \( p \) forces \( \beta \) is in \( \dot{q}(i) \). The definition of the dense sets implies that \( \langle a_i : i < \omega_1 \rangle \) is increasing, continuous, and cofinal in \( P_{\omega_1}(\alpha) \) for some \( \alpha \) in \( C \cap \text{cof}(\omega_1) \). By (4), for each \( \gamma \) in \( A \cap \alpha \), \( \{ a_i : i < \omega_1 \} \) is disjoint from \( C_\gamma \). Therefore \( \bigcup \{ C_\gamma : \gamma \in A \cap \alpha \} \) is non-stationary in \( P_{\omega_1}(\alpha) \), hence by the definition of \( A \), \( \alpha \) is in \( A \cap C \). So \( A \) is stationary.

\( \Box \)

5. **The Equiconsistency Result**

We now prove Theorem 0.1 establishing the consistency strength of each of the following statements to be exactly a Mahlo cardinal: (1) There does not exist a thin stationary subset of \( P_{\omega_1}(\omega_2) \). (2) There exists a disjoint club sequence on \( \omega_2 \). (3) There exists a fat stationary set \( S \subseteq \omega_2 \) such that any forcing poset which preserves \( \omega_1 \) and \( \omega_2 \) does not add a club subset to \( S \).

By \( [5] \), if there exists a thin stationary subset of \( P_{\omega_1}(\omega_2) \), then for any fat stationary set \( S \subseteq \omega_2 \), there is a forcing poset which preserves cardinals and adds a club subset to \( S \). So (2) and (3) both imply (1), which in turn implies there is no special Aronszajn tree on \( \omega_2 \). So \( \omega_2 \) is Mahlo in \( L \) by \( [8] \).

In the other direction assume that \( \kappa \) is a Mahlo cardinal. We will define a revised countable support iteration which collapses \( \kappa \) to become \( \omega_2 \) and adds a disjoint club sequence on \( \omega_2 \). At individual stages of the iteration we force with either a collapse forcing or the poset \( \text{ADD} \star \mathbb{P}(\dot{S}) \) from the previous section. To ensure that \( \omega_1 \) is
not collapsed we verify that \( \text{ADD}*P(\dot{S}) \) satisfies an iterable condition known as the \( \mathbb{I} \)-universal property. Our description of this construction is self-contained, except for the proof of Theorem 5.9 which summarizes the relevant properties of the RCS iteration. For more information on such iterations and the \( \mathbb{I} \)-universal property see [10].

**Definition 5.1.** A pair \( \langle T, I \rangle \) is a tagged tree if:

1. \( T \subseteq <\omega \text{On} \) is a tree such that each \( \eta \) in \( T \) has at least one successor,
2. \( I : T \to V \) is a partial function such that each \( I(\eta) \) is an ideal on some set \( X_\eta \) and for each \( \eta \) in the domain of \( I \), the set \( \{ \alpha : \eta^\frown \alpha \in T \} \) is in \( (I(\eta))^+ \),
3. for each cofinal branch \( b \) of \( T \), there are infinitely many \( n < \omega \) such that \( b \upharpoonright n \) is in the domain of \( I \).

If \( \eta \) is in the domain of \( I \), we say that \( \eta \) is a splitting point of \( T \). It follows from (1) and (3) that for every \( \eta \) in \( T \) there is \( \eta < \nu \) which is a splitting point.

**Definition 5.2.** Let \( I \) be a family of ideals and \( \langle T, I \rangle \) a tagged tree. Then \( \langle T, I \rangle \) is an \( I \)-tree if for each splitting point \( \eta \) in \( T \), \( I(\eta) \) is in \( I \).

Suppose \( T \subseteq <\omega \text{On} \) is a tree. If \( \eta \) is in \( T \), let \( T^{[\eta]} \) denote the tree \( \{ \nu \in T : \nu \subseteq \eta \text{ or } \eta \subseteq \nu \} \). A set \( J \subseteq T \) is called a front if for distinct nodes in \( J \), neither is an initial segment of the other, and for any cofinal branch \( b \) of \( T \) there is \( \eta \) in \( J \) which is an initial segment of \( b \).

**Definition 5.3.** Suppose \( \langle T, I \rangle \) is tagged tree. Let \( \theta \) be a regular cardinal such that \( \langle T, I \rangle \) is in \( H(\theta) \), and let \( <_\theta \) be a well-ordering of \( H(\theta) \). A sequence \( \langle N_\eta : \eta \in T \rangle \) is a tree of models for \( \theta \) provided that:

1. each \( N_\eta \) is a countable elementary substructure of \( \langle H(\theta), \in, <_\theta, \langle T, I \rangle \rangle \),
2. if \( \eta < \nu \) in \( T \), then \( N_\eta <_\text{elem} N_\nu \),
3. for each \( \eta \) in \( T \), \( \eta \) is in \( N_\eta \).

**Definition 5.4.** Suppose \( \langle T, I \rangle \) is an \( I \)-tree, and \( \theta \) is a regular cardinal such that \( H(\theta) \) contains \( \langle T, I \rangle \) and \( I \). A sequence \( \langle N_\eta : \eta \in T \rangle \) is an \( I \)-suitable tree of models for \( \theta \) if it is a tree of models for \( \theta \), and for every \( \eta \) in \( T \) and \( I \) in \( \mathbb{I} \cap N_\eta \), the set

\[
\{ \nu \in T^{[\eta]} : \nu \text{ is a splitting point and } I(\nu) = I \}
\]

contains a front in \( T^{[\eta]} \).

**Definition 5.5.** Let \( \langle T, I, \mathbb{I} \rangle \) and \( \theta \) be as in Definition 5.4. A sequence \( \langle N_\eta : \eta \in T \rangle \) is an \( \omega_1 \)-strictly \( \mathbb{I} \)-suitable tree of models for \( \theta \) if it is an \( \mathbb{I} \)-suitable tree of models for \( \theta \) and there exists \( \delta < \omega_1 \) such that for all \( \eta \) in \( T \), \( N_\eta \cap \omega_1 = \delta \).

If \( \langle N_\eta : \eta \in T \rangle \) is a tree of models and \( b \) is a cofinal branch of \( T \), write \( N_b \) for the set \( \bigcup \{ N_{b \upharpoonright n} : n < \omega \} \). Note that if \( \langle N_\eta : \eta \in T \rangle \) is an \( \omega_1 \)-strictly \( \mathbb{I} \)-suitable tree of models for \( \theta \), then for any cofinal branch \( b \) of \( T \), \( N_b \cap \omega_1 = N_\emptyset \cap \omega_1 \).

**Lemma 5.6.** Let \( \langle T, I, \mathbb{I} \rangle \) and \( \theta \) be as in Definition 5.4, and let \( \langle N_\eta : \eta \in T \rangle \) be an \( \omega_1 \)-strictly \( \mathbb{I} \)-suitable tree of models for \( \theta \). Suppose \( \eta < \nu \) in \( T \) and \( (N_\eta \cap \omega_2) \setminus N_\eta \) is non-empty. Let \( \gamma \) be the minimum element of \( (N_\nu \cap \omega_2) \setminus N_\eta \). Then \( \gamma \geq \text{sup}(N_\eta \cap \omega_2) \).

**Proof.** Otherwise there is \( \beta \) in \( N_\eta \cap \omega_2 \) such that \( \gamma < \beta \). By elementarity, there is a surjection \( f : \omega_1 \to \beta \) in \( N_\eta \). So \( f^{-1}(\gamma) \in N_\nu \cap \omega_1 = N_\eta \cap \omega_1 \). Hence \( f(f^{-1}(\gamma)) = \gamma \) is in \( N_\eta \), which is a contradiction. \( \square \)
Let $\mathcal{I}$ be a family of ideals. We say that $\mathcal{I}$ is \textit{restriction-closed} if for all $I$ in $\mathcal{I}$, for any set $A$ in $I^+$, the ideal $I \upharpoonright A$ is in $\mathcal{I}$. If $\mu$ is a regular uncountable cardinal, we say that $\mathcal{I}$ is $\mu$-	extit{complete} if each ideal in $\mathcal{I}$ is $\mu$-complete.

\textbf{Definition 5.7.} Suppose that $\mathcal{I}$ is a non-empty restriction-closed $\omega_2$-complete family of ideals and let $\mathbb{P}$ be a forcing poset. Then $\mathbb{P}$ satisfies the $\mathcal{I}$-universal property if for all sufficiently large regular cardinals $\theta$ with $\mathcal{I}$ in $\mathbb{H}(\theta)$, if $(N_\eta : \eta \in T)$ is an $\omega_1$-strictly $\mathcal{I}$-suitable tree of models for $\theta$, then for all $p$ in $N_\emptyset \cap \mathbb{P}$ there is $q \leq p$ such that $q$ forces that there is a cofinal branch $b$ of $T$ such that $N_0[\dot{G}] \cap \omega_1 = N_\emptyset \cap \omega_1$.

Definition 5.7 is Shelah’s characterization of the $\mathcal{I}$-universal property given in [10], Chapter XV 2.11, 2.12, and 2.13. Note that in the definition, $q$ is semigeneric for $N_\emptyset$. In 2.12 Shelah proves that there are stationarily many structures $N$ for which $N = N_\emptyset$ for some $\omega_1$-strictly $\mathcal{I}$-suitable tree of models $(N_\eta : \eta \in T)$. So by standard arguments if $\mathbb{P}$ satisfies the $\mathcal{I}$-universal property, then $\mathbb{P}$ preserves $\omega_1$ and preserves stationary subsets of $\omega_1$. Note that any semiproper forcing poset satisfies the $\mathcal{I}$-universal property.

\textbf{Theorem 5.8.} Let $\mathcal{I}$ be the family of ideals of the form $\NS_{\omega_2} \upharpoonright A$, where $A$ is a stationary subset of $\omega_2 \cap \cof(\omega)$. Let $\dot{S}$ be an Add-name for the set $(P_{\omega_1}(\omega_2) \setminus V)$. Then $\Add \ast \mathbb{P}(\dot{S})$ satisfies the $\mathcal{I}$-universal property.

\textbf{Proof.} Fix a regular cardinal $\theta \gg \omega_2$ and let $(N_\eta : \eta \in T)$ be an $\omega_1$-strictly $\mathcal{I}$-suitable tree of models for $\theta$. Let $p \ast \dot{q}$ be a condition in $(\Add \ast \mathbb{P}(\dot{S})) \cap N_\emptyset$. We find a refinement of $p \ast \dot{q}$ which forces that there is a cofinal branch $b$ of $T$ such that $N_0[\dot{G} \ast \dot{H}] \cap \omega_1 = N_\emptyset \cap \omega_1$.

We use an argument similar to the proof of Lemma 2.6 to define a sequence $(\eta_s, \xi_s : s \in \omega_2)$ satisfying:

1. each $\eta_s$ is in $T$, each $\xi_s$ is in $N_{\eta_s} \cap \omega_2$, and $s < t$ implies $\eta_s < \eta_t$,
2. if $s^* \in T$, then $\xi_{s^*}$ is not in $N_{\eta_s}$, and if $s^* \in T$, then $\xi_{s^*}$ is not in $N_{\eta_s}$.

Let $\eta_0 = \emptyset$ and $\xi_0 = 0$. Suppose $\eta_s$ is defined. Choose a splitting point $\nu_s$ in $T$ above $\eta_s$. Let $Z$ denote the set of $\alpha < \omega_2$ such that $\nu_s \in T$. Since $\nu_s$ is a splitting point, by the definition of $\mathcal{I}$ the set $Z$ is a stationary subset of $\omega_2 \cap \cof(\omega)$. For each $\alpha$ in $Z$, $\alpha$ is in $N_{(\nu_s \in \alpha)}$ and has cofinality $\omega$, so $N_{(\nu_s \in \alpha)} \cap \alpha$ is a countable cofinal subset of $\alpha$. Define $X_s$ as the set of $\xi$ in $\omega_2$ such that the set

$$\{\alpha \in Z : \xi \in N_{(\nu_s \in \alpha)} \cap \alpha\}$$

is stationary. An easy argument using Fodor’s Lemma shows that $X_s$ is unbounded in $\omega_2$. For all large enough $\alpha$ in $Z$, the set $(X_s \setminus \sup(N_{\nu_s} \cap \omega_2)) \cap \alpha$ has size $\omega_1$. So there is a stationary set $Z'_1 \subseteq Z$ and an ordinal $\xi_{s^*}$ in $X_s$ such that $\xi_{s^*}$ is larger than $\sup(N_{\nu_s} \cap \omega_2)$ and for all $\alpha$ in $Z'_1$, $\xi_{s^*}$ is not in $N_{(\nu_s \in \alpha)} \cap \alpha$. Let $Z'_0$ be the stationary set of $\alpha$ in $Z$ such that $\xi_{s^*}$ is in $N_{(\nu_s \in \alpha)} \cap \alpha$. Now define $Y_s$ as the set of $\xi$ in $\omega_2$ such that the set

$$\{\alpha \in Z'_1 : \xi \in N_{(\nu_s \in \alpha)} \cap \alpha\}$$

is stationary. Again we can find $Z_0 \subseteq Z'_0$ stationary and $\xi_{s^*}$ in $Y_s$ such that $\xi_{s^*}$ is larger than $\sup(N_{\nu_s} \cap \omega_2)$ and for all $\alpha$ in $Z_0$, $\xi_{s^*}$ is not in $N_{(\nu_s \in \alpha)} \cap \alpha$. Let $Z_1$ be the stationary set of $\alpha$ in $Z'_1$ such that $\xi_{s^*}$ is in $N_{(\nu_s \in \alpha)} \cap \alpha$.

Now define $\eta_{s^*}$ to be equal to $\nu_s \in \alpha$ for some $\alpha$ in $Z_0$ larger than $\xi_{s^*}$, and define $\eta_{s^*}$ to be $\nu_s \in \beta$ for some $\beta$ in $Z_1$ larger than $\xi_{s^*}$. By definition $\xi_{s^*}$ is in $N_{\eta_{s^*}} \cap \alpha$ and $\xi_{s^*}$ is in $N_{\eta_{s^*}} \cap \alpha$. 


We claim that if $\eta_0 \leq \nu$ in $T$, then $\xi_0^{-1}$ is not in $N_\nu$. Since $\alpha$ is in $Z_0$, $\xi_0^{-1}$ is not in $N_{(\eta_0, \omega)} \cap \alpha$. But $\xi_0^{-1} < \alpha$, so $\xi_0^{-1}$ is not in $N_{(\eta_0, \omega)}$. By Lemma 5.6 the minimum element of $N_\nu \cap \omega_2$ which is not in $N_{(\eta_0, \omega)}$, if such an ordinal exists, is at least $\sup(N_{(\eta_0, \omega)} \cap \omega_2) \geq \alpha > \xi_0^{-1}$. So $\xi_0^{-1}$ is not in $N_\nu$. Similarly if $\eta_0 \leq \nu$ in $T$, then $\xi_0^{-1}$ is not in $N_\nu$. This completes the definition. Conditions (1) and (2) are now easily verified.

Since $\mathbb{P}$ is $\omega_1$-c.c., the condition $p$ itself is generic for each $N_\nu$. Let $G$ be a generic filter for $\text{Add}$ over $V$ which contains $p$. Then for all $\eta$ in $T$, $N_\eta[G] \cap \omega_2 = N_\eta \cap \omega_2$.

So for any cofinal branch $b$ of $T$ in $V[G]$, $N_b[G] \cap \omega_2 = \bigcup\{N_b^{i_0} \cap \omega_2 : n < \omega\}$; in particular, $N_b[G] \cap \omega_1 = N_0 \cap \omega_1$.

Let $f : \omega \to 2$ be a function in $V[G] \setminus V$. Define $b_f = \{\eta_f[n] : n < \omega\}$. We prove that $N_{b_f} \cap \omega_2$ is not in $V$ by showing how to define $f$ inductively from this set. Suppose $f \upharpoonright n$ is known. Fix $j < 2$ such that $f(n) \neq j$. We claim that $\xi^* = \xi_{f(n)+1}$ is not in $N_{b_f} \cap \omega_2$. Otherwise there is $k > n$ such that $\xi^*$ is in $N_{\eta_{f(k)}}$. But $f \upharpoonright (n + 1) \subseteq f \upharpoonright k$. So by condition (2), $\xi^*$ is not in $N_{\eta_{f(k)}}$, which is a contradiction. So $f(n)$ is the unique $i < 2$ such that $\xi_{f(n) - i}$ is in $N_{b_f} \cap \omega_2$.

Let $\langle D_n : n < \omega \rangle$ enumerate all the dense subsets of $\mathbb{P}(S)$ lying in $N_{b_f[G]}$. Inductively define a sequence $\langle q_n : n < \omega \rangle$ by letting $q_0 = q$ and choosing $q_{n+1}$ to be a refinement of $q$ in $D_n \cap N_{b_f[G]}$. Let $\langle b_i : i < \gamma \rangle = \bigcup\{q_n : n < \omega\}$. Clearly $\bigcup\{b_i : i < \gamma\} = N_{b_f[G]} \cap \omega_2$. Since $N_{b_f[G]} \cap \omega_2$ is not in $V$, $r = \{b_i : i < \gamma\} \cap (N_{b_f[G]} \cap \omega_2)$ is a condition in $\mathbb{P}(S)$ below $q$ and $r$ is generic for $N_{b_f[G]}$. So $r$ forces $N_{b_f[G]}[H] \cap \omega_1 = N_{b_f[G]} \cap \omega_1 = N_0 \cap \omega_1$. Let $r$ be a name for $r$. Then $p \ast r \leq p \ast q$ is as required. 

We state without proof the facts concerning RCS iterations which we shall use. These facts follow immediately from [10] Chapter XI 1.13 and Chapter XV 4.15.

**Theorem 5.9.** Suppose $\langle \mathbb{P}_i, \mathbb{Q}_j : i \leq \alpha, j < \alpha \rangle$ is an RCS iteration. Then $\mathbb{P}_\alpha$ preserves $\omega_1$ if the iteration satisfies the following properties:

1. For each $i < \alpha$ there is $n < \omega$ such that $\mathbb{P}_{i+n} \Vdash |\mathbb{P}_i| \leq \omega_1$.

2. For each $i < \alpha$ there is an uncountable regular cardinal $\kappa_i$ and a $\mathbb{P}_{i+1}$-name $\hat{I}_i$ such that $\mathbb{P}_i$ is $\kappa_i$-c.c. and $\mathbb{P}_{i+1}$ forces $\hat{I}_i$ is a non-empty restriction-closed $\kappa_i$-complete family of ideals such that $\mathbb{Q}_i$ satisfies the $\hat{I}_i$-universal property.

**Theorem 5.10.** Let $\alpha$ be a strongly inaccessible cardinal. Suppose that $\langle \mathbb{P}_i, \mathbb{Q}_j : i \leq \alpha, j < \alpha \rangle$ is a revised countable support iteration such that $\mathbb{P}_\alpha$ preserves $\omega_1$ and for all $i < \alpha$, $|\mathbb{P}_i| < \alpha$. Then $\mathbb{P}_\alpha$ is $\alpha$-c.c.

Suppose $\kappa$ is a Mahlo cardinal and let $A$ be the stationary set of strongly inaccessible cardinals below $\kappa$. Define an RCS iteration $\langle \mathbb{P}_i, \mathbb{Q}_j : i \leq \kappa, j < \kappa \rangle$ by recursion as follows. Our recursion hypotheses will include that each $\mathbb{P}_\alpha$ preserves $\omega_1$, and is $\alpha$-c.c. if $\alpha$ is in $A$.

Suppose $\mathbb{P}_\alpha$ is defined. If $\alpha$ is not in $A$, then let $\mathbb{Q}_\alpha$ be a name for $\text{Coll}(\omega_1, |\mathbb{P}_\alpha|)$. Suppose $\alpha$ is in $A$. By the recursion hypotheses $\mathbb{P}_\alpha$ forces $\alpha = \omega_2$. Let $\mathbb{Q}_\alpha$ be a name for the poset $\text{Add} \ast \mathbb{P}(S)$.

If $\alpha$ is not in $A$, then choose some regular cardinal $\kappa_\alpha$ larger than $|\mathbb{P}_\alpha|$, and let $\hat{I}_\alpha$ be a name for some non-empty restriction-closed $\kappa_\alpha$-complete family of ideals on $\kappa_\alpha$. Then $\mathbb{P}_\alpha$ is $\kappa_\alpha$-c.c., and since $\mathbb{Q}_\alpha$ is proper, $\mathbb{P}_\alpha$ forces $\mathbb{Q}_\alpha$ satisfies the $\hat{I}_\alpha$-universal property. Suppose $\alpha$ is in $A$. Then let $\alpha = \kappa_\alpha$ and define $\hat{I}_\alpha$ as a name for the family of ideals on $\omega_2$ as described in Theorem 5.8. Then $\mathbb{P}_\alpha$ is $\kappa_\alpha$-c.c. and forces that $\mathbb{Q}_\alpha$ satisfies the $\hat{I}_\alpha$-universal property.
Suppose $\beta \leq \kappa$ is a limit ordinal and $\mathbb{P}_\alpha$ is defined for all $\alpha < \beta$. Define $\mathbb{P}_\beta$ as the revised countable support limit of $\langle \mathbb{P}_\alpha : \alpha < \beta \rangle$. By Theorem 5.9 and the recursion hypotheses, $\mathbb{P}_\beta$ preserves $\omega_1$. Hence if $\beta$ is in $A \cup \{\kappa\}$, then $\mathbb{P}_\beta$ is $\beta$-c.c. by Theorem 5.10.

This completes the definition. Let $G$ be generic for $\mathbb{P}_\kappa$. The poset $\mathbb{P}_\kappa$ is $\kappa$-c.c. and preserves $\omega_1$, so in $V[G]$ we have that $\kappa = \omega_2$ and $A$ is a stationary subset of $\omega_2 \cap \text{cof}(\omega_1)$. For each $\alpha$ in $A$ let $\mathcal{C}_\alpha$ be the club on $P_{\omega_1}(\alpha)$ introduced by $Q_{\alpha \beta}$. If $\alpha < \beta$ are in $A$, then $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ are disjoint since $\mathcal{C}_\beta$ is disjoint from $V[G \upharpoonright \beta]$. So $\langle \mathcal{C}_\alpha : \alpha \in A \rangle$ is a disjoint club sequence on $\omega_2$ in $V[G]$.

We conclude the paper with several questions.

(1) Assuming Martin’s Maximum, the poset $\text{ADD} \ast \mathbb{P}(\dot{S})$ is semiproper. Is this poset semiproper in general?

(2) Is it consistent that there exists a stationary set $A \subseteq \omega_2 \cap \text{cof}(\omega_1)$ such that neither $A \cup \text{cof}(\omega)$ nor $\omega_2 \setminus A$ can acquire a club subset in an $\omega_1$ and $\omega_2$ preserving extension?

(3) To what extent can the results of this paper be extended to cardinals greater than $\omega_2$? For example, is it consistent that there is a fat stationary subset of $\omega_3$ which cannot acquire a club subset by any forcing poset which preserves $\omega_1$, $\omega_2$, and $\omega_3$?

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