SHARP SOBOLEV INEQUALITIES
IN THE PRESENCE OF A TWIST

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Abstract. Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n \geq 3\). Let also \(A\) be a smooth symmetrical positive \((0, 2)\)-tensor field in \(M\). By the Sobolev embedding theorem, we can write that there exist \(K, B > 0\) such that for any \(u \in H^2_0(M)\),
\[
\left( \int_M |u|^2^\ast \, dv_g \right)^{2/2^\ast} \leq K \int_M A_x(\nabla u, \nabla u) \, dv_g + B \int_M u^2 \, dv_g
\]
where \(H^2_0(M)\) is the standard Sobolev space of functions in \(L^2\) with one derivative in \(L^2\). We investigate in this paper the value of the sharp \(K\) in the equation above, the validity of the corresponding sharp inequality, and the existence of extremal functions for the saturated version of the sharp inequality.

Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n \geq 3\). Also let \(A\) be a smooth symmetrical \((0, 2)\)-tensor field in \(M\). In a local chart, \(A = (A^{ij})\), \(i, j = 1, \ldots, n\). We assume that \(A\) is positive when acting on 1-forms in the sense that for any \(x \in M\), and any \(\eta\) in the cotangent space \(T_x^\ast(M)\), \(A_x = A(x)\) is such that \(A_x(\eta, \eta) > 0\) if \(\eta \neq 0\). Then, by the Sobolev embedding theorem, we can write that there exist \(K, B > 0\) such that for any \(u \in H^2_0(M)\),
\[
\left( \int_M |u|^2^\ast \, dv_g \right)^{2/2^\ast} \leq K \int_M A_x(\nabla u, \nabla u) \, dv_g + B \int_M u^2 \, dv_g
\]
where \(\nabla u = (\partial_i u)\) is the 1-form consisting (in local charts) of the first derivatives of \(u\), \(dv_g\) is the Riemannian volume element of \(g\), and \(H^2_0(M)\) is the standard Sobolev space consisting of functions in \(L^2\) with one derivative in \(L^2\). Clearly, the sharp constant \(B\) in (0.1) is \(V_n^{-2/n}\), where \(V_n\) is the volume of \((M, g)\), and the corresponding sharp inequality holds true since it holds true for the classical Sobolev inequality [and \(|\nabla u|^2\) is controlled by \(A_x(\nabla u, \nabla u)\)]. On the other hand, as is easily understood by the fact that \(A\) charges some parts of the space \(M\) more than others, it is expected that \(A\) will affect the sharp constant \(K\) in (0.1). Note (0.1) is associated to the operator \(\Delta^g_A u = -\operatorname{div}_g(A_x \nabla u)\) which appears in several places in mathematical and physics literature.

The questions we ask in this note are: what is the value \(K_s = K_s(g)\) of the sharp \(K\) in (0.1), does the corresponding sharp inequality hold true, and if yes, does its saturated version (where \(B\) is lowered to its minimum value under the constraint \(K = K_s\)) possess extremal functions? When \(A = g^{-1}\), we are back to the classical problem (dealing with the classical Sobolev inequality). Possible references in book
form for the classical problem are Druet and Hebey \[10\], and Hebey \[14\]. When \(A\) degenerates, the nature of (0.1) changes and we are led to inequalities studied such as in the very nice Beckner \[2\] where sharp inequalities involving the degenerate Grushin \[12\] operator are proved to hold.

When dealing with the general (0.1), in order to answer the above questions, we need to introduce some definitions. We define \(A_i, A_x = (A_{ij})\) in a local chart to be the smooth symmetrical \((2,0)\)-tensor field obtained from \(A\) by the \(g\)-musical isomorphism, so that \(A_{ij} = A^{a\beta}g_{a\beta}\). Then we define the twist function \(K_T\) of \(A\) and \(g\) by the equation

\[
K_T(x) = \sqrt{\frac{|A_{ij}(x)|}{|g(x)|}}
\]

where, in a local chart at \(x\), \(|A_{ij}(x)|\) stands for the determinant of the matrix \((A_{ij}(x))\), and \(|g(x)|\) stands for the determinant of the matrix \((g_{ij}(x))\). Let \(Ag\) be the \((1,1)\)-tensor field obtained by contracting one index of \(A\) with one index of \(g\) so that, in a local chart, \((Ag)_{ij} = A^{i\alpha}g_{\alpha j}\). For any \(x \in M\), \((Ag)_x = Ag(x)\) defines an isomorphism \(\Phi(x)\) of \(T_x(M)\) by \((\Phi(x).\mathbf{X})^i = (Ag_x)^i_aX^a\). Then, another (more intrinsic) equation for \(K\) is that \(K_T(x) = \sqrt{|Ag_x|}\), where \(|Ag_x|\) is the determinant of \(\Phi(x)\). We also define the twisted metric \(\tilde{g}\) by

\[
\tilde{g} = K_T^\frac{2}{n}\bar{g},
\]

where \(\bar{g}\) is the Riemannian metric in \(M\) such that \(\tilde{g}^{-1} = A\). In local coordinates the matrix consisting of the components \(\tilde{g}_{ij}\) of \(\tilde{g}\) is the inverse of the matrix \((A^{ij})\) consisting of the components of \(A\), so that \(A^{i\alpha}\tilde{g}_{\alpha j} = \delta^i_j\) at any point and for all \(i, j\). We let \(K_n\) be the sharp constant for the Euclidean Sobolev inequality \(\|u\|_{2^*} \leq K_n\|\nabla u\|_2\). Then, as is well known (see for instance Hebey \[14\]),

\[
K_n = \sqrt[1-n/2]{\frac{4}{n(n-2)\omega_n^{2/n}}},
\]

where \(\omega_n\) is the volume of the standard \(n\)-dimensional sphere. Our result states as follows.

**Theorem 0.1.** Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n \geq 3\), and let \(A = (A^{ij})\) be a smooth positive symmetrical \((0,2)\)-tensor field in \(M\). The value \(K_s(g)\) of the sharp constant \(K\) in (0.1) is \(K_s(g) = K_T^\frac{2}{n}\sqrt[n]{\min K_T}\), where \(\min K_T = \min_{x \in M} K_T(x)\), \(K_T\) is the twist function of \(A\) and \(g\) given by (0.2), and \(K_n\) is given by (0.4). Moreover, there exists \(B > 0\) such that for any \(u \in H^2(M)\) the sharp inequality

\[
\left(\int_M |u|^{2^*} dv_g\right)^{2/2^*} \leq \frac{K_T^2}{\sqrt[n]{\min K_T}} \int_M A_x(\nabla u, \nabla u) dv_g + B \int_M u^2 dv_g
\]

holds true. If \(B_0(g)\) stands for the lowest \(B\) in (0.3), then \(B_0(g) \geq V_g^{-2/n}\) and, if \(n \geq 4\), we also have that

\[
\frac{4(n-1)A^{2/(n-2)}}{(n-2)K_s(g)}B_0(g) \geq \max_{x \in \text{Min} K_T} \left[ S_\delta(x) + \frac{n-4}{n-2} \frac{\Delta_\delta K_T(x)}{K_T(x)} \right],
\]

where \(\text{Min} K_T\) is the subset of \(M\) consisting of the \(x\) in \(M\) which are such that \(K_T\) is minimum at \(x\), \(\Lambda = 1 / \min K_T\), \(\hat{g}\) is the twisted metric given by (0.3), \(\Delta_\delta = -\text{div}_g \nabla\).
sharp homogeneous Euclidean inequality with respect to $A$ | $\lambda > 0$.

Realize the equality in (0.7) possesses extremal functions, namely nontrivial (smooth positive) functions which realize the equality in (0.7).

When $A = g^{-1}$, we are back to the classical Sobolev inequality. The validity of the classical sharp inequality on arbitrary manifolds was proved in Hebey and Vaugon [15]. The existence of extremal functions for the classical sharp inequality (and the above result when $A = g^{-1}$) was studied in Dajadi and Druet [6]. Possible references in book form on the sharp classical Sobolev inequality are Druet and Hebey [10], and Hebey [14]. Extensions of the notions of weakly critical and critical functions (introduced in Hebey and Vaugon [16]) to inequalities like (0.7) are studied in Collion [4]. Results for 3-dimensional manifolds, in the spirit of those obtained by Druet [7, 8], are also available in Collion [4]. When $n = 3$, equations like (0.7) have to be replaced by an equation like $M_A(x) \leq 0$ for all $x \in \text{Min}K_T$, where $M_A(x)$ is the mass of a suitably chosen Shr¨odinger operator $\Delta g + h$, and the existence of extremal functions follows from equations like $M_A(x) < 0$ for all $x \in \text{Min}K_T$. Developments on the notions of weakly critical and critical functions may also be found in the papers Humbert and Vaugon [17], and Robert [19].

1. Proof of Theorem 0.1

We prove Theorem 0.1 in this section. As a preliminary remark, let $A = (A^{ij})$ be a positive symmetrical $(0, 2)$-tensor in $\mathbb{R}^n$. If $\delta$ stands for the Euclidean metric, and $u$ is smooth, define $\Delta_A u = -\text{div}_A (A \nabla u)$ so that $\Delta_A u = -A^{ij} \partial_{ij} u$. Also define $\Phi = 1/\sqrt{A}$ to be a $(1, 1)$-tensor ($\Phi$ is not unique) such that $A^{ij} \partial_{ij} = \delta - 1$ in the sense that $A^{\alpha\beta} \partial_{\alpha} \partial_{\beta} = \delta - 1$. We regard $\Phi$ as the isomorphism of $\mathbb{R}^n$ given by $(\Phi x)^i = \Phi^i_\alpha x^\alpha$, and if $u$ is a smooth function in $\mathbb{R}^n$, we define $u_A$ by the equation $u_A(x) = u(\Phi x)$. Then $u_A$ is a solution of $\Delta_A u_A = u_A^{2^*-1}$ in $\mathbb{R}^n$ if and only if $u_A$ is a solution of $\Delta_A u = u^{2^*-1}$ in $\mathbb{R}^n$, where $\Delta$ is the Euclidean Laplacian. In particular, by the results of Caffarelli-Gidas-Spruck [3], and also Obata [18], $u_A$ is a (positive) solution in $\mathbb{R}^n$ of $\Delta_A u_A = u_A^{2^*-1}$ if and only if

$$u_A(x) = \left( \frac{\lambda}{1 + \frac{\lambda^2 |\Phi x - a|^2}{n(n-2)}} \right)^{\frac{n-2}{2}}$$

for some $\lambda > 0$ and $a \in \mathbb{R}^n$. Let $A\delta$ be the isomorphism of $\mathbb{R}^n$ we get from $A$ by lowering one index with the $\delta$-musical isomorphism. Then, $|\Phi| = 1/\sqrt{|A\delta|}$, where $|A\delta|$ and $|\Phi|$ stand for the determinants of $A\delta$ and $\Phi$, and we can check that the sharp homogeneous Euclidean inequality with respect to $A$ reads as

$$\left( \int_{\mathbb{R}^n} |u|^2 \, dx \right)^{2/2^*} \leq \frac{K^*_n}{\sqrt{|A\delta|}} \int_{\mathbb{R}^n} A(\nabla u, \nabla u) \, dx$$

where $K_n$, as in (0.4), is the sharp constant for the classical homogeneous Euclidean Sobolev inequality $\|u\|_{2^*} \leq K_n \|\nabla u\|_2$. Moreover, as for the classical case where $A = \delta^{-1}$, extremal functions for (1.2) and positive solutions of the critical equation
\[ \Delta_A u = u^{2^*-1} \] are the same. Following the arguments in Hebey [14] (Proposition 4.2), it follows from (1.2) that for any compact Riemannian manifold \((M, g)\), and any \(B\), constants \(K\) in (1.1) are such that \(K \geq K_T^2 / \sqrt[4]{\min K_T}\). A closely related result is the following: for \((M, g)\) a smooth (compact) Riemannian manifold, and \(A = (A^i)\) a smooth symmetrical \((0, 2)\)-tensor field in \(M\), let \(\Delta_A^g = -\text{div}_g(A(x) \nabla)\), where \(\text{div}_g\) is the divergence with respect to \(g\). Then

\[ \Delta_A^g u = K_T \Delta u \]

for all smooth functions \(u\) in \(M\), where \(K_T\) is the twist function of \(A\) and \(g\) given by (0.2). \(\Delta_A^g\) is the twist metric given by (0.3), and \(\Delta g = -\text{div}_g \nabla\) is the Laplacian with respect to \(g\). Equation (1.3) holds true since

\[ \int_M A_x(\nabla u, \nabla u) dv_g = \int_M |\nabla u|^2 g dv_g \]

for all \(u \in H^2_1(M)\), where \(|\cdot|_g\) is the norm with respect to \(g\). Let \(f_T\) be the function given by the equation \(f_T^{(n-2)/2} K_T = 1\), and let \(h\) be a smooth function in \(M\). Noting that

\[ \int_M (A_x(\nabla u, \nabla u) + hu^2) dv_g = \int_M (|\nabla u|^2 + hu^2) dv_g \]

and that

\[ \int_M |u|^{2^*} dv_g = \int_M f_T |u|^{2^*} dv_g \]

for all \(u \in H^2_1(M)\), where \(\hat{h} = f_T h\), it follows from the result in Hebey and Vaugon [15] that we apply to the \(g\)-metric and that there exists \(B > 0\) such that

\[ \left( \int_M |u|^{2^*} dv_g \right)^{2/2^*} \leq \left( \max_M f_T \right)^{2/2^*} \int_M A_x(\nabla u, \nabla u) dv_g + B \int_M u^2 dv_g \]

for all \(u \in H^2_1(M)\). In particular, \(K_s(g) = K_2^2 / \sqrt[4]{\min K_T}\) is the sharp constant \(K\) in (0.1), and the sharp inequality (0.5) holds true on any compact Riemannian manifold. Equation (0.6) in Theorem 0.1 follows from (1.4), (1.5), and Aubin [1]. Then we are left with the proof that the saturated inequality (0.7) possesses extremal functions if the inequality in (0.6) is strict. By the definition of \(B_0(g)\), for any \(0 < \alpha < B_0(g)\) there exist \(u_\alpha \in C^\infty(M)\), \(u_\alpha > 0\), and \(\lambda_\alpha \in (0, K_s(g)^{-1})\) such that

\[ \Delta_A^g u_\alpha + \frac{\alpha}{K_s(g)} u_\alpha = \lambda_\alpha u_\alpha^{2^*-1} \]

and \(\int_M u_\alpha^{2^*} dv_g = 1\), where \(\Delta_A = -\text{div}_g(A(x) \nabla)\). The \(u_\alpha\)’s are bounded in \(H^2_1(M)\). Up to a subsequence, \(u_\alpha \rightharpoonup u\) weakly in \(H^2_1(M)\). If \(u \not\equiv 0\), then \(u\) is an extremal function for (1.7). By contradiction we assume that \(u \equiv 0\) so that, in particular, \(||u_\alpha||_\infty \to +\infty\) and \(\lambda_\alpha \to K_s(g)^{-1}\) as \(\alpha \to B_0(g)\). We define an \(A\)-bubble by the \(g\)-extension of equation (1.1) to sequences of functions. Namely we define an \(A\)-bubble as a sequence \((B_\alpha)\) of functions given by the equations

\[ B_\alpha(x) = \left( \frac{\mu_\alpha}{\mu_\alpha^2 + \frac{1}{d_1(x, x)^2} \frac{1}{n(n-2)}} \right)^{\frac{2}{n-2}} \]
where \((x_\alpha)\) is a convergent sequence of points in \(M\), and \((\mu_\alpha)\) is a sequence of positive real numbers such that \(\mu_\alpha \to 0\) as \(\alpha \to B_0(g)\). In what follows we let the \(x_\alpha\)'s and \(\mu_\alpha\)'s be given by the equations

\[
(1.8) \quad u_\alpha(x_\alpha) = \|u_\alpha\|_\infty, \quad \mu_\alpha^{-(n-2)/2} = \frac{\sqrt{\Lambda}}{K_s(g)^{(n-2)/2}} \|u_\alpha\|_\infty,
\]

where \(\Lambda = (\min K_T)^{-1}\) is as in Theorem 4.1. Up to a subsequence, the \(x_\alpha\)'s converge. Let \(x_0\) be their limit. Then we must have that \(x_0 \in \operatorname{Min}K_T\). We let also \(G\) be the Green’s function of the operator \(\Delta_A^p + K_s(g)^{-1}B_0(g)\) (or, equivalently, the Green’s function of \(\Delta_g + K_s(g)^{-1}B_0(g)\)), and we define \(\Phi\) to be the positive and continuous function in \(M \times M\) given by

\[
\Phi(x,y) = (n-2)\omega_{n-1}d_\beta(x,y)^{n-2}G(x,y)
\]

if \(x \neq y\), and \(\Phi(x,y) = 1\) if \(x = y\). Following the arguments developed in Druet, Hebey and Robert [11] (see Chapter 5, where minimum energy is discussed), we can write that

\[
(1.9) \quad \frac{\sqrt{\Lambda}u_0}{K_s(g)^{(n-2)/4}} = \left(\Phi(x_0, \cdot) + o(1)\right)B_\alpha,
\]

where \(o(1) \to 0\) in \(C^0(M)\) as \(\alpha \to B_0(g)\), \((B_\alpha)\) is given by (1.7), the \(x_\alpha\)'s and \(\mu_\alpha\)'s are given by (1.8), and \(x_0\) and \(\Lambda\) are as above. In particular, it follows from (1.9) that

\[
(1.10) \quad \lim_{\alpha \to B_0(g)} \frac{\int_{B(x_\alpha)}u_\alpha^2 dv_g}{\int_M u_\alpha^2 dv_g} = 1
\]

for all \(\delta > 0\) when \(n \geq 4\), but that (1.10) stops being true when \(n = 3\). By the local isoperimetric inequality in Druet [9], and the coarea formula, we can write that for any \(\varepsilon > 0\) there exists \(\delta_\varepsilon > 0\) such that for any smooth function \(u\) with compact support in \(B_{x_\alpha}(\delta_\varepsilon)\),

\[
(1.11) \quad \left(\int_M |u|^2 dv_g\right)^{2/2'} \leq K_s^2 \int_M |\nabla u|^2 dv_g + B_x \int_M u^2 dv_g
\]

where \(B_x = \frac{n-2}{4(n-1)}K_n^2(S_\beta(x_0) + \varepsilon)\). We fix \(\varepsilon > 0\), and let \(\eta\) be a smooth cutoff function such that \(\eta = 1\) in \(B_{x_\alpha}(\delta_\varepsilon/4)\), \(\eta = 0\) in \(M \setminus B_{x_\alpha}(\delta_\varepsilon/2)\), and \(0 \leq \eta \leq 1\). We plug \(\eta u_\alpha\) into (1.11). By (1.8), but also (1.3) and (1.10), we get that when \(n \geq 4\),

\[
(1.12) \quad \left(\int_M (\eta u_\alpha)^2 dv_g\right)^{2/2'} - \left(\frac{1}{\max f_T}\right) \frac{\eta^2}{\max f_T} \int_M u_\alpha^2 dv_g
\]

By Hölder’s inequality, writing that \(f_T \leq (\max f_T)^{(n-2)/n} f_T^{2/n}\), the right hand side in (1.12) is nonnegative. Choosing \(\varepsilon > 0\) sufficiently small, we get a contradiction if the first term in (1.12) is negative. In particular we get a contradiction if \(n = 4\) and the inequality in (1.6) is strict, or if \(n > 4\), the inequality in (1.6) is strict, and \(\Delta_g K_T(x) = 0\) for all \(x \in \operatorname{Min}K_T\). We assume in what follows that \(n \geq 5\). We
let $\Lambda_\alpha$ be the right hand side in (1.12). Writing that $f_T = f_T^{(1-2)/n} f_T^{2/n}$, and that $(1 + x)^p = 1 + (p + o(1)) x$, we get by Hölder’s inequality that

$$
(1.13) \quad \Lambda_\alpha \geq \left( \frac{2}{2^*} + o(1) \right) (\max f_T)^{1-\frac{2}{2^*}} \int_M |h_T(\eta u_\alpha)|^2 dv_g,
$$

where $h_T = \frac{f_T}{\max f_T} - 1$ (so that, in particular, $h_T \leq 0$). By (1.9),

$$
\int_M |h_T(\eta u_\alpha)|^2 dv_g
$$

$$
= \left(1 + \varepsilon_\delta\right) \left(\frac{K_s(g)(n-2)/4}{\sqrt{\Lambda}}\right)^{2^*} \int_{B_{\varepsilon_\delta}(\delta)} |h_T| B_{2}^{2} dv_g + o \left(\int_M u_\alpha^2 dv_g\right),
$$

where $\varepsilon_\delta \to 0$ as $\delta \to 0$. By the expansion of $h_T$ at $x_\alpha$ in geodesic normal coordinates, by (1.9) and (1.10), and also by Lemma 7 in Demengel and Hebey [5], we can write that

$$
\int_{B_{\varepsilon_\delta}(\delta)} h_T B_{0}^{2} dv_g = h_T(x_\alpha) \int_{B_{\varepsilon_\delta}(\delta)} B_{0}^{2} dv_g
$$

$$
- \frac{n(n-4)\Lambda}{8(n-1)K_s(g)(n-2)/2} (\Delta g h_T(x_0)) \int_M u_\alpha^2 dv_g + \varepsilon_\delta^2 \int_M u_\alpha^2 dv_g,
$$

where $\lim_{\delta \to 0} \limsup_{\alpha \to B_{\delta}(g)} |\varepsilon_\delta^2| = 0$. Plugging (1.13)–(1.15) into (1.12), and recalling $\Lambda_\alpha$ in (1.13) is the right hand side of (1.12), we get that

$$
\left(\frac{4(n-1)\Lambda^{2/(n-2)}}{(n-2)K_s(g)} B_{0}(g) - S_{\gamma}(x_0) - \frac{n-4}{n-2} \frac{\Delta g K_T(x_0)}{K_T(x_0)}\right) \int_M u_\alpha^2 dv_g
$$

$$
\leq C_1 \left(\varepsilon + o(1)\right) \int_M u_\alpha^2 dv_g + C_2 h_T(x_\alpha),
$$

where $C_1, C_2 > 0$ do not depend on $\alpha$. In particular, since $h_T(x_\alpha) \leq 0$, if the inequality in (1.12) is strict, then we get a contradiction with (1.16) by choosing $\varepsilon > 0$ sufficiently small. This proves Theorem 0.1.

If we assume that the points in $\text{Min} K_T$ are nondegenerate critical points for $K_T$, then $|h_T(x_\alpha)| \geq C d_\gamma(x_0, x_\alpha)^2$, where $C > 0$ does not depend on $\alpha$. In particular, if blow-up occurs, then we get with (1.16) (see also Collion [4] and Hebey [13]) that

$$
d_\gamma(x_0, x_\alpha) = o(\mu_\alpha)
$$

when $n \geq 5$, where, as above, $x_0 \in \text{Min} K_T$ is the limit of the $x_\alpha$’s.

**References**


