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NASH TYPE INEQUALITIES FOR FRACTIONAL POWERS OF NON-NEGATIVE SELF-ADJOINT OPERATORS

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Abstract. Assuming that a Nash type inequality is satisfied by a non-negative self-adjoint operator A, we prove a Nash type inequality for the fractional powers A^α of A. Under some assumptions, we give ultracontractivity bounds for the semigroup $(T_{t,\alpha})$ generated by $-A^\alpha$.

1. Introduction

Let (T_t) be a symmetric submarkovian semigroup acting on $L^2(X,\mu)$ with μ a σ-finite measure on X and let $(-A,\mathcal{D})$ be its generator. The following theorem is known and due to Varopoulos and Carlen, Kusuoka and Stroock (see [VSC], Thm. II.5.2 and references therein).

Theorem 1.1. *For $n>2$, the following conditions are equivalent:*

$$\| f \|_{2n/n-2}^2 \leq C(Af,f), \quad \forall f \in \mathcal{D}, \tag{1.1}$$

$$\| f \|_2^{2+4/n} \leq C_1(Af,f) \| f \|_1^{4/n}, \quad \forall f \in \mathcal{D} \cap L^1(X,\mu), \tag{1.2}$$

$$\| T_t \|_{1\to\infty} \leq C_2 t^{-n/2}, \quad \forall t>0. \tag{1.3}$$

In particular, using subordination, (1.2) implies that for all $\alpha \in (0,1)$:

$$\| f \|_2^{2+4\alpha/n} \leq C_3(A^\alpha f,f) \| f \|_1^{4\alpha/n}, \quad \forall f \in \mathcal{D}(A^\alpha) \cap L^1(X,\mu). \tag{1.4}$$

In [C] and [D], an equivalence of type (1.2)-(1.3) was proved in greater generality under some assumptions on the function $t \to \| T_t \|_{1\to\infty}$. In particular, in [C] the following condition (D) is used.

Definition. A differentiable function m: $\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfies condition (D) if the function $M(t) := -\log m(t)$ is such that:

$$\forall t>0, \quad \forall u \in [t,2t], \qquad M'(u) \geq cM'(t)$$

for some constant $c>0$.

Let m_1, m_2 be two functions from $]0,+\infty[$ to itself; we shall say that $m_1 \preceq m_2$ if there exist $C_1, C_2 > 0$ such that $m_1(t) \leq C_1 m_2(C_2 t)$, and that m_1, m_2 are equivalent ($m_1 \simeq m_2$) if $m_1 \preceq m_2$ and $m_2 \preceq m_1$. Note that if m_i, $i=1,2$, are

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decreasing differentiable bijections satisfying (D) and if $\Theta_i(x) = -m_i'(m_i^{-1}(x))$, then $m_1 \simeq m_2$ if and only if $\Theta_1 \simeq \Theta_2$. In the two following statements, the inequalities will be written modulo equivalence of functions.

Theorem 1.2. *Let m be a decreasing C^1 bijection of $\mathbb{R}_+$ satisfying (D) and set $\Theta(x) = -m'(m^{-1}(x))$. Then the following conditions are equivalent:*

$$\Theta(\| f \|_2^2) \leq (Af, f), \quad \forall f \in \mathcal{D}(A), \ \| f \|_1 = 1, \tag{1.5}$$

$$\| T_t \|_{1\to\infty} \leq m(t), \quad \forall t > 0. \tag{1.6}$$

We consider the following question: Assume that A satisfies the Nash type inequality (1.5). What kind of Nash inequality is satisfied by the operator A^α, the fractional power of A? In what follows, it will be convenient to write (1.5) in the equivalent form (see [BM])

$$\| f \|_2^2 \, B\left(\| f \|_2^2\right) \leq (Af, f), \quad \forall f \in \mathcal{D}(A), \ \| f \|_1 = 1, \tag{1.7}$$

where

$$B(x) = \sup_{t>0}(t \log x + tM(1/t)), \quad M(t) = -\log m(t). \tag{1.8}$$

In particular, $x \to B(x)$ is a non-decreasing function satisfying the following property:

$$\lim_{x\to\infty} \frac{B(x)}{\log x} = +\infty.$$

In some cases (non-ultracontractive semigroups) an inequality similar to (1.7) can be proved with $x \to B(x)$ being non-decreasing but not necessarily obtained from a function m by (1.8). For instance, the function $x \to B(x)$ with $B(x) = \log x$ may be relevant (see Section 5). We state our result with a very weak assumption on $x \to B(x)$ in order to take into account such cases. The main result of this note is the following theorem.

Theorem 1.3. *Let (X, μ) be a measure space with σ-finite measure μ. Let A be a non-negative self-adjoint operator with domain $\mathcal{D}(A) \subset L^2(X, \mu)$. Suppose that the semigroup $T_t = e^{-tA}$ acts as a contraction on $L^1(X, \mu)$ and satisfies the following Nash type inequality:*

$$\| f \|_2^2 \, B\left(\| f \|_2^2\right) \leq (Af, f), \quad \forall f \in \mathcal{D}(A), \ \| f \|_1 = 1, \tag{1.9}$$

where $B : [0, +\infty[\to [0, +\infty[$ is a non-decreasing function which tends to infinity at infinity. Then, for any $\alpha > 0$, the following Nash type inequality holds:

$$\| f \|_2^2 \left[B\left(\| f \|_2^2\right)\right]^\alpha \leq (A^\alpha f, f), \quad \forall f \in \mathcal{D}(A^\alpha), \ \| f \|_1 = 1. \tag{1.10}$$

Remark. Thus, if the function $x \to B(x)$ corresponds by (1.7) to the operator A, then the function $x \to [B(x)]^\alpha$ corresponds to the operator A^α. The function $x \to x^\alpha, x \geq 0$, $0 \leq \alpha \leq 1$, is a particular case of the so-called Bernstein function (see [BF]). The importance of Bernstein functions comes from the following property. If $-A$ is a Markov generator, then for any Bernstein function g, $-g(A)$ is again a Markov generator. More precisely, $-g(A)$ generates Markov semigroup (T_t^g) given by the following formula:

$$T_t^g = \int_0^{+\infty} T_s \, d\mu_t^g(s), \quad t > 0,$$

where (T_s) is the Markov semigroup generated by $-A$ and $(\mu_t^g)_{t\geq 0}$ is the one-sided-stable convolution semigroup on $\mathbb{R}_+$ (the subordinator) defined uniquely by its Laplace transform

$$\int_0^{+\infty} e^{-xs}\, d\mu_t^g(s) = e^{-tg(x)}, \quad x > 0.$$

In view of Theorem 1.3 one may wonder if the Nash inequality (1.9) for $-A$ implies the Nash inequality for $-g(A)$ in the form

$$\| f \|_2^2\, g \circ B\left(\| f \|_2^2\right) \leq (g(A)f, f)\,, \ \ \forall f \in \mathcal{D}(g(A)), \ \ \| f \|_1 = 1. \tag{1.11}$$

In general, the answer is not known. For instance, although we strongly suspect that for a minimal Bernstein function $g : x \to 1 - e^{-ax}, a > 0$, (1.9) does not imply (1.11), we have no proof of this fact at the present writing. It would be interesting to describe the set of Bernstein functions for which we can pass from Nash inequality (1.9) to Nash inequality (1.11). Theorem 1.3 states that this set contains all power functions $x \to x^\alpha$, $0 < \alpha \leq 1$.

2. Proof of Theorem 1.3 and related results

In this section, we will prove Theorem 1.3 in three steps. We prove (1.10) with $\alpha = \frac{1}{2}$. Then we iterate the result of step 1 to prove (1.10) for all α_n of the form $\alpha_n = \frac{1}{2^n}$, $n \in \mathbb{N}$. We give a convexity argument which will allow us to conclude for $0 < \alpha < 1$ and also for $\alpha \geq 1$.

Before embarking on the proof, we need some preparation. Let A be a non-negative self-adjoint operator on $L^2(X, \mu)$, where μ is a σ-finite measure. Since A is non-negative, its spectral decomposition has the form

$$A = \int_0^{+\infty} \lambda\, dE_\lambda.$$

In particular, the semigroup $T_t = \int_0^\infty e^{-t\lambda}\, dE_\lambda$ generated by $-A$ satisfies $\| T_t \|_{2\to 2} \leq 1$ for all $t > 0$. The fractional power A^α of A is defined by the formula

$$(A^\alpha f, f) = \int_0^{+\infty} \lambda^\alpha\, d(E_\lambda f, f)$$

on the domain $\mathcal{D}(A^\alpha) = \{f \in L^2 : \int_0^{+\infty} \lambda^{2\alpha}\, d(E_\lambda f, f) < +\infty\}$. The operators A^α are non-negative self-adjoint operators. The contraction semigroup generated by $-A^\alpha, 0 < \alpha < 1$, can be expressed in the form

$$T_{t,\alpha} = \int_0^{+\infty} T_s\, d\mu_t^\alpha(s),$$

where (μ_t^α) is the one-sided α-stable semigroup on $\mathbb{R}_+$. This semigroup can be characterized by its Laplace transform

$$\int_0^{+\infty} e^{-t\lambda}\, d\mu_t^\alpha(s) = e^{-t\lambda^\alpha}, \quad \lambda > 0.$$

We also denote $T_{t,\frac{1}{2}}$ by P_t and call (P_t) the Poisson semigroup associated to A.

2.1. The case $\alpha = \frac{1}{2}$.

Theorem 2.1. *Let A be a non-negative self-adjoint operator such that $T_t = e^{-tA}$ acts as a contraction on L^1 for all $t > 0$. Assume that there exists $B : \mathbb{R}^+ \to \mathbb{R}^+$, non-decreasing and such that*

$$\| f \|_2^2 \, B(\| f \|_2^2) \leq (Af, f), \quad \forall f \in \mathcal{D}(A), \ \ \| f \|_1 = 1. \tag{2.1}$$

Then, for all $\varepsilon \in (0, 1)$,

$$(1-\varepsilon^2)^{\frac{1}{2}} \| f \|_2^2 \left[B(\varepsilon \| f \|_2^2) \right]^{\frac{1}{2}} \leq (A^{\frac{1}{2}} f, f), \quad \forall f \in \mathcal{D}(A^{\frac{1}{2}}), \ \ \| f \|_1 = 1. \tag{2.2}$$

Proof. Let $g \in \mathcal{D}(A^{\frac{1}{2}})$ and $\| g \|_1 \leq 1$. Set $f = P_t g$. Then $f \in \mathcal{D}(A^n)$, for all $n \geq 1$. The semigroups (P_t) and (T_t) are related by the subordination formula

$$P_t g = \int_0^{+\infty} \mu_t^{\frac{1}{2}}(s) T_s g \, ds = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u} f \, du.$$

It follows that (P_t) is a contraction semigroup on L^1 and in particular

$$\| f \|_1 = \| P_t g \|_1 \leq \| g \|_1 \leq 1.$$

Since $f \in \mathcal{D}(A)$, the relation $Af = AP_t g = \frac{d^2}{dt^2} P_t g$ holds true. We apply (2.1) with $f = P_t g$:

$$\| P_t g \|_2^2 \, B(\| P_t g \|_2^2) \leq (AP_t g, P_t g). \tag{2.3}$$

Set $\phi(t) = \| P_t g \|_2^2$; then

$$\frac{d}{dt} \phi(t) = \dot{\phi}(t) = -2(A^{\frac{1}{2}} P_t g, P_t g)$$

and

$$\frac{d^2}{dt^2} \phi(t) = \ddot{\phi}(t) = 4(AP_t g, P_t g).$$

The inequality (2.3) can be written in the form

$$4\phi(t) B(\phi(t)) \leq \ddot{\phi}(t), \ \ t > 0. \tag{2.4}$$

Multiplying both sides in (2.4) by $-\dot{\phi} \geq 0$, we obtain

$$-4[\phi^2(t)]' B(\phi(t)) \leq -[\dot{\phi}^2]'(t), \quad \forall t > 0.$$

Fix $T > 0$ and integrate this inequality over $[0, T]$ to obtain

$$-4 \int_0^T B(\phi(s)) [\phi^2(s)]' \, ds \leq - \int_0^T [\dot{\phi}^2]'(s) \, ds.$$

The right hand side is clearly bounded by $[\dot{\phi}]^2(0) = 4(A^{\frac{1}{2}} g, g)^2$ for all $T > 0$. To deal with the left hand side, set $v(s) = \phi^2(s)$. Then it takes the form

$$-4 \int_0^T B(\sqrt{v(s)}) v'(s) \, ds = 4 \int_{v(T)}^{v(0)} B(\sqrt{x}) \, dx.$$

Thus finally we get the following inequality:

$$\int_{v(T)}^{v(0)} B(\sqrt{x}) \, dx \leq (A^{\frac{1}{2}} g, g)^2. \tag{2.5}$$

Let us assume that

$$\lim_{T \to +\infty} \| P_T \, g \|_2 = 0. \tag{2.6}$$

Below, we will see how to reduce the general case to this one. In the inequality (2.5), we take the limit as $T \to +\infty$ and obtain

$$\int_0^{v(0)} B(\sqrt{x})\, dx \leq (A^{\frac{1}{2}} g, g)^2.$$

Let $\varepsilon \in (0,1)$. Since B is non-decreasing,

$$(1-\varepsilon^2) v(0) B(\varepsilon \sqrt{v(0)}) \leq \int_{\varepsilon^2 v(0)}^{v(0)} B(\sqrt{x})\, dx \leq (A^{\frac{1}{2}} g, g)^2,$$

and finally,

$$(1-\varepsilon^2)^{\frac{1}{2}} \mid\mid g \mid\mid_2^2 \left[B\left(\varepsilon \mid\mid g \mid\mid_2^2\right)\right]^{\frac{1}{2}} \leq (A^{\frac{1}{2}} g, g). \tag{2.7}$$

This proves the theorem under the assumption (2.6). To consider the general case, define the operator $A_\rho = A + \rho I$, $\rho > 0$. A_ρ is non-negative and self-adjoint. It also satisfies (2.1). The property

$$\lim_{T \to +\infty} \mid\mid e^{-T\sqrt{A+\rho I}} \mid\mid_2 = 0$$

follows by spectral theory. We apply the inequality (2.7) with A_ρ instead of A. Since the left hand side of (2.7) is independent of $\rho > 0$, we can pass to the limit as $\rho \to 0$ in (2.7). The proof is now complete.

2.2. **Iteration.**

Proposition 2.2. *Under the same assumptions as in Theorem* 2.1*, for all* $n \in \mathbb{N}^*$ *there exists* $\alpha_n, \beta_n > 0$ *such that*

$$\alpha_n \mid\mid f \mid\mid_2^2 \left[B(\beta_n \mid\mid f \mid\mid_2^2)\right]^{1/2^n} \leq (A^{1/2^n} f, f), \quad \forall f \in \mathcal{D}(A^{1/2^n}), \ \mid\mid f \mid\mid_1 = 1. \tag{2.8}$$

Proof. We apply Theorem 2.1 and induction on n.

2.3. **The convexity argument.** We have already proved (1.10) for $\alpha = \alpha_n = 1/2^n$, $n \in \mathbb{N}^*$. To conclude that (1.10) holds true for all $\alpha \in (0,1)$ we need the following auxiliary result.

Proposition 2.3. *Let* $\Lambda : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *be a non-decreasing function. Assume that* A *is a non-negative self-adjoint operator that satisfies the inequality*

$$\mid\mid f \mid\mid_2^2 \Lambda(\mid\mid f \mid\mid_2^2) \leq (Af, f), \quad \forall f \in \mathcal{D}(A), \mid\mid f \mid\mid_1 = 1.$$

Then for any convex non-decreasing function $\Phi \geq 0$

$$\mid\mid f \mid\mid_2^2 \Phi \circ \Lambda(\mid\mid f \mid\mid_2^2) \leq (\Phi(A) f, f), \quad \forall f \in \mathcal{D}(\Phi(A)), \mid\mid f \mid\mid_1 = 1.$$

Proof. By renormalisation, $f \to f / \mid\mid f \mid\mid_1$, we have

$$\mid\mid f \mid\mid_2^2 \Lambda(\mid\mid f \mid\mid_2^2 / \mid\mid f \mid\mid_1^2) \leq \int_0^{+\infty} \lambda\, d(E_\lambda f, f).$$

For a fixed f denote $d\nu(\lambda) = d(E_\lambda f, f)$. Assume that $\mid\mid f \mid\mid_2 = 1$; then ν is a probability measure. Since Φ is convex non-decreasing function, Jensen's inequality yields

$$\Phi \circ \Lambda(1/ \mid\mid f \mid\mid_1^2) \leq \int_0^{+\infty} \Phi(\lambda)\, d(E_\lambda f, f),$$

that is,

$$\Phi \circ \Lambda(1/ \mid\mid f \mid\mid_1^2) \leq (\Phi(A)f, f), \quad \mid\mid f \mid\mid_2 = 1.$$

This obviously gives the result.

2.4. **End of the proof.** Let $0 < \alpha < 1$ be fixed and choose $n \in \mathbb{N}^*$ such that $\alpha_n = 1/2^n \leq \alpha$. We have

$$a_{\alpha_n} \mid\mid f \mid\mid_2^2 \left[B(b_{\alpha_n} \mid\mid f \mid\mid_2^2)\right]^{\alpha_n} \leq (A^{\alpha_n} f, f), \quad \forall f \in \mathcal{D}(A^{\alpha_n}), \ \mid\mid f \mid\mid_1 = 1.$$

Choose $\Phi(t) = t^{\alpha/\alpha_n}$ and let $\Lambda(t) = a_{\alpha_n} \left[B(b_{\alpha_n} t)\right]^{\alpha_n}$. Since $\alpha/\alpha_n \geq 1$, Φ is a non-decreasing convex function. Moreover,

$$\Phi \circ \Lambda(t) = a_\alpha \left[B(b_\alpha t)\right]^\alpha,$$

where $a_\alpha = (a_{\alpha_n})^{\alpha/\alpha_n}$, $b_\alpha = b_{\alpha_n}$. For $f \in \mathcal{D}(A^\alpha)$, $\Phi(A^{\alpha_n})f = A^\alpha f$ and Proposition 2.3 yields the result

$$a_\alpha \mid\mid f \mid\mid_2^2 \left[B(b_\alpha \mid\mid f \mid\mid_2^2)\right]^\alpha \leq (A^\alpha f, f), \quad \forall f \in \mathcal{D}(A^\alpha), \ \mid\mid f \mid\mid_1 = 1.$$

This finishes the proof of Theorem 1.3 for $0 < \alpha \leq 1$. In the case $\alpha > 1$, we just apply Proposition 2.3.

2.5. **Some generalizations.** We want to enlarge the class of functions treated in Theorem 1.3. Recall the notion of regularly varying function (see [BGT]). Function Φ defined on $[0, +\infty[$ is said to be a regularly varying function of index α if for any $\lambda \geq 1$,

$$\lim_{x \to \infty} \frac{\Phi(\lambda x)}{\Phi(x)} = \lambda^\alpha.$$

In the case $\alpha = 0$, Φ is called a slowly varying function. Any regularly varying function of index α can be represented in the form $\Phi(x) = x^\alpha \ell(x)$, where ℓ is a slowly varying function.

Example. The following functions illustrate the definition of regular variation of index α: $x \to cx^\alpha, cx^\alpha (\log x)^\beta, cx^\alpha (\log x)^\beta (\log\log x)^\gamma, cx^\alpha \exp\left[(\log x)^\delta\right]$, where $-\infty < \alpha, \beta, \gamma < +\infty$ and $0 < \delta < 1$.

Theorem 2.4. *Let A be a non-negative self-adjoint operator and $\Phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ a regularly varying function of index $\alpha > 0$. Assume that there exists $B : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $B(x) \nearrow \infty$ as $x \nearrow \infty$ and*

$$\mid\mid f \mid\mid_2^2 B(\mid\mid f \mid\mid_2) \leq (Af, f), \quad \forall f \in \mathcal{D}(A), \ \mid\mid f \mid\mid_1 = 1. \tag{2.9}$$

Then there exist $c, a > 0$ such that

$$c \mid\mid f \mid\mid_2^2 \Phi \circ B(\mid\mid f \mid\mid_2) \leq (\Phi(A)f, f), \quad \forall f \in \mathcal{D}(\Phi(A)), \ \mid\mid f \mid\mid_1 = 1, \mid\mid f \mid\mid_2 \geq a. \tag{2.10}$$

The proof of Theorem 2.4 is based on the following auxiliary results.

Proposition 2.5. *Let $\Phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be such that for some $N \geq 1$, the function $\varphi : x \longrightarrow \Phi(x^N)$ is eventually increasing and convex. Then* (2.9) *implies* (2.10).

To prove Proposition 2.5 we apply Theorem 1.3, the convexity argument of Proposition 2.3 and the following relation:

$$\Phi(A)f = \varphi(A^{\frac{1}{N}})f, \quad \forall f \in \mathcal{D}(A) \cap \mathcal{D}(\Phi(A)).$$

Proposition 2.6. *Let $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a regularly varying function of index $\alpha > 1$. Then there exists $\Phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ which is regularly varying of index α, increasing, convex and such that*

$$\lim_{x \longrightarrow +\infty} \frac{\Phi(x)}{\varphi(x)} = 1.$$

Proof. The function φ can be represented in the form

$$\varphi(x) = x^\alpha \ell(x), \quad x > 0,$$

where ℓ is slowly varying and non-negative. Define the following functions:

$$\tilde{\varphi}(x) = \alpha(\alpha - 1)x^{\alpha-2}\ell(x),$$
$$\Phi(x) = \int_0^x d\tau \int_0^\tau \tilde{\varphi}(s)\, ds.$$

It follows that Φ' and Φ'' are non-negative. Hence Φ is increasing and convex. The function $\tilde{\varphi}$ is regularly varying of index $\alpha - 2$. By Feller's theorem,

$$\int_0^\tau \tilde{\varphi}(s)\, ds \sim \frac{1}{\alpha - 1}\tau\tilde{\varphi}(\tau) = \alpha\tau^{\alpha-1}\ell(\tau), \quad \tau \to +\infty,$$

and

$$\int_0^x \left(\int_0^\tau \tilde{\varphi}(s)\, ds\right) d\tau \sim \frac{1}{\alpha} x \left(\alpha x^{\alpha-1}\ell(x)\right) = \varphi(x), \quad x \to +\infty.$$

This finishes the proof.

Proof of Theorem 2.4. For any fixed $N > \frac{1}{\alpha}$, the function $x \to \Phi(x^N)$ is regularly varying of index $\alpha' = \alpha N > 1$. By Proposition 2.6 there exists $\tilde{\Phi}$ regularly varying of index α', increasing, convex and such that $\Phi(x^N) \sim \tilde{\Phi}(x), \quad x \to +\infty$. Set $H(x) := \tilde{\Phi}(x^{\frac{1}{N}})$. Then there exists $a_1 > 0$ such that

$$\frac{1}{2}H(x) \leq \Phi(x) \leq 2H(x), \quad \forall x \geq a_1.$$

We now apply Proposition 2.5. For any $f \in L^2$ with $\| f \|_1 = 1$ we have

$$\begin{aligned}(\Phi(A)f, f) &= \int_0^\infty \Phi(\lambda)\, d(E_\lambda f, f) = \int_{a_1}^\infty \Phi(\lambda)\, d(E_\lambda f, f) + \int_0^{a_1} \Phi(\lambda)\, d(E_\lambda f, f) \\ &\geq \frac{1}{2}\int_{a_1}^\infty H(\lambda)\, d(E_\lambda f, f) - c_1 \| f \|_2^2 \geq \frac{1}{2}\int_0^\infty H(\lambda)\, d(E_\lambda f, f) - c_2 \| f \|_2^2 \\ &= \frac{1}{2}(H(A)f, f) - c_2 \| f \|_2^2 \geq \frac{1}{2}c_0 \| f \|_2^2 \; H\circ B(\| f \|_2) - c_2 \| f \|_2^2 .\end{aligned}$$

Since H and B approach to infinity as $x \to \infty$ we can find $a > a_1$ such that for $x \geq a, B(x) \geq a_1$ and $\Phi \circ B(x) \geq 4c_2/c$. Hence for $f \in L^1 \cap L^2$, such that $\| f \|_1 = 1, \| f \|_2 \geq a$,

$$(\Phi(A)f, f) \geq \frac{1}{4}c_0\, \Phi\circ B(\| f \|_2) - c_1 \| f \|_2^2 \geq \frac{1}{8}c_0\, \Phi\circ B(\| f \|_2).$$

This finishes the proof of the theorem.

3. Contraction properties of the semigroup $T_{t,\alpha}$

Let (T_t) be a semigroup acting on all $L^p, 1 < p < \infty$. (T_t) is said to be ultracontractive if for every $t > 0$, the operator T_t can be extended to a bounded operator from L^1 to L^∞. That is, there exists a non-decreasing function m from $\mathbb{R}_+$ to itself such that

$$|| T_t ||_{1\to\infty} \leq m(t), \qquad t > 0.$$

(T_t) is said to be hypercontractive if there exists $t > 0$ such that T_t is a bounded operator from L^2 to L^4. See [G].

In the following theorem, all inequalities will be understood in the sense of equivalent functions. See Section 1.

Theorem 3.1. *Let A be a non-negative self-adjoint operator such that the semigroup $T_t = e^{-tA}$, $t > 0$, acts as a contraction semigroup on L^1.*

1. *The following properties are equivalent:*
   (a) *There exists $\gamma > 0$, such that for any $t > 0$,*

$$|| T_t ||_{1\to\infty} \leq e^{t^{-\gamma}}. \tag{3.1}$$

   (b) *The following Nash inequality holds:*

$$|| f ||_2^2 \left[\log_+(|| f ||_2^2)\right]^{1+1/\gamma} \leq (Af, f), \quad f \in \mathcal{D}(A),\ || f ||_1 = 1. \tag{3.2}$$

2. *Assume that the equivalent properties 1(a) and 1(b) hold. Let $0 < \alpha \leq 1$; then the following inequality holds:*

$$|| f ||_2^2 \left[\log_+(|| f ||_2^2)\right]^{\alpha(1+1/\gamma)} \leq (A^\alpha f, f), \quad f \in \mathcal{D}(A^\alpha),\ || f ||_1 = 1. \tag{3.3}$$

   In particular, let $\alpha_c = \frac{\gamma}{\gamma+1}$; then
   (a) *If $\alpha > \alpha_c$, then $T_{t,\alpha} = e^{-A^\alpha t}$ is ultracontractive and $|| T_{t,\alpha} ||_{1\to\infty} \leq e^{t^{-\beta}}$, where $\beta = \frac{\alpha_c}{\alpha-\alpha_c}$.*
   (b) *If $\alpha \leq \alpha_c$, then $(T_{t,\alpha})$ may not be ultracontractive. See Section 4.*
   (c) *If $\alpha = \alpha_c$, and $-A$ is a Markov generator, the following logarithmic Sobolev inequality holds. There exists $C > 0$ such that*

$$\int f^2 \log\left(\frac{f}{|| f ||_2}\right) d\mu \leq C\left[(A^\alpha f, f) + || f ||_2^2\right], \quad f \in \mathcal{D}(A^\alpha). \tag{3.4}$$

   In particular, $(T_{t,\alpha})$ is hypercontractive.

Proof. Statement 1 is a consequence of Theorem 1.2. Statement 2(a) follows from Statement 1 and Theorem 1.3; $\beta = \frac{\alpha_c}{\alpha-\alpha_c}$ is the result of the integration of the Nash inequality (3.3); see [D]. For 2(b) we refer to property 1 of Theorem 4.1 below. In order to consider the case $\alpha = \alpha_c$, we need the following result from [BM]; see also [BCLS].

Proposition 3.2. *Suppose that $(\mathcal{E}, \mathcal{D})$ is a quadratic form in $L^2(X, \mu)$ which satisfies the following conditions:*

1. *For any non-negative $f \in \mathcal{D}$, $f_k = (f - 2^k)^+ \wedge 2^k \in \mathcal{D}$ for all $k \in \mathbb{Z}$.*
2. *$\sum_{k\in\mathbb{Z}} \mathcal{E}(f_k) \leq \mathcal{E}(f)$.*
3. *For any non-negative $f \in \mathcal{D}, || f ||_1 \leq 1$, $|| f ||_2^2 \log || f ||_2 \leq \mathcal{E}(f)$.*

Then there exists a constant $C > 0$ such that

$$\int f^2 \log\left(\frac{f}{\| f \|_2}\right) d\mu \leq C\left[\mathcal{E}(f) + \| f \|_2^2\right], \quad f \in \mathcal{D},\ f \geq 0. \tag{3.5}$$

For the sake of completeness, we give the proof of this statement.

Proof. Let $f \in \mathcal{D}, f \geq 0$. Without loss of generality, we assume that $\| f \|_2 = 1$. Let f_k be as above; then $\| f_k \|_1 < \infty$. For all $k \in \mathbb{Z}$ we have

$$\| f_k \|_2^2 \log(\| f_k \|_2 / \| f_k \|_1) \leq \mathcal{E}(f_k).$$

Since $\| f \|_2 = 1$ we have $\| f_k \|_2 / \| f_k \|_1 \geq 2^k$. Indeed,

$$\begin{aligned} \| f_k \|_1 = \int_{\{f_k > 0\}} f_k d\mu &\leq \mu(f_k > 0)^{\frac{1}{2}} \| f_k \|_2 = \mu(f > 2^k)^{\frac{1}{2}} \| f_k \|_2 \\ &\leq 2^{-k} \| f \|_2 \| f_k \|_2 \leq 2^{-k} \| f_k \|_2 . \end{aligned}$$

Markov inequality and the inequality above imply

$$(2^k)^2 \mu(f \geq 2^{k+1}) \log(2^{k-1}) \leq \mathcal{E}(f_k).$$

Let $A_k = \{2^{k+1} \leq f < 2^k\}$; then we have

$$\int_X f^2 \log f \, d\mu = \sum_{k \in \mathbb{Z}} \int_{A_k} f^2 \log f \, d\mu \leq \sum_{k \in \mathbb{Z}} (2^{k+2})^2 \mu(f \geq 2^{k+1}) \log(2^{k+2}).$$

This yields

$$\int_X f^2 \log f \, d\mu \leq 16 \sum_{k \in \mathbb{Z}} \mathcal{E}(f_k) + 12 \log 2 \left(\sum_{k \in \mathbb{Z}} (2^{k+1})^2 \mu(f \geq 2^{k+1}) \right).$$

We conclude by property 2 and by the fact that the last sum is comparable to $\| f \|_2^2$.

Proof of Theorem 3.1*(c).* Since $-A$ is a Markov generator, $\mathcal{E}(f) := (Af, f)$ is a Dirichlet form. Thus properties 1 and 2 of Proposition 3.2 hold true. Property 3 follows from the Nash inequality (3.3) with $\alpha = \frac{\gamma}{(1+\gamma)}$. Thus, by Proposition 3.2, the logarithmic Sobolev inequality (3.5) holds. This implies hypercontractivity of $(T_{t,\alpha})$; see [G, Theorem 3.7].

4. Invariant Dirichlet forms on the infinite dimensional torus

In this section, we consider the case where the measure space (X, μ) is the infinite dimensional torus $\mathbb{T}^\infty$, the product of countable many copies of $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. The topology on $\mathbb{T}^\infty$ is the product topology generated by cylindric sets. We regard $\mathbb{T}^\infty$ as a compact connected abelian group equipped with its (normalized) Haar measure μ and will focus on invariant strictly local Dirichlet forms $(\mathcal{E}, \mathcal{F})$ on the group $\mathbb{T}^\infty$. All the examples below are taken from [B] and [BSC], and the aim of this section is to illustrate the results of Sections 1, 2 and 3. We assume that both $\mathcal{F}$ and $\mathcal{E}$ are invariant under the action of translations on functions. Any such Dirichlet form can be described by a symmetric non-negative definite matrix $A = (a_{i,j})$ so that the associated Dirichlet form is given on smooth cylindric functions by the formula

$$\mathcal{E}(f, f) = \int_{\mathbb{T}^\infty} \sum_{i,j} a_{i,j} \partial_i f \partial_j f \, d\mu.$$

Yet another characterisation of $\mathcal{E}$ is that the L^2-generator L associated to $\mathcal{E}$ on smooth cylindric functions is given by the formula

$$Lf = -\sum_{i,j} a_{i,j}\partial_i\partial_j f.$$

Because of translation invariance, the associated semigroup $T_t := T_t^A$ is given by convolution with a Gaussian semigroup of measures (μ_t^A), that is, $T_t^A f = \mu_t^A * f$. See Heyer's book [He] for background on convolution semigroups of measures on locally compact groups.

4.1. **The product semigroup T_t^A.** Assume that A is a diagonal matrix with diagonal entries $a_{k,k} := a_k$. In this case, $\mu_t^A = \bigotimes_1^\infty \eta_{a_k t}$ is a product-measure, where $(\eta_s)_{s>0}$ is the standard Gaussian convolution semigroup on the torus $\mathbb{T}$. Since the operators T_t^A act as convolutions one can show that the semigroup (T_t^A) is ultracontractive if and only if the measures μ_t^A are absolutely continuous w.r.t. μ and admit continuous densities $x \to \mu_t^A(x)$. In this case

$$|| \, T_t^A \, ||_{L_1 \to L_\infty} = \mu_t^A(e), \quad e = (0, 0, \cdots).$$

Define the following function:

$$N_A(s) = \sharp\{k : a_k \leq s\}, \quad s > 0.$$

Then, the measures μ_t^A are absolutely continuous w.r.t. μ if and only if $\log N_A(s) = o(s)$ as $s \nearrow \infty$. In this case, the densities $x \to \mu_t^A(x)$ are continuous functions. Moreover, if we assume that N_A varies regularly of index $\gamma > 0$, then there exists $C_\gamma > 0$ such that

$$\log || \, T_t^A \, ||_{L_1 \to L_\infty} = \log \mu_t^A(e) \sim C_\gamma N_A\left(\frac{1}{t}\right), \quad t \searrow 0.$$

In particular, if $N_A(s) \sim s^\gamma$ as $s \nearrow \infty$, then

$$\log || \, T_t^A \, ||_{L_1 \to L_\infty} \sim C_\gamma t^{-\gamma}, \quad t \searrow 0.$$

4.2. **Contraction properties of the semigroup $T_{t,\alpha}^A$.** We now apply the results of Section 4.1 to the semigroup $T_{t,\alpha}^A$ generated by the operator $-(L_A)^\alpha$, $0 < \alpha < 1$. This clearly will illustrate the results of Section 3. In what follows, we assume that $N_A(s) \sim s^\gamma$ as $s \to \infty$. Hence μ_t^A is absolutely continuous w.r.t. μ and admits a continuous density $\mu_t^A(x)$ for all $t > 0$. Moreover condition (3.1) holds in very precise form

$$\log || \, T_t^A \, ||_{L_1 \to L_\infty} \sim C_\gamma t^{-\gamma}, \quad t \searrow 0.$$

Because of the subordination relation,

$$T_{t,\alpha}^A f = \mu_{t,\alpha}^A * f,$$

where $(\mu_{t,\alpha}^A)_{t>0}$ is a convolution semigroup of probability measures $\mu_{t,\alpha}^A$ given by the formula (see Section 2)

$$\mu_{t,\alpha}^A = \int_0^\infty \mu_s^A \, d\mu_t^\alpha(s).$$

Since μ_s^A is absolutely continuous w.r.t. μ for all $s > 0$, $\mu_{t,\alpha}^A$ is absolutely continuous w.r.t. μ for all $t > 0$ and admits a lower semi-continuous density $x \to \mu_{t,\alpha}^A(x)$. By symmetry

$$|| T_{t,\alpha}^A ||_{L_1 \to L_\infty} = \mu_{t,\alpha}^A(e), \quad t > 0.$$

These observations and the well-known asymptotic properties of the α-subordinator $(\mu_t^\alpha)_{t>0}$ (see [Z]) give the following result.

Theorem 4.1. *Assume that $N_A(s) \sim s^\gamma, \gamma > 0$. Let $\alpha_c = \frac{\gamma}{\gamma+1}$ be the critical exponent (see property 2 of Theorem 3.1). Then*

1. $\forall \alpha \in]0, \alpha_c[, \forall t > 0,$

$$|| T_{t,\alpha}^A ||_{L_1 \to L_\infty} = \mu_{t,\alpha}^A(e) = +\infty.$$

2. $\forall \alpha \in]\alpha_c, 1[, \forall t > 0,$

$$|| T_{t,\alpha}^A ||_{L_1 \to L_\infty} = \mu_{t,\alpha}^A(e) < +\infty$$

and

$$\log || T_{t,\alpha}^A ||_{L_1 \to L_\infty} = \log \mu_{t,\alpha}^A(e) \sim C_{\alpha,\gamma} t^{-\alpha_c/(\alpha-\alpha_c)}, \quad t \searrow 0.$$

3. $\alpha = \alpha_c$. *There exists $t = t_\gamma > 0$ such that*
   (a) $\forall t \in]0, t_\gamma[,$

$$|| T_{t,\alpha_c}^A ||_{L_1 \to L_\infty} = \mu_{t,\alpha_c}^A(e) = +\infty,$$

   (b) $\forall t \in]t_\gamma, \infty[,$

$$|| T_{t,\alpha_c}^A ||_{L_1 \to L_\infty} = \mu_{t,\alpha_c}^A(e) < +\infty.$$

Remark. Because we assume that $N_A(s) \sim s^\gamma$, $s \to \infty$, the generator $-L_A$ of the semigroup (T_t^A) satisfies the following Nash inequality:

$$|| f ||_2^2 \left(\log_+ || f ||_2^2\right)^{1+1/\gamma} \leq (L_A f, f), \quad || f ||_1 \leq 1,$$

which in this special case becomes sharp, i.e. for some sequence $\{f_n\}$ such that $|| f_n ||_1 \leq 1$ and $|| f_n ||_2 \to \infty$ as $n \to \infty$, the LHS and the RHS of the inequality above are comparable. This implies sharpness of the Nash inequality for $(L_A)^\alpha$:

$$|| f ||_2^2 \left(\log_+ || f ||_2^2\right)^{\alpha(1+1/\gamma)} \leq ((L_A)^\alpha f, f), \quad || f ||_1 \leq 1. \tag{4.1}$$

In particular, if $\alpha < \alpha_c = \frac{\gamma}{\gamma+1}$, $(L_A)^\alpha$ does not satisfy the log-Sobolev inequality (3.5). Indeed, the log-Sobolev inequality for $(L_A)^\alpha$ would imply the Nash inequality of the form

$$|| f ||_2^2 \log || f ||_2^2 \leq C ((L_A)^\alpha f, f), \quad || f ||_1 \leq 1,$$

for some $C > 0$. This is not possible since (4.1) is sharp. Hence, by [G], Theorem 3.7, $(T_{t,\alpha}^A)_{t>0}$, $\alpha < \alpha_c$, is not a hypercontractive semigroup.

5. Ornstein-Uhlenbeck semigroup

As an example where Theorem 1.3 applies we consider the Ornstein-Uhlenbeck semigroup (T_t) on $L^2(\gamma_n, \mathbb{R}^n)$, where γ_n is the standard Gaussian measure on $\mathbb{R}^n$

$$d\gamma_n(x) = (2\pi)^{-n/2} e^{-|x|^2/2} dx \ .$$

The Dirichlet form associated to (T_t) is defined as

$$\mathcal{E}(f) = \int_{\mathbb{R}^n} \mid \nabla f \mid^2 \ d\gamma_n,$$

where $\mid \nabla f \mid^2$ is the square of the gradient of f. Then the generator A of the form $\mathcal{E}$ is represented on $\mathcal{C}_c^\infty(\mathbb{R}^n)$ by the following equality:

$$-A = \Delta + x\nabla.$$

According to [G], Example 4.2, A satisfies the following log-Sobolev inequality:

$$\int f^2 \log (f/ \mid\mid f \mid\mid_2) \ d\gamma_n \leq \int \mid \nabla f \mid^2 \ d\gamma_n = (Af, f), \quad \forall f \in \mathcal{C}_c^\infty(\mathbb{R}^n). \tag{5.1}$$

Thus, (T_t) is hypercontractive, but not ultracontractive. Indeed, if $f(x) = x_i$ then $f \in L^1$, but $T_t f = f \notin L^\infty$. Inequality (5.1) implies

$$\mid\mid f \mid\mid_2^2 \log \mid\mid f \mid\mid_2 \leq (Af, f), \quad f \in \mathcal{D}(A), \ \mid\mid f \mid\mid_1 = 1. \tag{5.2}$$

Since $\mid\mid f \mid\mid_1 \leq \mid\mid f \mid\mid_2$, the LHS of this inequality is non-negative. Theorem 1.3 implies that for all $0 < \alpha < \infty$, the following Nash inequality holds:

$$\mid\mid f \mid\mid_2^2 (\log \mid\mid f \mid\mid_2)^\alpha \leq (A^\alpha f, f), \quad f \in \mathcal{D}(A^\alpha), \ \mid\mid f \mid\mid_1 = 1. \tag{5.3}$$

We claim that for any $0 < \alpha < 1$ the semigroup $(T_{t,\alpha})$ is not hypercontractive. Indeed, hypercontractivity of $(T_{t,\alpha})$ would imply the inequality

$$\mid\mid f \mid\mid_2^2 \left(\log \mid\mid f \mid\mid_2^2\right) \leq (A^\alpha f, f), \quad f \in \mathcal{D}(A^\alpha), \ \mid\mid f \mid\mid_1 = 1. \tag{5.4}$$

Then, by Theorem 1.3, the inequality (5.4) implies the following Nash inequality:

$$\mid\mid f \mid\mid_2^2 \left(\log \mid\mid f \mid\mid_2^2\right)^{\frac{1}{\alpha}} \leq (Af, f), \quad f \in \mathcal{D}(A), \ \mid\mid f \mid\mid_1 = 1. \tag{5.5}$$

Since $\frac{1}{\alpha} > 1$, the Nash inequality (5.5) implies ultracontractivity of (T_t)

$$\mid\mid T_t \mid\mid_{1\to\infty} \leq e^{t^{-\gamma}}, \ \ \gamma = \frac{\alpha}{1-\alpha}.$$

Contradiction. This proves the claim.

The reasons given above imply the following more general result.

Proposition 5.1. *Let $-A$ be a symmetric Markov generator. Assume that the semigroup (T_t) generated by $-A$ is not ultracontractive. Then, for any $0 < \alpha < 1$, the semigroup $(T_{t,\alpha})$ generated by $-(A^\alpha)$ is not hypercontractive.*

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