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THE VIRTUAL SPIVAK FIBER, DUALITY ON FIBRATIONS AND GORENSTEIN SPACES

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Abstract. In this paper we study a generalization of the homology of the Spivak fiber of a 1-connected space over any field and deduce consequences concerning Poincaré complexes, Gorenstein spaces and finiteness properties on fibrations.

1. Introduction

In [20] M. Spivak defined what is now called the Spivak fiber of a finite, simply connected complex: Let X be an n-dimensional simply connected complex and embed it in $\mathbb{R}^{n+k}$, with $k > n+1$. Let N be a regular neighborhood of X and consider the inclusion of its boundary $\partial N \hookrightarrow N$. The homotopy fiber of this inclusion F_X is, by definition, the Spivak fiber of X and it is, up to suspension, a well-defined invariant of the homotopy type of X. Among other interesting results, Spivak showed that F_X is a homotopy sphere if and only if X is a Poincaré complex.

For a general complex, there is a way of getting a picture of this homotopy invariant: In [5] the authors prove that given a 1-connected finite complex X, and up to suspension of a certain degree,

$$\widetilde{H}_*(F_X;k) \cong \mathrm{Ext}_{C^*(X;k)}\big(k, C^*(X;k)\big)$$

for any field k. The right-hand side of the above equality is a well-defined homotopy invariant for any space X, which may be seen as a generalization of the homology of the virtual Spivak fiber of X. We shall denote

$$HS(X,k) = \mathrm{Ext}_{C^*(X;k)}\big(k, C^*(X;k)\big)$$

and call it from now on the *virtual Spivak fiber of* X. The study of this invariant has had important consequences both in topology and local algebra [2, 5, 8, 9, 16]. This paper continues with this study along different directions: extending the concept of Poincaré duality, investigating the behavior of the virtual Spivak fiber on fibrations, and extracting finiteness properties of this invariant via the evaluation map.

There have been several successful attempts at locating Poincaré complexes in a more general duality framework [5, 10, 14], one of them being the study of Gorenstein spaces. Recall [5] that a space X is said to be Gorenstein over k if $HS(X;k)$ is a one-dimensional k-vector space. We say that X is Gorenstein if it is so over $\mathbb{F}_p$,

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p prime, and over $\mathbb{Q}$. Being Gorenstein generalizes being a Poincaré complex. In particular, for complexes of finite Lusternik-Schnirelman category (finite complexes for instance) a Gorenstein space over k is a space whose cohomology over k satisfies Poincaré duality.

On the other hand, it is well known how the Spivak fiber behaves with respect to fibrations. Gottlieb [11] proved that for a fibration $F \to E \to B$ of finite complexes we have $F_E \simeq F_F * F_B$. From here it is immediate that E is a Poincaré complex if and only if F and B are. However, with respect to the generalized Spivak fiber, there are answers only when the field k is of characteristic zero [2, 5, 16, 18]. Here we give a complete answer and prove:

Theorem 1.1. *Let $F \to E \to B$ be a fibration of simply connected spaces of finite type over the field k. Then there is an explicit morphism:*

$$HS(F;k) \otimes HS(B;k) \longrightarrow HS(E;k),$$

which is an isomorphism if $H^*(F;k)$ is finite dimensional.

This result has interesting consequences on Gorenstein spaces and duality. The first one, which is immediate, is the following generalization of the result of Gottlieb stated above:

Theorem 1.2. *Let $F \to E \to B$ be a fibration as in Theorem* 1.1 *in which $H^*(F;k)$ is finite dimensional. Then E is Gorenstein over k if and only if B and F are.*

Using this first consequence we can recover, and generalize, the original result of Spivak:

Theorem 1.3. *For a* 1*-connected finite complex X the following are equivalent:*

(i) *X is a Poincaré complex.*
(ii) *The Spivak fiber F_X has the homotopy type of a sphere.*
(iii) *The Spivak fiber F_X has the homotopy type of a finite complex.*

The contribution here is that it is enough to check whether F_X is a finite complex to deduce that X is a Poincaré complex. We also point out that a similar result for spectra has been proved in [15].

We now extract, via the study of the virtual Spivak fiber, new finiteness properties on fibrations: Associated to any space X and any field k there is a natural map called the *evaluation map*,

$$ev_{(X,k)} \colon HS(X;k) \longrightarrow H^*(X;k),$$

which carries important geometrical information. For instance, [9, 16], if $H_*(\Omega X;k)$ has polynomial growth, then $\mathrm{ev}_{(X,k)} \neq 0$ if and only if $H^*(X;k)$ is finite dimensional. Moreover, if we denote by $\mathcal{E}(X;k)$ the image of this map, it can be located in the following chain [8] (see §4 for a proper definition and details):

$$\mathcal{T}(X;k) \subset \mathcal{G}(X;k) \subset \mathcal{E}(X;k) \subset \mathcal{S}(X;k) \subset H^*(X;k).$$

Here, $\mathcal{T}(X;k)$ denotes the subspace of $H^*(X;k)$ generated by the "terminal classes", $\mathcal{G}(X;k)$ is the dual of the Gottlieb group, and $\mathcal{S}(X;k)$ is the socle of $H^*(X;k)$, i.e., the annihilator of $H^+(X;k)$.

Then we prove:

Theorem 1.4. *Let $F \to E \to B$ be a fibration as in Theorem* 1.1 *in which $H^*(F;k)$ is finite dimensional. Then $\mathcal{E}(B;k) \neq 0$ implies $\mathcal{E}(E;k) \neq 0$.*

This, together with the geometrical properties collected above, produces several finiteness results and gives rise to new obstructions for the existence of certain fibrations. As examples we have:

Corollary 1.5. *Let $F \to E \to B$ be a fibration in which $H^*(F;k)$ is finite dimensional and $\mathcal{E}(B) \neq 0$. Then, if $H_*(\Omega E;k)$ has polynomial growth, E is k-equivalent to a finite complex.*

Corollary 1.6. *Let $F \to E \to B$ be a fibration as in Theorem 1.1 in which $\mathcal{E}(E;k) = 0$. Then B has zero dual Gottlieb group and it cannot have a nontrivial terminal cell.*

This paper is organized as follows. In section 2 we present the notation, basic facts and algebraic tools we need to establish the necessary link between topology and algebraic fibrations. Section 3 is totally devoted to the proof of Theorems 1.1 and 1.3. Finally, in section 4 we prove Theorem 1.4 and its consequences.

2. Notation, basic facts and tools

In what follows the coefficient ring k will be a field of any characteristic although most of the results remain true over a good commutative ring. Hence "unadorned" tensor products mean with respect to k.

For us a topological space will be a space of the homotopy type of a simply connected CW-complex of finite type. Again most of what follows can be done for nilpotent spaces.

In this paper a differential graded algebra (DGA) is always an associative, connected, cochain algebra.

We shall be using basic facts from differential homological algebra based on the notion of *semifree modules* defined first in [1]. Good references are [6] or [7]. Here we just recall some definitions and basic facts.

Let A be a DGA. A differential graded module (DGM from now on) P is a *semifree extension* of another DGM M if there is an increasing sequence of submodules $M = P(0) \subset P(1) \subset \dots$ such that $P = \bigcup_{i\geq 0} P(i)$ and each $P(i)/P(i-1)$, $i \geq 1$ is A-free (considering A as a graded algebra with no differential) on a basis of cycles. If $M = 0$ we say that P is a *semifree A-module*.

Given a morphism of A-modules $f\colon M \to N$, a *semifree resolution* of f is a semifree extension P of M and a morphism $g : P \xrightarrow{\simeq} N$ inducing a homology isomorphism such that $g|_M = f$. A *semifree resolution* of an A-module M is a semifree extension of $0 \hookrightarrow M$, i.e., a quasi-isomorphism $P \xrightarrow{\simeq} M$ with P semifree.

Note that a semifree A-module M can be written as $M = A \otimes V$ for some vector space $V = \bigoplus_{j\geq 0} V(j)$ in which each $V(j)$ is a graded vector space, and the differential d in M satisfies: $dV(j) \subset A \otimes \big(\bigoplus_{i<j} V(i)\big)$, $d(a \otimes v) = da \otimes v + (-1)^{|a|} a.dv$.

We shall often use the *lifting lemma* for semifree modules: given morphisms of DGM $\varphi\colon Q \to M$ and $\eta\colon P \xrightarrow{\simeq} M$, with Q semifree, there exists a morphism $\psi\colon Q \to P$ (unique up to chain homotopy) such that $\eta\psi \sim \varphi$. Moreover, if η is surjective, ψ can be chosen so that $\eta\psi = \varphi$.

Definition 2.1. (i) Given differential graded left A-modules M, N define the k-vector space $\mathrm{Ext}_A\big(M,N\big) = H\big(\hom_A(M,N)\big)$, where $P \xrightarrow{\simeq} M$ is a semifree resolution of M.

(ii) In particular, for any DGA A, define the *virtual Spivak fiber* of A by $HS(A) = \mathrm{Ext}_A\big(k, A\big)$. For a given space X, its *virtual Spivak fiber* is then defined by $HS(X;k) = \mathrm{Ext}_{C^*(X;k)}\big(k, C^*(X;k)\big)$, or more generally, $HS(X;k) = HS(B)$ in which B is a DGA of the same homotopy type as $C^*(X;k)$.

Next we recall the basic link between algebra and topology we shall be using. This is based on the concept of a twisted tensor model of a fibration [4, 13].

Definition 2.2. A *twisted tensor product* of two DGA's A, C is again a DGA $A \odot C$ satisfying the following:

(i) $A \odot C = A \otimes C$ as k-vector spaces.

(ii) For all $a \in A, c \in C$,

$$(1 \otimes c) \cdot (a \otimes 1) = (-1)^{|a||c|} a \otimes c, \;\; (a \otimes 1) \cdot (1 \otimes c) = a \otimes c + \Phi, \;\; \Phi \in A^+ \otimes C^{<|c|}.$$

(iii) The sequence

$$A \xrightarrow{i} A \odot C \xrightarrow{\pi} C,$$

in which $i(a) = a \otimes 1$ and π is the projection over the ideal generated by A^+, is a sequence of DGA's.

The inclusion $A \xrightarrow{i} A \odot C$ is called a *twisted extension* and C is the *fiber* of the twisted extension.

Remark 2.3. (1) From now on the right and left A-module structures on $A \odot C$ (resp. $A \odot C$-module structures on C) is given by i (resp. by π). In particular, given $a \otimes c \in A \odot C$ and $b \in A$, $(a \otimes c)b = (a \otimes c)(b \otimes 1) = (-1)^{|c||b|} ab \otimes c$.

(2) Observe that the differential in $A \odot C$ satisfies: $d(a \otimes 1) = da \otimes 1$; $d(1 \otimes c) = 1 \otimes dc + \Psi$, $\Psi \in A^+ \otimes C$; $d(a \otimes c) = da \otimes c + (-1)^{|a|} a \otimes dc + (a \otimes 1) \cdot \Omega$, $\Omega \in A^+ \otimes C$.

Definition 2.4. A *twisted model* of a morphism of DGA's $\phi\colon R \to S$ is a homotopy commutative diagram of the form

$$\begin{array}{ccc} R & \xrightarrow{\phi} & S \\ \simeq\uparrow & & \uparrow\simeq \\ \bullet & \longrightarrow & \bullet \\ \simeq\downarrow & & \downarrow\simeq \\ A & \xrightarrow{i} & A \odot C \end{array}$$

in which i is a twisted extension. Often, we shall simply say that i is a *twisted model of* ϕ and that C is the *fiber of the twisted model*.

Not every morphism of DGA's admits a twisted model, but in the topological setting everything works fine:

Theorem 2.5 ([4])**.** *Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration. Then, the map $C^*(p)\colon C^*(B;k) \to C^*(E;k)$ admits a twisted model whose fiber has the homotopy type of $C^*(F;k)$. Such a model is called a twisted model of p.*

We finish by presenting the Serre spectral sequence of a given fibration $F \to E \to B$ at the level of its twisted model $A \to A \odot C$. For that, filter $A \odot C$ by $A^{\geq p} \otimes C$. After a straightforward computation one sees the following:

Proposition 2.6. *This is a filtration of $A \odot C$ as A-modules and the E_2-term of the resulting spectral sequence is $E_2^{p,q} \cong H^p(A) \otimes H^q(C)$.* $\square$

3. The virtual Spivak fiber along a fibration, Gorenstein spaces and Poincaré complexes

In view of the preceding section, Theorem 1.1 will be established once we prove the following:

Theorem 3.1. *Let $A \to A \odot C \to C$ be a twisted extension. Then, there is a natural morphism*

$$\varphi : HS(C) \otimes HS(A) \longrightarrow HS(A \odot C)$$

which is an isomorphism if $H^*(C)$ is finite dimensional.

In order to prove this result, and following the approach in [16], we consider two natural maps

$$\xi\colon \operatorname{Ext}_A\big(k, A\big) \longrightarrow \operatorname{Ext}_{A\odot C}\big(C, A \odot C\big),$$
$$\psi\colon \operatorname{Ext}_C\big(k, C\big) \longrightarrow \operatorname{Ext}_{A\odot C}\big(k, C\big),$$

defined as follows: If $\gamma\colon P \xrightarrow{\simeq} k$ is a semifree resolution of k as left A-modules, then the morphism

$$\lambda = 1_{A\odot C} \otimes \gamma\colon A \odot C \otimes_A P \xrightarrow{\simeq} A \odot C \otimes_A k \cong C$$

is also a quasi-isomorphism which exhibits $A \odot C \otimes_A P$ as a semifree resolution of C as left $A \odot C$-modules. Hence, if $f\colon P \to A$ represents an element $\alpha \in \operatorname{Ext}_A\big(k, A\big)$, then

$$A \odot C \otimes_A f\colon A \odot C \otimes_A P \to A \odot C \otimes_A A \cong A \odot C$$

represents an element $\xi(\alpha) \in \operatorname{Ext}_{A\odot C}\big(C, A \odot C\big)$.

On the other hand, if $R \xrightarrow{\simeq} k$ (resp. $Q \xrightarrow{\simeq} k$) is a semifree resolution of k as C-modules (resp. $A \odot C$-modules), then, as $R \xrightarrow{\simeq} k$ is also an $A \odot C$-morphism, via the lifting lemma, there exists an $A \odot C$-map $h\colon Q \to R$ such that the following diagram commutes up to chain homotopy:

$$\begin{tikzcd} & R \arrow[d, "\simeq"] \\ Q \arrow[ur, "h"] \arrow[r, "\simeq"'] & k. \end{tikzcd}$$

Thus, if $\beta \in \operatorname{Ext}_C\big(k, C\big)$ is represented by $g\colon R \to C$, define $\psi(\beta)$ as the gadget in $\operatorname{Ext}_{A\odot C}\big(k, C\big)$ represented by $g \circ h$.

Next, we combine ξ and ψ using a Yoneda composition

$$\mathcal{Y}\colon \operatorname{Ext}_{A\odot C}\big(k, C\big) \otimes \operatorname{Ext}_{A\odot C}\big(C, A \odot C\big) \longrightarrow \operatorname{Ext}_{A\odot C}\big(k, A \odot C\big)$$

defined in the following way: Let $Q \xrightarrow{\simeq} k$ and $\eta\colon S \xrightarrow{\simeq} C$ be semifree resolutions of $A \odot C$-modules, and let $\gamma \in \operatorname{Ext}_{A\odot C}\big(k, C\big)$ and $\omega \in \operatorname{Ext}_{A\odot C}\big(C, A \odot C\big)$ be represented by $f\colon Q \to C$ and $g\colon S \to A \odot C$ respectively. By the lifting lemma, there is a map $\widetilde{f}\colon Q \to S$ such that $\eta \circ h \sim \widetilde{f}$. Then define $\mathcal{Y}(\gamma \otimes \omega) = [g \circ \widetilde{f}]$.

Finally, the morphism

$$\varphi : HS(C) \otimes HS(A) \longrightarrow HS(A \odot C)$$

of Theorem 3.1 is defined by $\varphi(\alpha \otimes \beta) = \mathcal{Y}\big(\psi(\alpha) \otimes \xi(\beta)\big)$.

We now set all the technical tools for the proof.

As before, let $P \xrightarrow{\simeq} k$ be a semifree extension of $A \to k$ as left A-modules. Write $P \cong A \otimes V$, where $V = \bigoplus_{j \geq 0} V(j)$ with each $V(j)$ is a graded vector space, satisfying: $V(0) = k$; $dV(j) \subset A \otimes \big(\bigoplus_{i<j} V(i)\big)$; and $d(a \otimes v) = da \otimes v + (-1)^{|a|} a.dv$.

As we noted above, $\lambda\colon A \odot C \otimes V \cong A \odot C \otimes_A P \xrightarrow{\simeq} C$ is a semifree resolution of C as left $A \odot C$-modules.

Next, consider the $A \odot C$-morphism $A \odot C \otimes V \to k$ and build up a semifree $A \odot C$- extension of it, $Q \xrightarrow{\simeq} k$. Again, $Q \cong A \odot C \otimes V \otimes W$ for some vector space W. The differential D in Q induces, via $\lambda \otimes 1_W$ a differential $\overline{D}$ in $C \otimes W$: $\overline{D}(b \otimes w) = db \otimes w + (-1)^{|b|} b.\lambda(Dw)$. Call $R = (C \otimes W, \overline{D})$ and $h = \lambda \otimes 1_W\colon Q \to R$.

Lemma 3.2. *h is a quasi-isomorphism of $A \odot C$-modules.*

Proof. It is easily seen that h is an $A \odot C$-morphism. To see that it is in fact a quasi-isomorphism, filter Q and R respectively by

$$F^p = (A \odot C \otimes V)^{\geq p} \otimes W \quad \text{and} \quad G^p = C^{\geq p} \otimes W.$$

These are filtrations of $A \odot C$-DGM's and h is a morphism of filtered modules. The induced morphism $E_1(h)$ between the resulting spectral sequences at the E_1-term has the form

$$E_1(h) = H(\lambda) \otimes 1_W\colon H(A \odot C \otimes V) \otimes W \longrightarrow H(C) \otimes W.$$

Hence, since λ is a quasi-isomorphism so is $E_1(h)$. Therefore, by comparison, h is a quasi-isomorphism. □

Hence, since $H(Q) = k$, R is also contractible and we have:

Corollary 3.3. *The projection $R \xrightarrow{\simeq} k$ is a semifree resolution of k as C-modules.* □

We shall also need the following results:

Lemma 3.4. *The map*

$$\hom_C(R, C) \xrightarrow{\cong} \hom_{A \odot C}(A \odot C \otimes W, C),$$

given by $f \mapsto f \circ (\varepsilon \otimes 1_W)$, is an isomorphism. Here, as before, $R = C \otimes W$, $\varepsilon\colon A \odot C \to C$ denotes the projection of the twisted extension, and the structure of $A \odot C$-module on $A \odot C \otimes W$ is given by the isomorphism $A \odot C \otimes_{A \odot C} R \cong A \odot C \otimes W$.

Proof. First note that $f \circ (\varepsilon \otimes 1_W)$ is indeed an $A \odot C$-morphism. The map above is trivially injective and any $A \odot C$-morphism $A \odot C \otimes W \to C$ clearly factors through $C \otimes W$ via ε. □

In the following $A \otimes C$ shall denote the usual "untwisted" DGA structure of the tensor product of A and C.

Lemma 3.5.

$$H\big(\hom_{A \odot C}(A \odot C \otimes_A P, A \odot C)\big) \cong H\big(\hom_{A \otimes C}(A \otimes C \otimes_A P, A \otimes C)\big).$$

In other words,

$$\operatorname{Ext}_{A \odot C}\big(C, A \odot C\big) \cong \operatorname{Ext}_{A \otimes C}\big(C, A \otimes C\big) \cong \operatorname{Ext}_{A \otimes C}\big(C, A\big) \otimes H(C).$$

Proof. As before write $P = A \otimes V$ and define a linear map

$$\Phi\colon \hom_{A\odot C}(A \odot C \otimes_A P, A \odot C) \to \hom_{A\otimes C}(A \otimes C \otimes_A P, A \otimes C)$$

by $\Phi(f)(1 \otimes v) = f(1 \otimes v)$, $v \in V$, and then extend Φ to $A \otimes C \otimes_A P$ as an $A \otimes C$-morphism. Next, filter $\hom_{A\odot C}(A \odot C \otimes_A P, A \odot C)$ (respectively $\hom_{A\otimes C}(A\otimes C\otimes_A P, A\otimes C)$) by $F^p = \hom_{A\odot C}(A\odot C\otimes_A P, A^{\geq p}\odot C)$ (respectively $\mathcal{F}^p = \hom_{A\otimes C}(A \otimes C \otimes_A P, A^{\geq p} \otimes C)$. Although Φ may not commute with the differentials, it is easy to see that it does respect the filtration and, moreover, the induced map at the associated graded spaces

$$E_0(\Phi)\colon (E_0, d_0) \to (\mathcal{E}_0, d_0)$$

is a map of differential graded k-vector spaces. Next, using the filtration properties of the twisted tensor product stated in section 2 it is an exercise to check that at the E_2 term,

$$(E_2, d_2) = (\mathcal{E}_2, d_2) = H\big(\hom_{A\otimes C}(A \otimes C \otimes_A P, A \otimes C), d\big)$$

with d the usual differential, and $E_2(\Phi)$ is the identity. Finally observe that

$$H\big(\hom_{A\otimes C}(A \otimes C \otimes_A P, A \otimes C)\big) \cong H\big(\hom_{A\otimes C}(A \otimes C \otimes_A P, A)\big) \otimes H(C)$$

and, by the finiteness of $H(C)$, this means that the spectral sequences above are convergent. Then, by comparison, $E_\infty \cong \mathcal{E}_\infty$ and

$$H\big(\hom_{A\odot C}(A \odot C \otimes_A P, A \odot C)\big) \cong H\big(\hom_{A\otimes C}(A \otimes C \otimes_A P, A \otimes C)\big).$$

□

The following is a trivial observation:

Lemma 3.6. $\hom_A(P, A \otimes C) \cong \hom_{A\otimes C}(A \otimes C \otimes_A P, A \otimes C)$.

Finally, to proceed with the proof of Theorem 3.1, it is necessary to choose P, the A-resolution of k, of a particular form:

Lemma 3.7. *P can be chosen to be not just a semifree resolution of k as left A-modules, but also a right $A \odot C$-module.*

Proof. As in [13] we may consider $A = (T(X), d_X)$, $C = (T(Y), d_Y)$ and $A \odot C = (T\big(X \oplus Y \oplus s(Y \otimes X)\big), D)$ in which $(T(X), d_X)$ denotes a free tensor model for the DGA A. Then, one may construct an acyclic closure of $A \odot C$ (i.e., a contractible DGA) of the form

$$P = (T\big(X \oplus Y \oplus s(Y \otimes X) \oplus \overline{X} \oplus \overline{Y} \oplus \overline{s(Y \otimes X)}\big), D)$$

which contains $(T\big(X \oplus Y \oplus s(Y \otimes X)\big), D)$, and hence $(T(X), d_X)$, as a subalgebra. Therefore, P is a k-semifree resolution of k as A-module, and it is also a right $A \odot C$-module. □

Proof of Theorem 1.1. We have the following chain of isomorphisms:

$$\begin{aligned} HS(C)\otimes HS(A) &\overset{(1)}{\cong} H\big(\hom_C(R,C)\otimes \hom_A(P,A)\big) \overset{(2)}{\cong} \\ &\cong H\big(\hom_{A\odot C}(A\odot C\otimes W, C)\otimes \hom_A(P,A)\big) \overset{(3)}{\cong} \\ &\cong H\big(\hom_{A\odot C}(A\odot C\otimes W, \hom_{A\otimes C}(A\otimes C\otimes_A P, A\otimes C)\big) \overset{(4)}{\cong} \\ &\cong H\big(\hom_{A\odot C}(A\odot C\otimes W, \hom_{A\odot C}(A\odot C\otimes_A P, A\odot C)\big) \overset{(5)}{\cong} \\ &\cong H\big(\hom_{A\odot C}(A\odot C\otimes_A P\otimes_{A\odot C} A\odot C\otimes W, A\odot C)\big) \cong \\ &\cong H\big(\hom_{A\odot C}(Q, A\odot C)\big) = HS(A\odot C), \end{aligned}$$

given by: (1) Kunneth theorem; (2) Lemma 3.4; (3) Lemma 3.6; (4) Lemma 3.5 and (5) standard adjoint isomorphism considering, by Lemma 3.7, the $A\odot C$ right structure on P.

On the other hand observe that, with the particular resolutions chosen, the morphism φ of Theorem 3.1 can be seen as follows: let $\alpha\in HS(C)$, $\beta\in HS(A)$, represented by $g\colon R=C\otimes W\to C$ and $f\colon P\cong A\otimes V\to A$. In view of the lifting lemma, there exists an $A\odot C$-morphism $\widetilde{g}\colon Q\to A\odot C\otimes_A P$ such that $\lambda\circ\widetilde{g}=g\circ h$. Then,

$$\varphi(\alpha\otimes\beta)=[(A\odot C\otimes_A f)\circ\widetilde{g}].$$

However, it is easy to see that this is precisely the above chain of explicit isomorphisms. This finishes the proof of Theorem 3.1 and this section. □

Proof of Theorem 1.1. (i)⇒(ii): If X is a Poincaré complex it is Gorenstein over any field k; i.e., $HS(X;k)$ is a 1-dimensional vector space. But, since X is a finite complex, $HS(X)\cong H^*(F_X,k)$. Hence, $(F_X)_{(0)}\simeq (S^n)_{(0)}$ and $(F_X)_{(p)}\simeq (S^n)_{(p)}$ for any prime p and some n. Thus $F_X\simeq S^n$.

(ii)⇒(iii): Trivial.

(iii)⇒(i): Consider the fibration $F_X\to\partial N\to N\simeq X$. Next, since ∂N is an oriented closed manifold, it is a Poincaré complex and therefore it is Gorenstein. Hence, if F_X is a finite complex, apply Theorem 1.2 to deduce that X is Gorenstein and thus, it is a Poincaré complex. □

4. The evaluation map and finiteness properties of fibrations

For a given space X and a coefficient field k, the evaluation map [5, 8, 12, 17] is a linear morphism

$$\mathrm{ev}_{(X,k)}\colon HS(X;k)\longrightarrow H^*(X;k)$$

defined as follows. Let $\alpha\in HS(X;k)$ be represented by $f\colon P\to C^*(X;k)$, with P a semifree resolution of k as $C^*(X;k)$-modules. Then $\mathrm{ev}_{(X,k)}(\alpha)=[f(p)]$, with $p\in P$ a cycle representing $1\in k$. Note that the same process defines the evaluation map ev_A for a generic DGA A. We shall denote by $\mathcal{E}(X;k)$ (resp. $\mathcal{E}(A)$) the image of $\mathrm{ev}_{(X,k)}$ (resp. the image of ev_A).

The size of $\mathcal{E}(X;k)$ reflects interesting properties of the geometrical behavior of the space X. Let me state two known results which corroborate this fact:

Theorem 4.1 ([9]). *Let X be a space for which $H_*(\Omega X;k)$ has polynomial growth. Then, $\mathcal{E}(X;k)\neq 0$ if and only if $H^*(X;k)$ is finite dimensional.*

Theorem 4.2 ([8]). *For a given space X the following chain of inclusions holds:*

$$\mathcal{T}(X;k) \subset \mathcal{G}(X;k) \subset \mathcal{E}(X;k) \subset \mathcal{S}(X;k) \subset H^*(X;k).$$

Moreover, if X is k-formal, i.e., $C^*(X;k)$ has the homotopy type of its cohomology, then all of the gadgets above coincide.

We briefly recall the definition of these subspaces of $H^*(X;k)$:

• $\mathcal{T}(X;k)$ is the subspace of $H^*(X;k)$ generated by the "terminal classes": $\alpha \in H^n(X;k)$ is a *terminal class* if $X \simeq Y \cup_f e^n$ and α is in the image of the canonical map $H^n(X,Y) \to H^n(X)$, or equivalently, the restriction of α to $H^n(Y)$ is zero. The notation comes from the fact that the terminal classes correspond to the "terminal cells": A cell e of a CW-complex X is called *terminal* if the intersection of $\overline{e}$ with other cells only occurs in its boundary, or equivalently, the complement of e is a subcomplex. Therefore, note that every terminal cell determines a terminal class and every terminal class determines a homotopy equivalence $X \simeq Y \cup_f e$ in which e is obviously a terminal cell.

• $\mathcal{G}(X;k)$ is the dual of the Gottlieb group; i.e, it is generated by those classes $\alpha \in H^*(X;k)$ represented by $f\colon X \to K(k,n)$ such that the map $(1_X, f)\colon X \to X \otimes K(k,n)$ factors through $X \vee K(k,n)$.

• $\mathcal{S}(X;k)$ is the socle of $H^*(X;k)$, i.e., the annihilator of $H^+(X;k)$.

We now describe how the image of the evaluation map behaves along a given fibration by proving Theorem 1.4. This result is immediately deduced from its algebraic translation:

Theorem 4.3. *Let $A \to A \odot C \to C$ be a twisted extension of DGA's in which $H(C)$ is finite dimensional. Then $\mathcal{E}(A) \neq 0$ implies $\mathcal{E}(A \odot C) \neq 0$.*

Proof. Let n be the smallest integer for which $H^{>n}(C) = 0$. Then there is a natural map $\int_C\colon H^*(A \odot C) \to H^{*-n}(A) \otimes H^n(C)$ defined as follows:

As in section 2, consider the Serre spectral sequence of the twisted model filtering $A \odot C$ by the A-modules $A^{\geq p} \otimes C$. The E_2-term of the resulting spectral sequence is $E_2^{p,q} \cong H^p(A) \otimes H^q(C)$. By the finiteness property of the fiber, we have

$$E_\infty^{*,n} \hookrightarrow \cdots \hookrightarrow E_2^{*,n} \cong H^*(A) \otimes H^n(C).$$

Also, if we denote by G^p the induced filtration in the cohomology $H(A \odot C)$, one has

$$E_\infty^{p-n,n} = (G^{p-n}/G^{p-n+1})^p = H^p(A \odot C)/G^{p-n+1,n-1}.$$

Then, we define

$$\int_C H^p(A \odot C) \xrightarrow{\pi} E_\infty^{p-n,n} \hookrightarrow \cdots \hookrightarrow E_2^{p-n,n} \cong H^{p-n}(A) \otimes H^n(C),$$

where π is just the projection. We use this notation since this map is, when dealing with fiber bundles, nothing but the integration over the fiber defined by Spanier; see [3, 19].

Next we show that the following diagram commutes:

$$\begin{tikzcd} HS^n(C) \otimes HS(A) \arrow[r, "\varphi"] \arrow[d, "\mathrm{ev}_C \otimes \mathrm{ev}_A"'] & HS(A \odot C) \arrow[d, "\mathrm{ev}_{A \odot C}"] \\ H^n(C) \otimes H(A) & H(A \odot C) \arrow[l, "\tau \circ \int_C"] \end{tikzcd} \tag{4.1}$$

where $\tau(a \otimes b) = b \otimes a$. For that, as in the proof of Theorem 3.1, choose special resolutions

$$\begin{aligned} &P \cong A \otimes V \xrightarrow{\simeq} k, \\ &\lambda\colon A \odot C \otimes_A P \cong A \odot C \otimes V \xrightarrow{\simeq} C, \\ &Q \cong A \odot C \otimes V \otimes W \xrightarrow{\simeq} k, \\ &R = C \otimes W \xrightarrow{\simeq} k. \end{aligned}$$

Thus, given $\alpha \in HS(C)$ and $\beta \in HS(A)$ represented by $g\colon R \to C$ and $f\colon P \to A$, $\varphi(\alpha \otimes \beta) = [(A \odot C \otimes_A f) \circ \widetilde{g}]$, in which $\widetilde{g}$ is such that $\lambda \circ \widetilde{g} = g \circ h$, with $h\colon Q \xrightarrow{\simeq} R$. Hence, observe that $\widetilde{g}(1) = 1 \otimes g(1) + \Phi + \Psi$, $\Phi \in A^+ \odot C$, $\Psi \in A \odot C \otimes V^+$. Then

$$\begin{aligned} \tau \circ \int_C \circ \mathrm{ev}_{A \odot B} \circ \varphi(\alpha \otimes \beta) &= \tau \circ \int_C [(A \odot C \otimes_A f)\big(\widetilde{g}(1)\big)] = \\ &= \tau \circ \int_C [f(1) \otimes g(1) + (A \odot C \otimes_A f)(\Phi) + (A \odot C \otimes_A f)(\Psi)] \overset{(1)}{=} \\ &= \tau([f(1)] \otimes [g(1)]) = \mathrm{ev}_C(\alpha) \otimes \mathrm{ev}_A(\beta). \end{aligned}$$

Note that (1) holds since both $(A \odot C \otimes_A f)(\Phi)$ and $(A \odot C \otimes_A f)(\Psi)$ live in $A \odot C^{<n}$.

Finally, we shall use the commutativity of diagram (4.1) to complete the proof. First, observe that $H^n(C)$ is not trivial and is contained in $\mathcal{E}(C)$: indeed, consider the DGA C/I where I is the acyclic ideal formed by $C^{>n} \oplus B^n$, B^n being the complement of the n-cocycles of C. Then, the projection $C \to C/I$ induces a homology isomorphism. On the other hand, any element in $(C/I)^n$ is a cycle which annihilates $(C/I)^+$, and therefore, by [17], it lives in $\mathcal{E}(C/I)$. Then, $\mathcal{E}(C/I) = \mathcal{E}(C)$ contains all of $H^n(C)$.

To finish, if $\mathrm{ev}_A \neq 0$, in view of diagram (4.1), $\mathrm{ev}_{A \odot C}$ cannot be zero and the theorem holds. $\square$

In view of the geometrical behavior of the evaluation map described at the beginning of this section, this result gives easy criteria for the existence of certain kinds of fibrations which cannot be obtained by classical methods. For instance, Corollary 1.5 follows immediately from Theorems 1.4 and 4.1 while Corollary 1.6 is easily deduced from Theorems 1.4 and 4.2.

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