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NONZERO DEGREE MAPS BETWEEN CLOSED ORIENTABLE THREE-MANIFOLDS

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Abstract. This paper adresses the following problem: Given a closed orientable three-manifold M, are there at most finitely many closed orientable three-manifolds 1-dominated by M? We solve this question for the class of closed orientable graph manifolds. More precisely the main result of this paper asserts that any closed orientable graph manifold 1-dominates at most finitely many orientable closed three-manifolds satisfying the Poincaré-Thurston Geometrization Conjecture. To prove this result we state a more general theorem for Haken manifolds which says that any closed orientable three-manifold M 1-dominates at most finitely many Haken manifolds whose Gromov simplicial volume is sufficiently close to that of M.

1. Introduction

1.1. **Statement of the general problem.** We deal here with nonzero degree maps between closed orientable 3-manifolds. Recall that a 3-manifold is termed *geometric* if it admits one of the eight uniform geometries classified by W. P. Thurston. Denote by $\mathcal{G}$ the set of closed geometric and Haken manifolds union the connected sums of such manifolds. Note that the Poincaré-Thurston Geometrization Conjecture asserts that $\mathcal{G}$ represents all closed orientable 3-manifolds. Thus a 3-manifold of $\mathcal{G}$ will be termed a *Poincaré-Thurston* 3-*manifold*. According to [1], given two closed orientable 3-manifolds M, N, we say that M d-dominates N ($M \geq_{(d)} N$) if there is a map $f\colon M \to N$ of degree $d \neq 0$. A motivation for studying nonzero degree maps comes from the observation that they seem to give a way to measure the topological complexity of 3-manifolds and of knots in $\mathbf{S}^3$. For instance, Y. Rong proved in [16] that degree-one maps define a partial order on the set $\mathcal{G}$, up to homotopy equivalence. In the same way one can define a partial order on the set $\mathcal{K}$ of knots in $\mathbf{S}^3$, up to knots equivalence. Given two knots K and K' in $\mathcal{K}$ we say that K 1-dominates K' if the complement E_K of K properly 1-dominates $E_{K'}$. Then it follows from [23] combined with the fact that knots in $\mathbf{S}^3$ are determined by their complement, see [3], that $(\mathcal{K}, \geq_{(1)})$ is a partially ordered set (a *poset*). This paper addresses the following question, which is closely related to the partial order induced by degree-one maps (see also Kirby's Problem List [9, Problem 3.100]):

Question 1. Given a closed orientable 3-manifold M, are there at most finitely many 3-manifolds N in $\mathcal{G}$ (up to homeomorphism) 1-dominated by M?

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Note that in this question the targets are 3-manifolds of $\mathcal{G}$ because of the Poincaré Conjecture. Indeed if there is a fake 3-sphere K, then one can get infinitely many reducible homotopy 3-spheres by doing connected sums of finitely many copies of K, and since there always exists a degree-one map from a closed orientable 3-manifold M to a homotopy 3-sphere we have to exclude this kind of 3-manifold. On the other hand, in Question 1, we consider always degree-one maps to avoid some easy counterexamples. For instance, for any spherical Lens space $L(p,q)$ there always exists a nonzero degree map (actually a finite covering) from the 3-sphere $\mathbf{S}^3$ to $L(p,q)$.

1.2. **The main result.** In this paper we solve Question 1 when the domain M is a closed orientable graph manifold. More precisely our main result is stated as follows.

Theorem 1.1. *Any closed orientable graph manifold* 1*-dominates at most finitely many closed orientable Poincaré-Thurston* 3*-manifolds.*

This result comes from a more general theorem which gives an affirmative answer to Question 1 when the targets are closed Haken manifolds whose Gromov simplicial volume, denoted by Vol(.), is sufficiently close to that of the domain M. More precisely:

Theorem 1.2. *For any closed orientable* 3*-manifold M there exists a constant $c \in (0,1)$, which depends only on M, such that M* 1*-dominates at most finitely many closed Haken manifolds N satisfying* $\mathrm{Vol}(N) \geq (1-c)\mathrm{Vol}(M)$.

Recall that there are many important results related to Question 1 obtained when the targets are restricted. More precisely the known answers can be summarized as follows.

Theorem 1.3 ([5], [18], [24], [15])**.** *Any closed orientable* 3*-manifold* 1*-dominates at most finitely many orientable closed geometric* 3*-manifolds.*

Notice that in some cases the degree of the maps need not be bounded. This is true in particular when the targets admit a hyperbolic or an $\mathbf{H}^2 \times \mathbf{R}$-structure. Thus a useful consequence of the proof of Theorem 1.3 is the following result.

Corollary 1.4 ([18], [24])**.** *Any orientable* 3*-manifold M properly dominates at most finitely many orientable geometric* 3*-manifolds with nonempty boundary.*

Then the following step is to study Question 1 when the targets are Haken manifolds (a Haken manifold is not geometric in general but it admits a decomposition into geometric 3-manifolds). This is the purpose of Theorem 1.2.

We end this section by giving an interpretation of Theorem 1.1 for the subclass $\mathcal{G}_0$ of $\mathcal{G}$ which consists of graph manifolds. The purpose of this remark is to study the *local finiteness* of the poset $(\mathcal{G}_0, \geq_1)$, up to homotopy equivalence. Recall that a poset $(\mathcal{P}, \geq)$ is *locally finite* if for any x, y in $\mathcal{P}$ with $x \leq y$ the interval $[x,y] = \{z \in \mathcal{P}, x \leq z \leq y\}$ is finite (many results on posets require this condition). Then Theorem 1.1 implies the following.

Corollary 1.5. *The poset of closed orientable graph manifolds partially ordered, up to homotopy equivalence, by degree-one maps is locally finite.*

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2. Notation and terminology

2.1. **The degree of a map.** Let $f\colon M \to N$ be a map between orientable compact connected n-manifolds. We say that f is proper if $f^{-1}(\partial N) = \partial M$. Suppose f is proper. Then f induces homomorphisms $f_*\colon \pi_1 M \to \pi_1 N$, $f_\sharp\colon H_*(M, \partial M) \to H_*(N, \partial N)$, $f^\sharp\colon H^*(N; \mathbf{R}) \to H^*(M; \mathbf{R})$. The degree of f, $\deg(f)$, is given by the equation $f_\sharp([M]) = \deg(f)[N]$, where $[M] \in H_n(M, \partial M; \mathbf{Z})$, $[N] \in H_n(N, \partial N; \mathbf{Z})$ are the chosen fundamental classes of M and N. On the other hand the Gromov simplicial volume $\mathrm{Vol}(M)$ of the pair $(M, \partial M)$ is the infimum of the l^1-norms $\sum_{j=1}^{k} |\lambda_j|$ of all cycles $z = \sum_{j=1}^{k} \lambda_j \sigma_j$, with $\sigma_j\colon \Delta^n \to M$ singular n-simplexes of M, $\lambda_j \in \mathbf{R}$, representing the fundamental class $[M] \in H_n(M, \partial M; \mathbf{Z})$ (see [4, Sect. 1.1]). We recall the following well-known and useful result on nonzero degree maps.

Proposition 2.1. *Suppose $f\colon M \to N$ is a proper nonzero degree map between compact orientable 3-manifolds. Then the following properties hold:*

(i) the index of $f_*(\pi_1 M)$ in $\pi_1 N$ divides $\deg(f)$*,*

(ii) the induced homomorphism $f_\sharp\colon H_*(M, \partial M; \mathbf{R}) \to H_*(N, \partial N; \mathbf{R})$ is surjective and by duality $f^\sharp\colon H^*(N; \mathbf{R}) \to H^*(M; \mathbf{R})$ is a monomorphism,

(iii) $\mathrm{Vol}(M) \geq \deg(f)\mathrm{Vol}(N)$.

Sketch of proof. Point (i) comes directly from a covering space argument as in the proof of Lemma 15.12 in [6]. Point (ii) comes from the Poincaré Duality combined with the naturality of cap products. Point (iii) can be obtained directly using the definition of Gromov simplicial volume combined with the definition of the degree of a map given in paragraph 2.1. □

2.2. **Haken manifolds and sewing involutions.** An orientable compact irreducible 3-manifold is called a *Haken manifold* if it contains an orientable proper incompressible surface. Given a closed Haken manifold N we denote by $\mathcal{T}_N$ the Jaco-Shalen-Johannson family of canonical tori of N and by $\mathcal{H}(N)$ (resp. $\mathcal{S}(N)$) the disjoint union of the hyperbolic (resp. Seifert) components of $N \setminus \mathcal{T}_N \times (-1, 1)$ so that $N \setminus \mathcal{T}_N \times (-1, 1) = \mathcal{H}(N) \cup \mathcal{S}(N)$, where $\mathcal{T}_N \times [-1, 1]$ is identified with a regular neighborhood of $\mathcal{T}_N$ in such a way that $\mathcal{T}_N \simeq \mathcal{T}_N \times \{0\}$ (see [7], [8] and [22] for the statement and the proof of this decomposition). On the other hand, we denote by $\Sigma(N) = (\Sigma(N), \emptyset)$ the *characteristic Seifert pair of* N in the sense of [7] and [8].

Let N be a Haken manifold. Consider the 3-manifold N^* obtained after splitting N along $\mathcal{T}_N$. There is an involution $s\colon \partial N^* \to \partial N^*$ defined as follows. Let $r\colon N^* \to N$ be the canonical identification map. For any component T of ∂N^* we denote by T' the unique component of ∂N^* distinct from T such that $r(T') = r(T)$. Let $s_T\colon T \to T'$ be the unique homeomorphism such that $(r|T') \circ s_T = r|T$. Define s by setting $s|T = s_T$ for any $T \in \partial N^*$. The map s will be termed the *sewing involution* for N.

Consider now two Haken manifolds N_1 and N_2 with sewing involutions s_1 and s_2. We say that the two ordered pairs $(N_1^*, s_1), (N_2^*, s_2)$ are *equivalent* if there is a homeomorphism $\eta\colon N_1^* \to N_2^*$ such that $\eta \circ s_1$ and $s_2 \circ \eta$ are isotopic. Using this notation, then two Haken manifolds N_1 and N_2 are homeomorphic if and only if the two ordered pairs (N_1^*, s_1) and (N_2^*, s_2) are equivalent. On the other hand we will say, for convenience, that two Haken manifolds N_1 and N_2 are *weakly equivalent* if there is a homeomorphism $\eta\colon N_1^* \to N_2^*$.

2.3. **Haken manifolds, graph manifolds and simplicial volume.** Recall that it follows from [21] that if H is a complete finite volume hyperbolic manifold, then

$$\mathrm{Vol}(H) = \frac{\mathrm{Vol}_{\mathrm{int}}(H)}{v_3},$$

where $\mathrm{Vol}_{\mathrm{int}}(H)$ is the volume associated to the complete hyperbolic metric in $\mathrm{int}(H)$ and v_3 is a constant which depends only on the dimension. On the other hand it follows from [4] that $\mathrm{Vol}(S) = 0$ when S is a Seifert fibered space. Then using the Cutting-off Theorem of M. Gromov ([4]) we get

$$\mathrm{Vol}(N) = \sum_{H \in \mathcal{H}(N)} \mathrm{Vol}(H).$$

A 3-manifold G is termed a *graph manifold* if there is a collection $\mathcal{T}$ of disjoint embedded tori in G such that each component of $G \setminus \mathcal{T}$ is Seifert. Note that the Gromov simplicial volume gives a characterization of graph manifolds in the following way:

Theorem 2.2 ([20]). *A closed orientable* 3*-manifold N is a graph manifold if and only if N is an element of $\mathcal{G}$ with zero Gromov simplicial volume.*

We end this section with the following convenient definition. Given a closed Haken manifold N, a zero codimensional submanifold G of N which is the union of some geometric (resp. Seifert) components of N will be termed a *canonical* (resp. *graph) submanifold of N.*

3. Main steps of the proof of Theorem 1.2 and statement of the intermediate results

Let M be a closed orientable 3-manifold and let N be a closed Haken manifold 1-dominated by M. First note that we may assume, throughout the proof of Theorem 1.2, that the target satisfies the following condition:

(I) N is a closed nongeometric Haken manifold.

This condition comes from Theorem 1.3. On the other hand the constant $c \in (0,1)$ of Theorem 1.2 is given by a result of T. Soma in [19, Theorem 1] which implies the following.

Theorem 3.1 ([19]). *Let M be a closed orientable* 3*-manifold. There exists a constant $c \in (0,1)$, which depends only on M, satisfying the following property. If $f\colon M \to N$ denotes a nonzero degree map to a closed Haken manifold N whose Gromov simplicial volume satisfies* $\mathrm{Vol}(N) \geq (1-c)\mathrm{Vol}(M)$, *then* $\mathrm{Vol}(M) = \deg(f)\mathrm{Vol}(N)$.

Hence, in order to state Theorem 1.2 we will prove the following general result on nongeometric closed Haken manifolds.

Proposition 3.2. *Let M be a closed orientable* 3*-manifold. Then there are at most finitely many closed nongeometric Haken manifolds N such that there exists a degree-one map $f\colon M \to N$ satisfying* $\mathrm{Vol}(M) = \mathrm{Vol}(N)$.

The proof of Proposition 3.2 contains two steps. In the first one, we show that there are at most finitely many homeomorphism classes for N^* (when N runs over the target manifolds) and in the second one, we prove that there are at most finitely many equivalence classes of pairs (N^*, s), where s is the sewing map which produces the target N from its geometric decomposition N^*. We now give the key results of these two steps.

3.1. **First step: Control of the geometric decomposition of the targets.** According to the paragraph above, the purpose of this step is to prove the following result:

Proposition 3.3. *Let M be a closed orientable 3-manifold. Then there are at most finitely many classes of weakly equivalent nongeometric closed Haken manifolds N such that there exists a nonzero degree map $f\colon M \to N$ satisfying* $\mathrm{Vol}(M) = \deg(f)\mathrm{Vol}(N)$.

The proof of Proposition 3.3 depends on the following key result, which says that a nonzero degree map f into a Haken manifold N has a kind of *canonical standard form* with respect to the geometric decompostion of N.

Lemma 3.4 (Standard Form)**.** *Any closed orientable 3-manifold M admits a finite set $\mathcal{H} = \{M_1, ..., M_k\}$ of closed Haken manifolds satisfying the following property. For any nonzero degree map $g\colon M \to N$ into a closed nongeometric Haken manifold N containing no embedded Klein bottles and satisfying* $\mathrm{Vol}(M) = \deg(g)\mathrm{Vol}(N)$ *there exists at least one element M_i in $\mathcal{H}$ and a map $f\colon M_i \to N$ with the same degree as g such that:*

(i) $\mathrm{Vol}(M_i) = \deg(f)\mathrm{Vol}(N)$, *and*

(ii) f induces a finite covering between $\mathcal{H}(M_i)$ and $\mathcal{H}(N)$, and

(iii) for any geometric component Q in N^* the preimage $f^{-1}(Q)$ is a canonical submanifold of M.

Remark 3.5. It will follow from the proof of Lemma 3.4 that if Q is a Seifert piece of N, then $f^{-1}(Q)$ is a graph submanifold of M_i and if Q is a hyperbolic piece, then each geometric component of $f^{-1}(Q)$ is a hyperbolic piece of M_i.

Recall that in [19, Key Lemma], T. Soma proves the following result for complete finite volume hyperbolic 3-manifolds without any condition on the Gromov simplicial volume:

Lemma 3.6 (T. Soma)**.** *Any closed orientable 3-manifold M admits a finite set $\mathcal{F} = \{F_1, ..., F_n\}$ of 3-manifolds such that for any closed Haken manifold N dominated by M, then any component H of $\mathcal{H}(N)$ is properly dominated by at least one element F_i of $\mathcal{F}$.*

Since a closed Haken manifold contains at most finitely many canonical submanifolds, then point (iii) of Lemma 3.4 gives a version of Lemma 3.6 for Seifert fibered manifolds with an additional condition on the Gromov simplicial volume. First of all, note that in the proof of Lemma 3.4 as well as in the proof of Lemma 3.6, it can be shown that there is no loss of generality in assuming that M is a closed Haken manifold. With this assumption, recall that the proof of Soma of Lemma 3.6 uses the geometry of the hyperbolic space and, in particular, the isotropy of hyperbolic geometry is crucial for "locally hyperbolizing" certain simplicial subcomplexes of M. This method cannot be adaptedin the Seifert case since the geometry is not isotropic (indeed there is an invariant direction corresponding to the Seifert fibration).

In the proof of Lemma 3.4 the condition on the Gromov simplicial volume is essential. More precisely the proof of Lemma 3.4 is based on the observation that when $\mathrm{Vol}(M) = \deg(f)\mathrm{Vol}(N)$, then we can "control" the "essential part" of $f^{-1}(\mathcal{T}_N)$. Actually one can show, up to homotopy, that this essential part is a subfamily of $\mathcal{T}_M$, which is crucial in our proof since this ensures that the genus of the essential components of $f^{-1}(\mathcal{T}_N)$ is bounded independently of N. This control cannot be accomplished when $\mathrm{Vol}(M) \gg \deg(f)\mathrm{Vol}(N)$. Indeed, consider for example a degree-one map from a closed hyperbolic 3-manifold M to a graph manifold N. (This kind of example can be built by taking a hyperbolic null-homotopic knot k in a graph manifold N and by gluing a solid torus along $\partial(N \setminus k)$ in such a way that the resulting manifold M is hyperbolic. Then the degree of the canonical decomposition map $f : M \to N$ is one; see [1] for details on this construction.) In this case one can clearly not control the genus of the components of $f^{-1}(\mathcal{T}_N)$.

The family $\mathcal{H}$ of Haken manifolds in Lemma 3.4 comes from a finite family of canonical submanifolds $\mathcal{A}$ of M after some Dehn fillings. Note that to get a family $\hat{\mathcal{A}}$ of Haken manifolds whose elements satisfy conditions (i), (ii) and (iii) one can use a construction of Rong in [16]. But this construction does not guarantee the finiteness of the family $\hat{\mathcal{A}}$ (actually the construction of Rong does not allow us to control the slopes of the Dehn fillings performed along the components of $\mathcal{A}$ to obtain $\hat{\mathcal{A}}$). Thus we have to modify this construction to avoid this problem. To this purpose we will define and construct the *maximal essential part* of M (see Section 5.3).

3.2. **Second step: Control of the sewing involutions of the targets.** In this step we complete the proof of Proposition 3.2. Thus the key result of this section is stated as follows.

Proposition 3.7. *Let M be a closed orientable 3-manifold. Let N_i be a sequence of weakly equivalent nongeometric closed Haken manifolds such that there exists a degree-one map $g_i\colon M \to N_i$ satisfying $\mathrm{Vol}(M) = \mathrm{Vol}(N_i)$. For each $i \in \mathbf{N}$, we denote by $s_i\colon \partial N_i^* \to \partial N_i^*$ the sewing involution corresponding to N_i. Then the sequence $\{(N_i^*, s_i), i \in \mathbf{N}\}$ is finite, up to equivalence of pairs.*

Throughout the proof of Proposition 3.7 we will use the collection of closed Haken manifolds $\mathcal{H}$ given by Lemma 3.4. Points (i), (ii) and (iii) say that the elements of $\mathcal{H}$ dominate the manifolds N_i in a convenient way. Roughly speaking, the core of the proof of Proposition 3.7 is to show that the sewing involution associated to each Haken manifold of $\mathcal{H}$ does fix the sewing involution s_i that produces N_i from N_i^*. Note that in this step the condition on the Gromov simplicial volume is still crucial in our proof.

3.3. **Organization of the paper.** This paper is organized as follows. Section 4 is devoted to the statement of a *mapping result* for maps from Seifert fibered spaces to Haken manifolds. This result is only of technical interest and will be used in Sections 5 and 6. Section 5 is devoted to the proof of Proposition 3.3 and in Section 6 we prove Proposition 3.7 to complete the proof of Proposition 3.2. Section 7 is devoted to the proof of Theorem 1.1, which is a consequence of Theorems 1.2 and 1.3.

4. On the characteristic pair theorem of W. Jaco and P. Shalen

We start by recalling a main consequence of the Characteristic Pair Theorem of W. Jaco and P. Shalen (see [7, Chapter V]) which allows one to control a nondegenerate map from a Seifert fibered space into a Haken manifold. We first give the definition of degenerate maps in the sense of W. Jaco and P. Shalen.

Definition 4.1. Let (S, F) be a connected Seifert pair, and let (N, T) be a connected 3-manifold pair. A map $f\colon (S, F) \to (N, T)$ is said to be *degenerate* if either

(0) the map f is inessential as a map of pairs, or
(1) the group $\mathrm{Im}(f_*\colon \pi_1 S \to \pi_1 N) = \{1\}$, or
(2) the group $\mathrm{Im}(f_*\colon \pi_1 S \to \pi_1 N)$ is cyclic and $F = \emptyset$, or
(3) the map $f|\gamma$ is homotopic in N to a constant map for some fiber γ of (S, F).

Then the Characteristic Pair Theorem of Jaco and Shalen implies the following result.

Theorem 4.2 (Jaco, Shalen)**.** *If f is a nondegenerate map of a Seifert pair $(S, \emptyset)$ into a Haken manifold pair $(N, \emptyset)$, then there exists a map f_1 of S into N, homotopic to f, such that $f_1(S) \subset \mathrm{int}(\Sigma(N))$.*

The purpose of this section is to give a kind of *mapping lemma* for a certain class of degenerate maps. More precisely we show here the following result, which will be used in the proof of Theorem 1.2.

Lemma 4.3. *Let $f\colon M \to N$ be a map between closed Haken manifolds and suppose that N is nongeometric and contains no embedded Klein bottles. Let S and S' be two components of $\mathcal{S}(M)$ which are adjacent in M along a subfamily $\mathcal{T}$ of $\mathcal{T}_M$. Assume that S and S' satisfy the following hypothesis:*

(i) $f(S') \subset \mathrm{int}(B')$, where B' is a component of $\Sigma(N)$, and
(ii) $f_*(t_S) \neq 1$, where t_S denotes the homotopy class of the regular fiber of S.

Then there exists a component B of $\Sigma(N)$, with regular fiber h, and a homotopy $(f_t)_{0\leq t\leq 1}$ which is constant outside of a regular neighborhood of S such that $f_0 = f$ and $f_1(S) \subset \mathrm{int}(B)$. Moreover if $(f_1)_*(t_S)$ is not conjugate to a nontrivial power of h, then one can choose $B = B'$ and thus $f_1(S \cup_{\mathcal{T}} S') \subset \mathrm{int}(B')$. See Figure 1.

Proof. Let T be a canonical torus of M such that $T \in \partial S \cap \partial S'$ and denote by t_S the regular fiber of S represented in T. It follows from the hypothesis of the lemma that there exists a Seifert piece B' of $\Sigma(N)$ such that $f(S') \subset B'$ and thus $f_*(t_S) \in \pi_1 B' \setminus \{1\}$. Fix a base point x in T in such a way that the groups $\pi_1 S$ and $\pi_1 S'$ are always considered with base point x and denote by $y = f(x)$ a base point in B'.

Case 1. If $f_*(\pi_1 S)$ is nonabelian, since $f_*(t_S) \neq \{1\}$, then $f|S\colon S \to N$ is a nondegenerate map. Hence the Characteristic Pair Theorem implies that there exists $B \in \Sigma(N)$ such that $f(S) \subset \mathrm{int}(B)$. Moreover since $f_*(\pi_1 S)$ is nonabelian, then $f_*(t_S)$ has nonabelian centralizer and [7, Addendum to Theorem VI.I.6] implies that $f_*(t_S) \in \langle h\rangle$, where h denotes the regular fiber of B. This proves the lemma when $f_*(\pi_1 S)$ is nonabelian.

Assume that $f_*(\pi_1 S)$ is abelian. Since $\pi_1 N$ is torsion free, and since N is an aspherical 3-manifold, then the subgroup $f_*(\pi_1 S)$ of $\pi_1 N$ must have cohomological

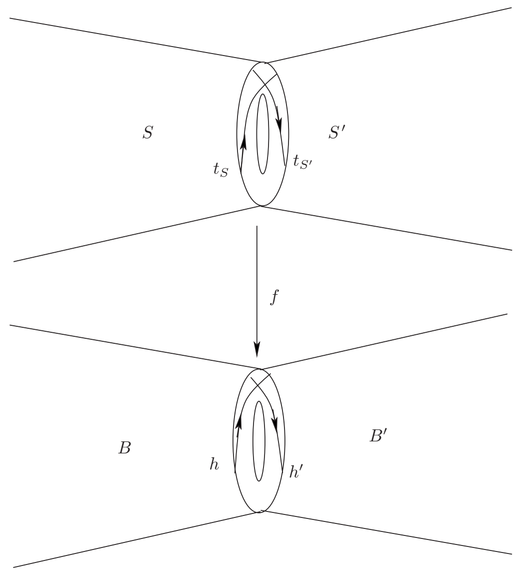

FIGURE 1.

dimension at most 3 and thus it is isomorphic to either $\mathbf{Z}$ or $\mathbf{Z}\times\mathbf{Z}$ or $\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$. The case $\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$ is excluded since N is a nongeometric closed Haken manifold.

Case 2. Thus assume first that $f_*(\pi_1 S)\simeq \mathbf{Z}\times\mathbf{Z}$. In this case $f|S\colon S\to N$ is still a nondegenerate map and the Characteristic Pair Theorem implies that there exists a component $B\in\Sigma(N)$, with regular fiber h, adjacent to B' in N such that $f(S)\subset \mathrm{int}(B)$, after a homotopy on f. Suppose that $f_*(t_S)\notin\langle h\rangle$. Thus by [7, Addendum to Theorem VI.I.6] we know that the centralizer $\mathcal{Z}_{|\pi_1(B,y)}(f_*(t_S))$ of $f_*(t_S)$ in $\pi_1(B,y)$ is necessarily abelian. Let c be an element of $\pi_1 S$. Then $f_*(c)\in\mathcal{Z}_{|\pi_1(B,y)}(f_*(t_S))$. Denote by h' the regular fiber of B' represented in a component of $B\cap B'$ in such a way that

$$h'\in\pi_1(B,y)\cap\pi_1(B',y).$$

Since $f_*(t_S)\in\pi_1(B',y)\cap\pi_1(B,y)$ (recall that $t_S\in\pi_1(S,x)\cap\pi_1(S',x)$) then h' commutes with $f_*(t_S)$, and since $h'\in\pi_1(B',y)\cap\pi_1(B,y)$ then $h'\in\mathcal{Z}_{|\pi_1(B,y)}(f_*(t_S))$. Thus, since $\mathcal{Z}_{|\pi_1(B,y)}(f_*(t_S))$ is abelian this implies that $f_*(c)\in Z(h')$. Since c is an arbitrary element in $\pi_1 S$ then $f_*(\pi_1 S)\subset Z(h')$. This implies that $f_*(\pi_1 S)$ is conjugate to a subgroup of $\pi_1(B',y)$. Then after a homotopy on f we may assume that $f(S)\subset\mathrm{int}(B')$. This prove the lemma when $f_*(\pi_1 S)\simeq\mathbf{Z}\times\mathbf{Z}$.

Case 3. Assume now that $f_*(\pi_1 S)\simeq\mathbf{Z}$. Then there exists an element $c\in\pi_1 S$ such that $f_*(\pi_1 S)=\langle f_*(c)\rangle$ and in particular there exists $n\in\mathbf{Z}^*$ such that $f_*(t_S)=(f_*(c))^n$. In the following $[a,b]$ denotes the commutator $aba^{-1}b^{-1}$ of a and b. Since in this case the Characteristic Pair Theorem does not apply, since $f|S\colon S\to N$ is a degenerate map, we first prove that there exists $B\in\mathcal{S}(N)$ such that $f(S)\subset\mathrm{int}(B)$, after a homotopy on f.

Subcase 3.1. Assume that $[f_*(c),h']=1$. In this case $f_*(c)$, and hence $f_*(\pi_1 S)$, is in the centralizer of h' and thus one can deform f on a regular neighborhood of S such that $f(S)\subset\mathrm{int}(B')$.

Subcase 3.2. Assume that $[f_*(c), h'] \neq 1$. Since $f_*(c)$ and h' are in the centralizer $Z(f_*(t_S))$ of $f_*(t_S)$ then the group $Z(f_*(t_S))$ is nonabelian. Then by [7, Addendum to Theorem VI.I.6] we know that $f_*(t_S)$ is conjugate to a power of the regular fiber h of a Seifert piece B of $\mathcal{S}(N)$. Thus one can deform f on a regular neighborhood of S such that $f(S) \subset \text{int}(B)$. Note that since a power of $f_*(c)$ lies in $\langle h \rangle$ then by [7, Lemma II.4.2], $f_*(c) = c_i^{\alpha_i}$, where c_i denotes the homotopy class of a fiber in B and $\alpha_i \in \mathbf{Z}^*$.

To complete the proof of the lemma in Case 3 it is sufficient to apply the same argument as in Case 2. $\square$

5. Control of the geometric pieces of the targets

This section is devoted to the proof of Proposition 3.3. To this purpose we first give a proof of Lemma 3.4. Let M be a closed orientable 3-manifold and let $f\colon M \to N$ be a nonzero degree map into a closed nongeometric Haken manifold which contains no embedded Klein Bottles such that $\text{Vol}(M) = \deg(f)\text{Vol}(N)$. First we claim that to prove Lemma 3.4 there is no loss of generality assuming that M is a closed Haken manifold. Indeed, consider the Milnor decomposition of M into prime manifolds $M = M_1 \sharp ... \sharp M_k$ (see [12]). Since $\pi_2(N)$ is trivial, there exists, for each $i \in \{1, ..., k\}$, a map $f_i\colon M_i \to N$ such that $\deg(f_1) + ... + \deg(f_k) = \deg(f)$. Note that when $\deg(f_i) \neq 0$, then M_i is necessarily a closed Haken manifold. On the other hand if $\text{Vol}(M) = \deg(f)\text{Vol}(N)$, then the Cutting-off Theorem of M. Gromov, [4], implies that there exists $i \in \{1, ..., k\}$ such that $f_i\colon M_i \to N$ has nonzero degree and satisfies $\text{Vol}(M_i) = \deg(f_i)\text{Vol}(N)$. Then from now on we assume that M is a closed Haken manifold.

5.1. **A convenient alternative to Lemma 3.4.**

5.1.1. *Sections of Seifert fibered spaces.* Let S be an orientable Seifert fibered space. Fix a Seifert fibration for S, denote by B its 2-orbifold basis and by $\eta\colon S \to B$ the canonical projection map. If S has exceptional fibers $C_1, ..., C_r$, let $D_1, ..., D_r$ be pairwise disjoint 2-cell neighborhoods of $\eta(C_1), ..., \eta(C_r)$ in $\text{int}(B)$. Let $B' = B \setminus \bigcup_i \text{int}(D_i)$ and $S' = \eta^{-1}(B')$. Then $\eta|S'\colon S' \to B'$ is the orientable circle bundle over B'. Choose a cross section $s_0\colon B' \to S'$ of the circle bundle. We may choose standard generators of $\partial S'$, with respect to this choice of a cross section, in the following way. Denote $\partial S' = \partial S \cup U_1 \cup ... \cup U_r$, where $U_j = \partial \eta^{-1}(D_j)$. Then for each component U_j (resp. T_i of ∂S) we choose generators t, q_j (resp. t, $\delta(S, T_i)$) where t is represented by a regular fiber and q_j (resp. $\delta(S, T_i)$) is the boundary curve of the cross section s_0 in U_j (resp. in T_i). In the following the curve $\delta(S, T_i)$ will be termed a section of T_i (with respect to the fixed Seifert fibration of S). Notice that if we replace the section s_0 by another one $s\colon B' \to S'$, then the section $\delta(S, T_i)$ of T_i is replaced by $\delta(S, T_i)t^m$, $m \in \mathbf{Z}$.

5.1.2. *Dehn fillings.* Let Q be a compact oriented three-manifold whose boundary is composed of tori $T_1, ..., T_k$. For each $i = 1, ..., k$ we fix generators l_i, m_i of $\pi_1 T_i$. Let $\mathcal{P}_*$ be the subset of $\mathbf{S}^2 = \mathbf{C} \cup \{\infty\}$ defined by

$$\mathcal{P}_* = \{(p, q) \in \mathbf{Z} \times \mathbf{Z}, \gcd(p, q) = 1\} \cup \{\infty\},$$

where $\gcd(p, q)$ denotes the greatest common divisor of p and q. We will denote by $Q_{d_1, ..., d_k}$ the 3-manifold obtained from Q by gluing to each T_i, $i = 1, ..., k$, a solid torus $S^1 \times D^2$ identifying a meridian $m = \{z_0\} \times \partial D^2$ with $p_i l_i + q_i m_i$ when

$d_i = (p_i, q_i) \in \mathcal{P}_* \setminus \{\infty\}$. When $d_i = \infty$ the torus T_i is cut out. On the other hand recall that the manifolds obtained in this way depend, up to diffeomorphism, only on the pair of integers (p_i, q_i) with $\gcd(p_i, q_i) = 1$. Let M be closed Haken manifold. From now on we adopt the following convention.

For each T in $\partial\mathcal{S}(M)$ we fix a Seifert fibered space S adjacent to T and a basis (h_T, δ_T) of $\pi_1(T)$, where h_T corresponds to the generic fiber $h(S)$ of S and δ_T is a section $\delta(S,T)$ of T with respect to the Seifert fibration of S as defined in paragraph 5.1.1. If S is adjacent to a Seifert fibered space S' along T we denote by $(h(S'), \delta(S', T))$ another basis for $\pi_1 T$ with respect to S' in the same way as for S. We denote by $d_T = (a_T, b_T)$ the element of $\mathcal{P}_*$ such that $h(S') = a_T h_T + b_T \delta_T$. Note that $b_T \neq 0$ by the minimality property of $\mathcal{T}_M$. Denote by $\mathcal{P}_*^0$ the finite subset of $\mathcal{P}_*$ defined by

$$\mathcal{P}_*^0 = \{(a_T, b_T), T \in \partial\mathcal{S}(M) \setminus \partial\mathcal{S}(M) \cap \partial\mathcal{H}(M), (1,0)\}.$$

Then to prove Lemma 3.4 it is sufficient to state the following result.

Lemma 5.1. *Let M be a closed Haken manifold and let N be a closed nongeometric Haken manifold that contains no embedded Klein bottles. If $f\colon M \to N$ denotes a nonzero degree map satisfying $\mathrm{Vol}(M) = \deg(f)\mathrm{Vol}(N)$, then there exists a canonical submanifold G_N of M whose boundary is composed of some components of $\partial\mathcal{S}(M) \setminus \partial\mathcal{S}(M) \cap \partial\mathcal{H}(M)$ and such that if $T_1, ..., T_k$ denote the components of ∂G_N, then there exist $d_1, ..., d_k$ in $\mathcal{P}_*^0$ satisfying the following properties:*

(a) $(G_N)_{d_1,...,d_k}$ is a closed Haken manifold, and

(b) there exists a map $g\colon (G_N)_{d_1,...,d_k} \to N$ with the same degree as f satisfying points (i), (ii) and (iii) of Lemma 3.4.

5.2. **Nonzero degree maps preserving the Seifert part of the domain.** In this section we prove that Lemma 5.1 is true for nonzero degree maps $f : M \to N$ such that $\mathrm{Vol}(M) = \deg(f)\mathrm{Vol}(N)$ and satisfying $f(\mathcal{S}(M)) \subset \mathrm{int}(\Sigma(N))$.

Lemma 5.2. *Let $f : M \to N$ be a nonzero degree map between nongeometric Haken manifolds such that $\mathrm{Vol}(M) = \deg(f)\mathrm{Vol}(N)$. If $f(\mathcal{S}(M)) \subset \mathrm{int}(\Sigma(N))$, then there exists a map homotopic to f which satisfies the conclusion of Lemma* 3.4.

Proof. First of all note that using the construction of T. Soma in [17] one can modify f by a homotopy fixing $f|\mathcal{S}(M)$ in such a way that $f(\mathcal{H}(M), \partial\mathcal{H}(M)) \subset (\mathcal{H}(N), \partial\mathcal{H}(N))$ and $f|\mathcal{H}(M) : \mathcal{H}(M) \to \mathcal{H}(N)$ is a $\deg(f)$-fold covering.

Let $T \in \mathcal{T}_N$. Using standard cut-and-paste arguments and the fact that $\partial\mathcal{S}(M)$ and $\partial\mathcal{H}(M)$ are incompressible we can modify f by a homotopy fixing $f|\mathcal{S}(M) \cup \mathcal{H}(M)$, so that $f^{-1}(T)$ is a collection of 2-sided incompressible surfaces in M. Since $f^{-1}(T) \subset M \setminus (\mathcal{S}(M) \cup \mathcal{H}(M))$ it must be a union of parallel copies of some tori in $\mathcal{T}_M \times (-1, 1)$. We can arrange f in its homotopy class so that for any $U \in \mathcal{T}_M$, a regular neighborhood $U \times [-1, 1]$ of U contains at most one component of $f^{-1}(T)$. Indeed, suppose that X and X' are two consecutive components of $f^{-1}(T) \cap (U \times [-1, 1])$. Then X and X' bound a region Q in $U \times [-1, 1]$ which is homeomorphic to $\S^1 \times \S^1 \times I$ and there is a Seifert piece B in $\mathcal{S}(N)$ so that $f(Q, \partial Q) \subset (B, \partial B)$. Then by [16, Lemma 2.8], $f|Q$ is homotopic, mod ∂Q, to a map f_1 such that $f_1(Q) \subset \partial B$, unless $B \simeq \S^1 \times \S^1 \times I$ which is excluded since N is not a geometric 3-manifold. So we can eliminate X and X' by pushing Q into

$N-B$. After repeating this operation a finite number of times we may assume that $f^{-1}(T)\cap(U\times[-1,1])$ has at most one component.

Note that since $f: M\to N$ is a nonzero degree map then $f_*(\pi_1 M)$ has finite index in $\pi_1 N$ and thus for any S in $\mathcal{S}(N)$ there exists at least one component of $\mathcal{S}(M)$ which is sent into $\mathrm{int}(S)$ via f. So $f^{-1}(S)$ consists of some components of $\mathcal{S}(M)$ union some $T\times[-1,1]$ for T in $\mathcal{T}_M$ (precisely when $f^{-1}(\mathcal{T}_N)\cap(T\times[-1,1])=\emptyset$). So each component of $f^{-1}(S)$ is a canonical graph submanifold of M. This proves Lemma 5.2. □

5.3. **Proof of Lemma 5.1: the general case.** We first realize a kind of *factorization* on the map f which is inspired from a construction of Y. Rong in [16] to have a reduction to the case of Lemma 5.2. If S is a component of $\mathcal{S}(M)$, we denote by t_S the homotopy class of the regular fiber in S. Let B_0 be the union of all S in $\mathcal{S}(M)$ such that $f|S$ is degenerate in the sense of Definition 4.1. If $f|S$ is a degenerate map, then one of the following cases holds.

Case 1: $f_*(\pi_1 S)=\{1\}$.
Case 2: $f_*(\pi_1 S)=\mathbf{Z}$.
Case 3: Since $\pi_1(N)$ is torsion free, $(f|S)_*:\pi_1 S\to\pi_1 N$ factors through $\pi_1 V$, where V is the base 2-manifold of the Seifert fibered space S.

Set $G_0=\overline{M-B_0}$. Define a subset of B_0 by setting

$$\mathcal{S}_0=\{S\in B_0\setminus(B_0\cap\mathcal{T}_M)\text{ s.t. } S \text{ is adjacent to } G_0 \text{ and } f_*(t_S)\neq 1\},$$

and set $B_1=B_0-\mathcal{S}_0$ and $G_1=\overline{M-B_1}$. We continue this process by setting

$$\mathcal{S}_1=\{S\in B_1\setminus(B_1\cap\mathcal{T}_M)\text{ s.t. } S \text{ is adjacent to } G_1 \text{ and } f_*(t_S)\neq 1\}$$

to construct an increasing sequence $G_0\subset G_1\subset\ldots\subset G_i\subset G_{i+1}\subset\ldots$ of canonical submanifolds of M. We claim that this sequence satisfies the following conditions:

(1) the number of connected components n_i of G_i satisfies $n_{i+1}\leq n_i$,
(2) for any i, $\mathrm{int}(G_i)$ contains $\mathcal{H}(M)$ and $f|\partial\mathcal{H}(M)\colon\ \partial\mathcal{H}(M)\to N$ is a nondegenerate (i.e. π_1-injective) map,
(3) for any i there exists a nonzero degree map $\beta_i:\hat{G}_i\to N$ such that $\deg(\beta_i)=\deg(f)$, where $\hat{G}_i$ denotes the space obtained from G_i after performing some Dehn fillings along the components of ∂G_i.

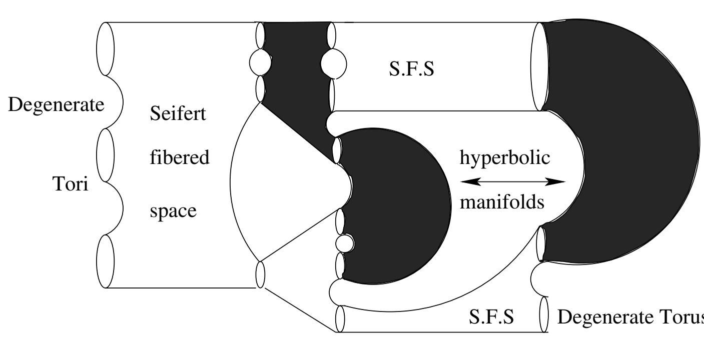

FIGURE 2. Essential submanifold G_0 of M with respect to $f: M\to N$

For this reason G_i is called an *essential part* of M with respect to f (see Figure 2). We define an integer n_0 by setting:

$$n_0 = \min\{n \geq 0 \text{ such that } G_n = G_{n+1}\}.$$

We prove point (3) for G_{n_0}, which will be termed a *maximal essential part* of M. The proof for the $G_i \subset G_{n_0}$ works in the same way and the proof of points (1) and (2) follows directly from the construction and from the equality $\mathrm{Vol}(M) = \deg(f)\mathrm{Vol}(N)$ (see [17, Lemma 2]). Denote by $B_{\mathbf{Z}}$ the subset of $G_{n_0} \setminus (G_{n_0} \cap \mathcal{T}_M)$ that consists of the Seifert fibered spaces (S.F.S.) that are degenerate under f. Note that it follows from the construction that for any S in $B_{\mathbf{Z}}$, $f_*(\pi_1 S)$ is necessarily infinite cyclic. Set $B = B_0 - B_{\mathbf{Z}}$. We have $G_{n_0} = \overline{M - B}$.

Let Q be a geometric piece in G_{n_0} such that $\partial Q \cap \partial G_{n_0} \neq \emptyset$. Then it follows from the construction that Q is a Seifert fibered space and it is adjacent along each component of $\partial Q \cap \partial G_{n_0}$ to a degenerate Seifert piece in M whose fibers are sent trivially into $\pi_1 N$. For any S in B, define a group π_S to be one of the following:

Case 1: $\{1\}$;
Case 2: $\mathbf{Z}$;
Case 3: $\pi_1 V$.

Define a three-dimensional space $D_S = K(\pi_S, 1)$. Since D_S and N are both $K(\pi, 1)$ there exist maps $\alpha : S \to D_S$ and $\beta : D_S \to N$ such that $f|S$ is homotopic to $\beta \circ \alpha$ and satisfies the following convenient conditions: for each $T \subset \partial S$, let $\{\lambda, \mu\}$ be a base of $\pi_1 T$ with $\alpha_*(\lambda) = 1$. Note that it follows from the construction that for any T in $\partial B = \partial G_{n_0}$, $\lambda = h_S(T)$, where S is the Seifert fibered manifold of B containing T in its boundary and where $h_S(T)$ denotes the regular fiber of S represented in T. Parametrize T by $T = \S^1 \times \S^1$ with $[\S^1 \times *] = \lambda$ and $[* \times \S^1] = \mu$. Then $\alpha(x, y) = \alpha_1(y)$ for some embedding $\alpha_1 : \S^1 \to D_S$. Denote the knot $\alpha_1(\S^1)$ by l_T. We may also assume that $l_{T_1} \cap l_{T_2} = \emptyset$ for different components T_1 and T_2 of ∂S. We extend the homotopy on $f|S$ over M, we replace f by the new map and we do this for each component S of B. Set $D_B = \bigcup_{S \in B} D_S$. Then the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "f|B"] \arrow[d, "\alpha"'] & N \\ D_B \arrow[ur, "\beta"'] & \end{tikzcd}$$

Let $\hat{G}_{n_0}$ be the closed 3-manifold obtained from G_{n_0} by attaching a solid torus V_T to G_{n_0} along each component T of $\partial G_{n_0} = \partial B$ so that the meridian of V_T is identified with the curve λ defined above. Let l'_T be the core of V which has the same orientation as μ. Let $X = D_B \cup_\tau \hat{G}_{n_0}$ where τ identifies each l_T in D_B with l'_T in $\hat{G}_{n_0}$ (preserving orientation). Define $\alpha : G_{n_0} \to \hat{G}_{n_0} = G_{n_0} \cup \left(\bigcup_{T \in \partial G_{n_0}} V_T\right)$ to be the map such that

$$\alpha|G_{n_0} \setminus \partial G_{n_0} : G_{n_0} \setminus \partial G_{n_0} \to \hat{G}_{n_0} \setminus \bigcup l'_T$$

is a homeomorphism and each $T \subset \partial G_{n_0}$ is sent onto l'_T. Now the map $\alpha : M = B \cup G_{n_0} \to X$ is a well-defined continuous map. Since $\alpha|G_{n_0} \setminus \partial G_{n_0} : G_{n_0} \setminus \partial G_{n_0} \to \hat{G}_{n_0} \setminus \bigcup l'_T$ is a homeomorphism we can define $\beta|\hat{G}_{n_0} \setminus \bigcup l'_T = f \circ \alpha^{-1}_{|\hat{G}_{n_0} \setminus \bigcup l'_T}$. So we get a map $\beta : X \to N$ such that the following diagram commutes:

$$\begin{tikzcd} M = B \cup G_{n_0} \arrow[r, "f"] \arrow[d, "\alpha"'] & N \\ X = D_B \cup_\tau \hat{G}_{n_0} \arrow[ur, "\beta"'] & \end{tikzcd}$$

More precisely let $T_1, ..., T_l$ be the components of $\partial G_{n_0} = \partial B$ and let $S_1, ..., S_l$ (resp. $B_1, ..., B_l$) be the Seifert pieces (not necessarily pairwise distinct) in G_{n_0} (resp. in B) such that for each $i = 1, ..., l$, B_i and S_i are adjacent along T_i. Denote by

$$(h(B_i), \delta(B_i, T_i)) \quad \text{resp.} \quad (h(S_i), \delta(S_i, T_i))$$

a system of generators of $\pi_1 T_i$, where $h(B_i)$ (resp. $h(S_i)$) denotes the generic fiber of B_i (resp. S_i) represented in T_i and $\delta(B_i, T_i)$ (resp. $\delta(S_i, T_i)$) is a section of T_i (with respect to B_i, resp. S_i) as defined in Section 4.1.1. We know from the construction that $f_*(h(B_i)) = 1$. Let $(a_{T_i}(S_i), b_{T_i}(S_i))$ denote the element of $\mathcal{P}_*$ such that

$$h(B_i) = a_{T_i}(S_i).h(S_i) + b_{T_i}(S_i).\delta(S_i, T_i).$$

If

$$(h_{T_i}, \delta_{T_i}) = (h(B_i), \delta(B_i, T_i)),$$

we set $d_i = (1, 0) \in \mathcal{P}_*^0$ and if

$$(h_{T_i}, \delta_{T_i}) = (h(S_i), \delta(S_i, T_i)),$$

then $(a_{T_i}(S_i), b_{T_i}(S_i)) = (a_{T_i}, b_{T_i}) \in \mathcal{P}_*^0$ and we set $d_i = (a_{T_i}, b_{T_i}) \in \mathcal{P}_*^0$ (see paragraph 4.1.2 for the notation). Thus we get $\hat{G}_{n_0} = (G_{n_0})_{d_1,...,d_l}$. See Figure 3.

Denote by f_1 the map $\beta \circ i\colon \hat{G}_{n_0} \to N$, where $i\colon \hat{G}_{n_0} \to X$ is the inclusion. Note that since $H_3(D_B) = 0$ then a Mayer-Vietoris argument shows that f_1 is a nonzero degree map equal to $\deg(f)$.

Remark 5.3. Let $G^1, ..., G^m$ be the components of $\hat{G}_{n_0}$. Up to re-indexing we may assume that there exists $1 \le u \le m$ such that $\deg(f_1|G^i) \ne 0$ for $i = 1, ..., u$ and $\deg(f_1|G^i) = 0$ for $i = u+1, ..., m$. Set $\hat{G} = G^1 \cup ... \cup G^u$. There is no loss of generality in assuming that $\hat{G}_{n_0} = \hat{G}$.

Thus to complete the proof of Lemma 5.1 it remains to check, in view of Lemma 5.2, the following claim.

Claim 5.4. *The space $\hat{G}_{n_0}$ is a Haken manifold and the map $f_1 : \hat{G}_{n_0} \to N$ satisfies $\deg(f_1) = \deg(f)$ and $\mathrm{Vol}(\hat{G}_{n_0}) = \deg(f_1)\mathrm{Vol}(N)$. Moreover there exists a map $g : \hat{G}_{n_0} \to N$ homotopic to f_1 such that $g(\mathcal{S}(\hat{G}_{n_0})) \subset \mathrm{int}(\Sigma(N))$.*

Proof of Claim. Each component G^i of $\hat{G}_{n_0}$ is $\hat{M}_i = (M_i)_{d_1,...,d_{j_i}}$, where M_i is a union of some hyperbolic pieces and some Seifert fibered pieces of M connected by some $T \times I$ in $\mathcal{T}_M \times I$. Note that it follows from the construction that for each i the minimal torus decomposition of M_i gives in an obvious way the minimal torus decomposition of $G^i = \hat{M}_i$ in the sense that there exists a subfamily $\mathcal{T}$ of $\mathcal{T}_M \cap \mathrm{int}(G_{n_0})$ such that $\alpha(\mathcal{T}) = \mathcal{T}_{\hat{G}_{n_0}}$ (recall that $\alpha|G_{n_0} \setminus \partial G_{n_0} : G_{n_0} \setminus \partial G_{n_0} \to \hat{G}_{n_0} \setminus \bigcup l'_T$ is a homeomorphism). We describe precisely the torus decomposition of $\hat{G}_{n_0}$.

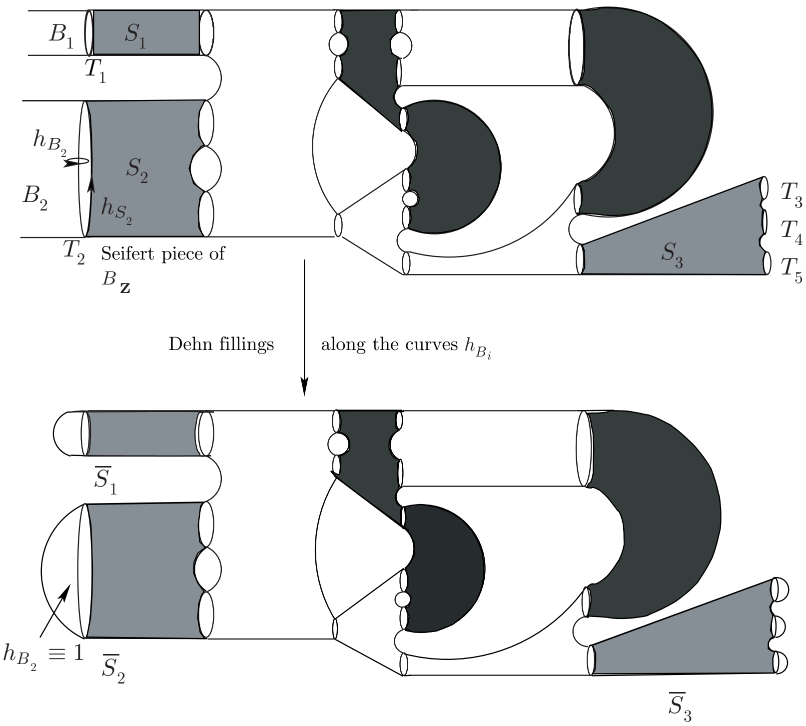

FIGURE 3. Maximal essential submanifold $G = G_{n_0}$ of M

Case 1: Let S be a component of $\mathcal{S}(M) \cap G_{n_0}$ such that $\partial S \cap \partial G_{n_0} \neq \emptyset$ and such that $f|S : S \to N$ is a nondegenerate map. Set $\hat{S} = \alpha(S)$ in $\hat{G}_{n_0}$. Then $\hat{S}$ admits a Seifert fibration which extends that of $S \subset G_{n_0}$ and $\hat{S}$ is not homeomorphic to a solid torus, $\partial\hat{S}$ (if nonempty) is incompressible and $f_1|\hat{S} : \hat{S} \to N$ is a nondegenerate map. Indeed let T be a component of $\partial S \cap \partial G_{n_0}$ and let λ be the primitive curve of T defined as before. Since λ is not a fiber of S, by the definition of nondegenerate maps, the Seifert fibration of S extends to a Seifert fibration of $S \cup_{\lambda=m} V_T$, where V_T denotes a solid torus glued along T by identifying λ with the meridian m of V_T. Now since $\pi_1(\hat{S}) = \pi_1 S/\langle\lambda\rangle$ maps onto $\pi_1 S/\ker(f_*) \simeq f_*(\pi_1 S)$, which is not cyclic by the definition of nondegenerate maps, then $\hat{S}$ is not a solid torus and so $\partial\hat{S}$ is incompressible. Moreover notice that if a torus T connects two nondegenerate Seifert pieces S_1 and S_2 in M_i, then T also connects $\alpha(S_1) = \hat{S}_1$ and $\alpha(S_2) = \hat{S}_2$ and the fibers of $\hat{S}_1$ and $\hat{S}_2$ do not match up along T and thus $T \in \mathcal{T}_{\hat{G}_{n_0}}$.

Case 2: Consider now the case of a component S of $G_{n_0} \cap B_{\mathbf{Z}}$ such that $\partial S \cap \partial G_{n_0} \neq \emptyset$ and denote by h the regular fiber of S and set $\hat{S} = \alpha(S)$. Since $f_*(h) \neq 1$ in $\pi_1 N$, then the same argument as before implies that the Seifert fibration of S extends to a Seifert fibration of $\hat{S}$. But since $f_*(\pi_1(S)) = \mathbf{Z}$, then $\hat{S}$ can be homeomorphic to a fibered solid torus.

If $\hat{S}$ is a solid torus V_T, then this means that S has a single component T in $\mathrm{int}(G_{n_0})$ and $\partial S - T$ is adjacent to Seifert fibered pieces in B. Let S' be the Seifert fibered piece in G_{n_0} which is adjacent to S along T. It follows from the construction that the regular fiber of $\hat{S}' = \alpha(S')$ represented in T is not free homotopic to the meridian of $\partial\hat{S} = T = \partial V_T$. Consider the space $\overline{S} = \hat{S}' \cup_T \hat{S}$. Thus the Seifert fibration of $\hat{S}'$ extends to a Seifert fibration of $\overline{S}$. If $\overline{S}$ is not a solid torus, then $\partial\overline{S}$ is incompressible and we have a reduction to the first case. This is true in particular when $f|S' : S' \to N$ is a nondegenerate map. If $\overline{S}$ is still a solid torus, then we reiterate the same process. This process must stop. To see this it is sufficient to check the following.

Claim 5.5. *If G^i is a component of $\hat{G}_{n_0}$ that is sent via f_1 into N with nonzero degree, then G^i contains at least one nondegenerate Seifert piece.*

Proof of Claim 5.5. Suppose that each Seifert piece of G^i is degenerate under $f_1|G^i$. Then each Seifert piece S of G^i satisfies $(f_1)_*(\pi_1 S) \simeq \mathbf{Z}$ by the construction. This implies, using condition (2), that G^i is a graph manifold and that the canonical tori of G^i are degenerate under f_1. Now, since $f_1|G^i\colon\ G^i \to N$ has nonzero degree, then using the same construction as above, one can show that there exists a Seifert fibered space $\hat{S}$, obtained from a Seifert piece S in G^i after Dehn filling, and a nonzero degree map $\hat{f}_1\colon\ \hat{S} \to N$ such that $f_1|S \simeq \hat{f}_1 \circ \alpha$, where $\alpha\colon\ S \to \hat{S}$ denotes the canonical quotient map. Since $\hat{f}_1$ has nonzero degree and since $(\hat{f}_1)_*(\pi_1\hat{S})$ is cyclic this means that $\pi_1 N$ contains a cyclic finite index subgroup. This is impossible since N is a nongeometric closed Haken manifold. □

This proves that $\hat{G}_{n_0}$ is still a Haken manifold with a torus decomposition induced from that of M. Moreover,

$$\mathrm{Vol}(\hat{G}_{n_0}) \geq \deg(f_1)\mathrm{Vol}(N) = \deg(f)\mathrm{Vol}(N) = \mathrm{Vol}(M)$$

and by condition (2),

$$\mathrm{Vol}(\hat{G}_{n_0}) = \mathrm{Vol}(\hat{G}_{n_0} \cap \mathcal{H}(M)) = \mathrm{Vol}(\mathrm{int}(\mathcal{H}(M))) = \mathrm{Vol}(M);$$

thus $\mathrm{Vol}(\hat{G}_{n_0}) = \deg(f_1)\mathrm{Vol}(N)$. Applying Theorem 4.2 to the set of nondegenerate Seifert pieces $\mathcal{S}_0(\hat{G}_{n_0})$ of $\hat{G}_{n_0}$, we may assume after a homotopy supported on a regular neighborhood of $\mathcal{S}_0(\hat{G}_{n_0})$ that $f_1(\mathcal{S}_0(\hat{G}_{n_0})) \subset \mathrm{int}(\Sigma(N))$. Let S be a degenerate Seifert piece in $\hat{G}_{n_0}$ adjacent along a canonical torus T to an element S' in $\mathcal{S}_0(\hat{G}_{n_0})$. After a homotopy on a small regular neighborhood of S we may assume, by Lemma 4.3, that $f(S) \subset \mathrm{int}(\Sigma(N))$. Since each component G^i of $\hat{G}_{n_0}$ satisfies $\deg(f_1|G^i : G^i \to N) \neq 0$ (see Remark 5.3) then it contains some nondegenerate Seifert fibered pieces and thus we may assume by repeating our argument that $f(B_{\mathbf{Z}}) \subset \mathrm{int}(\Sigma(N))$. Hence $f_1(\mathcal{S}(\hat{G}_{n_0})) \subset \mathrm{int}(\Sigma(N))$. This ends the proof of Claim 5.4 and completes the proof of Lemma 5.1. The proof of Lemma 3.4 follows directly from Lemma 5.1. □

5.4. **Proof of Proposition 3.3.** Let $(N_i)_{i\in\mathbf{N}}$ be a sequence of nongeometric closed Haken manifolds such that for each $i \in \mathbf{N}$ there exists a nonzero degree map $g_i\colon\ M \to N_i$ with $\mathrm{Vol}(M) = \deg(g_i)\mathrm{Vol}(N_i)$.

Throughout the proof of Proposition 3.3 one can assume, without loss of generality, that the targets satisfy the following condition:

(II) for any $i \in \mathbf{N}$, each Seifert piece of N_i has orientable orbifold base and admits an $\mathbf{H}^2 \times \mathbf{R}$-geometry.

It is well known that each Seifert piece S of N_i with nontrivial JSJ-decomposition admits $\mathbf{H}^2 \times \mathbf{R}$-geometry unless S is the twisted I-bundle over the Klein bottle, which is indeed also homeomorphic to the orientable $\S^1$-bundle over the Moebius band.

If S is a geometric piece of N_i with a Seifert fibration over a nonorientable orbit surface, then S has a double cover $\tilde{S}$ corresponding to the orientation double cover of its orbit surface. Note that this double cover is trivial on the boundary, and thus the components of ∂S lift to this cover. Then by taking a copy of this double cover for each component of $\mathcal{S}(N)$ that admits a Seifert fibration over a nonorientable surface, taking two copies of each component otherwise and identifying these along their torus boundary via the sewing involution between the components of $N_i \setminus \mathcal{T}_{N_i}$ (since the boundary components of each component of $N_i \setminus \mathcal{T}_{N_i}$ lift then so does the sewing involution) we obtain a double cover $p_i\colon\ \tilde{N}_i \to N_i$ satisfying condition (II) above. We have to check the following claim (the notation is the same as above).

Claim 5.6. *If the family $\{N_i, i \in \mathbf{N}\}$ is infinite, up to homeomorphism, then so is the family $\{\tilde{N}_i, i \in \mathbf{N}\}$.*

Proof. The claim is a direct corollary of a finiteness result on the conjugacy classes of finite group actions on those N_i (see [2] or [25], both using [13]). □

The index of $(g_i)_*^{-1}(\pi_1\tilde{N}_i)$ in $\pi_1 M$ is 1 or 2. Let $\tilde{M}_i$ be the finite cover of M corresponding to $(g_i)_*^{-1}(\pi_1\tilde{N}_i)$ and let $\tilde{g}_i\colon\ \tilde{M}_i \to \tilde{N}_i$ be the nonzero degree map that covers $g_i\colon\ M \to N_i$. Since any finitely presented group has only finitely many subgroups of given index then there are only finitely many homeomorphism types among $\tilde{M}_i$ when $i \in \mathbf{N}$. On the other hand, if $\deg(g_i)\mathrm{Vol}(N_i) = \mathrm{Vol}(M)$, then $\deg(\tilde{g}_i)\mathrm{Vol}(\tilde{N}_i) = \mathrm{Vol}(\tilde{M}_i)$. Then there is no loss of generality in assuming that the targets satisfy condition (II).

Now since the N_i's satisfy condition (II) then the targets contain no embedded Klein bottles and one can apply Lemma 3.4 to the sequence of nonzero degree maps $g_i\colon\ M \to N_i$. Hence, possibly after passing to a subsequence, we can assume that there exists a closed Haken manifold M_1 which admits nonzero degree maps $f_i\colon\ M_1 \to N_i$ satisfying properties (i), (ii) and (iii) of Lemma 3.4 for $i \in \mathbf{N}$.

Note that the Haken number of the N_i's is bounded by that of M_1 and then the number of connected components of $(N_i^*, \mathcal{T}_{N_i})$ is bounded by a constant which only depends on M_1. Then combining Corollary 1.4 with point (iii) of Lemma 3.4 we conclude that there are at most finitely many topological types for N_i^* when $i \in \mathbf{N}$. This completes the proof of Proposition 3.3.

6. Control of the sewing involutions of the targets

6.1. **Statement of the Key Result for the proof of Proposition 3.7.** The purpose of this section is to complete the proof of Proposition 3.2. To do this it remains to prove Proposition 3.7. Let $(N_i)_{i\in\mathbf{N}}$ be a sequence of weakly equivalent nongeometric closed Haken manifolds such that for each $i \in \mathbf{N}$ there exists a degree-one map $g_i\colon\ M \to N_i$ with $\mathrm{Vol}(M) = \deg(g_i)\mathrm{Vol}(N_i)$. As in paragraph 5.4 one can assume that the N_i's satisfy condition (II). Possibly after passing to a subsequence, one can assume, using Lemma 3.4, that there exists a closed Haken manifold M_1 and a degree-one map $f_i\colon\ M_1 \to N_i$ satisfying the following properties:

(i) $\mathrm{Vol}(M_1) = \deg(f_i)\mathrm{Vol}(N_i)$,

(ii) the map f_i induces a homeomorphism between $\mathcal{H}(M_1)$ and $\mathcal{H}(N_i)$,

(iii) for any Q in N_i^* each component of $(f_i)^{-1}(Q)$ is a canonical submanifold of M_1.

Remark 6.1. For convenience one requires the following additional condition for point (iii) of Lemma 3.4. Over all maps homotopic to $f_i\colon M_1 \to N_i$ satisfying point (iii) we choose always the maps such that the number of connected components of $(f_i)^{-1}(N_i^*)$ is minimal. This minimality condition implies the following properties:

Let T_i be a canonical torus in N_i and let A_i and A_i' denote the geometric pieces of N_i adjacent to the sides of T_i (A_i and A_i' are not necessarily distinct). Let G_i be a component of $(f_i)^{-1}(A_i)$ and let $\mathcal{T}_i = (f_i|G_i)^{-1}(T_i)$. Then each geometric piece of $M_1 \setminus G_i$ adjacent to G_i along some component of $\mathcal{T}_i$ is sent onto A_i' by f_i.

On the other hand, since N_i is nongeometric this condition together with Lemma 4.3 implies that if B_i is a component of $\mathcal{S}(N_i)$ and if W_i is a component of $(f_i)^{-1}(B_i)$, then W_i contains at least one geometric piece Q_i such that $(f_i)_*(t_{Q_i}) \in \langle h_{B_i}\rangle$, where t_{Q_i} (resp. h_{B_i}) denotes the regular fiber in Q_i (resp. in B_i).

A map between closed Haken manifolds satisfying points (i), (ii), (iii) and the minimality property of Remark 6.1 will be termed in *standard form*.

Denote by $Q_1, ..., Q_l$ the component of the N_i^*'s and by $T_1^k, ..., T_{n_k}^k$ the boundary components of Q_k, for $k = 1, ..., l$. Each sewing involution s_i, $i \in \mathbf{N}$, induces a fixed point free bijection denoted by s_i^* on the set $\{T_1^k, ..., T_{n_k}^k, 1 \leq k \leq l\}$ by setting

$$s_i^*\colon\ (v,\nu) \mapsto (w,\mu) \ \text{ if } \ T_\nu^v \ \text{ is identified with } \ T_\mu^w.$$

Thus, passing to a subsequence we may assume that

(III) for any i, j in $\mathbf{N}$, $s_i^* = s_j^*$.

Moreover, throughout the proof of Proposition 3.7, we claim that there is no loss of generality in assuming that the targets satisfy the following condition:

(IV) any connected component of N_i^*, $i \in \mathbf{N}$, has at least two boundary components.

Condition (IV) comes from the following result.

Lemma 6.2. *Let $\{N_i\}_{i\in\mathbf{N}}$ be a sequence of weakly equivalent nongeometric Haken manifolds satisfying conditions (II) and (III). Then there exists an integer $d > 0$ such that for any $i \in \mathbf{N}$ there exists a finite regular α_i-fold covering $p_i\colon \tilde{N}_i \to N_i$ such that*

(i) each component of $\tilde{N}_i \setminus \mathcal{T}_{\tilde{N}_i}$ has at least two boundary components and $\alpha_i \leq d$,

(ii) the family $\{\tilde{N}_i, i \in \mathbf{N}\}$ is a sequence of weakly equivalent Haken manifolds satisfying conditions (II) and (III),

(iii) if the family $\{N_i, i \in \mathbf{N}\}$ is infinite, up to homeomorphism, then so is $\{\tilde{N}_i, i \in \mathbf{N}\}$.

We will use the following terminology for convenience. Let $\mathcal{T}$ be a 2-manifold whose components are all tori and let m be a positive integer. A covering space $\tilde{\mathcal{T}}$ of $\mathcal{T}$ will be termed *$m\times m$-characteristic* if each component of $\tilde{\mathcal{T}}$ is equivalent to the covering space of some component T of $\mathcal{T}$ associated to the characteristic subgroup H_m of index $m \times m$ in $\pi_1 T$ (if we identify $\pi_1 T$ with $\mathbf{Z} \times \mathbf{Z}$, then $H_m = m\mathbf{Z} \times m\mathbf{Z}$).

Proof of Lemma 6.2. Since $\{N_i\}_{i\in\mathbf{N}}$ is a sequence of weakly equivalent nongeometric closed Haken manifolds then N_i^* is homeomorphic to N_j^* for any $i, j \in \mathbf{N}$. Then we denote by $Q_1, ..., Q_l$ the components of N_i^*, $i \in \mathbf{N}$. Since each N_i satisfies condition (II) then using Theorems 2.4 or 3.2 of [10], according to whether Q_j is Seifert fibered or hyperbolic, we know that there is a prime q such that for every $j = 1, ..., l$, there is a finite regular covering $p_j\colon \tilde{Q}_j \to Q_j$ such that for any component T of ∂Q_j then $(p_j)^{-1}(T)$ consists of more than one component and for any component $\tilde{T}$ of $\partial\tilde{Q}_j$ over T then $p_j|\tilde{T}\colon \tilde{T} \to T$ is the $q \times q$-characteristic covering. Denote by η_j the degree of $p_j\colon \tilde{Q}_j \to Q_j$. Then $(p_j)^{-1}(T)$ consists of exactly $\eta_j/q^2 \geq 2$ copies of a torus. Let $m = \text{l.c.m.}(\eta_1/q^2, ..., \eta_l/q^2)$. Take $t_j = m/(\eta_j/q^2)$ copies of $\tilde{Q}_j$, $j = 1, ..., l$, and glue the component of $\coprod_{j=1,l}(\coprod_{1,t_j} \tilde{Q}_j)$ together via lifts of the sewing involution s_i of N_i in the following way: let T be a component of ∂Q_j and T' be a component of ∂Q_k such that T is identified to T' in N_i via $s_i|T\colon T \to T'$. Note that by Condition (III), the couple (T, T') does not depend on $i \in \mathbf{N}$. Let $\tilde{T}, \tilde{T}'$ be components of $(p_j)^{-1}(T), (p_k)^{-1}(T')$. Since both $p_j|\tilde{T}\colon \tilde{T} \to T$ and $p_k|\tilde{T}'\colon \tilde{T}' \to T'$ are the $q \times q$-characteristic covering then there is a sewing involution $\tilde{s}_i$ such that $\tilde{s}_i|\tilde{T}\colon \tilde{T} \to \tilde{T}'$ covers $s_i|T\colon T \to T'$. This gives an m-fold covering $\tilde{N}_i$ of N_i satisfying properties (i) and (ii). Now for each $i \in \mathbf{N}$ one can choose a finite covering $\hat{N}_i \to \tilde{N}_i$ such that the composition $P_i\colon \hat{N}_i \to \tilde{N}_i \to N_i$ is an α_i-fold regular covering of N_i with $\alpha_i \leq m!$. Using [25], this completes the proof of the lemma. □

In order to complete the proof of Proposition 3.7 we first state the following technical key result which shows that the sewing involution of the domain "fix" is, in a certain sense, the sewing involution of the targets.

Lemma 6.3 (Gluing Lemma). *Let M_1 be a closed Haken manifold and let $\{N_i, i \in \mathbf{N}\}$ be a sequence of weakly equivalent nongeometric closed Haken manifolds satisfying conditions (II), (III), and (IV) such that there exist degree-one maps $f_i\colon M_1 \to N_i$ in standard form satisfying $\mathrm{Vol}(M_1) = \deg(f_i)\mathrm{Vol}(N_i)$. Let A and B be two components of (N_i^*, s_i) such that s_i connects a component T_A of ∂A with a component T_B of ∂B for any $i \in \mathbf{N}$. Denote by T the component of $\mathcal{T}(N_i)$ obtained by identifying T_A with T_B via s_i. Then, possibly after passing to a subsequence, the following properties hold:*

(i) if A and B are both hyperbolic pieces of N_i, then the maps $\{s_i|T_A\colon T_A \to T_B\}_{i\in\mathbf{N}}$ are in the same isotopy class,

(ii) if A and B are both Seifert pieces of N_i, then there exist two elements (a, b) and (c, d) of $\mathcal{P}_*$, which depend on T, such that $bd \neq 0$ and a sequence $\{\delta_A^i\}_{i\in\mathbf{N}}$ (resp. $\{\delta_B^i\}_{i\in\mathbf{N}}$) of sections for T_A (resp. for T_B), with respect to the Seifert fibration of A (resp. of B) such that

$$(s_i)_*(h_A) = h_B^a(\delta_B^i)^b \quad \text{and} \quad (s_i)_*(h_B) = h_A^c(\delta_A^i)^d,$$

where h_A (resp. h_B) denotes the regular fiber of A (resp. of B),

(iii) if B is a Seifert piece and if A is hyperbolic, then there exists a basis (λ_A, μ_A) of $\pi_1 T_A$ and a sequence $\{\delta_i, i \in \mathbf{N}\}$ of sections of T_B with respect to the Seifert fibration of B such that

$$(s_i)_*(\lambda_A) = h_B^{\pm 1} \quad \text{and} \quad (s_i)_*(\mu_A) = \delta_i$$

for any $i \in \mathbf{N}$.

6.2. **Proof of Lemma 6.3.** Denote by T the component of $\mathcal{T}_{N_i}$ obtained by sewing A and B via $s_i|T_A\colon T_A \to T_B$.

Case 1: A and B are hyperbolic manifolds. Denote by $W(T) = T \times [-1, 1]$ a regular neighborhood of T so that $T \times \{-1\} = T_A$ and $T \times \{+1\} = T_B$. There exist two components H_A and H_B of $\mathcal{H}(M_1)$ (independant of $i \in \mathbf{N}$, possibly after passing to a subsequence) such that $f_i|H_A\colon H_A \to A$ and $f_i|H_B\colon H_B \to B$ are homeomorphisms. Possibly after passing to a subsequence, we may assume that

$(f_i)^{-1}(T_A) = \mathcal{T}_A$ and $(f_i)^{-1}(T_B) = \mathcal{T}_B$ are independant of $i \in \mathbf{N}$. It follows from the fact that $f_i^{-1}(A) = H_A$, $f_i^{-1}(B) = H_B$, $f_i|H_A \colon H_A \to A$ and $f_i|H_B \colon H_B \to B$ are homeomorphisms and from Remark 6.1 that $(f_i)^{-1}(W(T))$ is a connected characteristic graph submanifold G_T of M with exactly two boundary components U_A and U_B such that U_A is identified with $\mathcal{T}_A$ and U_B is identified with $\mathcal{T}_B$ in M. Thus after a homotopy we may assume that $f_i^{-1}(A) = H_A \cup_{\mathcal{T}_A = U_A} G_T = \overline{H}_A$ and $f_i^{-1}(B) = H_B$. Then to prove Lemma 6.3 in the first case it is sufficient to show the following result.

Claim 6.4. *Let $f_i \colon (H_A, \partial H_A) \to (A, \partial A)$ be a sequence of proper nonzero degree maps to an orientable complete finite volume hyperbolic manifold. Let T_A be a component of ∂A and assume that $f_i^{-1}(T_A) = \mathcal{T}_A$ is a connected component of ∂H_A so that $f_i|\mathcal{T}_A \colon \mathcal{T}_A \to T_A$ is a homeomorphism. Let l be a simple closed curve in $\mathcal{T}_A$ and let l_T^i be the simple closed curve in T_A such that $(f_i)_*([l]) = [l_T^i]$. Then the set of curves $\{l_T^i, i \in \mathbf{N}\}$ generates at most finitely many isotopy classes of curves in T_A. See Figure* 4.

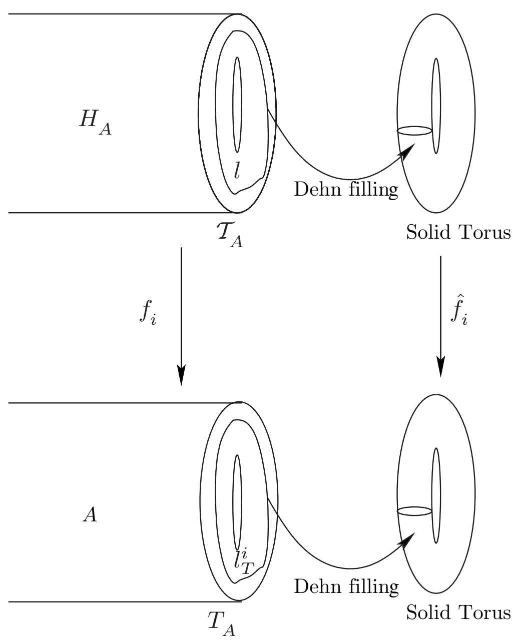

FIGURE 4.

Proof of Claim 6.4. Consider the 3-manifold A_i obtained after performing a Dehn filling on A by identifying the meridian of a solid torus with l_T^i and denote by $r_i \colon A \to A_i$ the canonical quotient map. Denote by H_A^l the 3-manifold obtained from H_A by performing a Dehn filling along $\mathcal{T}_A$ identifying the meridian of a solid torus with l. Using the Mayer–Vietoris exact sequence it is easily checked that $f_i|H_A \colon H_A \to A$ induces a proper nonzero degree map $\hat{f}_i \colon H_A^l \to A_i$ such that $\deg(\hat{f}_i) = \deg(f_i|H_A)$. Assume that the curves l_T^i generate infinitely many isotopy classes of curves in T_A. Thus when the length of the curve l_T^i is sufficiently large then the formulae established in [14] imply that

$$\mathrm{Vol}(A_i) \approx \mathrm{Vol}(A) - \pi^2 \frac{\mathcal{A}(T_A)}{\big(\mathrm{length}(l_T^i)\big)^2},$$

where $\mathcal{A}(T_A)$ denotes the area of the torus T_A with respect to the Euclidean structure induced by the complete hyperbolic structure on $\mathrm{int}(A)$ and where $\mathrm{length}(l_T^i)$ is the length of the curve l_T^i on the torus T_A with respect to this Euclidean structure. Then we may assume, passing to a subsequence, that $\{\mathrm{Vol}(A_i), i \in \mathbf{N}\}$ is a striclty increasing sequence such that

$$\lim_{i\to\infty} \mathrm{Vol}(A_i) = \mathrm{Vol}(A) \qquad (\star)$$

and that the A_i's are complete finite volume hyperbolic 3-manifolds by the Hyperbolic Surgery Theorem of W. P. Thurston, [21].

Then the latter equality implies that H_A^l dominates infinitely many hyperbolic manifolds (these manifolds can be distinguished by their volume), which contradicts Corollary 1.4.

$\square$

End of proof of Lemma 6.3*, point (i).* Let $\tilde{c} \in \pi_1 U_B$ be a primitive element. Since $f_i|\overline{H}_A\colon \overline{H}_A \to A$ induces a homeomorphism from U_B to T_A then there exists a primitive curve c_i in $\pi_1 T_A$ such that $(f_i)_*(\tilde{c}) = c_i$. Using Claim 6.4 we may assume, after passing to a subsequence, that c_i is independant of i and we denote c_i by c. The sewing involution σ_1 in M_1 identifies $\tilde{c}$ with an element $\tilde{l}$ in $\mathcal{T}_B$, where $\mathcal{T}_B$ is the component of ∂H_B such that $\mathcal{T}_B = \sigma_1(U_B)$. Using Claim 6.4 then, after passing to a subsequence, we may assume that there exists a simple closed curve l_T in T_B such that $(f_i|B)_*([\tilde{l}]) = [l_T]$ when $i \in \mathbf{N}$. This proves, possibly after passing to a subsequence, that $(s_i)_*(c) = [l_T]$ for any $i \in \mathbf{N}$, where $\tilde{c}$ is an arbitrary element of $\pi_1 U_B$. Since $f_i|U_B\colon U_B \to T_A$ is a homeomorphism, this completes the proof of Lemma 6.3, point (i). $\square$

Case 2: A and B are Seifert fibered spaces.

Fix a component Q_A^i of the preimage of A such that $f_i|Q_A^i\colon Q_A^i \to A$ has nonzero degree and let Q_B^i denote the components of the preimage of B which are adjacent to Q_A^i along $(f_i|Q_A^i)^{-1}(T_A)$. Passing to a subsequence we may assume that Q_A^i and Q_B^i are independant of i and we denote them by Q_A and Q_B. Note that, by Remark 3.5, Q_A and Q_B are graph manifolds.

6.2.1. *First Step.* For convenience, we perform the following modification on Q_A. Let S_B^j be a Seifert piece of Q_B adjacent to Q_A along a component of $(f_i|Q_A)^{-1}(T_A)$ such that $(f_i)_*(h_B^j) \notin \langle h_B\rangle$, where h_B^j denotes the regular fiber of S_B^j and h_B is the regular fiber of B. Then by Lemma 4.3 one can perturb f_i slightly by a homotopy, which is constant outside of a regular neighborhood of S_B^j, so that $Q_A^{i,j} = Q_A \cup S_B^j$ is a component of $(f_i)^{-1}(A)$. We do that for any Seifert piece S_B^j of Q_B adjacent to Q_A along a component of $(f_i|Q_A)^{-1}(T_A)$ such that $(f_i)_*(h_B^j) \notin \langle h_B\rangle$. Denote by $Q_A^{i,\mathrm{new}}$ (resp. $Q_B^{i,\mathrm{new}}$) the *new preimage* of A (resp. of B). After repeating this process a finite number of times we may assume that each Seifert piece W of $Q_B^{i,\mathrm{new}}$ adjacent to a Seifert piece of $Q_A^{i,\mathrm{new}}$ along a component of $(f_i|Q_A^{i,\mathrm{new}})^{-1}(T_A)$ satisfies $f_*(h_W) \in \langle h_B\rangle$, where h_W denotes the homotopy class of the regular

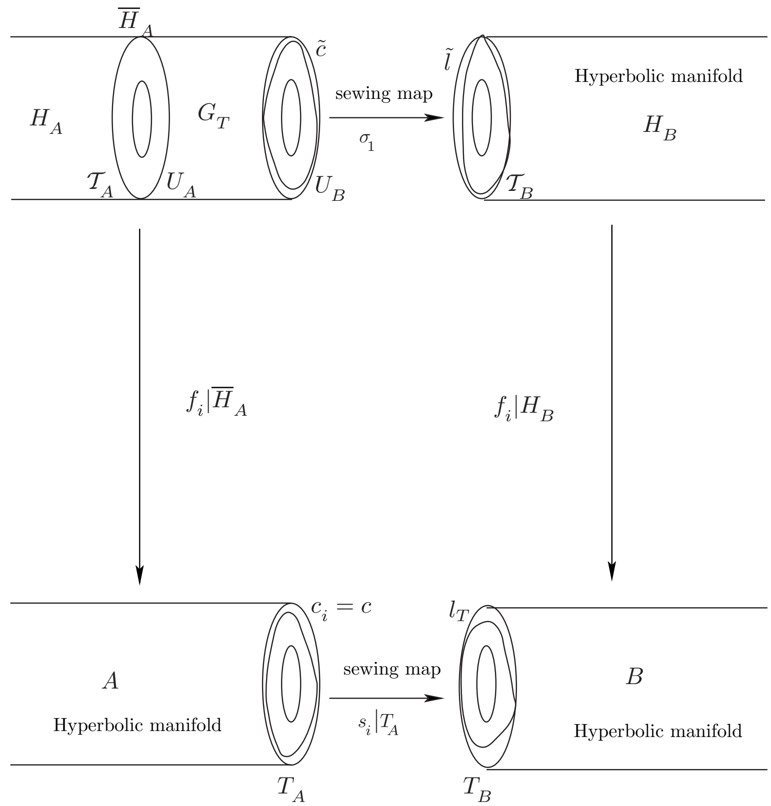

FIGURE 5.

fiber in W. Passing to a subsequence we may assume that $Q_A^{i,\text{new}}$ and $Q_B^{i,\text{new}}$ are independant of $i \in \mathbf{N}$, and we denote them by Q_A^{new} and Q_B^{new}. Note that $Q_B^{\text{new}} \neq \emptyset$ by Remark 6.1. Denote by $\mathcal{T}_A^{\text{new}} = \{U_A^1, ..., U_A^l\}$ the components of $(f_i|Q_A^{\text{new}})^{-1}(T_A)$ and by $\mathcal{T}_B^{\text{new}} = \{U_B^1, ..., U_B^l\}$ the components of $\partial Q_B^{\text{new}}$ adjacent to $\mathcal{T}_A^{\text{new}}$. For each $v \in \{1, ..., l\}$, choose a simple closed curve t_B^v in U_B^v which represents the regular fiber of the Seifert fibered space in Q_B^{new} containing U_B^v. By construction we know that $(f_i)_*([t_B^v]) \in \langle [h_B] \rangle$. Using the sewing involution σ_1 of M_1 the family of curves $\{t_B^1, ..., t_B^l\}$ defines a family of curves $\{c_A^1, ..., c_A^l\}$ in $\mathcal{T}_A^{\text{new}}$ defined by $c_A^v = \sigma_1(t_B^v)$. It follows from our construction that for any $i \in \mathbf{N}$ there exists a simple closed curve l_T^i in T_A such that $(f_i)_*([c_A^v]) \in \langle [l_T^i] \rangle$ for $v = 1, ..., l$ and $i \in \mathbf{N}$. See Figure 5.

6.2.2. *Second Step.* Consider the 3-manifold A_i obtained after performing a Dehn filling on A by identifying the meridian of a solid torus with l_T^i and denote by $r_i \colon A \to A_i$ the canonical quotient map. Denote by $\mathcal{D}(Q_A^{\text{new}})$ the 3-manifold obtained from Q_A^{new} after gluing l solid tori along $\mathcal{T}_A^{\text{new}}$ by identifying each meridian with c_A^v when $v \in \{1, ..., l\}$. This gives proper nonzero degree maps $\hat{f}_i \colon \mathcal{D}(Q_A^{\text{new}}) \to A_i$ with $\partial A_i \neq \emptyset$ and $\partial \mathcal{D}(Q_A^{\text{new}}) \neq \emptyset$ by condition (IV).

Fix a section δ_0 on T_A with respect to the Seifert fibration of A. Let (c_i, d_i) be a sequence of coprime integers such that

$$l_T^i = h_A^{c_i} \delta_0^{d_i}, \quad \text{for } i \in \mathbf{N}.$$

Note that $d_i \neq 0$ when $i \in \mathbf{N}$ by minimality of the JSJ decomposition and thus A_i is still a Seifert fibered space.

We first claim that the sequence $\{d_i, i \in \mathbf{N}\}$ is finite. Indeed, if the sequence $\{d_i, i \in \mathbf{N}\}$ is infinite, then we get infinitely many pairwise nonhomeomorphic Seifert fibered spaces $\{A_i, i \in \mathbf{N}\}$ properly dominated by $\mathcal{D}(Q_A^{\text{new}})$. These Seifert fibered spaces can be distinguished, for example, by the order of the exceptional fiber generated by performing the Dehn fillings along T_A. This gives a contradiction with Corollary 1.4 since the A_i have nonempty boundary (with geometry $\mathbf{H}^2 \times \mathbf{R}$). Thus from now on one can assume, passing to a subsequence, that d_i is a constant denoted by d.

Consider now the sequence $\{c_i, i \in \mathbf{N}\}$. We know that A_i is homeomorphic to A_j if and only if $c_i = c_j \bmod(d)$. Then using the same argument as above, we may assume, possibly after passing to a subsequence, that there exists an integer c and a sequence of integers of $\{m_i, i \in \mathbf{N}\}$ such that $c_i = c + m_i d$, for any $i \in \mathbf{N}$. Hence we get

$$l_T^i = h_A^{c+m_i d}\delta_0^d = h_A^c \delta_i^d, \quad \text{for } i \in \mathbf{N},$$

where δ_i is the section of T_A defined by $\delta_i = \delta_0 h_A^{m_i}$. Thus $(s_i)_*(h_B) = h_A^c\delta_i^d$, $i \in \mathbf{N}$. The proof of point (ii) of Lemma 6.3 follows by permuting the roles of Q_A and Q_B. Indeed it is sufficient to choose a component of $(f_i)^{-1}(B)$ which dominates B and to proceed in the same way as above.

Case 3: B is Seifert and A is a hyperbolic manifold. (See Figure 6.) Let Q_B be a component of the preimage of B such that $f_i|Q_B\colon Q_B \to B$ has nonzero degree, and let Q_A be the preimage of A which is adjacent to Q_B along $\mathcal{T}_B = (f_i|Q_B)^{-1}(T_B)$. Denote by $\mathcal{T}_A$ the components of ∂Q_A identified with $\mathcal{T}_B$ in M_1 (as in the paragraphs above we can assume that $Q_A, Q_B, \mathcal{T}_A$ and $\mathcal{T}_B$ are independant of $i \in \mathbf{N}$). Note that Q_A is a hyperbolic manifold and that $\mathcal{T}_A$ and $\mathcal{T}_B$ are connected. We now state the following.

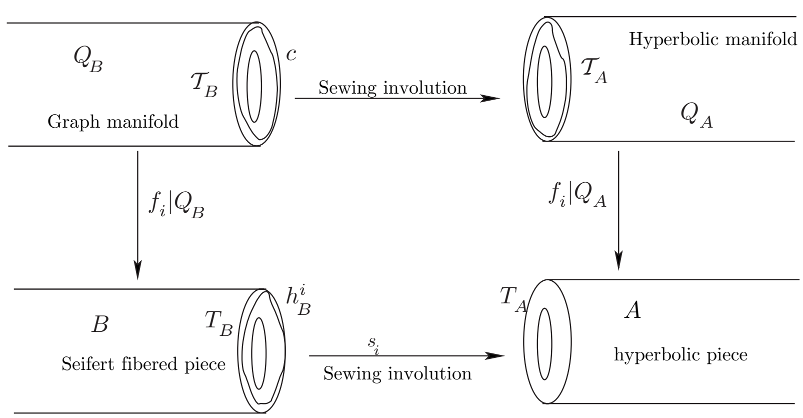

Figure 6.

Claim 6.5. *Let c be a simple closed curve in $\mathcal{T}_B$, and let h_B^i be the simple closed curve in T_B such that $(f_i|Q_B)_*([c]) = [h_B^i]$. Then, possibly after passing to a subsequence, the following properties hold: there exists an element (a, b) in $\mathcal{P}_*$ and* *a sequence of sections* $\{\delta_i, i \in \mathbf{N}\}$ *of* T_B *with respect to the Seifert fibration on* B *such that*

$$h_B^i = h_B^a \delta_i^b,\ \ i \in \mathbf{N},$$

where h_B *denotes the homotopy class of the regular fiber on* B.

Proof. The proof of this claim follows directly from the arguments of paragraph 6.2.2. □

End of Proof of Lemma 6.3*, point (iii).* Let $(\tilde{\lambda}_A, \tilde{\mu}_A)$ be a basis of $\pi_1 \mathcal{T}_A$. One can define a basis $(\tilde{\lambda}_B, \tilde{\mu}_B)$ of $\pi_1 \mathcal{T}_B$ with $\tilde{\lambda}_B = \sigma_1(\tilde{\lambda}_A)$ and $\tilde{\mu}_B = \sigma_1(\tilde{\mu}_A)$ (where σ_1 is the sewing involution of M_1). By Claim 6.4, possibly after passing to a subsequence, one can assume that there exists a basis $\langle h_A, k_A\rangle$ of $\pi_1 T_A$ such that

$$(f_i|Q_A)_*(\tilde{\lambda}_A) = [h_A] \text{ and } (f_i|Q_A)_*(\tilde{\mu}_A) = [k_A] \text{ for any } i \in \mathbf{N}.$$

On the other hand, using Claim 6.5, we know that there exist two elements (a, b) and (c, d) of $\mathcal{P}_*$ and two sequences of sections $\{\delta_i, i \in \mathbf{N}\}$, $\{d_i, i \in \mathbf{N}\}$ of T_B (with respect to the Seifert fibration of B) such that

$$(f_i|Q_B)_*(\tilde{\lambda}_B) = h_B^a \delta_i^b \text{ and } (f_i|Q_B)_*(\tilde{\mu}_B) = h_B^c d_i^d.$$

Let $\{n_i, i \in \mathbf{N}\}$ denote the sequence of integers such that $d_i = \delta_i h_B^{n_i}$ for $i \in \mathbf{N}$. With this notation we get $(f_i|Q_B)_*(\tilde{\mu}_B) \in \langle h_B^{c+dn_i} \delta_i^d\rangle$. Note that since $f_i|\mathcal{T}_B \colon \mathcal{T}_B \to T_B$ is a homeomorphism then, possibly after passing to a subsequence, we may assume that $\Delta_i = ad - bc - bdn_i = 1$ when $i \in \mathbf{N}$. It follows from the construction that

$$(s_i)_*(h_A) = h_B^a \delta_i^b \ \text{ and } \ (s_i)_*(k_A) = h_B^{c+dn_i} \delta_i^d.$$

Subcase 1: Assume first that $bd = 0$. Then either $b = 0$ and $(s_i)_*(h_A) = h_B^a$ where $a = \pm 1$, or $d = 0$ and $(s_i)_*(k_A) = h_B^{c+dn_i}$ where $c + dn_i = c = \pm 1$. Then we set $(\lambda_A = h_A, \mu_A = k_A)$ or $(\lambda_A = k_A, \mu_A = h_A)$ depending on whether $b = 0$ or $d = 0$.

If $b = 0$, then $d = \pm 1$, and $(s_i)_*(\mu_A) = h_B^{c+dn_i} \delta_i^{\pm 1}$.

If $d = 0$, then b and c are equal to ± 1 and thus $(s_i)_*(\mu_A) = h_B^a \delta_i^{\pm 1}$.

Subcase 2 : Assume now that $bd \neq 0$. Note that since Δ_i is constant when $i \in \mathbf{N}$ then in this case the sequence $\{n_i, i \in \mathbf{N}\}$ is necessarily constant. Hence we denote n_i by n_0 (when $i \in \mathbf{N}$). Consider the element of $\pi_1 T_A$ given by $\overline{\lambda}_A = h_A^d k_A^{-b}$. Then $(s_i)_*(\overline{\lambda}_A) = h_B^{\Delta_i} = h_B$. Choose a primitive element μ_A in $\pi_1 T_A$ in such a way that $(\overline{\lambda_A}, \mu_A)$ is a basis of $\pi_1 T_A$. We know that there exist coprime integers x and y such that $\mu_A = h_A^x k_A^y$. Then

$$(s_i)_*(\mu_A) = h_B^{ax+yc+ydn_0} \delta_i^{bx+dy}.$$

Since $(\overline{\lambda_A}, \mu_A)$ is a basis of $\pi_1 T_A$, since s_i induces an isomorphism between $\pi_1 T_A$ and $\pi_1 T_B$ and since $(s_i)_*(\overline{\lambda_A}) = h_B^{\pm 1}$ then there exists an integer v such that $(s_i)_*(\mu_A) = h_B^v \delta_i^{\pm 1}$. This ends the proof of Lemma 6.3. □

6.3. **Proof of Proposition 3.7.** Let $\{N_i, i \in \mathbf{N}\}$ be a sequence of weakly equivalent nongeometric closed Haken manifolds such that there exist degree-one maps $f_i \colon M \to N_i$ satisfying $\mathrm{Vol}(M) = \mathrm{Vol}(N_i)$.

By paragraph 5, we may assume that M is a closed Haken manifold. By paragraphs 5.4 and 6.1 we may assume that the N_i's satisfy conditions (II), (III) and (IV) and that the f_i's are in standard form.

For each Seifert piece A of N_i we denote by c_i the "attaching curves" on ∂A for the N_i obtained in Lemma 6.3. On the other hand, it follows from the proof of Lemma 6.3 that there exists a graph manifold B obtained by attaching curves, denoted by c, on ∂B (independent of $i \in \mathbf{N}$) such that $f_i|(B,c)\colon (B,c) \to (A,c_i)$ are degree-one maps when $i \in \mathbf{N}$. By the proof of Lemma 6.3, extend these maps via Dehn filling to degree-one maps $\hat{f}_i\colon \hat{B} \to A_i$, where $\hat{B}$ (resp. A_i) is obtained from B (resp. A) by Dehn fillings along the curves c (resp. c_i). Then applying Theorem 1.3 to the maps $\hat{f}_i$ one can get the finiteness of the pairs (A,c_i) when $i \in \mathbf{N}$. Thus combining this fact with points (i) and (iii) of Lemma 6.3 we obtain the finiteness of the gluing. This completes the proof of Proposition 3.7.

Combining Proposition 3.7 with Proposition 3.3 we have completed the proof of Proposition 3.2. This ends the proof of Theorem 1.2.

7. Proof of Theorem 1.1

Let M be a closed orientable graph manifold and let $(N_i)_{i\in\mathbf{N}}$ be a sequence of closed orientable Poincaré-Thurston 3-manifolds dominated by M via degree-one maps $f_i\colon M \to N_i$. Since M is a graph manifold then $\mathrm{Vol}(M) = 0$, and since $\mathrm{Vol}(N_i) \leq \mathrm{Vol}(M)$ then the N_i's are graph manifolds by Theorem 2.2. Denote by n_i the number of prime factors of each N_i and let $N_i = N_i^1 \sharp ... \sharp N_i^{n_i}$ denote the prime decomposition of N_i (see [12]). First note that the sequence $(n_i)_{i\in\mathbf{N}}$ is finite. Indeed since M 1-dominates N_i then by point (i) of Proposition 2.1 applied to degree-one maps we know that $r(M) \geq r(N_i)$, where $r(M)$ denotes the rank of the fundamental group of M. On the other hand, it follows from Grushko's Theorem ([11]) that $r(N_i) = r(N_i^1) + ... + r(N_i^{n_i})$. This implies that the number of prime factors of N_i is bounded by the rank of $\pi_1 M$. Thus to complete the proof of Theorem 1.1 it is sufficient to show that the sequence of prime factors (N_i^j), when $(i,j) \in \mathbf{N} \times \{1,...,n_i\}$, is finite up to homeomorphism. For each $(i,j) \in \mathbf{N} \times \{1,...,n_i\}$, consider the projection map $p_i^j\colon N_i \to N_i^j$. Such a map is a 0-pinch and has degree one. On the other hand, the Cutting-off-Theorem of M. Gromov applied to the prime decomposition of each N_i implies that $\mathrm{Vol}(N_i^j) = \mathrm{Vol}(M) = 0$ for all i,j. Then to complete the proof of Theorem 1.1 we apply Theorem 1.2 or Theorem 1.3 to the sequence of degree-one maps defined by $p_i^j \circ f_i$ depending on whether N_i^j is Haken or not.

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