GEOMETRY AND ERGODIC THEORY
OF NON-HYPERBOLIC EXPONENTIAL MAPS

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Abstract. We deal with all the maps from the exponential family \( \{ \lambda e^z \} \) such that the orbit of zero escapes to infinity sufficiently fast. In particular all the parameters \( \lambda \in (1/e, +\infty) \) are included. We introduce as our main technical devices the projection \( F_\lambda \) of the map \( f_\lambda \) to the infinite cylinder \( Q = \mathbb{C}/2\pi i \mathbb{Z} \) and an appropriate conformal measure \( m \). We prove that \( J_r(F_\lambda) \), essentially the set of points in \( Q \) returning infinitely often to a compact region of \( Q \) disjoint from the orbit of \( 0 \in Q \), has the Hausdorff dimension \( h_\lambda \in (1, 2) \), that the \( h_\lambda \)-dimensional Hausdorff measure of \( J_r(F_\lambda) \) is positive and finite, and that the \( h_\lambda \)-dimensional packing measure is locally infinite at each point of \( J_r(F_\lambda) \). We also prove the existence and uniqueness of a Borel probability \( F_\lambda \)-invariant ergodic measure equivalent to the conformal measure \( m \). As a byproduct of the main course of our considerations, we reprove the result obtained independently by Lyubich and Rees that the \( \omega \)-limit set (under \( f_\lambda \)) of Lebesgue almost every point in \( \mathbb{C} \), coincides with the orbit of zero under the map \( f_\lambda \). Finally we show that the the function \( \lambda \mapsto h_\lambda \), \( \lambda \in (1/e, +\infty) \), is continuous.

1. Introduction

Let \( f_\lambda = \lambda \exp(z) \), \( \lambda \in \mathbb{C} \), \( \lambda \neq 0 \) be a family of exponential maps. In this paper we deal with a set of parameters \( \lambda \) for which the trajectory of the singular value 0 tends to infinity exponentially fast. More precisely, let

\[
\beta_n = f_\lambda^n(0), \quad \alpha_n = \text{Re} \beta_n.
\]

We say that the parameter \( \lambda \) is super-growing if \( \alpha_n \to +\infty \) and there exists a constant \( c > 0 \) such that for all \( n \) large enough

\[
\alpha_{n+1} \geq c e^{\alpha_n} = \frac{c}{|\lambda|} |\beta_{n+1}|.
\]

Notice that this implies

\[
|\beta_{n+1}| \geq |\lambda| \exp \left( \frac{c}{|\lambda|} |\beta_n| \right).
\]

It is known that for these parameters \( J(f_\lambda) = \mathbb{C} \); moreover it follows from \( \text{We} \) that the Hausdorff dimension of the set of super-growing parameters equals 2.

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In the papers [UZ1] and [UZ2] we have dealt with geometry and dynamics of the set \( J_r \) of points non-escaping to infinity under the iteration of a hyperbolic exponential map. In this paper we go beyond hyperbolicity, allowing in particular the singular value 0 to belong to the Julia set. We consider the projection \( F_\lambda \) of the map \( f_\lambda \) to the infinite cylinder \( Q = \mathbb{C}/2\pi i \mathbb{Z} \), and we define \( J_r(F_\lambda) \), essentially the set of points in \( Q \) returning infinitely often to a compact region of \( Q \) disjoint from the orbit of 0 ∈ \( Q \). This set turns out to carry on a reach geometric structure and intriguing dynamics. Its Hausdorff dimension \( h_\lambda \) lies strictly between 1 and 2, its \( h_\lambda \)-dimensional Hausdorff measure is positive and finite and its \( h_\lambda \)-dimensional packing measure is locally infinite at each point of \( J_r(f_\lambda) \). The former fact, interesting itself, also provides a transparent geometric interpretation of the \( h_\lambda \)-conformal measure, the object defined by purely dynamical means. The latter fact is more interesting than it seems at first glance. The reason is that it forms the main ingredients in the proofs of the following two results. That \( \text{HD}(J_r(f_\lambda)) < 2 \) and that, consequently, the \( \omega \)-limit set (under \( f_\lambda \)) of Lebesgue almost every point in \( \mathbb{C} \) coincide with the orbit of zero under the map \( f_\lambda \). So, as a byproduct of the main course of our considerations, we have reproved the celebrated result of M. Lyubich and M. Rees (see [Ly], [Re]). In the last section we show that the function \( \lambda \mapsto h_\lambda, \lambda \in (1/e, +\infty) \), is continuous. We also study the metric dynamics of the map \( F_\lambda \), and starting of with M. Martens’ approach (see [Ma]) and using a very useful old result of Hayman (see [Ha]), we prove the existence and uniqueness of a Borel probability \( F_\lambda \)-invariant ergodic measure equivalent to the conformal measure \( m \), or equivalently to the Hausdorff measure \( \mathcal{H}^h|_{J_r(f_\lambda)} \). The just mentioned conformal measure \( m \) forms the basic tool to exhibit both geometrical and dynamical features of the set \( J_r(f_\lambda) \). Already proving its existence (via tightness) requires new ideas and careful estimates. Other results described in this introduction also require very technical considerations and fresh methods.

In what follows, we shall frequently use the Koebe distortion theorem: given \( r < 1 \) there exists a constant \( K_r \) such that for every univalent function \( f \) defined in \( B(0, 1) \) and for every \( x, y \in B(0, r) \) we have \( \left| \frac{f'(x)}{f'(y)} \right| \leq K_r \). We shall denote by \( K \) the Koebe constant \( K_r \).

We shall use the notation \( a \leq b \) to compare the variables \( a \) and \( b \); \( a \leq b \) if there is a constant \( C \) such that \( a \leq C b \).

2. Preliminaries

From now on throughout the entire paper we fix a super-growing parameter \( \lambda \in \mathbb{C} \setminus \{0\} \), and we denote the map \( f_\lambda : \mathbb{C} \to \mathbb{C} \) by \( f \). We define the equivalence relation \( \sim \) on \( \mathbb{C} \times \mathbb{C} \) by saying that \( z \sim w \) if and only if there exists \( k \in \mathbb{Z} \) such that \( z - w = 2\pi i k \). We denote the quotient space \( \mathbb{C}/\sim \) by \( Q \), which is an infinite cylinder, and by \( \pi : \mathbb{C} \to Q \) we denote the corresponding quotient map, i.e. \( \pi(z) \) is the equivalence class of \( z \) with respect to the equivalence relation \( \sim \). Since the maps \( f : \mathbb{C} \to \mathbb{C} \) and \( \pi \circ f : \mathbb{C} \to Q \) are constant on equivalence classes of the relation \( \sim \), they canonically induce respective conformal maps \( f : Q \to \mathbb{C} \) and \( F : Q \to Q \).

**Definition 2.1.** For every \( n \geq 0 \) we put
\[
\beta_n = f^n(0), \quad \alpha_n = \text{Re}\beta_n, \quad \beta_n^\infty = \{\beta_k : k \geq n\}, \quad \hat{\beta}_n = \pi(\beta_n), \quad \text{and} \quad \hat{\beta}_n^\infty = \{\hat{\beta}_k : k \geq n\}.
\]
Fix $M > 0$ and consider two sets

$$Q_M = \{ z \in Q : |\text{Re} z| \leq M \}$$

and

$$J_M = \{ z \in Q : F^n(z) \in Q_M \text{ for all } n \geq 0 \} = \bigcap_{n \geq 0} F^{-n}(Q_M).$$

Obviously the set $J_M$ is compact and forward invariant under $F$. Since $\lambda$ is super-growing, $\beta_n \to \infty$. Thus $0 \notin J_M$ and

$$\delta_M = \frac{1}{2} \text{dist}(J_M, \beta_0) > 0.$$

Since $\delta_M > 0$, for every $z \in J_M$ and every $n \geq 1$, all the holomorphic inverse branches of $F^{-n}$ are well-defined on the ball $B(z, 2\delta_M)$. We shall prove the following.

**Lemma 2.2.** For every $M > 0$ there exists $n_M \geq 1$ such that $|(F^k)'(x)| \geq 2$ for all $x \in J_M$ and all $k \geq n_M$.

**Proof.** Suppose on the contrary that there exist a sequence $\{x_i\}_{i=1}^\infty \subset J_M$ and $\{n_i\}_{i=1}^\infty$, an unbounded increasing sequence of positive integers such that

$$|(F^{n_i})'(x_i)| \leq 2. \quad (2.1)$$

Consider inverse branches $F_{x_i}^{-n_i} : B(F^{n_i}(x_i), 2\delta_M) \to Q$ of $F^{n_i}$, sending $F^{n_i}(x_i)$ to $x_i$. Let $y \in Q$ be an accumulation point of the sequence $\{F^{n_i}(x_i)\}_{i=1}^\infty$. Passing to a subsequence, we may assume that $|y - F^{n_i}(x_i)| < \delta_M/2$ for all $i \geq 1$. Then all the inverse branches $F_{x_i}^{-n_i}$ are well-defined on $B(y, 3\delta_M/2)$, and applying Koebe’s distortion theorem, it follows from (2.1) that $|(F_{x_i}^{-n_i})'(y)| \geq \kappa$ for some $\kappa > 0$ and all $i \geq 1$. Now applying the $\frac{1}{4}$-Koebe distortion theorem we see that there exists a non-empty open set $B \subset Q$ such that $F_{x_i}^{-n_i}(B(y, 3\delta_M/2)) \supset B$ for all $i \geq 1$. Hence $F^{n_i}(B) \subset B(y, 3\delta_M/2)$, which means that $f^{n_i}(B) \subset \bigcup_{n=-\infty}^{n=\infty} (B(y, 3\delta_M/2) + 2k\pi i)$ (recall that $F^n = \pi \circ f^n \circ \pi^{-1}$). In particular the family $\{F^{n_i} : B \to \mathbb{C}\}$ is normal. This is a contradiction of the fact that $J(f) = \mathbb{C}$, and we are done. \hfill \Box

### 3. The existence of a conformal measure

Given $t \geq 0$ a Borel probability measure $m$ on $Q$ is said to be $t$-conformal (or: conformal with exponent $t$) if and only if

$$m(F(A)) = \int_A |F'|^t \, dm \quad (3.1)$$

for every Borel set $A \subset Q$ such that $F|_A$ is 1-to-1. Our main goal in this section is to prove the existence of a conformal measure. Obviously, one conformal (but infinite) measure already exists; this is simply the Lebesgue measure. We shall construct another measure, which will be finite and conformal with an exponent smaller than 2. First, following [Du1], for every $M > 0$ we shall build a probability Borel measure $m_M$, with support contained in $J_M$, which will be “almost conformal” for some $t_M \geq 0$, i.e.

$$m_M(F(A)) \geq \int_A |F'|^{t_M} \, dm_M \quad (3.2)$$

for every Borel set $A \subset Q$ such that $F|_A$ is 1-to-1 and (3.1) holds if we assume in addition that $A \cap \{ z \in Q : |\text{Re} z| \geq M \} = \emptyset$. In the sequel, we shall need to refer
to some details of the construction, so we make them more specific now. So, let $\delta_M$ be defined as above. For every $M > 0$ we choose a collection of points $E^M = \{x_1, \ldots, x_{q_M}\} \subset J_M$ such that the balls $B(x_i, \delta_M)$ cover the set $J_M$. Consider the function formula (3.3)

$$c_M(t) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{x \in E^M} \sum_{w \in F_{-1}^M x} |(F^n)'|^{-t}(w).$$

(Notice that the summation is taken over only those preimages of $x$ which are in $J_M$.) The function $t \mapsto c_M(t)$ has three important properties. First, notice that it follows from the Hölder inequality that it is convex in $\mathbb{R}$, so it is continuous. Next, it follows easily from Lemma 2.2 that it is strictly decreasing. Finally, it follows from [UZ1], Theorem 2.1, that one can construct an expanding Cantor repeller whose limit set $X$ is invariant under $F^2$, contained in $J_M$ for $M$ large, and $c_M(0) \geq \frac{1}{2} h_{top}(F^2_{|X}) > 0$. Thus, we conclude that there exists a unique value $t = t_M$ such that $c_M(t_M) = 0$. Following the general construction in [DU1] (see also [PU], Chapter 10) and using the sets $E_n = F_{-1}^M(E^M)$, we construct the measure $m_M$, for which $m_M(J_M) = 1$ and which is “almost conformal” with exponent $t_M$.

We start with the following.

**Lemma 3.1.** It holds that $\text{HD}(J_M) \geq t_M$.

**Proof.** Fix a point $x \in J_M$ and an integer $n \geq 1$. Let $F_x^{-n} : B(F^n(x), 2\delta_M) \to Q$ be the holomorphic inverse branch of $F^n$ sending $F^n(x)$ to $x$. Now applying the $\frac{1}{r}$-Koebe distortion theorem and the standard Koebe distortion theorem, it follows from (3.2) that

$$m_M \left( B(x, \frac{1}{4}|(F^n)'(x)|^{-1}\delta_M) \right) \leq m_M \left( F_x^{-n}(B(F^n(x), \delta_M)) \right) \leq K^{t_M}|(F^n)'(x)|^{-t_M} m_M(B(F^n(x), \delta_M)) \leq (4K\delta_M)^{t_M} \left( \frac{1}{4}\delta_M|(F^n)'(x)|^{-1} \right)^{t_M}. \tag{3.3}$$

Since, by Lemma 2.2, $\lim_{n \to \infty} |(F^n)'(x)| = \infty$ uniformly in $J_M$, we conclude that for every $r > 0$ small enough there exists $n \geq 1$ such that

$$\frac{1}{4}\delta_M|(F^{n+1})'(x)|^{-1} \leq r \leq \frac{1}{4}\delta_M|(F^n)'(x)|^{-1}. $$

Using (3.3), we therefore get

$$m_M(B(x, r)) \leq m_M \left( B(x, \frac{1}{4}\delta_M|(F^n)'(x)|^{-1}) \right) \leq (4K\delta_M)^{t_M} (4^{-1}\delta_M|(F^{n+1})'(x)|^{-1})^{t_M} |F'(F^n(x))|^{t_M} \leq \left(4KT\delta_M^{t_M} \right)^{t_M},$$

where $T = \sup \{ |F'(y)| : y \in J_M \}$ is finite since $J_M$ is bounded. This inequality implies in a standard way that $\text{HD}(J_M) \geq t_M$ (see e.g. [PU]). \qed

**Lemma 3.2.** For every $M$ large enough there exists $p_0 > 0$ such that for all $p > p_0$, $\text{HD}(J_M) \leq t_{M+p}$.
Proof: It easily follows from Lemma 2.2 and the absence of critical points of \( F \) in \( Q \) that
\[
L = \inf \{ |(F^n)'(w)| : w \in J_M, n \geq 1 \} > 0 \text{ and } \lim_{n \to \infty} |(F^n)'(z)| = \infty
\]
for all \( z \in J_M \). Let us fix \( p > 0 \) so that \( KL^{-1} < p \) and let us consider the set \( J_{M+p} \). Following the construction described above, we choose a finite collection of points \( E^{M+p} \subset J_{M+p} \) such that the balls \( B(x, \delta_{M+p}), x \in E^{M+p} \), cover the set \( J_{M+p} \). Let \( y \in J_M \subset J_{M+p} \). Given \( n \geq 0 \) there exists \( x \in E^{M+p} \) such that \( F^n(y) \in B(x, \delta_{M+p}) \). By our definition of \( \delta_{M+p} \), all holomorphic branches \( F^{-i} \) are defined in \( B(x, 2\delta_{M+p}) \). Let \( F^{-i}_y \) be the branch sending the point \( F^n(y) \) to \( F^{-i}(y) \). Then, by Koebe distortion theorem, for all \( z \in B(x, \delta_{M+p}) \) we get
\[
\frac{|(F^{-i}_y)'(z)|}{|(F^{-i}_y)'(F^n(y))|} < K.
\]
So, \( |(F^{-i}_y(z))'| \leq K|(F^{-i}_y)'(F^n(y))| \leq KL^{-1} \) since \( F^{(n-i)}(y) \in J_M \). Thus, by integrating, we conclude that \( d(F^{-i}_y(x), F^{-i}(y)) < KL^{-1} < p \) and, finally, \( F^{-i}_y(x) \in Q_{M+p} \) for all \( 0 \leq i \leq n \) (since \( F^{(n-i)}(y) \in Q_M \)). This implies that the point \( w = F^{-n}_y(x) \) belongs to the set \( J_{M+p} \), i.e. \( w \in \{ F^{-n}_{J_{M+p}}(x) \} \). Let \( F^n(x) \) be the collection of branches \( F^-n \) on \( B(x, \delta_{M+p}) \), satisfying \( F^-n(x) \in J_{M+p} \). It follows from the above considerations that
\[
J_M \subset \bigcup_{x \in E^{M+p}} \bigcup_{F_n(x)} F^{-n}_x(B(x, \delta_{M+p})).
\]
Moreover, \( \text{diam}(F^{-n}_x(B(x, \delta_{M+p}))) \to 0 \) uniformly as \( n \to \infty \) since \( F_{|J_{M+p}} \) is expanding, and

\[
\sum_{x \in E^{M+p}} \sum_{F_n(x)} (\text{diam}(F^{-n}_x(B(x, \delta_{M+p}))))^t \leq \sum_{x \in E^{M+p}} \sum_{w \in F_{|J_{M+p}}^{-n}(x)} \frac{1}{|(F^n)'(w)|^t}.
\]

Now fix an arbitrary \( t > t_{M+p} \). Then \( c_{M+p}(t) < 0 \), so there exists \( \varepsilon > 0 \) such that
\[
\sum_{x \in E^{M+p}} \sum_{w \in F_{|J_{M+p}}^{-n}(x)} \frac{1}{|(F^n)'(w)|^t} < \exp(-n\varepsilon)
\]
for all \( n \) large enough. Using (3.4) we conclude that \( H_t(J_M) = 0 \) for all \( t > t_{M+p} \) and, consequently, \( \text{HD}(J_M) \leq t_{M+p} \).

\[ \square \]

Corollary 3.3. There exists \( s > 1 \) such that \( t_M > s \) for all \( M \) large enough.

Proof. It follows from Theorem 2.1 in [12] that \( \text{HD}(J_M) > 1 \) for all \( M \) large enough. Fix one such \( M \). Choose \( p \) as in the preceding lemma. Then \( t_{M+q} \geq \text{HD}(J_M) = s > 1 \) for all \( q \geq p \), and we are done.

\[ \square \]

Given \( M > 0 \) we set
\[
Y^+_M = \{ z \in Q : \text{Re} z > M \}, \quad Y_M = \{ z \in Q : |\text{Re} z| > M \}, \quad Y^-_M = \{ z \in Q : \text{Re} z < -M \}.
\]
The main technical result of this section is the following.

Recall (see [13]) that a family \( F \) of Borel probability measures on a Polish space \( X \) is said to be tight provided that for every \( \varepsilon > 0 \) there exists a compact set \( G \subset X \)

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such that \( \mu(G) > 1 - \varepsilon \) for every \( \mu \in \mathcal{F} \). Prokhorov’s theorem states that each tight family is relatively weakly compact. We shall prove the following.

**Lemma 3.4.** The sequence of measures \( \{m_n\}_{n=1}^{\infty} \) is tight on \( Q \).

**Proof.** We check that for every \( \varepsilon > 0 \) there exists \( M > 0 \) such that for all \( n \), \( m_n(Y_M) < \varepsilon \). We first estimate from above that \( m_n(Y_M^+) \) in essentially the same way as in \( [UZ1] \). For the needs of the proof of Theorem 3.8 we shall establish a slightly more general result. Fix a Borel set \( G \subset Q \). This set is mapped one to one by \( f \) onto some set in \( \mathbb{C} \). But \( F \) is no longer one-to-one in \( G \) since two points in the image are identified if they differ by \( 2k\pi i \) for some \( k \in \mathbb{Z} \). We have

\[
m_n(G \cap Y_M^+) = m_n(\{ z \in G \cap Y_M^+ : f(z) \in Y_{\text{exp}(M/2)} \}) \\
+ m_n(\{ z \in G \cap Y_M^+ : f(z) \in Q_{\text{exp}(M/2)} \}).
\]

To estimate the first summand, let us write \( \{ \text{Re}z > \exp(M/2) \} \cap f(G) = \bigcup S_k \) where \( S_k = \{ \text{Re}z > \exp(M/2) \} \cap f(G) \cap \{ 2k\pi \leq \text{Im}z < 2(k+1)\pi \} \). Then the map \( F \) is one-to-one on each set \( G \cap f^{-1}(S_k) \), and the derivative \( |F'| \) on this set can be estimated from below by \( \inf_{w \in S_k} |w| \geq |\exp(M/2) + 2k\pi i| \) if \( k \) is non-negative and by \( |\exp(M/2) + 2(k+1)\pi i| \) if \( k \) is negative. Thus,

\[
m_n(F^{-1}(Y_{\text{exp}(M/2)} \cap f(G))) \\
\leq m_n(Y_{\text{exp}(M/2)} \cap f(G)) \cdot 2 \sum_{k=0}^{+\infty} |\exp(M/2) + 2k\pi i|^{-s} \\
\leq m_n(F(G)) \cdot 2 \sum_{k=0}^{+\infty} |\exp(M/2) + 2k\pi i|^{-s} \\
\leq m_n(F(G)) \int_{\text{exp}(M/2)}^{+\infty} x^{-s} dx \\
\leq (\exp(M/2))^{1-s} m_n(F(G)),
\]

where \( s > 1 \) is the number produced in Corollary 3.3. In order to estimate the second summand put

\[
A = \{ z \in Y_M^+ \cap G : f(z) \in Q_{\text{exp}(M/2)} \}.
\]

Let

\[
Z_A = \{ k \in \mathbb{Z} : f(A) \cap \{ z \in \mathbb{C} : 2k\pi \leq \text{Im}z < 2(k+1)\pi \} \neq \emptyset \}.
\]

Now, if \( z \in A \), then \( \text{Re}z > M \) and therefore \( |f(z)| = |\lambda|e^{\text{Re}z} \geq |\lambda|e^M \). Hence, if \( k \in Z_A \), then

\[
|\lambda|^2 e^{2M} \leq |f(z)|^2 \leq (\exp(M/2))^2 + 4\pi^2 \max \{ |k + 1|^2, |k - 1|^2 \}
\]

and therefore

\[
(\max \{ |k + 1|, |k - 1| \})^2 \geq \frac{1}{4\pi^2} (|\lambda|^2 e^{2M} - e^M) = \frac{1}{4\pi^2} e^{2M} (|\lambda|^2 - e^{-M}) \geq e^{2M}
\]

assuming that \( M \) is large enough. Thus \( \max \{ |k + 1|, |k - 1| \} \geq e^M \), and in consequence \( |k| \geq e^M \). Hence

\[
m_n(A) \leq m_n(F(A)) \sum_{|k| \geq e^M} |k|^{-s} \leq e^{M(1-s)} m_n(F(A)) \leq e^{M(1-s)} m_n(F(G)).
\]
Combining this and (3.3), we get

\[(3.6) \quad m_n(Y_M^- \cap G) \leq \exp \left( \frac{1}{2} M (1 - s) \right) m_n(f(G)) \leq \exp \left( \frac{1}{2} M (1 - s) \right). \]

In particular,

\[(3.7) \quad m_n(Y_M^+) \leq \exp \left( \frac{1}{2} M (1 - s) \right). \]

We shall now estimate \(m_n(Y_M^-)\). This will be more complicated, since the set \(Y_M^-\) is mapped by \(f\) onto the ball \(B(0, |\lambda| \exp(-M))\) and \(|F'(z)| = |F(z)|\). This means that even if we bound the measure \(m_n\) of \(B(0, |\lambda| \exp(-M))\) by the radius \(\exp(-M)\) raised to power \(t_n\), this will not be enough to conclude that \(m_n(Y_M^-)\) is small. But, actually, due to our super-growing condition, the measure \(m_n\) of the ball \(B(0, |\lambda| \exp(-M))\) is much smaller that \(\exp(-t_n(M))\), and we shall estimate it carefully. Keep the same set \(G\). It follows from (1.1) that for every \(k \geq 0\)

\[(3.8) \quad |\beta_{k+1}| = |\lambda e^{\beta_k}| = |\lambda| |e^{\alpha_k}| \geq |\lambda| \exp \left( \frac{c}{|\lambda|} |\beta_k| \right). \]

Consider now the balls \(B_k = B(\hat{\beta}_k, |\beta_{k-1}|-1)\). Then for all \(k \geq 1\) large enough \(2B_k\) (the ball with the same center as \(B_k\) and the radius twice as big) is a topological disc, and it follows from Koebe’s distortion theorem followed by (1.1) that for all \(k \geq 1\) large enough

\[
f(B_k \cap G) \subset B(\beta_{k+1}, K|\beta_{k+1}|^1|\beta_{k-1}|-1) \cap f(G)\]

\[
\subset \{ z \in \mathbb{C} : \text{Re} z \geq \alpha_{k+1} - K|\beta_{k+1}|^1|\beta_{k-1}|-1 \} \cap f(G)\]

\[
\subset \{ z \in \mathbb{C} : \text{Re} z \geq \alpha_{k+1}(1 - K|\lambda| e^{-1}|\beta_{k-1}|-1) \} \cap f(G)\]

\[
\subset \{ z \in \mathbb{C} : \text{Re} z > \alpha_{k+1}/2 \} \cap f(G).\]

The map \(F|_{B_k \cap G}\) is no longer one-to-one but, since the height of the image \(f(B_k)\) is bounded from above by \(K|\beta_{k+1}|^1|\beta_{k-1}|-1\), every point in \(F|_{B_k \cap G}\) has at most \(\frac{1}{2\pi} K|\beta_{k+1}|^1|\beta_{k-1}|-1\) preimages in \(B_k \cap G\). Using (3.8), we therefore get

\[
m_n(B_k \cap G) \leq m_n \left( Y_{\alpha_{k+1}/2}^+ \cap f(G) \right) \left( \inf \{|f'|_{|\beta_k|}\} \right)^{-t_n} \left( |\beta_{k+1}|^1|\beta_{k-1}|-1 \right)^{-t_n} \exp \left( \frac{2\alpha_{k+1}}{4} (1 - s) \right) \left( \inf \{|f'|_{|\beta_k|}\} \right)^{-t_n} \left( |\beta_{k+1}|^1|\beta_{k-1}|-1 \right)^{-t_n} m_n(F^2(G)).\]

Since \(f(B_k) \subset B(\beta_{k+1}, K|\beta_{k+1}|^1|\beta_{k-1}|-1)\), using (3.3) and Koebe’s distortion theorem, we conclude that for all \(k\) large enough \(\inf \{|f'|_{|\beta_k|}\} \geq |\beta_{k+1}|\). Therefore, using the “equality” part of (3.8), we obtain the following:

\[(3.9) \quad m_n(B_k \cap G) \leq \exp \left( \frac{\alpha_{k+1}}{4} (1 - s) \right) |\beta_{k+1}|^1|\beta_{k-1}|-t_n |\beta_{k+1}|^1|\beta_{k-1}|-t_n m_n(F^2(G))\]

\[
\leq |\beta_{k+1}|^{1-s} |\beta_{k-1}|^{-1} \left( \frac{|\beta_{k+1}|}{|\lambda|} \right) \frac{1}{m_n(F^2(G))} \leq |\beta_{k+2}|^{-\kappa} m_n(F^2(G)) \quad \text{for} \quad \kappa = (s-1)/4.\]

We now consider the holomorphic inverse branch \(F_0^{-k} : 2B_k \to Q\) sending \(\hat{\beta}_k = F^k(0)\) to \(0\). It follows from Koebe’s distortion theorem that

\[(3.10) \quad K^{-1}(|\beta_1| \cdot |\beta_2| \cdot \ldots \cdot |\beta_k|)^{-1} \leq |(F_0^{-k})'(z)| \leq K(|\beta_1| \cdot |\beta_2| \cdot \ldots \cdot |\beta_k|)^{-1}\]
for all \( z \in B_k \), and
\[
\tilde{B}_k = F_0^{-k}(B_k) \subset B(\tilde{0}, K(\lvert \beta_1 \rvert \cdot \lvert \beta_2 \rvert \cdots \lvert \beta_k \rvert)^{-1} \lvert \beta_{k-1} \rvert^{-1}).
\]
Applying in turn Koebe’s \( \frac{1}{4} \)-distortion theorem, we get
\[
\tilde{B}_k \supset B(\tilde{0}, 4^{-1}(\lvert \beta_1 \rvert \cdot \lvert \beta_2 \rvert \cdots \lvert \beta_k \rvert)^{-1} \lvert \beta_{k-1} \rvert^{-1}).
\]
Using (3.10) and (3.9), we obtain
\[
m_n(F_0^{-k}(B_k \cap G)) \leq K^n(\lvert \beta_1 \rvert \cdot \lvert \beta_2 \rvert \cdots \lvert \beta_k \rvert)^{-\gamma}m_n(B_k \cap G)
\]
\[
\leq (\lvert \beta_1 \rvert \cdot \lvert \beta_2 \rvert \cdots \lvert \beta_k \rvert)^{-\gamma}m_n(B_k \cap G) \leq |\beta_{k+2}|^{-\kappa}m_n(F_2(G)).
\]
Notice that, in particular, in this way we get the estimate of the measure of \( \tilde{B}_k \) by \( |\beta_{k+2}|^{-\kappa} \). Now looking at (3.11) and (3.12) with \( k \) replaced by \( k+1 \) we conclude for all \( k \) large enough
\[
c\tilde{B}_{k+1} \subset \tilde{B}_k.
\]
Let \( W_k \) be the unbounded connected component of \( F^{-1}(\tilde{B}_k) \) and let
\[
V_k = W_k \setminus \overline{W_{k+1}}.
\]
In view of (3.12) with \( k \) replaced by \( k+1 \), in view of (3.13), and in view of Lemma 3.1 we can estimate as follows:
\[
m_n(V_k) \leq \left( \inf_{V_k} \{ |f'| \} \right)^{\gamma}m_n(\tilde{B}_k \setminus \tilde{B}_{k+1})
\]
\[
\leq (4|\lambda|(|\beta_1| \cdot |\beta_2| \cdots |\beta_k||\beta_{k+1}|)|\beta_k|)^{\gamma}m_n(\tilde{B}_k)
\]
\[
\leq (4|\lambda|(|\beta_1| \cdot |\beta_2| \cdots |\beta_k||\beta_{k+1}|)|\beta_k|)^{\gamma}2|\beta_{k+2}|^{-\kappa}
\]
\[
\leq |\beta_{k+2}|^{-\gamma}
\]
for an arbitrary \( \gamma \in (0, \kappa) \) and all \( k \geq 1 \) sufficiently large (depending on \( \gamma \)). The latter follows from the following simple

**Lemma 3.5.** \textit{If the sequence \( (\alpha_i)_{i=1}^{\infty} \) satisfies \( \alpha_i \to +\infty \) and \( \alpha_{n+1} > c \exp \alpha_n \) for some positive \( c \), then for every \( \varepsilon > 0 \) there exists \( n_0 \) such that for every \( n > n_0 \), \( \alpha_1 + \cdots + \alpha_n < \varepsilon \alpha_{n+1} \).}

For every \( M > 0 \) let \( l(M) \geq 1 \) be the largest integer such that
\[
Y_M = \bigcup_{k=l(M)}^{\infty} V_k.
\]
Since \( \lim_{M \to +\infty} l(M) = +\infty \), it follows from (3.14) that for all \( n \geq 1 \) and all \( M > 0 \) sufficiently large
\[
m_n(Y_M) \leq \sum_{k=l(M)}^{\infty} |\beta_{k+2}|^{-\gamma} \to 0 \quad \text{as} \quad M \to +\infty.
\]
The proof of tightness is finished. \( \square \)
Theorem 3.6. The weak-limit measure \( m \) is an \( h \)-conformal for the map \( F : Q \to Q \).

Let us now introduce the main set \( J_r = J_r(F) \) we will be dealing with throughout the rest of this paper.

Definition 3.7. \( J_r(f) \subset \mathbb{C} \) is the set of those points \( z \in \mathbb{C} \) for which there exists an unbounded sequence \( \{n_k(z)\}^\infty_{k=1} \) such that
\[
\text{dist}(\{f^{n_k(z)}(z)\}^\infty_{k=1}, \beta^\infty) > 0
\]
and the set \( \text{Re}\{f^{n_k(z)}(z)\}^\infty_{k=1} \) is bounded. The set \( J_r = J_r(F) \subset Q \) is defined to be \( \pi(J_r(f)) \).

Given \( M > 0 \) we define the set \( J_{r,M} \) to consist of those points \( z \in Q \) for which there exists an unbounded sequence \( \{n_k(z)\}^\infty_{k=1} \) such that
\[
\text{dist}(\{f^{n_k(z)}(z)\}^\infty_{k=1}, \beta^\infty) > 0
\]
and \( \{F^{n_k(z)}(z)\}^\infty_{k=1} \subset Q_M \). Obviously, \( J_r = \bigcup J_{r,M} \). We shall now prove our second basic result.

Theorem 3.8. If \( m \) is a \( t \)-conformal probability measure for \( F : Q \to Q \) with \( t > 1 \), then \( m(J_r) = 1 \). Even more, there exists \( M > 0 \) such that \( m(J_{r,M}) = 1 \).

Proof. Fix \( M > 0 \) and define
\[
E_M = \{x \in Y_M : \forall k \geq 0 \text{ if } F^k(x) \in Y_M^+ \Rightarrow F^{k+1}(x) \in Y_M\}.
\]
We shall first show that there exists \( M > 0 \) arbitrarily large such that \( m(E_M) = 0 \). Notice that \( F(Y_M^+ \cap E_M) \subset E_M \) and therefore \( Y_M^+ \cap E_M \subset F^{-1}(E_M) \). Thus the same argument as in formula (3.5) gives us the following:
\[
(3.15) \quad m(E_M \cap Y_M^+) \leq m(F^{-1}(E_M)) \lesssim \sum_{k=-\infty}^{+\infty} |M + 2k\pi i|^{-t} m(Y_M \cap E_M)
\]
\[
\lesssim M^{1-t} m(Y_M \cap E_M) \leq \frac{1}{4} m(E_M)
\]
for all \( M > 0 \) large enough. It remains to show that there exists an arbitrarily large \( M \) such that
\[
m(E_M \cap Y_M^-) \leq \frac{1}{4} m(E_M).
\]
This task requires a much more involved reasoning. Again, as in the proof of tightness, the difficulty is caused by the fact that the set \( E_M \cap Y_M^- \) is mapped by \( F \).
into a neighbourhood of 0 with small derivative. Thus, we shall need to carefully estimate the measure $m(F(E_M \cap Y^-_M))$. For every $n \geq 2$ put

$$M_n = \log 4 + \sum_{j=1}^{n} \log |\beta_j| + \log |\beta_{n-1}| = \log 4 + (n + 1) \log |\lambda| + \sum_{j=0}^{n-1} \alpha_j + \alpha_{n-2}.$$ 

Similar to (3.19) we can show that

$$m(B_n \cap G) \leq |\beta_{n+2}|^{-\kappa} m(F^2(G))$$

for $\kappa = \frac{t-1}{4}$, for all Borel sets $G \subset Q$ and all $n$ (independent of $G$) large enough. We shall now check that for all $n \geq 2$ large enough

$$F(Y^-_{M_n}) \subset \tilde{B}_n \quad \text{and} \quad F^{n+1}(Y^-_{M_n}) \subset Y^+_{M_n}.$$ 

Indeed,

$$F(Y^-_{M_n}) \subset B(\tilde{0}, e^{-M_n}) = B\left(\tilde{0}, \frac{1}{4} \prod_{j=1}^{n} |\beta_j|^{-1}|\beta_{n-1}|^{-1}\right),$$

and applying (3.12) we see that the first inclusion in the formula (3.17) is satisfied. Also

$$F^n(\tilde{B}_n) = B_n \subset Y_{n+1} - \beta_{n+1}^{-1} \subset Y^+_{M_n}$$

for all $n \geq 2$ large enough (again, this follows easily from Lemma 3.5). Combining this and the first inclusion in (3.17) we see that the second inclusion in (3.17) is also satisfied. Now fix $k \geq 2$ so large that (3.16) and (3.17) hold for all $n \geq k$. It follows from the second inclusion in (3.17) and from the definition of $E_{M_k}$ that

$$F^{n+1}(E_{M_k} \cap Y^-_{M_n}) \subset E_{M_k} \cap Y^+_{M_n}.$$ 

For every $n \geq k$ put

$$Z_n = Y^-_{M_n} \setminus Y^-_{M_{n+1}}.$$ 

It then follows from the first inclusion in (3.17) and from (3.18) that

$$m(E_{M_k} \cap Z_n) \leq \left(\inf\{|F'(z)|^t : z \in E_{M_k} \cap Z_n\}\right)^{-1} m(F(E_{M_k} \cap Z_n)) \leq \exp(M_{n+1} t) m(\tilde{B}_n \cap F^{-n}(E_{M_k}))$$

$$= \left(4 \prod_{j=1}^{n+1} |\beta_j| \cdot |\beta_{n+1}|\right)^t m(\tilde{B}_n \cap F^{-n}(E_{M_k})).$$

Notice that the estimate (3.13) (which was established for measures $m_n$) is also valid for an arbitrary $t$-conformal measure $t$ (with $\kappa = \frac{t-1}{4}$). Applying (3.13) with $m_n$ replaced by $m$, $k$ replaced by $n$, and with $G = B_n \cap E_{M_k}$, we get

$$m(\tilde{B}_n \cap F^{-n}(E_{M_k})) = m(F^{-n}(B_n \cap E_{M_k})) \leq |\beta_{n+2}|^{-\kappa} m(F^2(B_n \cap E_{M_k})),$$

and consequently

$$m(E_{M_k} \cap Z_n) \leq \left(4 \prod_{j=1}^{n+1} |\beta_j| \cdot |\beta_{n+1}|\right)^t |\beta_{n+2}|^{-\kappa} m(F^2(B_n \cap E_{M_k}))$$

$$\leq |\beta_{n+2}|^{-\kappa} m(F^2(B_n \cap E_{M_k})).$$
for \( u = \frac{3}{2} \) and for all \( n \) large enough. The latter easily follows again from Lemma 3.5.

We shall now check that \( F^2(B_n \cap E_M) \subset E_M \). Indeed, by Koebe’s distortion

\[
F(B_n) \subset B(\hat{\beta}_{n+1}, K|\beta_{n+1}| |\beta_n|^{-1}) \subset Y_{n+1}^+ \subset Y_M^+. 
\]

Thus, it follows from the definition of Lemma 3.9. Fix a point \( z \in E_M \) for all \( k \) large enough, we get

\[
F(B_n \cap E_M) \subset E_M \cap Y_{Mn}^+, 
\]

and consequently

\[
F^2(B_n \cap E_M) \subset F(E_M \cap Y_{Mn}^+) \subset Y_M. 
\]

Therefore, estimate (3.19) we conclude that

\[
m(E_M \cap Z_n) \leq |\beta_{n+2}|^{-u} m(E_M) 
\]

for all \( k \geq 2 \) large enough and all \( n \geq k \). Summing up these inequalities over all \( n \geq k \) and using the fact that \( \bigcup_{n \geq k} Z_n = Y_M \), we obtain

\[
m(E_M \cap Y_{M-k}) \leq \sum_{n=k}^{\infty} |\beta_{n+2}|^{-u} m(E_M) \leq \frac{1}{4} m(E_M) 
\]

for all \( k \geq 2 \) large enough. Combining this and (3.13), where \( M = M_k \) (\( k \) large enough), we get

\[
m(E_M) \leq \frac{1}{4} m(E_M), 
\]

which implies that \( m(E_M) = 0 \). Thus, by

\[
\text{conformality of the measure } m 
\]

\[
m \left( \bigcup_{j=0}^{\infty} F^{-j} (E_M) \right) = 0. 
\]

In order to complete the proof, it therefore suffices to prove

**Lemma 3.9.**

\[
Q \setminus \bigcup_{j=0}^{\infty} F^{-j} (E_M) \subset J_{r,M_k}. 
\]

**Proof.** Fix a point \( z \in Q \setminus \bigcup_{j=0}^{\infty} F^{-j} (E_M) \). Then either

Case 1: There exists \( j_0 = j_0(z) \geq 0 \) such that \( |\text{Re}(F^j(z))| \leq M_k \) for all \( j \geq j_0 \) or

Case 2: There are infinitely many \( j \)'s such that \( |\text{Re}(F^j(z))| > M_k \) and \( |\text{Re}(F^{j+1}(z))| \leq M_k \).

Consider first Case 1, \( |\text{Re}f^j(z)| < M_k \) for \( j > j_0 \).

Since the sequence \( \{\beta_n\}_{n=1}^{\infty} \) diverges to \( \infty \), we therefore conclude that

\[
\text{dist}(\{f^n(z)\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}) > 0. 
\]

Thus, \( z \in J_{r,M_k} \), and we are done in this case.

Now consider Case 2.

Fix \( j \geq 1 \) such that \( |\text{Re}f^j(z)| = |\text{Re}(F^j(z))| > M_k \) and \( |\text{Re}f^{j+1}(z)| = |\text{Re}(F^{j+1}(z))| \leq M_k \).

It follows from the definition of \( M_k \) that \( \alpha_0, \alpha_1, \ldots, \alpha_{k-1} < M_k - 1 \), and therefore

\[
\pi^{-1}(\{\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_{k-1}\}) \cap B(f^j(z), 1) = \emptyset. 
\]

Since \( \beta_n \in Y_{M_{k+1}}^+ \) for all \( k \geq 2 \) large enough and all \( n \geq k + 1 \), we also get

\[
f^{j+1}(z) \notin B(\{\beta_n\}_{n=k+1}^{\infty}, 1). 
\]

By Koebe’s \( \frac{1}{4} \)-distortion theorem,

\[
f(B(f^j(z), 1)) \supset B(f^{j+1}(z), 4^{-1}|\lambda| e^{M_k}). 
\]
On the other hand, it follows from (3.20) that $B(f^j(z),1) \cap f^{-1}\{\beta_1,\ldots,\beta_n\} = \emptyset$. Thus, $f^{j+1}(z) \notin B(\{\beta_n\}_{n=1}^k, 4^{-1}|\lambda|e^{M_k})$. Combining this and (3.21) we see that

$$f^{j+1}(z) \notin B(\{\beta_n\}_{n=1}^\infty, \min\{1, 4^{-1}|\lambda|e^{M_k}\}).$$

Obviously, we can also write

$$f^{j+1}(z) \notin B(\{\beta_n\}_{n=0}^\infty, \min\{1, 4^{-1}|\lambda|e^{M_k}\})$$

since $|f^{j+1}(z) - \beta_0| = |f^{j+1}(z)| \geq |\lambda|e^{M_k}$. Since we are in Case 2, $F^{j+1}(z) \in Q_{M_k}$. Thus $z \in J_{r,M}$, and we are done. \hfill \Box

Now fix $M > 0$. For every $z \in J_{r,M}$ we fix one sequence $\{n_k(z)\}_{k=1}^\infty$ for which the condition in the definition of the set $J_{r,M}$ (Definition 3.7) is satisfied. Since $f$ restricted to the ball centered at $f^{n_k(z)}(z)$ with radius $\min\{\pi, \text{dist}(\{\{f^{n_k(z)}(z)\}_{k=1}^\infty, \beta_0^\infty)\}$ is univalent and since $|f'(f^{n_k(z)}(z))| = |f(f^{n_k(z)}(z))| = |\lambda| \exp(\text{Re}(f^{n_k(z)}(z))) \geq |\lambda|e^{-M}$, it follows from Koebe’s distortion theorem that there exists a unique holomorphic branch $f^{-s_k(z)} : B(f^{s_k(z)}(z), 4\delta_z)$ of $f^{-s_k(z)}$ sending $f^{s_k(z)}(z)$ to $z$, where $s_k(z) = n_k(z) + 1$ and

$$\delta_z = \delta(M) = \frac{1}{16}|\lambda|e^{-M} \min\{\pi, \text{dist}(\{f^{n_k(z)}(z)\}_{k=1}^\infty, \beta_0^\infty)\}.$$ 

Remark 3.10. Note that from (3.23) one can deduce that the radius $\delta(M_k)$ in the above construction can be chosen to be independent of a point $z \in Q \setminus \bigcup_{j=0}^\infty F^{-j}(E_{M_k})$. We shall use this property in the proof of Theorem 5.2.

Let $z \in J_{r,M}$. Then $f^{n_k(z)}(z) \in Q_{M}$ for all $k \geq 1$. Hence

$$|f^{s_k(z)}(z)| = |f(f^{s_k(z)}(z))| = |\lambda| \exp(\text{Re}(f^{n_k(z)}(z))) \leq |\lambda|e^{M}.$$ 

Therefore, assuming $M$ to be large enough, we have that

$$B(f^{s_k(z)}(z), 4\delta_z) \subset B(0, |\lambda|e^{2M}).$$ 

Definition 3.11. Let $z \in J_r = \bigcup_{M} J_{r,M}$. Passing to a subsequence, we may assume that the limit $\lim_{k \to \infty} f^{s_k(z)}(z)$ exists and belongs to $B(0, |\lambda|e^{M})$ if $z \in J_{r,M}$. This limit will be denoted by $y(z)$.

We shall prove now the following.

Lemma 3.12. For every $M > 0$ and every $z \in J_{r,M}$

$$\lim_{k \to \infty} |(f^{n_k(z)})'(z)| = \lim_{k \to \infty} |(f^{s_k(z)})'(z)| = +\infty.$$ 

Proof. The idea of this proof is the same as that of the proof of Lemma 2.2. Put $s_k = s_k(z), k \geq 1$. Suppose on the contrary that $\liminf_{k \to \infty} |(f^{s_k})'(z)| < +\infty$. Without loss of generality we may assume that $\tau = \lim_{k \to \infty} |(f^{s_k})'(z)| < +\infty$ and that $f^{s_k}(z) \in B(y(z), \delta_z)$ for all $k \geq 1$. Consider the family $\{f^{-s_k} : B(y(z), 3\delta_z) \to \mathbb{C}\}_{k \geq 1}$ of holomorphic inverse branches of $f^{s_k}$ sending $f^{s_k}(z)$ to $z$. Applying 4-Koebe’s distortion theorem, we see that $f^{-s_k}(B(y(z), 3\delta_z)) \supset B(z(8r)^{-1})$ for all $k \geq 1$ large enough. Thus $f^{s_k}(B(z(8r)^{-1})) \subset B(y(z), 3\delta_z)$, and consequently the family of maps $f^{s_k} : B(z(8r)^{-1}) \to \mathbb{C}$ is normal, which contradicts the fact that $z \in J(F)$ and shows that $\lim_{k \to \infty} |(f^{s_k})'(z)| = +\infty$. Since $|f'|$ is uniformly bounded
on \( Q_M \) and since \( s_k = n_k + 1 \), we also conclude that \( \lim_{k \to \infty} |(f^{n_k})'(z)| = +\infty \). We are done. \qed

**Lemma 3.13.** Let \( \nu \) be an arbitrary conformal measure for \( F \). Then for every non-empty open set \( U \subset Q \), we have
\[
\lim_{n \to \infty} \nu(F^n(U)) = 1.
\]

**Proof.** Let \( \tilde{U} \) be a connected component of \( \pi^{-1}(U) \). Since periodic points of \( f \) are dense in \( \mathbb{C} \), \( \tilde{U} \) contains a repelling periodic point \( \omega \). Denote the period of \( \omega \) by \( p \). There then exists an open ball \( W \subset \tilde{U} \) centered at \( \omega \) such that \( f^p(W) \supset W \). Since \( \omega \in \mathbb{C} = J(f^p) \), \( \bigcup_{n \geq 0} f^n(W) \supset \mathbb{C} \setminus \{0\} \). Hence for every \( n \geq 1 \)
\[
F^n(\pi(W)) = \pi(f^n(W)) \supset \pi(W)
\]
and
\[
\bigcup_{n \geq 0} F^n(\pi(W)) = \bigcup_{n \geq 0} \pi(f^n(W)) = \pi \left( \bigcup_{n \geq 0} f^n(W) \right) \supset \pi(\mathbb{C} \setminus \{0\}) = Q.
\]
Thus
\[
\lim_{n \to \infty} \nu(F^n(\pi(W))) = \nu \left( \bigcup_{n \geq 0} F^n(\pi(W)) \right) = \nu(Q) = 1.
\]
Since \( \pi(W) \subset \pi \tilde{U} = U \), we are done. \qed

**Corollary 3.14.** If \( \nu \) is a conformal measure for \( F \) and \( U \subset Q \) is an arbitrary open set, then \( \nu(U) > 0 \).

Now, we are ready to conclude the following theorem. After the above preparation, the proof is rather standard. It follows the idea of the analogous theorem in [13]. However, we present it here for the sake of completeness and since some details are different.

**Theorem 3.15.** The \( h \)-conformal measure \( m \) is a unique probability \( t \)-conformal measure for \( F : J(F) \to J(F) \) with \( t > 1 \). In addition, \( m \) is conservative and ergodic.

**Proof.** For all \( s, l \geq 1 \) put
\[
Z_{s,l} = \{ z \in J_r,a : \delta_z(s) \geq 1/l \}
\]
(see [12] for the definition of \( \delta_z(s) \)). Fix \( z \in Z_{s,l} \). Recall that, by Definition [12], \( y(z) = \lim_{k \to \infty} f^{s_k}(z) \). Without loss of generality we may assume that \( |y(z) - f^{s_k}(z)| < (4Kl)^{-1} \) for all \( k \geq 1 \). Consider the holomorphic inverse branches \( F_z^{-s_k(z)} : B(\pi(y(z)), 3/l) \to Q \) sending \( F^{s_k}(z) \) to \( z \). Suppose that \( \nu \) is an arbitrary \( t \)-conformal measure with \( t > 1 \). Since, by the \( \frac{1}{2t} \)-Koebe distortion theorem and the standard version of Koebe’s distortion theorem,
\[
F_z^{-s_k(z)}(B(\pi(y(z)), 3/l)) \supset F_z^{-s_k(z)}(B(F^{s_k}(z), 3/l)) \supset B \left( z, \frac{1}{2t} |(F^{s_k}(z))'(z)|^{-1} \right)
\]
and
\[ F_{z}^{-s_{k}(z)} \left( B \left( \pi(y(z)), \frac{1}{4Kl} \right) \right) \subset F_{z}^{-s_{k}(z)} \left( B \left( F^{s_{k}(z)}(z), \frac{1}{2Kl} \right) \right) \subset B \left( z, \frac{1}{2l} |(F^{s_{k}(z)})'(z)|^{-1} \right), \]

using the conformality of the measure \( \nu \) along with the standard version of Koebe’s distortion theorem, and the fact that \( \inf \{ \nu(B(w,(2Kl)^{-1})) : w \in B(0,|\lambda e^{s}) \} > 0 \), we deduce that
\[ B(\nu, l, s)^{-1}r_{k}(z)^{t} \leq \nu(B(z, r_{k}(z))) \leq B(\nu, l, s)r_{k}(z)^{t}, \tag{3.26} \]
where \( r_{k}(z) = (2l)^{-1} |(F^{s_{k}(z)})'(z)|^{-1} \) and \( B(\nu, l, s) \) depends only on \( \nu, l \) and \( s \). Now fix \( E \), an arbitrary bounded Borel set contained in \( Z_{s,l} \). Since \( m \) is regular, for every \( x \in E \) there exists a radius \( r(x) > 0 \) of the form \( r_{k}(x) \) such that
\[ m(\bigcup_{x \in E} B(x, r(x)) \setminus E) < \varepsilon. \tag{3.27} \]

Now by the Besicović theorem (see \[ \text{[Gu]} \]) we can choose a countable subcover \( \{ B(x_{i}, r(x_{i})) \}_{i=1}^{\infty}, r(x_{i}) \leq \varepsilon, \) from the cover \( \{ B(x, r(x)) \}_{x \in E} \) of \( E \), of multiplicity bounded by some constant \( C \geq 1 \), independent of the cover. Therefore by (3.26) and (3.27), we obtain
\[ \nu(E) \leq \sum_{i=1}^{\infty} \nu(B(x_{i}, r(x_{i}))) \leq B(\nu, l, s) \sum_{i=1}^{\infty} r(x_{i})^{t} \]
\[ \leq B(\nu, l, s)B(m, l, s) \sum_{i=1}^{\infty} r(x_{i})^{t-h} m(B(x_{i}, r(x_{i}))) \]
\[ \leq B(\nu, l, s)B(m, l, s)C \varepsilon^{t-h} m\left( \bigcup_{i=1}^{\infty} B(x_{i}, r(x_{i})) \right) \]
\[ \leq CB(\nu, l, s)B(m, l, s) \varepsilon^{t-h}(\epsilon + m(E)). \tag{3.28} \]

In the case when \( t > h \), letting \( \varepsilon \searrow 0 \) we obtain \( \nu(Z_{s,l}) = 0 \). Since \( J_{t} = \bigcup_{s \geq 1} \bigcup_{l \geq 1} Z_{s,l} \), we thus conclude that \( \nu(J_{t}) = 0 \). This contradiction shows that \( t \leq h \). If \( t < h \), then exchanging the role of \( \nu \) and \( m \) in the above reasoning \( , \) we would get \( m(J_{t}) = 0 \). Thus \( t = h \). Then (3.28) with exchanged roles of measures \( m \) and \( \nu \) show that the measures \( \nu \) and \( m \) are equivalent.

Let us now prove that any \( h \)-conformal measure \( \nu \) is ergodic. Indeed, suppose to the contrary that \( F^{-1}(G) = G \) for some Borel set \( G \subset J(F) \) with \( 0 < m(G) < 1 \). But then the two conditional measures \( \nu_{G} \) and \( \nu_{J(F) \setminus G} \),
\[ \nu_{G}(B) = \frac{\nu(B \cap G)}{\nu(G)}, \quad \nu_{J(F) \setminus G}(B) = \frac{\nu(B \cap (J(F) \setminus G))}{\nu(J(F) \setminus G)}, \]
would be \( h \)-conformal and mutually singular; a contradiction.

If now \( \nu \) is again an arbitrary \( h \)-conformal measures, then by a simple computation based on the definition of conformal measures we see that the Radon-Nikodym derivative \( \phi = d\nu/dm \) is constant on grand orbits of \( F \). Therefore by ergodicity of \( m \) we conclude that \( \phi \) is constant \( m \)-almost everywhere. As both \( m \) and \( \nu \) are probability measures, it implies that \( \phi = 1 \) a.e., hence \( \nu = m \).
It remains to show that \( m \) is conservative. We shall first prove that every forward invariant \((F(E) \subset E)\) subset \( E \) of \( J(F) \) is either of measure 0 or 1. Indeed, suppose to the contrary that \( 0 < m(E) < 1 \). In view of the second part of Theorem 3.3, it suffices to show that

\[
m(E \cap J_{r,M}) = 0,
\]

where \( M \) comes from Theorem 3.3. Let

\[
Z = \left\{ z \in E \cap J_{r,M} : \lim_{r \to 0} \frac{m(B(z,r) \cap E \cap J_{r,M})}{m(B(z,r))} = 1 \right\}.
\]

In view of the Lebesgue density theorem (see for example Theorem 2.9.11 in [Fe]), \( m(Z) = m(E) \). Since \( m(E) > 0 \) we find at least one point \( z \in Z \). Let \( \{u_k(z)\}_{k=1}^\infty \) be a sequence associated to \( z \) by virtue of the definition of the set \( J_{r,M} \). Let \( \delta_z \) be the number defined in formula (3.24) and let \( y(z) \) be defined as in Definition 3.11. Put \( \eta = \delta_z/8 \). Suppose that \( m(B(y(z),\eta) \setminus E) = 0 \). By conformality of \( m \), \( m(F(Y)) = 0 \) for all Borel sets \( Y \) such that \( m(Y) = 0 \). Hence,

\[
0 = m(F^n(B(y(z),\eta) \setminus E)) \geq m(F^n(B(y(z),\eta)) \setminus F^n(E))
\]

\[
\geq m(F^n(B(y(z),\eta) \setminus E)) \geq m(F^n(B(y(z),\eta)) - m(E).
\]

It therefore follows from Lemma 3.13 that \( 0 \geq 1 - m(E) \), which is a contradiction. Consequently \( m(B(x,\eta) \setminus E) > 0 \). Hence for every \( j \geq 1 \) large enough,

\[
m(B(F^{n_j}(z),2\eta) \setminus E) \geq m(B(y(z),\eta) \setminus E) > 0.
\]

Therefore, as \( F^{-1}(J(F) \setminus E) \subset J(F) \setminus E \), the standard application of Koebe’s Distortion Theorem shows that

\[
\limsup_{r \to 0} \frac{m(B(z,r) \setminus E)}{m(B(z,r))} > 0
\]

which contradicts (3.29). Thus either \( m(E) = 0 \) or \( m(E) = 1 \).

Now conservativity is straightforward. One needs to prove that for every Borel set \( B \subset J(F) \) with \( m(B) > 0 \) one has \( m(G) = 0 \), where

\[
G = \{ x \in J(F) : \sum_{n \geq 0} \chi_B(f^n(x)) < +\infty \}.
\]

Indeed, suppose that \( m(G) > 0 \), and for all \( n \geq 0 \) let

\[
G_n = \{ x \in J(F) : \sum_{k \geq n} \chi_B(f^n(x)) = 0 \} = \{ x \in J(F) : f^k(x) \notin B \text{ for all } k \geq n \}.
\]

Since \( G = \bigcup_{n \geq 0} G_n \), there exists \( n_0 \geq 0 \) such that \( m(G_{n_0}) > 0 \). Since all the sets \( G_n \) are forward invariant we conclude that \( m(G_{n_0}) = 1 \). But on the other hand all the sets \( F^{-n}(B) \), \( n \geq k \), are of positive measure and are disjoint from \( G_{n_0} \). This contradiction finishes the proof of conservativity of \( m \).

\[\square\]

4. INVARINT MEASURE

In this section we show the existence and uniqueness of a probability \( F \)-invariant measure equivalent to \( m \). We first prove the following.
Lemma 4.1. Up to a multiplicative constant there exists a unique \( F \)-invariant, \( \sigma \)-finite measure \( \mu \), which is conservative, ergodic and equivalent to the \( h \)-conformal measure \( m \).

The idea of the proof of Lemma 4.1 is to apply a general sufficient condition for the existence of a \( \sigma \)-finite absolutely continuous invariant measure proven in [Ma]. In order to formulate this condition suppose that \( X \) is a \( \sigma \)-compact metric space, \( m \) is a Borel probability measure on \( X \), positive on open sets, and that a measurable map \( T : X \to X \) is given with respect to which measure \( m \) is quasi-invariant, i.e. \( m \circ T^{-1} \ll m \). Moreover we assume the existence of a countable partition \( \alpha = \{ A_n : n \geq 0 \} \) of subsets of \( X \) which are all \( \sigma \)-compact and of positive measure \( m \). We also assume that \( m(X \setminus \bigcup_{n \geq 0} A_n) = 0 \), and if additionally for all \( m, n \geq 1 \) there exists \( k \geq 0 \) such that

\[
(4.1) \quad m(T^{-k}(A_m) \cap A_n) > 0,
\]

then the partition \( \alpha \) is called irreducible. Martens’ result comprising Proposition 2.6 and Theorem 2.9 of [Ma] is as follows.

Theorem 4.2. Suppose that \( \alpha = \{ A_n : n \geq 0 \} \) is an irreducible partition for \( T : X \to X \). Suppose that \( T \) is conservative and ergodic with respect to the measure \( m \). If for every \( n \geq 1 \) there exists \( K_n \geq 1 \) such that for all \( k \geq 0 \) and all Borel subsets \( A \) of \( A_n \)

\[
(4.2) \quad K_n^{-1} m(A) \leq \frac{m(T^{-k}(A))}{m(T^{-k}(A_n))} \leq K_n \frac{m(A)}{m(A_n)},
\]

then \( T \) has a \( \sigma \)-finite \( T \)-invariant measure \( \mu \) absolutely continuous with respect to \( m \). Additionally \( \mu \) is equivalent to \( m \), conservative and ergodic, and unique up to a multiplicative constant.

Proof of Lemma 4.1 (sketch). Since in the sequel we will not only need Lemma 4.1 but a bit more, namely the way in which the invariant measure claimed in Theorem 4.2 is produced, we shall also describe this procedure briefly. Following Martens, one considers the following sequences of measures:

\[
S_k m = \sum_{i=0}^{k-1} m \circ T^{-i} \quad \text{and} \quad Q_k m = \frac{S_k m}{S_k m(A_0)}.
\]

It is proven in [Ma] that each weak limit \( \mu \) of the sequence \( Q_k m \) has the properties required in Theorem 4.2, where a sequence \( \{ \nu_k : k \geq 1 \} \) of measures on \( X \) is said to converge weakly if for all \( n \geq 1 \) the measures \( \nu_k \) converge weakly on all compact subsets of \( A_n \). In fact it turns out that the sequence \( Q_k m \) converges and

\[
\mu(F) = \lim_{n \to \infty} Q_k m(F)
\]

for every Borel set \( F \subseteq X \). Of course \( \mu(A) \leq 1 < \infty \). Making use of (4.1) and (4.2) one proves (see Lemma 2.4 in [Ma]) the following two lemmas.

Lemma 4.3. For every \( n \geq 0 \) we have \( 0 < \mu(A_n) < \infty \), even more, the Radon-Nikodym derivative \( \frac{d\mu}{dm} \) is bounded on \( A_n \).
Lemma 4.4. For all $i, j \geq 0$ there exists a constant $\kappa > 0$ such that
\[
\frac{S_{n,m}(D)}{S_{n,m}(E)} \leq \kappa \frac{m(D)}{m(E)}
\]
for all $n \geq 1$ and all Borel sets $D \subset A_i$ and $E \subset A_j$.

Let us now pass to our map $F : Q \to Q$. The ergodicity and conservativity of the measure $m$ is proven in Theorem 3.13. Thus, we only need to construct an irreducible partition $\alpha$ with property (4.2). Indeed, set $Y = J(F) \setminus \hat{\beta}_1^\infty$. For every $y \in Y$ consider a ball $B(y, r(y)) \subset Q$ such that $r(y) > 0, m(\partial B(y, r(y))) = 0$, and $r(y) < (1/2) \text{dist}(y, \hat{\beta}_1^\infty)$. The balls $B(y, r(y)), y \in Y$, cover $Y$, and, obviously, one can choose a countable cover, say $\{\hat{A}_n : n \geq 0\}$, from them. We may additionally require that the family $\{\hat{A}_n : n \geq 0\}$ is locally finite; that is that each point $x \in Y$ has an open neighborhood intersecting only finitely many balls $\hat{A}_n, n \geq 0$. We now define the family $\alpha = \{A_n : n \geq 0\}$ inductively setting
\[
A_0 = \hat{A}_0 \quad \text{and} \quad A_{n+1} = \hat{A}_{n+1} \setminus \bigcup_{k=1}^{n} \hat{A}_n
\]
(and throwing away empty sets). Obviously $\alpha$ is a disjoint family and
\[
\bigcup_{n \geq 1} A_n \supset J(F) \setminus (\hat{\beta}_1^\infty \cup \bigcup_{n \geq 0} \partial \hat{A}_n).
\]
Hence, in view of the last assumption of our theorem, $m(\bigcup_{n \geq 0} A_n) = 1$. The distortion condition (4.2) now follows from Koebe’s distortion theorem with all constants $K_n = K$, and irreducibility of partition $\alpha$ follows from openness of the sets $A_n$ and Lemma 4.13.

For the proof of Theorem 4.6, the main result of this section, we will need the following.

Lemma 4.5. There exists $R \in (0, \pi/2)$ such that for every $t > 0$ there exists a constant $C(t) > 0$ such that
\[
m(B(\hat{\beta}_n, r)) \leq C(t)r^t|(F^n)'(0)|^{-h-t}
\]
for all $n \geq 0$ and all $r \in (0, 1)$.

Proof. Combining (3.12) and (3.13), we see that for every $k \geq 2$
\[
m(B(0, r_k)) \leq |\beta_{k+2}|^{-\kappa},
\]
where $\kappa$ is the fixed positive number, introduced in (3.9) and $r_k = (4|\beta_1||\beta_2| \cdots |\beta_k|)^{-1}|\beta_{k-1}|^{-1}$. Consider an arbitrary radius $r \in (0, r_2]$. Then $r_{k+1} < r \leq r_k$ for some $k \geq 2$. Hence, using our super-growing condition (4.1) and Lemma 3.5, we easily get
\[
(4.3) \quad m(B(0, r)) \leq m(B(0, r_k)) \leq |\beta_{k+2}|^{-\kappa} \leq C_0(t)r_{k+1}^t \leq C_0(t)r^t
\]
for some constant $C_0(t)$, and we are done with the case $n = 0$. Since $\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} |\beta_n| = +\infty$, the set $\hat{\beta}_n$ has no accumulation point in $Q$, and there exists $R > 0$ such that holomorphic inverse branches $F_0^{-n} : B(\hat{\beta}_n, 2R) \to Q$ of $F^n$ sending $\hat{\beta}_n$ to 0 are well-defined for all $n \geq 1$. Hence, using Koebe’s distortion theorem and $h$-conformality of the measure $m$, we get for every $r \in (0, R)$ that
\[
m(B(0, K|(F^n)'(0)|^{-1} r)) \geq m(B_0^{-n}(B(\hat{\beta}_n, r))) \geq K^{-h-1}|(F^n)'(0)|^{-h}m(B(\hat{\beta}_n, r)).
\]
Hence
\[ m(B(\hat{\beta}_n, r)) \leq K^n |(F^n)'(0)| h m(B(\hat{0}, K |(F^n)'(0)|^{-1}r)) \]
\[ \leq K^n C_0(t)(|F^n)'(0)| h K r |(F^n)'(0)|^{-1} t = C(t)r^t |(F^n)'(0)| h^{-t}, \]
where \( C(t) = K^{h+t}C_0(t) \).

Our next, technically most involved, goal is to prove the following main result of this section.

**Theorem 4.6.** The \( \sigma \)-finite, \( F \)-invariant measure \( \mu \) equivalent to the \( h \)-conformal measure \( m \), produced in Lemma 4.5, is finite.

**Proof.** Let \( \{ A_n \}_{n \geq 0} \) be the irreducible partition constructed just before Lemma 4.5. We may assume without loss of generality that \( A_0 = B(z', \xi) \subset B(0, R) \) for some \( z' \in Q \) and some \( \xi \in (0, R) \). Fix \( r \in (0, R] \). Decreasing \( r > 0 \) if necessary, we may assume that
\[ \pi^{-1}(z') \cap \bigcup_{k=0}^{\infty} B(\beta_k, r) = \emptyset. \]
We shall write \( z' + 2\pi ij \) to denote the unique point in \( \pi^{-1}(z') \) with imaginary part in the interval \([2\pi j, 2\pi(j + 1)) \). Let
\[ D_{n,j} := B(2\pi ij, |4\beta_n|). \]
Now take \( n \geq 1 \) so large that for every \( j \in \mathbb{Z} \)
\[ D_{n,j} \cap \{ \beta_k : k \geq n + 1 \} = \emptyset. \]
The last property is guaranteed by the super-growing condition (see (1.1) and (1.2)). Fix a point
\[ z'' \in B(2\pi ij, 2|\beta_n|) \setminus \bigcup_{k=0}^{n} B(\beta_k, r). \]
It then follows from Lemma 3, p. 152 in [Ha] that there exists a simply connected open set \( D'_{n,j} \subset D_{n,j} \) such that \( z'_j + 2\pi ij, z'' \in D'_{n,j}, \beta_1, \beta_2, \ldots, \beta_n \notin D'_{n,j} \) and for all \( n \geq 1 \) large enough
\[ \rho_h(z'_j + 2\pi ij, z''; D'_{n,j}) \leq \rho_h(z'_j + 2\pi ij, z''; D_{n,j}) + \frac{1}{2} A \left( n + \log \frac{1}{r} \right) \]
\[ \leq A + \frac{1}{2} A \left( n + \log \frac{1}{r} \right) \]
\[ \leq A \left( n + \log \frac{1}{r} \right), \]
where \( A > 0 \) is some universal constant and \( \rho_h \) is the hyperbolic metric in respective domain \( D'_{n,j} \) or \( D_{n,j} \). Consequently, it follows from Koebe’s distortion theorem that if \( H : D'_{n,j} \rightarrow \mathbb{C} \) is a univalent holomorphic function, then
\[ \frac{|H'(z'_j + 2\pi ij)|}{|H'(z'')|} \leq \exp \left( 6A \left( n + \log \frac{1}{r} \right) \right) \leq e^{6An \left( 1 \right)} \]
\[ \leq e^{6An \left( \frac{1}{r} \right)} \]
Now consider an arbitrary geometric square \( S \subset Q \) with diagonal of length \( \leq r \) and the center \( w \notin B(\hat{\beta}_0^\infty, r) \). In particular \( 2S \cap \hat{\beta}_0^\infty = \emptyset. \)
Definition 4.7. Let $n = n(S)$ be the least integer such that $S \subset Q_{\alpha_n(S)}$.

Notice that all holomorphic branches of all backward iterates of $F$ are well defined on $2S$. Fix one such branch $F_{\nu}^{-k} : 2S \to Q$. Then $F_{\nu}^{-k} = \pi \circ f_{\nu}^{-k} \circ \pi_{\nu}^{-1}$, where $\pi_{\nu}^{-1}$ is an appropriate holomorphic inverse branch of $\pi$ defined on $2S$, and $f_{\nu}^{-k} : \pi_{\nu}^{-1}(2S) \to \mathbb{C}$ is an appropriate holomorphic inverse branch of $f^k$. Write $\pi_{\nu}^{-1}(w) = w + 2\pi ij$ for some $j \in \mathbb{Z}$ and note that $\pi_{\nu}^{-1}(w) \in \pi_{\nu}^{-1}(2S) \subset B(2\pi ij, 2|\beta_n(S)|)$ and $\pi_{\nu}^{-1}(w) \notin B(\beta_0, r)$. Now apply considerations leading to (4.4) with $z' = \pi_{\nu}^{-1}(w)$. Since $D'_{n,j}$ is simply connected, since $D'_{n,j} \cap \beta_0 = \emptyset$ and since $\pi_{\nu}^{-1}(w) \in \pi_{\nu}^{-1}(2S) \cap D'_{n,j}$, there exists a unique holomorphic continuation $f_{\nu}^{-k} : D'_{n,j} \to \mathbb{C}$ of the branch $f_{\nu}^{-k} : \pi_{\nu}^{-1}(2S) \cap D'_{n,j}$. So,

$$f_{\nu}^{-k}|_{\pi_{\nu}^{-1}(2S) \cap D'_{n,j}} = f_{\nu}^{-k}|_{\pi_{\nu}^{-1}(2S) \cap D'_{n,j}}.$$ 

In particular $f_{\nu}^{-k}(\pi_{\nu}^{-1}(w)) = f_{\nu}^{-k}(\pi_{\nu}^{-1}(w))$. Since $(2A_0 + 2\pi ij) \cap \beta_0 = \emptyset$ and since $z' + 2\pi ij \in (A_0 + 2\pi ij) \cap D'_{n,j}$, there exists a unique holomorphic inverse branch $f_{\nu}^{-k} : 2A_0 + 2\pi ij \to \mathbb{C}$ such that $f_{\nu}^{-k} \big|_{(A_0 + 2\pi ij) \cap D'_{n,j}} = f_{\nu}^{-k} \big|_{(A_0 + 2\pi ij) \cap D'_{n,j}}$. In particular $f_{\nu}^{-k}(z' + 2\pi ij) = f_{\nu}^{-k}(z' + 2\pi ij)$. Let $F_{\nu}^{-k} : 2A_0 \to Q$ be defined by the formula $F_{\nu}^{-k} = \pi \circ f_{\nu}^{-k} \circ \pi_{\nu}^{-1}$, where $\pi_{\nu}^{-1}(z) = z + 2\pi ij$. Notice that the mapping $F_{\nu}^{-k} \to F_{\nu}^{-k}$ is a bijection between the set of all holomorphic inverse branches of $F^k$ defined on $2S$ and those defined on $2A_0$. Applying (4.4) to the map $f_{\nu}^{-k} : D'_{n,j} \to \mathbb{C}$, and using Koebe’s distortion theorem to the maps $f_{\nu}^{-k} : \pi_{\nu}^{-1}(2S) \to \mathbb{C}$ and $f_{\nu}^{-k} : 2A_0 + 2\pi ij \to \mathbb{C}$, we obtain for all $x \in S$ and all $y \in A_0$ that

$$\frac{|(F_{\nu}^{-k})'(x)|}{|(F_{\nu}^{-k})'(y)|} \leq K \frac{|(F_{\nu}^{-k})'(w)|}{|(F_{\nu}^{-k})'(z')|} = K^2 \frac{|(F_{\nu}^{-k})'(\pi_{\nu}^{-1}(w))|}{|(F_{\nu}^{-k})'(z' + 2\pi ij)|} \leq e^{6A_0(S)} \left(\frac{1}{r}\right)^{6A}.$$ 

Since $m(F_{\nu}^{-k}(S)) \leq (\sup_S \{|(F_{\nu}^{-k})'|\})^h m(S)$ and since

$$m(F_{\nu}^{-k}(A_0)) \geq \left(\inf_{A_0} \{|(F_{\nu}^{-k})'|\}\right)^h m(A_0),$$

we therefore get

$$\frac{m(F_{\nu}^{-k}(S))}{m(F_{\nu}^{-k}(A_0))} \leq \left(\frac{\sup_S \{|(F_{\nu}^{-k})'|\}}{\inf_{A_0} \{|(F_{\nu}^{-k})'|\}}\right)^h \frac{m(S)}{m(A_0)} \leq e^{6A_0(S)} \left(\frac{1}{r}\right)^{6Ah} m(S).$$

Summing over all branches $\nu$, we thus get

$$m(F^{-k}(S)) \leq \exp(6Ah(S)) \left(\frac{1}{r}\right)^{6Ah} m(F^{-k}(A_0)) m(S).$$

This immediately implies that

$$(4.5) \quad \mu(S) \leq \exp(6Ah(S)) \left(\frac{1}{r}\right)^{6Ah} m(S).$$

Assume now $R \in (0, 1)$ to be so small that all the balls $B(\beta_n, 2R)$, $n \geq 0$, are mutually disjoint. Our goal now is to estimate the measure $\mu$ of the neighbourhood of $\beta_n$. To do so, we divide each ball $B(\beta_n, R)$ into geometric annuli

$$B(\beta_n, \frac{R}{2r}) \setminus B(\beta_n, \frac{R}{2r+1}).$$
Obviously, each annulus can be covered by a bounded (independent of \(k\) and \(n\)) number of balls \(B_{n,k}\) with radius equal to \(R2^{-(k+1)}\). So, consider an arbitrary ball \(B_{n,k} \subset B(\beta_n, \frac{R}{2^n})\), \(k \geq 0\), with radius equal to \(R2^{-(k+1)}\) and the center at the distance from \(\beta_n\) exactly equal to \(2^{-k}R\). It then follows from (4.5) and Lemma 4.5 that
\[
\mu(B_k) \leq C(t) \cdot \exp(6Ahn) \left( \frac{2k}{R} \right)^{6Ah} \left( \frac{R}{2^n} \right)^t \left| (f^n)'(0) \right|^h \leq C(t) \left| (f^n)'(0) \right|^{-u} 2^{-uk}
\]
with an arbitrarily large \(u\) assuming \(t\) to be large enough. Notice now that there exists an integer \(L \geq 1\) so large that appropriately choosing for each \(k \geq 0\), \(L\) balls of the form \(B_{n,k}\), we will cover the punctured disk \(B(\beta_n, R) \setminus \{\beta_n\}\). Hence
\[
\sum_{n \geq 0} \mu(B(\beta_n, R)) \leq L \sum_{n \geq 0} \sum_{k=0}^{\infty} \left| (f^n)'(0) \right|^{-u} 2^{-uk} \leq \sum_{n \geq 0} \left| (f^n)'(0) \right|^{-u} < \infty.
\]
Obviously, there exists an integer \(T \geq 1\) such that each set
\[
W_n = \{z \in Q : \alpha_n \leq \text{Re}z \leq \alpha_{n+1}\} \setminus \bigcup_{k=0}^{\infty} B(\beta_k, R), \ n \geq 0,
\]
can be covered by a family \(\mathcal{F}\) of geometric squares with lengths of diagonals \(\leq R\) and centers lying in \(W_n\). In addition, we may require this family to be of multiplicity bounded above by 4. Then applying (4.5) and (4.7) with \(m_n\) replaced by \(m\) (remember also that each of these squares is contained in \(Y_n^{\pm}\)), we obtain
\[
\mu(W_n) \leq \sum_{S \in \mathcal{F}} \mu(S) \leq \sum_{S \in \mathcal{F}} \exp(6Ah(n+1)) \left( \frac{2}{R} \right)^{6Ah} m(S)
\]
\[
= \exp(6Ah(n+1)) \left( \frac{2}{R} \right)^{6Ah} \sum_{B \in \mathcal{F}} m(S)
\]
\[
\leq \exp(6Ah(n+1)) \left( \frac{2}{R} \right)^{6Ah} 4m \left( \bigcup_{S \in \mathcal{F}} m(S) \right)
\]
\[
\leq 4 \exp(6Ah(n+1)) \left( \frac{2}{R} \right)^{6Ah} m \left( \{z \in Q : \alpha_n \leq \text{Re}z \leq \alpha_{n+1}\} \right)
\]
\[
\leq 4 \exp(6Ah(n+1)) \left( \frac{2}{R} \right)^{6Ah} \exp\left( \frac{1}{2}(1-s)\alpha_n \right)
\]
\[
\leq e^{-u\alpha_n}
\]
for some positive \(u\). The last inequality follows from super-growing condition (1.1). Since the measure \(\mu\) is \(F\)-invariant and since there exists \(N > 0\) so large that \(F(Y_N) \subset B(0, R)\), we conclude from (4.6) that \(\mu(Y_1^-) < \infty\). Combining this along with (4.6) and (4.7), we deduce that
\[
\mu \left( Y_1^- \cup \bigcup_{n=0}^{\infty} (B(\beta_n, R) \cup W_n) \right) < \infty.
\]
Since it is obvious that the complement of this set has a finite \(\mu\)-measure, we are done.
5. Haussdorff and Packing Measures

Let $H^h$ and $P^h$ be respectively the $h$-dimensional Haussdorff and packing measures (see [TT], comp. [PU] for example, for its definition and some basic properties). The results of this section provide in a sense a complete description of the geometrical structure of the sets $J_r(F)$ and $J_r(f)$ and also they exhibit the geometrical meaning of the $h$-conformal measure $m$. The short proof of the first result improves on the argument from the proof of Proposition 4.9 from [UZ1].

**Proposition 5.1.** We have $P^h(J_r(f)) = P^h(J_r(F)) = \infty$. In fact $P^h(G) = \infty$ for every open non-empty subset of $J_r(f)$.

**Proof.** Since $m(J_r(F) \cap Y^-) > 0$ for every $M \in \mathbb{R}$, it follows from Birkhoff’s ergodic theorem, Theorem 4.6 and Theorem 3.15 that there exists a set $E \subset J_r(F)$ such that $m(E) = 1$ and $\liminf_{n \to \infty} \Re F^n(z) = -\infty$ for every $z \in E$. Fix $z \in E$. The above formula means that there exists an unbounded increasing sequence $(n_k)_k$, depending on $z$, such that

\[
\lim_{k \to \infty} \Re(F^{n_k}(z)) = -\infty.
\]

Since $\lim_{n \to \infty} \alpha_n = +\infty$, the balls $B(F^{n_k}(z), 1)$ are for all $k$ large enough, say $k \geq q$, disjoint from the set $\beta_0^{-\infty}$. Fix $k \geq q$ and consider the ball $B(z, K^{-1} |(F^{n_k})'(z)|^{-1})$. Then

\[
B(z, K^{-1} |(F^{n_k})'(z)|^{-1}) \subset F_{-n}^{-1}(B(F^{n_k}(z), 1)),
\]

where $F_{-n}^{-1}$ is the holomorphic inverse branch of $F^n$, defined on $B(F^n(z), 1)$ and mapping $F^n(z)$ to $z$. Applying Koebe’s distortion theorem and conformality of the measure $m$, we obtain

\[
m(B(z, K^{-1} |(F^{n_k})'(z)|^{-1})) \leq K h |(F^{n_k})'(z)|^{-h} m(B(F^{n_k}(z), 1)) \leq K^h (K^{-1} |(F^{n_k})'(z)|^{-1})^h m(Y_{\Re F^{n_k}(z)-1}).
\]

Since by (5.1), $\lim_{k \to \infty} m(Y_{\Re F^{n_k}(z)-1}) = 0$, we see that

\[
\liminf_{r \to 0} \frac{m(B(z, r))}{r^h} = 0.
\]

Since $m(G \cap J_r(F)) > 0$ for every non-empty open subset of $J_r(F)$, this implies (see an appropriate Converse Frostman’s Type Theorem in [DU2] or [PU]) that $P^h(G) = \infty$. Since $J_r(f) = \pi^{-1}((J_r(F)))$ and $\pi$ is a local isometry, we are therefore done. □

**Theorem 5.2.** The $h$-dimensional Hausdorff measure restricted to the set $J_r(F)$ is positive, finite, and equal to the $h$-conformal measure $m$ up to a multiplicative constant.

**Proof.** We shall show first that $H^h|_{J_r(F)}$ is absolutely continuous with respect to $m$ and finite. Fix $z \in J_{r,M}(F)$. This implies (see the considerations after Remark 3.10) that there exists an increasing unbounded sequence $(s_k)_k$ and a positive number $\delta_z(M)$ such that $f^{s_k}(z) \to y(z)$, $|y(z)| < |x|e^M$ and a holomorphic inverse branch of $f^{-s_k}$ sending $F^{s_k}(z)$ to $z$ is well defined on the ball $B(F^{s_k}(z), \delta_z(M))$. Denote by $\pi^{-1}_k : B(F^{s_k}(z), \pi) \to \mathbb{C}$ the holomorphic inverse branch of the projection $\pi : Q \to \mathbb{C}$, which sends $F^{s_k}(z)$ to $F^{s_k}(z)$. Then the composition $F_{-s_k}^{-1} = \pi \circ f_{-s_k}^{-1} \circ \pi^{-1}_k : B(F^{s_k}(z), \delta_z(M)) \to \mathbb{C}$ is well-defined and forms a holomorphic inverse branch of
$F^{s_k}$ sending $F^{s_k}(z)$ to $z$. In this way, taking an appropriate component of preimage $F^{-s_k}(B(F^{s_k}(z), \delta_z(M)))$ we get a neighbourhood of $z$, contained in a ball of radius $K|(F^{s_k})'(z)|^{-1}\delta_z(M)$. This allows us to construct, for every point $z \in J_{r,M}$ a sequence of balls of radii $r_k(z)$ converging to 0 (see Lemma 3.12) for which we can estimate, using conformality of the measure $m$:

$$m(B(z, r_k(z))) \geq r_k^h \inf_{\Re w \leq |\exp(M)|} m(B(w, \delta_z(M))).$$

This is enough to conclude (see an appropriate Converse Frostman’s Type Theorem in [DU2] or [PU]) that for every $M$ the measure $H^h_{J_{r,M}}$ is absolutely continuous with respect to $m$ and, consequently, $H^h_{J_{r,F}}$ is absolutely continuous with respect to $m$. Now, there exists $M = M_k$ so that $m(J_{r,M}) = 1$ (see Theorem 5.5). Moreover, there exists a positive number $\delta_M$ such that for $m$ almost every point $z \in J_{r,M}$, $\delta_z(M) \geq \delta_M$ (see Remark 5.10). So, there exists a set $H \subset J_{r,M}$ such that $m(H) = 1$ and for each point $z \in H$ one can find a sequence of radii $r_k(z) \to 0$ such that

$$m(B(z, r_k(z))) \geq r_k^h \inf_{\Re w \leq |\exp(M)|} m(B(w, \delta_M)) \geq C r_k^h,$$

where the constant $C$ does not depend on $z$. This implies in a standard way that the $h$-dimensional measure of the set $H$ is finite. Since $m(J_{r,M} \setminus H) = 0$, we conclude that $H^h(J_{r,M} \setminus H) = 0$ (we already know that $H_k$ is absolutely continuous with respect to $m$). Consequently, $H^h(J_{r,M}) = H^h(H) < \infty$. Finally, we know from Theorem 5.5 that $m(J_{r,F} \setminus J_{r,M}(F)) = 0$. Therefore (again, by absolute continuity)

$$H^h(J_{r,F} \setminus J_{r,M}(F)) = 0.$$

This proves that the $h$-dimensional Hausdorff measure of $J_{r,F}$ is finite.

We shall now prove that $H^h(J_{r,F}) > 0$. Since $\beta_n$ tends to infinity fast (see (1.1)), there exists $1 > \theta > 0$ such that $|\beta_j - \beta_i| > \theta$ for $i \neq j$. So, fix $z \in J_r$ and $r \in (0, \pi)$. Since, by Lemma 3.12, $\lim \sup_{n \to \infty} |(f^n)'(z)| = +\infty$, there exists a least $n = n(z, r) \geq 0$ such that

$$r|{(f^{n+1})}'(z)| \geq \theta(32K)^{-1}.$$

Thus

$$(5.2) \quad r|{(f^n)'(z)| < \theta(32K)^{-1}.$$
and therefore $|f^k(z)| < 32Kr|(f^k)'(z)|$. On the other hand, since $r|(f^{k-1})'(z)| < (32K)^{-1}$, we conclude that
\[
32Kr|(f^k)'(z)| = 32Kr|(f^{k-1})'(z)| \cdot |f'(f^{k-1}(z))| = 32Kr|(f^{k-1})'(z)| \cdot |f^k(z)| < \theta |f^k(z)| < |f^k(z)|.
\]
This contradiction shows that the holomorphic inverse branch
\[
f_z^n : B(f^n(z), 32r|(f^n)'(z)|) \to \mathbb{C}
\]
of $f^n$ sending $f^n(z)$ to $z$, is well-defined. Now, the map $f$, restricted to $B(f^n(z), 32r|(f^n)'(z)|)$, is 1-to-1, and by Koebe’s $\frac{1}{4}$-Theorem
\[
f(B(f^n(z), 32r|(f^n)'(z)|)) \supset B(f^{n+1}(z), 8r|(f^{n+1})'(z)|).
\]
Hence there exists a unique holomorphic inverse branch
\[
f_z^{-(n+1)} : B(f^{n+1}(z), 8r|(f^{n+1})'(z)|) \to \mathbb{C}
\]
of $f^{n+1}$ mapping $f^{n+1}(z)$ to $z$. Applying Koebe’s $\frac{1}{4}$-Theorem again, we see that
\[
f_z^{-(n+1)}(B(f^{n+1}(z), 4r|(f^{n+1})'(z)|)) \supset B(z, r).
\]
Since the ball $B(f^{n+1}(z), 4r|(f^{n+1})'(z)|)$ intersects at most $\frac{1}{4\pi} 4r|(f^{n+1})'(z)| + 1 \leq r|(f^{n+1})'(z)|$ horizontal strips of the form $2\pi i j + (\mathbb{R} \times [0, 2\pi])$, $j \in \mathbb{Z}$, using (5.3) Koebe’s Distortion Theorem, $h$-conformality of the measure $m$ and, at the end, (5.2), we get
\[
r^{-h}(m(B(z, r))) \leq r^{-h}K^h|(f^{n+1})'(z)|^{-h}(r|(f^{n+1})'(z)|)
\times m(\pi(B(f^{n+1}(z), 4r|(f^{n+1})'(z)|)))
\leq r^{-h}K^h|(f^{n+1})'(z)|^{-h}(r|(f^{n+1})'(z)|)
= K^h(r|(f^{n+1})'(z)|)^{1-h} \leq K^h(\theta)^{1-h}(32K)^{h-1}.
\]
That $H^h(J_r(F)) > 0$ follows now from an appropriate Converse Frostman’s Type Theorem in [DU2] or [PU]. Since the measure $H^h$ is $h$-conformal, the last assertion of our theorem follows from Theorem 3.15. 

**Theorem 5.3.** We have $HD(J_r(f)) = HD(J_r(F)) = h \in (1, 2)$.

**Proof.** It follows immediately from Theorem 5.2 that $HD(J_r(f)) = HD(J_r(F)) = h$. We know already that $h > 1$. In order to prove that $h < 2$, let us recall that it follows from Proposition 5.1 that for every open set $Z$ such that $Z \cap J_r(f) \neq \emptyset$ we have $P^h(J_r(f) \cap Z) = \infty$. In particular, if $Z$ is a ball, assuming $h = 2$ we get $\infty = P^2(Z \cap J_r(f)) \leq P^2(Z)$, thus $P^2(Z) = \infty$, which is a contradiction.

Finally, let us notice that the following result of Lyubich (Ly) and Rees (Re) can be deduced as a corollary:

**Corollary 5.4.** If $\lambda$ is a super-growing parameter, then for Lebesgue almost every point $z \in \mathbb{C}$, $\omega(z) = \beta_0^\infty \cup \{\infty\}$.

**Proof.** Since $HD(J_r(f)) < 2$, the complement of $J_r$ is a set of full Lebesgue measure. Fix a point $z \notin J_r(f)$. By the definition of $J_r(f)$, this implies that $\omega(z) \subset \{\infty\} \cup \beta_0^\infty$. We only have to check that, actually, the equality holds for almost every point. So, assume that $\omega(z) = \infty$. The set of such points has Lebesgue measure 0; actually, this is true for a large class of maps; see e.g. [McM] or [EL]. Next, assume that
\( \omega(z) = \{ \infty \} \cup \beta_k^\infty \) for some \( k > 0 \). Thus, there exists an infinite sequence of integers \( s_i \) such that \( f^{s_i}(z) \to \beta_k \). Then, denoting \( n_i = s_i - 1 \), we see that \( \text{Re} f^{n_i}(z) \to \text{Re} \beta_k \) and, moreover, \( \text{dist}(f^{n_i}(z), \beta_k^\infty) > 0 \). Consequently, \( z \in J_r(f) \), a contradiction. We are done.

Remark 5.5. Actually, the result in Lyubich’s paper is stated for \( \lambda = 1 \), but his proof extends in a straightforward way to all maps \( f_\lambda \) with super-growing parameter \( \lambda \). So, the statement of the corollary is not new.

It is natural to ask about the dependence of the dimension \( \text{HD}(J_r(f_\lambda)) \) on \( \lambda \). Below, we sketch a proof of one partial result: Denote by \( M_{c,N} \) the set of parameters for which the condition (1.1) holds with a constant \( c \) and for all \( n > N \). Let \( \lambda \in M_{c,N} \). Using the arguments of [UZ1], one can easily check that there exists a neighbourhood \( U \) of \( \lambda \in \mathbb{C} \) and a constant \( s > 1 \) such that \( \text{HD}(J_M(f_\mu)) > s \) for every \( \mu \in U \). This, in turn implies that \( h_\mu = \text{HD}(J_r(f_\mu)) > s \). Following the way of proof of Lemma 5.4, we can show that the family of conformal measures \( m_\mu, \mu \in M_{c,N} \cap U \) is tight. If \( \lambda_n \in M_{c,N} \) and \( \lambda_n \to \lambda \), then one can choose a subsequence \( n_k \) such that \( m_{\lambda_n} \) converges weakly to some measure \( m \). One can also assume that \( h_{\lambda_n} \to h \geq s \). Then one can check that this limit measure is \( h \)-conformal for the limit map \( f_\lambda \). It now follows from the uniqueness of conformal measure that \( h = h_\lambda \). Thus, the following theorem is true (we omit details of the proof).

Theorem 5.6. The function \( \lambda \mapsto \text{HD}(J_r(f_\lambda)) \) is continuous in the set \( M_{c,N} \). In particular, it is continuous in \( \{ \lambda \in \mathbb{R}, \lambda > 1/2 \} \).

References


[Ma] M. Martens, The existence of \( \sigma \)-finite invariant measures, Applications to real one-dimensional dynamics, Preprint.


[Re] M. Rees, The exponential map is not recurrent, Math. Z. 191 (1986), 593-598. MR 0832817 (87g:58053)


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