ON QUADRATIC DERIVATIVE SCHRÖDINGER EQUATIONS
IN ONE SPACE DIMENSION

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Abstract. We consider the Schrödinger equation with derivative perturbation terms in one space dimension. For the linear equation, we show that the standard Strichartz estimates hold under specific smallness requirements on the potential. As an application, we establish existence of local solutions for quadratic derivative Schrödinger equations in one space dimension with small and rough Cauchy data.

1. Introduction

In this paper, we are interested in the existence and uniqueness theory for certain quadratic derivative Schrödinger equations with low regularity Cauchy data in one space dimension. For the sake of clarity, we prefer to concentrate on the models (1.1) and (1.2) below, although our methods are directly applicable to related problems. The derivative Schrödinger equations that we consider are

\begin{align}
\frac{\partial}{\partial t} u - i \frac{\partial^2}{\partial x^2} u \pm u u_x &= 0 \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^1, \\
\frac{\partial}{\partial t} u - i \frac{\partial^2}{\partial x^2} u \pm (u_x)^2 &= 0 \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^1,
\end{align}

(1.1) (1.2)

These equations have been studied extensively by many authors; for the most recent developments, we refer to [2], [3], [14], [15]. Most of these works were inspired by a pseudodifferential calculus approach pioneered by Doi, [7], [8]. This is due to the basic difficulty present in both (1.1) and (1.2), namely the lack of classical energy estimates. Moreover, one observes a loss of derivatives due to the presence of the $u_x$ term in the nonlinearity. To deal with this problem, Constantin-Saut, [5], [6], have shown a local smoothing effect for the Schrödinger semigroup $e^{it\Delta}$ in the spaces $L^2_{loc,tx}$, which was later improved to by Kenig-Ponce-Vega, [12], [13], who have shown the smoothing estimates in the mixed Lebesgue spaces $L_x^\infty L_t^2$.

These developments led to the successful resolution of the initial value problem for the Schrödinger equation with derivative nonlinearities of the form $F(u, \bar{u}, \partial u, \partial \bar{u})$. However, in the case of quadratic nonlinearities and one space dimension, all the known local well-posedness results require that the space of initial data be a weighted Sobolev space, rather than the usual Sobolev spaces; see Theorem 4.2 in [13] (for small data) and Theorem 2 in [3] for arbitrary large data. It is worth noting that Chihara, [3], treats the case of (1.2) as well, with the same smoothness requirements.

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Theorem 1.1 (Chihara, Theorem 2, [3]). For any $f \in H^5(\mathbb{R}^1) \cap H^{3,2}(\mathbb{R}^1)$, there exists a positive time $T = T(\| f \|_{H^5} + \| f \|_{H^{3,2}})$ and a unique classical solution to (1.1) and (1.2), belonging to $C([0,T); H^5 \cap H^{3,2})$.

An elementary scaling analysis shows that $u_{\lambda}(t,x) = \lambda u(\lambda^2 t, \lambda x)$ solves (1.1), whenever $u$ solves it, and similarly $u_{\lambda}(t,x) = u(\lambda^2 t, \lambda x)$ solves (1.2), whenever $u$ is a solution of the same. This implies the scaling number for (1.1) $s_c = -1/2$, while for (1.2), $s_c = 1/2$. Thus, at least on the level of this heuristical analysis, one expects that (1.1) and (1.2) will be locally ill-posed, when the initial data is in $H^s(\mathbb{R}^1)$, and $s$ is below their corresponding scaling numbers.

Note that by Chihara’s theorem, we are guaranteed the (local) existence of classical solutions of (1.1) for data which is both smooth and decaying. We have

Theorem 1.2 (local well-posedness in $H^1$ with small total disturbance). There exists an absolute $\varepsilon : 0 < \varepsilon < 1$ and absolute constants $C_0, C_1$, so that

- The solution map for (1.1) $U(t) : H^2(\mathbb{R}^1) \cap H^{3,2}(\mathbb{R}^1) \to H^2(\mathbb{R}^1) \cap H^{3,2}(\mathbb{R}^1)$ can be continuously extended to $H^1(\mathbb{R}^1) \cap L^1(\mathbb{R}^1) \cap \{ f : \sup_x | f^2 - f(y)dy | \leq \varepsilon \}$ for all times $0 \leq t \leq T$, where $T = T(\| f \|_{L^2})$. There is the $L^2$ a priori estimate

$$\| U(t)f \|_{L^\infty(0,T)L^2} \leq C_0 \| f \|_{L^2}.$$
• If $f \in H^1 \cap L^1 \cap \{ f : \sup_x |\int_{-\infty}^x f(y)dy| \leq \varepsilon \}$, $U(t)f$ is a weak solution of \eqref{1.1}. Moreover, there is the a priori estimate
\begin{equation}
\|U(t)f\|_{C^0(0,T_1)H^s_x} \leq C_0 \|f\|_{H^s_x},
\end{equation}
for all $s \geq 1$, which is valid for some $T_1 = T_1(\|f\|_{H^1})$.

• $U(t) : H^1(\mathbb{R}^1) \cap L^1(\mathbb{R}^1) \cap \{ f : \sup_x |\int_{-\infty}^x f(y)dy| \leq \varepsilon \} \to C^0(0,T)H^1_x$ is a continuous mapping, and the mapping is $L^2$ Lipschitz, that is,
\begin{equation}
\|U(t)f - U(t)g\|_{L^2} \leq C_1 \|f - g\|_{L^2}
\end{equation}
for some $T = T(\|f\|_{H^1}, \|g\|_{H^1})$.

Our next theorem concerns \eqref{1.2}. This is a local existence result, which requires smallness of $\|f\|_{B^1_{2,1}}$.

**Theorem 1.3** (Local well-posedness for \eqref{1.2} with $\|f\|_{B^1_{2,1}} << 1$). There exists an absolute $\varepsilon > 0$ and an absolute constant $C_0$, so that the solution map for \eqref{1.2}, $V(t)$ initially defined on $\left\{ H^5(\mathbb{R}^1) \cap H^{3,2}(\mathbb{R}^1), f : \|f\|_{B^1_{2,1}} \leq \varepsilon \right\} \to H^1(\mathbb{R}^1)$ for all times $0 < t < T$, where $T = T(\|f\|_{H^1})$.

Moreover, there are the a priori estimates
\begin{equation}
\|V(t)f\|_{C^0(0,T)H^s} \leq C_0 \|f\|_{H^s},
\end{equation}
and for all $s \geq 2$
\begin{equation}
\|V(t)f\|_{C^0(0,T)H^s} \leq C_0 \|f\|_{H^s_x}.
\end{equation}

Finally, $V(t) : H^2(\mathbb{R}^1) \cap \{ f : \|f\|_{B^1_{2,1}} \leq \varepsilon \} \to C^0(0,T)H^2_x(\mathbb{R}^1)$ is continuous, $V(t)f$ is a weak solution to \eqref{1.2}, when $f \in H^2 \cap \{ f : \|f\|_{B^1_{2,1}} \leq \varepsilon \}$, and for $T = T(\|f\|_{H^1}, \|g\|_{H^1})$, there is
\begin{equation}
\|V(t)f - V(t)g\|_{L^2} \leq C_0 \|f - g\|_{L^2}.
\end{equation}

Note that in both Theorem \ref{1.2} and Theorem \ref{1.3} the smallness assumptions are scale invariant. More specifically, for \eqref{1.1}, the natural scaling $u^\lambda(t,x) = \lambda u(\lambda^2t, \lambda x)$ preserves $\|\int_{-\infty}^x u(t,y)dy\|_{L^\infty}$, while for \eqref{1.2} the natural scaling is $u^\lambda(t,x) = u(\lambda^2t, \lambda x)$ and it preserves $\|u\|_{B^1_{2,1}}$.

The proofs of both Theorem \ref{1.2} and Theorem \ref{1.3} require a Strichartz type estimates for the solutions of the linear derivative Schrödinger equation in one space dimension. These have been recently studied by Burq and Planchon \cite{8}, but we provide our own version, which will be used in the sequel.

To be more specific, the linear derivative Schrödinger equation with time dependent potential $A$ is
\begin{equation}
\begin{cases}
\partial_t u - i\partial_x^2 u + A(t,x)u_x = F & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^1, \\
u(0,0) = f(x).
\end{cases}
\end{equation}

Under some smoothness conditions on $f$, $F$ and smoothness and decay assumptions on $A$, smooth solutions of \eqref{1.8} are known to exist globally; see for example Proposition 3.1 in Section 3 as well as earlier work of Doi, \cite{7}, \cite{9}. These are all results based on $L^2$ a priori estimates for the corresponding equations.

\footnote{This is guaranteed by Chihara’s theorem.}
As far as Strichartz estimates are concerned, recall that in the case \( A = 0 \), the Strichartz estimates are classical and well-known. Namely, for every \( C^3 \),

\[
\|u\|_{L^q_tL^r_x} \leq C(\|f\|_{L^2_t} + \|F\|_{L^1_tL^2_x}).
\]

Our next theorem shows that these can be extended in the case of nonzero potential \( A \) under some smallness and integrability/smoothness assumptions on \( A \).

**Theorem 1.4** (Strichartz estimates for the linear derivative Schrödinger equation). There exists an \( \varepsilon : 0 < \varepsilon < 1 \), so that whenever \( A : \mathbb{R}^+ \times \mathbb{R}^1 \to \mathbb{C} \) with

- \( A \in L^{\infty}_tL^1_x, (\partial_t - i\partial_y^2)A(t,x) \in L^1_tL^1_x, \)
- \( \| \int_{-\infty}^{x} A(t,y)dy \|_{L^\infty_t} \leq \varepsilon, \)
- \( \| \int_{-\infty}^{x} (\partial_t - i\partial_y^2)A(t,y)dy \|_{L^1_tL^\infty_x} \leq \varepsilon, \)

the Strichartz estimates for \( 1.8 \) hold true, that is, there exists an absolute constant \( C_0 \) so that for any \( (q,r) : q,r \geq 2, 2/q + 1/r = 1/2 \) and every \( T > 0 \),

\[
\|u\|_{L^q(0,T)L^r} \leq C_0(\|f\|_{L^2_t} + \|F\|_{L^1(0,T)L^2_x}).
\]

**Remarks.**

- The smoothness constants for \( A \) do not enter the Strichartz estimates in any way. Note that under those assumptions, the quantities \( \| \int_{-\infty}^{x} A(t,y)dy \|_{L^\infty_t} \) etc. are properly defined.
- All the smallness requirements are invariant under the natural scaling \( A \to A^\lambda(t,x) = \lambda A(\lambda^2 t, \lambda x) \).

We give a reformulation of Theorem 1.4 which will be more useful for us. Namely, we need a version which allows us to treat local in time solutions of (1.1) and (1.2).

**Proposition 1.5.** There exist an absolute \( \varepsilon : 0 < \varepsilon < 1 \) and an absolute constant \( C_0 \) with the property that whenever \( A : \mathbb{R}^+ \times \mathbb{R}^1 \to \mathbb{C} \) with \( A \in L^{\infty}_tL^1_x \) and \((\partial_t - i\partial_y^2)A(t,x) \in L^1_tL^1_x \) and \( T \) be the maximal time for which

\[
\| \int_{-\infty}^{x} A(t,y)dy \|_{L^\infty^T(0,T)\L^\infty_x} \leq \varepsilon, \| \int_{-\infty}^{x} (\partial_t - i\partial_y^2)A(t,y)dy \|_{L^1(0,T)\L^\infty_x} \leq \varepsilon,
\]

then for every \( (q,r) : q,r \geq 2, 2/q + 1/r = 1/2 \),

\[
\|u\|_{L^q(0,T)\L^r} \leq C_0(\|f\|_{L^2_t} + \|F\|_{L^1(0,T)\L^2_x}).
\]

We refer the reader to the work of Tataru, [19], [20], for a similar argument in the context of variable coefficient quasilinear wave equations, given an appropriate set of Strichartz estimates as in Proposition 1.5.

2. Notations and preliminaries

Define the Fourier transform by

\[
\mathcal{F}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx
\]

and its inverse by

\[
f(x) = \int_{\mathbb{R}^n} \mathcal{F}(\xi) e^{2\pi i x \cdot \xi} d\xi.
\]

Two notable pairs that will be used in the paper are \((q,r) = (4, \infty)\) and \((q,r) = (\infty, 2)\).
For a positive, smooth and even function $\chi : \mathbb{R}^1 \to \mathbb{R}^1$, supported in $\{ \xi : |\xi| \leq 2 \}$ and $\chi(\xi) = 1$ for all $|\xi| \leq 1$, define $\varphi(\xi) = \chi(\xi) - \chi(2\xi)$, which is supported in the annulus $1/2 \leq |\xi| \leq 2$. Clearly $\sum_{k \in \mathbb{Z}} \varphi(2^{-k}\xi) = 1$ for all $\xi \neq 0$.

The $k^{th}$ Littlewood-Paley projection is $\hat{P}_k f(\xi) = \varphi(2^{-k}\xi)\hat{f}(\xi)$. Define also $P_{<k} = \sum_{l<k} P_l$, and similarly for $P_{\leq k}, P_{>k}$ etc.

Note that the kernels of $P_k, P_{<k}$ are integrable and thus $P_k, P_{<k} : L^p \to L^p$ for $1 \leq p \leq \infty$ and $\|P_k\|_{B^p \to B^p}, \|P_{<k}\|_{L^p \to L^p} \leq \|\hat{\chi}\|_{L^1}$ and thus have bounds independent of $k$. Their kernels are smooth, and $P_k, P_{<k}$ commute with differential operators. The (homogeneous) Besov spaces are defined via

$$
\|f\|_{B^p_{k,q}} = \left( \sum_{k \in \mathbb{Z}} 2^{kj} \|P_k f\|_{L^p}^q \right)^{1/q}.
$$

We will make use of the endpoint Sobolev embedding result $B^{n/2}_{2,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$.

The weighted Sobolev spaces $H^{m,l}$ are defined via completion of test functions in the norm

$$
\|f\|_{H^{m,l}} = \sum_{|\alpha| \leq m} \left( \int (1 + |x|)^{2l} \|\partial^\alpha f\|_{L^2}^2 \right)^{1/2}.
$$

We briefly discuss the properties of products under the action of $P_k$. For any two (Schwartz) functions $f, g$

$$
P_k(fg) = \sum_{l \geq k-2} P_k(f_l g_{l-2} \leq l+2) + \text{symmetric term}
+ P_k(f_{k-4} g_{k-1} \leq k+1) + \text{symmetric term}
= f_{k-4} g_k + [P_k, f_{k-4}] g_{k-1} \leq k+1 + \text{symmetric terms}
+ \sum_{l \geq k-2} P_k(f_l g_{l-2} \leq l+2) + \text{symmetric term}.
$$

In the particular case, when $g = v_x$, write

$$
P_k(uv_x) = u_{k-4} \partial_x v_k + E^k(u, v),
$$
where

$$
E^k(u, v) = [P_k, u_{k-4}] \partial_x v_{k-1} \leq k+1 + \sum_{l \geq k-2} P_k(u_l \partial_x v_{l-2} \leq l+2)
+ \sum_{l \geq k-2} P_k(u_{l-2} \leq l+2 \partial_x v_l) + P_k(u_{k-1} \leq k+1 \partial_x v_{k-4}).
$$

In other words $P_k(uv_x) = u_{k-4} \partial_x v_k$ modulo the error term $E^k(u, v)$, which will be easy to control. The following lemma is standard and it appears in various forms throughout the literature, but we include its precise statement and proof for completeness.

**Lemma 2.1.** For every $s > 0$,

$$
\left( \sum_{k=-\infty}^{+\infty} 2^{ks} \|E^k(u, v)\|_{L^2}^2 \right)^{1/2} \leq C(\|u_x\|_{L^\infty}\|v\|_{H^s} + \|u\|_{H^s}\|\partial_x v\|_{L^\infty}).
$$
More generally, for every \( j \in \mathbb{Z} \), we have
\[
\left( \sum_{k \geq j} 2^{ks} \left\| P_k (u, v) \right\|_{L^2}^2 \right)^{1/2} \lesssim \| P_{\geq j-3} u_x \|_{L^\infty} \| v \|_{H^s} + \| u_x \|_{L^\infty} \| P_{\geq j-3} v \|_{H^s} + \| P_{\geq j-3} u \|_{H^s} \| \partial_x v \|_{L^\infty} + \| u \|_{H^s} \| \partial_x P_{\geq j-3} v \|_{L^\infty}
\]
for some constant \( C = C(s) \).

**Proof.** Note that the first inequality is a corollary of the second inequality by taking \( j \to -\infty \).

Recall the Calderón commutator estimate, stating that for \( 1 \leq p, q, r \leq \infty \) and \( 1/p = 1/q + 1/r \), \( \| [P_k, f] \nabla g \|_{L^r} \leq C \| \nabla f \|_{L^p} \| g \|_{L^q} \).

We estimate various terms arising in (2.1). For the commutator term, we have
\[
\left( \sum_{k \geq j} 2^{ks} \left\| [P_k, u_{\leq k-4}] \partial_x v_{k-1 \leq k+1} \right\|_{L^2}^2 \right)^{1/2} \leq \left( \sum_{k \geq j} 2^{ks} \left\| \partial_x u_{\leq k-4} \right\|_{L^\infty}^2 \left\| v_{k-1 \leq k+1} \right\|_{L^2}^2 \right)^{1/2} \leq \sup_k \left\| \partial_x u_{\leq k-4} \right\|_{L^\infty} \left( \sum_{k \geq j} 2^{ks} \left\| v_{k-1 \leq k+1} \right\|_{L^2}^2 \right)^{1/2} \leq \| u_x \|_{L^\infty} \| v_{\geq j-3} \|_{H^s}.
\]
The next two terms in (2.1) are similar, so we just discuss one of them. To obtain the desired estimate for the term
\[
\left( \sum_{k \geq j} 2^{ks} \left\| P_k \left( \sum_{l \geq k-2} u_l \partial_x v_{l-2 \leq l+2} \right) \right\|_{L^2}^2 \right)^{1/2},
\]
it will suffice by interpolation to show the \( l^1 \) estimate
\[
\sum_{k \geq j} 2^{ks} \left\| P_k \left( \sum_{l \geq k-2} u_l \partial_x v_{l-2 \leq l+2} \right) \right\|_{L^2} \leq C_s \| u_x \|_{L^\infty} \sum_{l \geq j-2} 2^{ls} \| v_l \|_{L^2}
\]
and the \( l^\infty \) estimate
\[
\sup_{k \geq j} 2^{ks} \left\| P_k \left( \sum_{l \geq k-2} u_l \partial_x v_{l-2 \leq l+2} \right) \right\|_{L^2} \leq C_s \| u_x \|_{L^\infty} \sup_{l \geq j-2} 2^{ls} \| v_l \|_{L^2}.
\]
The \( l^1 \) estimate is
\[
\sum_{k \geq j} 2^{ks} \left\| \sum_{l \geq k-2} u_l \partial_x v_{l-2 \leq l+2} \right\|_{L^2} \leq C_s \sum_{l \geq j-2} 2^l \| u_l \|_{L^\infty} 2^{ls} \| v_{l-2 \leq l+2} \|_{L^2} \leq C_s \| u_x \|_{L^\infty} \sum_{l \geq j-2} 2^{ls} \| v_l \|_{L^2},
\]
and the \( l^\infty \) estimate
\[
\sup_l 2^l \| u_l \|_{L^\infty} \sum_{l \geq j-2} 2^{ls} \| v_l \|_{L^2} \leq C_s \| u_x \|_{L^\infty} \sum_{l \geq j-2} 2^{ls} \| v_l \|_{L^2}.
\]
while the $l^\infty$ estimate is

$$\sup_{k \geq j} 2^{ks} \left\| \sum_{l \geq k-2} u_l \partial_x v_{l-2 \leq l \leq l+2} \right\|_{L^2} \lesssim \sup_l 2^l \| u_l \|_{L^\infty} \sup_{k \geq j} 2^{ks} \sum_{l \geq k-2} \| v_{l-2 \leq l \leq l+2} \|_{L^2}$$

$$\lesssim \| u_x \|_{L^\infty} \sup_{k \geq j} \sum_{l \geq k-2} 2^{(k-l)s} (2^{ls} \| v_l \|_{L^2}) \leq C_s \| u_x \|_{L^\infty} \sup_{l \geq j-2} 2^{ls} \| v_l \|_{L^2}.$$  

Finally, the last term in (2.1) is estimated by

$$\left( \sum_{k \geq j} 2^{2ks} \| u_{k-1 \leq k \leq k+4} \|_{L^2}^2 \right)^{1/2} \leq C \left( \sum_{k \geq j} 2^{2ks} \| u_{k-1 \leq k+1} \|_{L^2}^2 \right)^{1/2},$$

$$\times \sup_k \| \partial_x v_{k-4} \|_{L^\infty} \leq C \| u_{j-3} \|_{H^m} \| v_x \|_{L^\infty}.$$  

The constants $C$ that will be used throughout the text will generally have different values per each appearance, but they will be absolute constants, independent of anything. We will frequently use the notation $A \lesssim B$ ($A \gtrsim B$) when there is an absolute constant $C$, so that $A \leq CB$ ($A \geq CB$ respectively).

For the rest of the paper, whenever we write a pair of exponents $(q, r)$, we mean a generic Strichartz pair in one dimension, that is, $\infty \geq q, r \geq 2 : 2/q + 1/r = 1/2$.

## 3. Strichartz estimates for the linear equation

In this section, we establish Theorem 1.4. We consider only $f, F$ that are sufficiently smooth and potentials $A$ that are both sufficiently smooth and decaying (see Proposition 3.1 below). However, our estimates will be independent of these smoothness bounds and in fact depend only on the smallness assumptions made in Theorem 1.4. By approximation by smooth $f, F$ and smooth and decaying $A$, the Strichartz estimates will be valid for all $A$ satisfying the smallness assumptions in Theorem 1.4 and for all functions $f \in L^2, F \in L^1 L^2$.

There is the following $L^2$ existence theorem, due to Kenig, Ponce and Vega, which we state in the case under consideration.

### Proposition 3.1 (Kenig-Ponce-Vega, [14]). For the equation (1.8), there is a unique global solution, provided $A \in C^N, \{Im A \leq \langle x \rangle^{-m} \}$ for some large integers $N, m$. Moreover the solutions are smooth, provided $f, F$ smooth and for every $T > 0$

$$(3.1) \quad \| u \|_{L^\infty(0,T) L^2} \leq C_{T,A}(\| f \|_{L^2} + \| F \|_{L^1(0,T) L^2}).$$

A somewhat more convenient way of stating Proposition 3.1 is the following. Let the linear operator $U_A(t, s) : L^2(\mathbb{R}^1) \to L^2(\mathbb{R}^1)$ be the solution to the homogeneous equation

$$\begin{cases}
\partial_t u - i\partial^2_x u + A(t, x)u_x = 0 & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^1, \\
u(s, x) = f(x).
\end{cases}$$

Clearly, the solution to the corresponding inhomogeneous equation is given by

$$u(t, x) = U_A(t, 0)f + \int_0^t U_A(t, s)F(s, \cdot)ds.$$
These are all identities that hold in appropriate $L^2$ sense as guaranteed by Proposition 3.1. Our goal is to show

$$\|U_A(t,0)f\|_{L^qL^r} \leq C\|f\|_{L^2},$$

$$\left\| \int_0^t U_A(t,s)F(s,\cdot)ds \right\|_{L^qL^r} \leq C\|F\|_{L^1L^2}.$$

Since by Proposition 3.1 we know how to construct $(L^2)$ solutions of (1.8), our task consists of proving the relevant estimates to validate Theorem 1.4. The strategy of the proof is to construct an approximate solution (parametrix) for (1.8). We have the following lemma, which will be used in a continuity argument in the sequel. Denote

$$\mathcal{L}\psi := \partial_t\psi - i\psi_{xx} + A\psi_x.$$

**Lemma 3.2.** Let $A$ be as in Proposition 3.1. Then for a fixed integer $k_0$, there exists a time $T_0 = T_0(k_0, A) \leq \infty$, so that whenever $0 < T < T_0$, $\psi$ is in the Schwartz class

$$\|P_{<k_0}\psi\|_{L^q(0,T)L^r} \leq C(T, k_0, A)(\|\psi(0,\cdot)\|_{L^2} + \|\mathcal{L}\psi\|_{L^1(0,T)L^2}).$$

Moreover, $C(T, k_0, A)$ depends on $T$ in a continuous way.

**Proof.** We use the standard Strichartz estimates for the linear Schrödinger equation by treating the term $A\psi_x$ as a perturbation of the linear Schrödinger equation.

We have

$$\|P_{<k_0}\psi\|_{L^q(0,T)L^r} \leq C(\|\psi(0)\|_{L^2} + \|A\partial_x P_{<k_0}\psi\|_{L^1(0,T)L^2} + \|\mathcal{L}P_{<k_0}\psi\|_{L^1L^2})$$

$$\leq C_1(\|\psi(0)\|_{L^2} + T\|A\|_{L^\infty} \|\partial_x P_{<k_0}\psi\|_{L^\infty(0,T)L^2})$$

$$+ C_1(\|P_{<k_0}\mathcal{L}\psi\|_{L^1L^2} + \|P_{<k_0,A}\mathcal{L}\psi\|_{L^1L^2})$$

$$\leq C_1(\|\psi(0)\|_{L^2} + \|\mathcal{L}\psi\|_{L^1L^2}) + C_1 T(\|A\|_{L^\infty} + \|A_x\|_{L^\infty})\|\psi\|_{L^\infty(0,T)L^2}.$$  

According to (3.1), $\|\psi\|_{L^\infty(0,T)L^2} \leq C_T A(\|\psi(0)\|_{L^2} + \|\mathcal{L}\psi\|_{L^1(0,T)L^2})$ and therefore

$$\|P_{<k_0}\psi\|_{L^q(0,T)L^r} \leq C_{k_0,T,A}(\|\psi(0)\|_{L^2} + \|\mathcal{L}\psi\|_{L^1L^2}).$$

$\square$

Fix an integer $k_0$ and an absolute and small $\varepsilon > 0$, to be chosen below. Set $0 < T^* \leq \infty$ be the maximal time, so that for all $0 < T < T^*$ and for all Schwartz functions $\psi$ one has

$$\sup_{q,r:2/q + 1/r = 1/2} \|P_{<k_0}\psi\|_{L^q(0,T)L^r} \leq \varepsilon^{-1}(\|\psi(0)\|_{L^2} + \|\mathcal{L}\psi\|_{L^1(0,T)L^2}).$$

We will show that for a suitable choice of $\varepsilon$ and for $A$ satisfying the conditions of Theorem 1.4, with the said $\varepsilon$, it follows that $T^* = \infty$ for all $k_0$. Thus, we will have established Theorem 1.4.

The proof is actually based on a continuity argument, which iterates (3.3) to a global estimate via a relatively simple parametrix construction, which we state next.
Lemma 3.3. Let $\varepsilon > 0$ and $A$ satisfies the assumptions in Theorem 1.4. Then for every Schwartz function $f$, one can find a smooth function $v : [0, \infty) \times \mathbb{R}^1 \to \mathbb{C}$ so that

\begin{align}
(3.4) \quad & \|v(0, x) - f\|_{L^2} \leq C\varepsilon\|f\|_{L^2}, \\
(3.5) \quad & \|v\|_{L^6 L^\infty} \leq C\|f\|_{L^2}, \\
(3.6) \quad & \|L v\|_{L^1 L^2} \leq C\varepsilon\|f\|_{L^2}
\end{align}

for some absolute constant $C$.

Assuming Lemma 3.3 we show that $T^*$ in (3.3) must be infinite. To that end, we follow an idea of Rodnianski and Tao, [17].

Consider the homogeneous equation $Lg = 0$, $g(0, x) = f(x)$, where $f$ is a Schwartz function and $g$ is defined via $g = U_A(t, 0) f$. By virtue of Lemma 3.3 we construct a function $v$. We have for every $T < T^*$ by (3.3)

\[
\|P_{<k_0} g\|_{L^6(0, T), L^\infty} \leq \|P_{<k_0} (g - v)\|_{L^6(0, T), L^\infty} + \|P_{<k_0} v\|_{L^6(0, T), L^\infty} \\
\leq \varepsilon^{-1}(\|v(0, x) - f(x)\|_{L^2} + \|L (g - v)\|_{L^1(0, T), L^2}) + C\|v\|_{L^6(0, T), L^\infty} \\
\leq \varepsilon^{-1}(C\varepsilon\|f\|_{L^2}) + C\|f\|_{L^2} \leq C\|f\|_{L^2}.
\]

That is, we have shown

\[
\|P_{<k_0} U_A(t, 0) f\|_{L^6(0, T), L^\infty} \leq C\|f\|_{L^2},
\]

with some absolute constant $C$. More generally, for every $0 < s < t < T$, we have

\[
(3.7) \quad \|P_{<k_0} U_A(t, s) f\|_{L^6_0(0, T), L^\infty} \leq C\|f\|_{L^2},
\]

with the same absolute constant $C$. This is done by solving for fixed $s$ the homogeneous problem $L g_s = 0$, $g_s(s, x) = f(x)$ forward in time.

Next, consider the inhomogeneous problem $L w = F, w(0, x) = 0$. By Duhamel’s formula

\[
w(t, x) = \int_0^t U_A(t, s) F(s, \cdot) ds,
\]

whence using Minkowski’s inequality and (3.7)

\[
\|P_{<k_0} w\|_{L^6(0, T), L^\infty} = \left\| \int_0^t P_{<k_0} U_A(t, s) F(s, \cdot) ds \right\|_{L^6(0, T), L^\infty} \\
\leq \int_0^T \|P_{<k_0} U_A(t, s) F(s, \cdot)\|_{L^6(0, T), L^\infty} ds \leq C \int_0^T \|F(s, \cdot)\|_{L^2} ds \leq C\|F\|_{L^1(0, T), L^2}.
\]

Therefore, for the solution of $L u = F$, $u(0, x) = f$, we get the estimates

\[
\|P_{<k_0} u\|_{L^6(0, T), L^\infty} \leq C_0(\|f\|_{L^2} + \|F\|_{L^1(0, T), L^2}),
\]

with some absolute constant $C_0$. Thus, since $\psi$ is a solution to $L u = L \psi$ and $u(0, x) = \psi(0, x)$, we have

\[
\|P_{<k_0} \psi\|_{L^6(0, T), L^\infty} \leq C_0(\|\psi(0, x)\|_{L^2} + \|L \psi\|_{L^1(0, T), L^2}).
\]
Thus, we have

This is a contradiction with the maximality of $T^*$ and $T^* < \infty$, if $C_0 < \varepsilon^{-1}$. Choose $\varepsilon = \min(1/(2C_0), 1)$. We have therefore established $T^* = \infty$. It follows that

$$\|P_{<k_0}\psi\|_{L^q(0,\infty)L^r} \leq 2C_0(\|\psi(0,x)\|_{L^2} + \|\mathcal{L}\psi\|_{L^q(0,\infty)L^r})$$

for every $k_0$. Taking $k_0 \to \infty$ yields the desired inequality (1.9).

It thus remains to prove Lemma 3.3.

### 3.1. Existence of the parametrix.

In this subsection, we prove Lemma 3.3. The construction of the parametrix $v$ is similar in spirit to the works of Rodnianski and Tao, [17], in the wave equation context and of Roux and Yafaev, [21]. Actually, in the one dimensional case, matters greatly simplify, and we have to select only an appropriate phase shift function.

Concretely, we look for $v$ in the form

$$v(t, x) = \int_{\mathbb{R}^1} e^{\sigma(t,x,\xi)} e^{-4it\pi^2\xi^2} e^{2\pi i\xi x} \hat{f}(\xi) d\xi.$$

Clearly taking $\sigma = 0$ provides a solution of the free Schrödinger equation with initial data $f$. Compute

$$\mathcal{L}v(t, x) = v_t - iv_{xx} + Av_x$$

where

$$[\sigma_t - i\sigma_{xx} + 2\pi \xi (\sigma_x + iA) - i(\sigma_x)^2 + A\sigma_x] e^{-4it\pi^2\xi^2} e^{2\pi i\xi x} \hat{f}(\xi) d\xi.$$

Choose $\sigma = 2\pi \xi (\sigma_x + iA) = 0$, that is,

$$\sigma(t, x) = -i \int_{-\infty}^{x} A(t, y) dy.$$

Note that such a choice makes the phase correction $\sigma$ a $\xi$ independent function.\[3\]

Thus, we have

$$v(t, x) = e^{-i\int_{-\infty}^{x} A(t, y) dy} e^{it\partial^2_x} f,$$

$$\mathcal{L}v(t, x) = ( - i \int_{-\infty}^{x} (A_t - iA_{yy})dy)v(t, x).$$

Now it is easy to verify the properties of the parametrix (3.4), (3.5), (3.6).

Indeed for $(3.4)$, use $\|\sigma\|_{L^\infty_{x,t}} = \|\int_{-\infty}^{x} A(t, y)dy\|_{L^\infty_x} \leq \varepsilon$ to get

$$\|v(0, \cdot) - f(\cdot)\|_{L^2_x} = \left\| (e^{\sigma(0, \cdot)} - 1)f\right\|_{L^2_x} \leq \|\sigma\|_{L^\infty_{x,t}} \|f\|_{L^2_x} \leq \varepsilon \|f\|_{L^2_x}.$$

For $(3.5)$,

$$\|v\|_{L^4_{x,t} L^4_{x}} \leq \left\| e^{\sigma(t,x)} \right\|_{L^\infty_{x,t}} \left\| e^{it\partial^2_x} f\right\|_{L^4_{x,t} L^4_{x}} \leq C\|\sigma(t,x)\|_{L^\infty_x} \|f\|_{L^4_x} \leq 3C\|f\|_{L^2_x},$$

where $C$ is the Strichartz constant for the free Schrödinger equation in 1D.

Finally, using $(3.5)$ and

$$\left\| \int_{-\infty}^{x} (A_t - iA_{yy})dy\right\|_{L^1_{t} L^\infty_{x}} \leq \varepsilon,$$

we have

$$\|\mathcal{L}v\|_{L^1_{t} L^2_{x}} \leq \left\| \int_{-\infty}^{x} (A_t - iA_{yy})dy\right\|_{L^1_{t} L^\infty_{x}} \|v\|_{L^\infty_{x} L^2_{t}} \leq 3C\varepsilon \|f\|_{L^2_{x}}.$$

---

\[3\]The $\xi$ independence cannot be achieved in dimensions $n \geq 2$, which accounts for the extra difficulties in this case.
4. Well-posedness for quadratic nonlinear Schrödinger equations

4.1. Strategy of the proof. In this section, we prove Theorem 1.2 and Theorem 1.3. But first, let us quickly sketch the argument that we employ in order to solve (1.1) and (1.2).

We solve (locally) nonlinear equations in the form

\[ u_t - iu_{xx} + A(u, u_x)u_x = 0 \]

as follows. Take \( \varepsilon \) as in Proposition 1.5. We first ensure that initially,

\[ \left\| \int_{-\infty}^{x} A(u(0, y), u_y(0, y))dy \right\|_{L^\infty} \leq \varepsilon/10. \tag{4.1} \]

If the data \( f \) is sufficiently smooth and decaying, the solution will persist for some time, as guaranteed by the Chihara existence result, Theorem 1.1. By Proposition 1.5, we have an existence of the solution \( u \) up to time \( T \), enjoying the a priori estimate

\[ \|u\|_{L^q(0, T; L^r)} \leq C\|f\|_{L^2} \]

so long as

\[ \left\| \int_{-\infty}^{x} A(u(t, y), u_y(t, y))dy \right\|_{L^\infty(0, T; L^\infty)} \leq \varepsilon, \tag{4.3} \]

\[ \left\| \int_{-\infty}^{x} (\partial_t - i\partial_y^2)A(u(t, y), u_y(t, y))dy \right\|_{L^1(0, T; L^\infty)} \leq \varepsilon. \tag{4.4} \]

To show existence, it will suffice to show that given initial data so that (4.1) holds, and smooth solution \( u \) to (1.1) defined up to time \( T = T(\|f\|_{L^2}) \), satisfying (4.2), then one has (4.3) and (4.4).

4.2. Well-posedness for \( u_t - iu_{xx} \pm uu_x = 0 \). For technical reasons, we must rescale first to have the initial data \( f : \|f\|_{L^2} \leq \varepsilon/100 \), where \( \varepsilon \) is as in Proposition 1.5. Indeed, this is possible since \( u^\lambda(t, x) := \lambda u(\lambda^2 t, \lambda x) \) is a solution to (1.1), if \( u \) is, and moreover \( \|u^\lambda\|_{L^2} = \lambda^{1/2}\|u\|_{L^2} \).

4.2.1. Existence of solutions and \( L^2 \) a priori estimates. Following the strategy described in the previous section, take \( A(t, x) = \pm u \). Note that the quantities in (4.1) and (4.3) remain unchanged under the scaling transformation \( u \rightarrow u^\lambda \), that is, (4.1), (4.3) are scale invariant and the assumption is

\[ \sup_x \left| \int_{-\infty}^{x} f(y)dy \right| \leq \varepsilon/10. \]

Denote \( B(t, x) = \int_{-\infty}^{x} u(t, y)dy \) and in this notation, \( \|B(0, x)\|_{L^\infty} \leq \varepsilon/10. \)

We verify first that (4.3) holds. Compute

\[ B_t - iB_{xx} = \int_{-\infty}^{x} (u_t - iu_{yy})dy = \int_{-\infty}^{x} (u_t - iu_{yy})dy = \pm \frac{u^2(t, x)}{2}. \]

It follows by Duhamel’s formula that

\[ B(t, x) = e^{i\xi_0^2/2} B(0, x) + \int_{0}^{t} e^{i(t-s)}\xi_0^2 u^2(s, \cdot)/2ds. \]
By the $L^1 \to L^\infty$ dispersive estimate for $e^{it\partial_x^2}$,
\[
\left\Vert \int_0^t e^{i(t-s)\partial_x^2} u_s^2(s, \cdot) / 2ds \right\Vert_{L^\infty(0,T)L^2_x} \leq C \int_0^T \left\Vert u_s^2(0,T)L^2_x \right\Vert ds \leq C \sqrt{T} \left\Vert u \right\Vert_{L^\infty(0,T)L^2_x}^2.
\]

It follows by the a priori estimate (4.2)
\[
\left\Vert B \right\Vert_{L^\infty(0,T)L^\infty_x} \leq \left\Vert e^{it\partial_x^2} B(0, x) \right\Vert_{L^\infty(0,T)L^\infty_x} + C \sqrt{T} \left\Vert f \right\Vert_{L^2_x}^2.
\]

Thus, (4.3) would be verified as long as $T \leq (\varepsilon/(2C))^2 \left\Vert f \right\Vert_{L^2_x}^{-4}$ and provided one can show $\left\Vert e^{it\partial_x^2} B(0, x) \right\Vert_{L^\infty(0,T)L^\infty_x} \leq \varepsilon/2$. For that, we estimate
\[
\left\Vert e^{it\partial_x^2} B(0, x) \right\Vert_{L^\infty(0,T)L^\infty_x} \leq \left\Vert e^{it\partial_x^2} P_{<0} B(0, x) \right\Vert_{L^\infty_x} + \left\Vert e^{it\partial_x^2} P_{\geq0} B(0, x) \right\Vert_{L^\infty_x}.
\]

For the first term, we have
\[
\left\Vert e^{it\partial_x^2} P_{<0} B(0) \right\Vert_{L^\infty_x(0,T)} \leq \left\Vert (e^{it\partial_x^2} - 1) P_{<0} B(0) \right\Vert_{L^\infty_x} + \left\Vert P_{<0} B(0) \right\Vert_{L^\infty_x} \\
\leq \left\Vert t \partial_x^2 P_{<0} B(0) \right\Vert_{L^\infty_x} + \left\Vert B(0, x) \right\Vert_{L^\infty_x} \leq T \left\Vert \partial_x B(0) \right\Vert_{L^2_x} + \varepsilon/10 \\
= T \left\Vert f \right\Vert_{L^2_x} + \varepsilon/10 \leq T \varepsilon/100 + \varepsilon/10.
\]

For the term $\left\Vert e^{it\partial_x^2} P_{\geq0} B(0, x) \right\Vert_{L^\infty_x(0,T)L^2_x}$, use the Sobolev embedding to estimate by
\[
\left\Vert e^{it\partial_x^2} P_{\geq0} B(0, x) \right\Vert_{L^\infty_x(0,T)L^2_x} \leq \left\Vert e^{it\partial_x^2} \partial_x B(0, x) \right\Vert_{L^\infty_x(0,T)L^2_x} = \left\Vert f \right\Vert_{L^2_x} \leq \varepsilon/100.
\]

Clearly, we can achieve $\left\Vert e^{it\partial_x^2} B(0, x) \right\Vert_{L^\infty_x(0,T)L^\infty_x} \leq \varepsilon/2$, by choosing $T \lesssim 1$.

For the proof of (4.4), note that by (4.2)
\[
\left\Vert \int_{-\infty}^x (\partial_t - i\partial_y^2)u(t, y)dy \right\Vert_{L^1(0,T)L^\infty_x} = \left\Vert \int_{-\infty}^x \partial_y(u^2(t, y)/2)dy \right\Vert_{L^1(0,T)L^\infty_x} \\
\leq C \left\Vert u \right\Vert_{L^2(0,T)L^\infty_x}^2 \leq C \sqrt{T} \left\Vert u \right\Vert_{L^4(0,T)L^\infty_x}^2 \leq C \sqrt{T} \left\Vert f \right\Vert_{L^2_x}^2.
\]

We need $T \leq (\varepsilon/C)^2 \left\Vert f \right\Vert_{L^4_x}^{-4}$ to conclude (4.4) holds.

We have thus shown that the solutions constructed by the Strichartz estimates of Proposition 1.3 have a life span $T = T(\left\Vert f \right\Vert_{L^2_x})$, together with the a priori estimate (4.2).

4.2.2. $H^s$ a priori estimates. Next, we concentrate on the a priori estimate (1.3). We project the equation (1.1) in the $k$th Littlewood-Paley frequency to get
\[
0 = \partial_t u_k - i\partial_{xx}^2 u_k \pm P_k(\bar{u}u_x) = \partial_t u_k - i\partial_{xx}^2 u_k \pm u_{<k-4} \partial_x u_k \pm E^k(u, u),
\]
where $u_k = P_k u$ and $E^k(u, u)$ is as in (2.1). It follows that $u_k$ satisfies (1.8) with $A = \pm u_{<k-4}$ with data $f_k$ and right hand side $\pm E^k(u, u)$. According to our construction of the solution $u$, $A = \pm u$ was chosen to satisfy the smallness
assumptions in Proposition 1.5 up to the time \( T = T(\| f \|_{L^2}) \). For \( A = \pm u_{<k-4} \) and a time \( T_1 < T \), we have
\[
\left\| \int_{-\infty}^{x} P_{<k-4} u(t,y) dy \right\|_{L^\infty} \leq \left\| P_{<k-4} \right\|_{L^\infty \to L^\infty} \left\| \int_{-\infty}^{x} u(t,y) dy \right\|_{L^\infty(0,T_1) L^\infty_x},
\]
\[
\left\| \int_{-\infty}^{x} (\partial_t - i\partial_y^2) P_{<k-4} u(t,y) dy \right\|_{L^\infty(0,T_1) L^\infty_x} \leq \left\| P_{<k-4} \right\|_{L^\infty \to L^\infty} \left\| \int_{-\infty}^{x} (\partial_t - i\partial_y^2) u(t,y) dy \right\|_{L^\infty(0,T_1) L^\infty_x}.
\]
Select \( T_1 : \left\| \int_{-\infty}^{x} u(t,y) dy \right\|_{L^\infty(0,T_1) L^\infty_x} \leq \left\| P_{<k-4} \right\|_{L^\infty \to L^\infty} \leq \|\tilde{\chi}\|_{L^1} = C. \) This is possible by the constructions in the previous section. With this choice of \( T_1 = T_1(\| f \|_{L^2}) \), we conclude that the linear derivative Schrödinger equation with \( A = \pm u_{<k-4} \) satisfies the smallness assumptions in Proposition 1.5 and has a solution in at least the time interval \((0,T_1)\). In particular, we have that the a priori estimates \((1.10)\) hold. Thus,
\[
\left\| u_k \right\|_{L^1(0,T_1) L^r_x} \leq C(\| f_k \|_{L^2} + \| E_k^1(u,u) \|_{L^1(0,T_1) L^2_x}).
\]
Multiplying by \(2^{ks}\) and square-summing in \( k \) yields
\[
\left( \sum_k 2^{ks} \left\| u_k \right\|_{L^1(0,T_1) L^r_x}^2 \right)^{1/2} \leq C \| f \|_{\dot{H}^s} + C \left( \sum_k 2^{ks} \| E_k^1(u,u) \|_{L^1(0,T_1) L^2_x}^2 \right)^{1/2}.
\]
According to Lemma 2.1, we have
\[
\left( \sum_k 2^{ks} \| E_k^1(u,u) \|_{L^1(0,T_1) L^2_x}^2 \right)^{1/2} \leq \sqrt{T_1} \left( \sum_k 2^{ks} \| E_k^1(u,u) \|_{L^2_x}^2 \right)^{1/2} \left\| u \right\|_{L^\infty(0,T_1) H^s}.\]
Note that
\[
\left\| u \right\|_{L^\infty(0,T_1) H^s} \lesssim \sup_{q,r} \left( \sum_{k \geq 0} 2^{2ks} \left\| u_k \right\|_{L^q(0,T_1) L^r_x}^2 \right)^{1/2} + \left\| u \right\|_{L^\infty(0,T_1) L^2_x}.
\]
Let \( s = 1 \) and put everything together to get
\[
\left( \sum_k 2^{2k} \left\| u_k \right\|_{L^2(0,T_1) L^2}^2 \right)^{1/2} \leq C \| f \|_{\dot{H}^1} + C T^{3/4} \| u_x \|_{L^4(0,T) L^\infty_x} \| u_x \|_{L^\infty(0,T) L^2},
\]
for all \( T < T_1 \). Select \( T_2 \leq T_1 \), so that \( T_2^{3/4} \| f \|_{\dot{H}^1} << 1 \). We can then obviously hide the second term on the right hand side behind the left hand side to get
\[
\left( \sum_k 2^{2ks} \left\| u_k \right\|_{L^4(0,T_2)}^2 \right)^{1/2} \leq C \| f \|_{\dot{H}^1}.
\]
For \( s \geq 1 \), we obtain for the same \( T_2 \)
\[
\left( \sum_k 2^{2ks} \left\| u_k \right\|_{L^4(0,T_2)}^2 \right)^{1/2} \leq C \| f \|_{\dot{H}^s} + C T_2^{3/4} \| u_x \|_{L^4(0,T_2) L^\infty_x} \| u \|_{L^\infty(0,T_2) H^s}.
\]
whence by the choice of $T_2$ and (4.2)
\[
\|u\|_{L^\infty(0,T_2)H'} \leq \left( \sum_{k \geq 0} 2^{2ks} \|u_k\|_{L^\infty(0,T_2)L^r}^2 \right)^{1/2} + \|u\|_{L^\infty(0,T_2)L^2} \leq C\|f\|_{H'}.
\]

4.2.3. $U(t) : H^1 \to L^2$ is Lipschitz. We aim at establishing the $L^2$ Lipschitz bound (1.4). Let $u, v$ be two solutions corresponding to initial data $f, g \in H^1$.

According to the previous step, there will be some nontrivial common lifespan $T \geq \min(T(\|f\|_{H^1}), T(\|g\|_{H^1}))$. Let $z = u - v$, which satisfies
\[
0 = z_t - iz_{xx} \pm (u + v)z_x/2 \pm z(u_x + v_x)/2.
\]
That is, $z$ satisfies (4.8) with $A = \pm(u + v)/2$, with initial data $f - g$ and right hand side $\pm z(u_x + v_x)/2$. It follows from (4.10) that
\[
\|z\|_{L^1(0,T)L^r} \leq C(\|f - g\|_{L^2} + \|z(u_x + v_x)\|_{L^1(0,T)L^2}).
\]

But
\[
\|z(u_x + v_x)\|_{L^1(0,T)L^2} \leq CT^{3/4}(\|u_x\|_{L^\infty(0,T)L^2} + \|v_x\|_{L^\infty(0,T)L^2})\|z\|_{L^1(0,T)L^\infty}.
\]

For $T_2 : T_2^{3/4}(\|f\|_{H^1} + \|g\|_{H^1}) << 1$, we can hide the term $C\|z(u_x + v_x)\|_{L^1(0,T)L^2}$ behind the left hand side of (4.10) to obtain
\[
\|u - v\|_{L^1(0,T_2)L^r} = \|z\|_{L^1(0,T_2)L^r} \leq C\|f - g\|_{L^2},
\]
which establishes the Lipschitz bound (1.4).

4.2.4. $U(t)f$ is weak solution, if $f \in H^1$, $\| \int_{-\infty}^x f(y)dy \|_{L^\infty} < \varepsilon/10$. Start with a sequence of smooth and decaying data $f^n$, with $\| \int_{-\infty}^x f^n(y)dy \|_{L^\infty} < \varepsilon$, which converges to $f \in H^1$.

According to Chihara’s theorem, the corresponding (classical) solutions $u^n$ exist, and moreover by the results in the previous sections, they have a nontrivial common lifespan $T = T(\|f\|_{H^1})$ for $n >> 1$. By the $L^2$ Lipschitz estimate, we have
\[
u^n(t, x) = U(t)f^n \to U(t)f = u(t, x)
\]
in $L^2$ sense.

Since $u^n$ is a classical solution, for any test function $\psi \in C_0^\infty((-T, T) \times \mathbb{R}^1)$
\[
\int u^n(x, t)\psi_t(x, t)dxdt + i \int u^n\psi_{xx}dxdt \pm \int (u^n)^2\psi_xdxdt = 0.
\]
Taking limit $n \to \infty$ yields
\[
\int u(x, t)\psi_t(x, t)dxdt + i \int u\psi_{xx}dxdt \pm \int u^2\psi_xdxdt = 0,
\]
whence $u$ is a weak solution in the time interval $(-T, T)$.

4.2.5. $U(t) : H^1(\mathbb{R}^1) \cap L^1(\mathbb{R}^1) \cap \{ f : \sup_\delta \int_{-\infty}^x f(y)dy \leq \varepsilon \} \to C^0(0, T)H^1_x$ is continuous. We will show that for every $\delta > 0$, $f \in H^1$ and every sequence $f^n : \|f^n - f\|_{H^1} \to 0$, there exists a time $T = T(\|f\|_{H^1})$ and $N = N(f, \delta)$, so that whenever $n > N$, we have
\[
\|U(t)f - U(t)f^n\|_{L^\infty(0,T)H^1} < \delta.
\]
First of all, it is clear by the proof presented above, that \( T = T(\|f\|_{H^1}) \) is a continuous function of \( \|f\|_{H^1} \), and therefore, since \( f^n \to H^1 \), we have a nontrivial common lifespan for all solutions \( u^n \) and \( u \), provided \( n \) is sufficiently large.

For \([1,7]\), we borrow an idea, due to T. Tao, \([18]\) For an integer \( j \), write

\[
\|U(t)f - U(t)f^n\|_{L^\infty(0,T;H^1)} \leq \|P_{<j}(U(t)f - U(t)f^n)\|_{L^\infty(0,T;H^1)} + \|P_{\geq j}(U(t)f^n)\|_{L^\infty(0,T;H^1)}.
\]

By the Lipschitz bound \((1.4)\), we have

\[
\|P_{<j}(U(t)f - U(t)f^n)\|_{L^\infty(0,T;H^1)} \lesssim 2^j \|U(t)f - U(t)f^n\|_{L^\infty(0,T;L^2)} \lesssim 2^j \|f - f^n\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty.
\]

For the other two terms, say \( \sum_{k \geq j} P_k U(t)f \), we have that \( u_k \) satisfies \((4.5)\), which is an equation with \( A = \pm u \leq k-4 \) with a forcing term \( E^k \), which unfortunately depends on \( u_{<k}, u_{>k} \). This is the inevitable “leakage” that occurs from projecting the equation in a fixed dyadic frequency. More precisely, the evolution of \( u_k \) is driven not only by \( u_k \), but also on the forward frequencies \( u_{>k} \) and the backward frequencies \( u_{<k} \). This effect however is (usually) very weak, but to at least heuristically motivate the argument, let us ignore it for a second.

Then, \( u_k \) satisfies an equation, which is driven by nonlinearities containing only \( u_k \), and thus only by the initial data \( f_k \). Then, it must be that \( \|P_{>j} U(t)f\|_{H^1} \leq C\|f_{>j}\|_{H^1} \), which should go to zero, as \( j \to \infty \) and similarly for \( \|P_{>j} U(t)f^n\|_{H^1} \).

The estimates in \([18]\) are performed in the envelope spaces \( H^1_c \), which makes the above argument rigorous.

We take a slightly different, but essentially equivalent approach. Fix \( j > 0 \). Since \( P_k u \) satisfies \((1.6)\), by the \textit{a priori} estimates \((1.10)\)

\[
\|P_{>j} u\|_{L^\infty(0,T_2;H^1)} \lesssim \left( \sum_{k \geq j} 2^{2k} \|P_k u\|_{L^4(0,T_2;L^4)}^2 \right)^{1/2} \leq C\|P_{>j} f\|_{H^1}.
\]

By Lemma 2.41 it follows that

\[
\left( \sum_{k \geq j} 2^{2k} \|E^k(u,u)\|_{L^1(0,T_2;L^2)}^2 \right)^{1/2} \leq CT^3_2 \|P_{>j-3} u_x\|_{L^4(0,T_2;L^\infty)} \|u\|_{L^\infty H^1} + C T^3_2 \|u_x\|_{L^1(0,T_2;L^\infty)} \|P_{>j-3} u\|_{L^\infty H^1}.
\]

\footnote{The arguments in \([18]\) concerned the Benjamin-Ono equation, a closely related dispersive equation.}
Fix a small positive number \( \delta \), to be specified below. Putting the last two inequalities together yields

\[
\left( \sum_{k>j} 2^{2k} \| P_k u \|_{L^r(T_2)}^2 \right)^{1/2} \lesssim \| P_{>j} f \|_{H^1},
\]

\[
+ T_2^{3/4} \| P_{>j-3} u_x \|_{L^4(T_2)} \| u \|_{L^\infty H^1} + T_2^{3/4} \| u_x \|_{L^4(T_2)} \| P_{>j-3} u \|_{L^\infty H^1}
\]

\[
\leq C \| P_{>j} f \|_{H^1} + \delta \| P_{>j-3} u_x \|_{L^4(T_2)} + \| P_{>j-3} u \|_{L^\infty H^1},
\]

provided \( T_2^{3/4} \| f \|_{H^1} \lesssim \delta \).

Let \( a_j = \sup_{q,r} \text{Strichartz}(\sum_{k>j} 2^{2k} \| P_k u \|_{L^q(T_2)}^2)^{1/2} \). Since \( \| P_{>j-3} u \|_{L^\infty H^1} \), \( \| P_{>j-3} u_x \|_{L^4(T_2)} \leq a_{j-3} \), we arrive at

\[
a_j \leq C \| f_{>j} \|_{H^1} + C \delta a_{j-3}.
\]

Iterating the last inequality yields

\[
a_j \leq (C \delta)^{\lfloor j/3 \rfloor - 1} a_0 + C \sum_{k=0}^{\lfloor j/3 \rfloor} (C \delta)^k \| f_{>j-3k} \|_{H^1}.
\]

Fix \( \delta : C \delta = 1/2 \). Split the sum in \( k \geq j/6 \) and \( k < j/6 \). Recall that according to the \( H^1 \) a priori estimates \( a_0 \leq C \| f \|_{H^1} \), we get

\[
a_j \leq C 2^{-\lfloor j/6 \rfloor} \| f \|_{H^1} + C \| f_{\geq j/2} \|_{H^1} \sum_{k=0}^{\infty} 2^{-k} + C \| f \|_{H^1} \sum_{k \geq j/6} 2^{-k}.
\]

All in all, we obtain

\[
(4.8) \quad a_j \leq C 2^{-j/6} \| f \|_{H^1} + C \| f_{\geq j/2} \|_{H^1} \to 0 \quad \text{as} \quad j \to \infty.
\]

It follows that

\[
\| P_{>j} u \|_{L^\infty(T_2)} \lesssim \sup_{q,r} \left( \sum_{k>j} 2^{2k} \| P_k u \|_{L^q(T_2)}^2 \right)^{1/2} \to 0 \quad \text{as} \quad j \to \infty.
\]

Similarly, take \( f^n \) and the corresponding solution \( u^n := U(t) f^n \). Since \( f^n \to_{H^1} f \), it follows that \( \sup_n \| f^n \|_{H^1} < \infty \) and \( \lim_{j \to \infty} \limsup_n \| P_{>j} f^n \|_{H^1} = 0 \). By (4.3),

\[
\limsup_n \| P_{>j} u^n \|_{L^\infty(T_2)} \lesssim \limsup_{q,r} \left( \sum_{k>j} 2^{2k} \| P_k u^n \|_{L^q(T_2)}^2 \right)^{1/2} \leq C 2^{-j/6} \sup \| f^n \|_{H^1} + C \limsup_n \| f_{\geq j/2} \|_{H^1} \to 0.
\]

The continuity of the map \( U(t) : H^1(\mathbb{R}^1) \cap L^1(\mathbb{R}^1) \cap \{ f : \sup_x | \int_{-\infty}^{\infty} f(y)dy| \leq \varepsilon \} \to C^0(0,T)H^1_x \) is established.

4.3. Well-posedness for \( u_t - iu_{xx} \pm (u_x)^2 = 0 \). Let \( f \) be an initial data with \( \| f \|_{B^{1/2}_{2,1}} \leq \varepsilon/C_1 \), with \( C_1 \) an absolute constant to be specified below.
4.3.1. **Existence of solutions and $L^2, H^1$ a priori estimates.** We start with the observation that if $u$ satisfies (1.2), then $v = u_x$ satisfies

$$v_t - iv_{xx} + 2u_xv_x = 0. \tag{4.9}$$

In the context of Proposition 1.5 and by the remarks in Section 4.1, we have that $u$ satisfies a derivative Schrödinger equation with $A = \pm u_x$.

Our goal is again to verify (4.3) and (4.4), for smooth solutions of (1.2) satisfying the a priori estimate (4.2), and since $v$ satisfies (4.9) (with $A = \pm 2u_x$), then it also satisfies the a priori estimate

$$\|u_x\|_{L^q(0,T)L^r} = \|v\|_{L^q(0,T)L^r} \leq C\|f_x\|_{L^2}, \tag{4.10}$$

with an eventually smaller time $T = T(\|f\|_{H^1})$.

We verify that (4.3) holds. Note that $\int^t_{-\infty} A(t, y) dy = \pm u(t, x)$ and we need only check that $\|u\|_{L^\infty(0,T) L^\infty} \leq \varepsilon$ for $T = T(\|f\|_{H^1})$.

By (1.2), we have that

$$u(t, x) = e^{it\partial_x^2} f \mp \int_0^t e^{i(t-s)\partial_x^2} u_x^2(s, \cdot) ds,$$

whence by the Sobolev embedding $B^{1/2}_{2,1}(\mathbb{R}^1) \hookrightarrow L^\infty(\mathbb{R}^1)$ and by the $L^1 \to L^\infty$ dispersive estimate for $e^{it\partial_x^2}$

$$\|u\|_{L^\infty(0,T) L^\infty} \leq \left\|e^{it\partial_x^2} f\right\|_{L^\infty(0,T) L^\infty} + \left\|\int_0^t e^{i(t-s)\partial_x^2} u_x^2(s, \cdot) ds\right\|_{L^\infty(0,T) L^\infty} \leq C\left\|e^{it\partial_x^2} f\right\|_{L^\infty(0,T) B^{1/2}_{2,1}(\mathbb{R}^1)} + C\int_0^T \left\|u_x\right\|_{L^2(0,T) L^2}^2 \frac{ds}{\sqrt{t-s}} \leq C\|f\|_{B^{1/2}_{2,1}(\mathbb{R}^1)} + 2C\sqrt{T}\|u_x\|_{L^\infty(0,T) L^2}^2.

By $\|f\|_{B^{1/2}_{2,1}} \leq \varepsilon/C_1$ and (4.10), the resulting inequality is

$$\|u\|_{L^\infty(0,T) L^\infty} \leq C\varepsilon/C_1 + C\sqrt{T}\|f_x\|_{L^2}^2.$$

Clearly, letting $C_1 := 2C$ and $T \leq \varepsilon^2/(4C^2\|f_x\|_{L^2}^4)$ will guarantee $\|u\|_{L^\infty(0,T) L^\infty} \leq \varepsilon$, as required by (4.3).

On the other hand, (4.4) is an easy consequence of the a priori estimate (4.10). Indeed

$$\int_{-\infty}^x (\partial_t - i\partial_y^2) A dy = (\partial_t - i\partial_y^2) u = \mp u_x^2.$$

Therefore,

$$\left\|\int_{-\infty}^x (\partial_t - i\partial_y^2) A dy\right\|_{L^1(0,T)L^\infty}^2 = \left\|u_x\right\|_{L^2(0,T)L^\infty}^2 \leq T^{1/2}\|u_x\|_{L^4(0,T)L^\infty}^2.$$


The last expression is dominated by $CT^{1/2}\|f_x\|_{L^2}^2$ according to (4.10) and thus choosing $T: T \leq \varepsilon^2/(C^2\|f_x\|_{L^2}^4)$ will guarantee that (4.4) is satisfied as well.

4.3.2. A priori estimates in $H^s$, $s \geq 2$. Note that in the previous section, we have already proved (1.3), so now we concentrate on (1.6).

This follows from Theorem (1.2) since $v = u_x$ satisfies $v_t - iv_{xx} + 2vv_x = 0$, with initial data $v(0, x) = f_x$. It follows from (1.3) that for all $a \geq 1$,

$$\|u_x\|_{C^0(0, T_1)H^s_x} = \|v\|_{C^0(0, T_1)H^s_x} \leq C_0\|f_x\|_{H^s_x}.$$ 

Set $s = a + 1 \geq 2$. By the $L^2$ a priori estimate for $u$ and the last estimate, we have

$$\|u\|_{C^0(0, T_1)H^s_x} \leq C(\|u\|_{C^0(0, T_1)L^2} + \|u_x\|_{C^0(0, T_1)H^s_x}) \leq C(\|f\|_{L^2} + \|f_x\|_{H^s_x}) \leq C\|f\|_{H^s_x}.$$ 

4.3.3. $L^2$ Lipschitz estimates for $V(t)$. In this section, we show (1.7). Let $u, v$ be two solutions to (1.2) with data $f$ and $g$ respectively.

Denote $z = u - v$, which satisfies $z_t - iz_{xx} \pm (u_x + v_x)z_x = 0$ with data $z(0, x) = f(x) - g(x)$, so that $\|f\|_{B_{1/2}^2} + \|g\|_{B_{1/2}^2} << \varepsilon$. It follows from the $L^2$ a priori estimates (1.10) in Proposition (1.3) that

$$\|u - v\|_{L^\infty(0, T_0)L^2_x} = \|z\|_{L^\infty(0, T_0)L^2_x} \leq \|f - g\|_{L^2}.$$ 

The above inequality holds for all $T_0$, so that

$$\|u\|_{L^\infty(0, T_0)L^\infty_x} + \sqrt{T_0}\|u_x\|_{L^4(0, T_0)L^\infty_x} << \varepsilon,$n
$$\|v\|_{L^\infty(0, T_0)L^\infty_x} + \sqrt{T_0}\|v_x\|_{L^4(0, T_0)L^\infty_x} << \varepsilon.$$ 

Therefore by the results in the previous section, it suffices to take

$$T_0 : \sqrt{T_0}(\|f_x\|_{L^2}^2 + \|g_x\|_{L^2}^2) << \varepsilon.$$ 

4.3.4. Continuity of $V(t) : H^2(R_1) \cap \{f : \|f\|_{B_{1/2}^2} \leq \varepsilon\} \to C^0(0, T)H^2(R_1)$. Similar to the proof of continuity for $U(t)$ and having in mind (1.7), it will suffice to prove $\|V(t)f^n - V(t)f\|_{H^2_x} \to 0$ as $\|f^n - f\|_{H^2_x} \to 0$.

Denote $u^n = V(t)f^n, u = V(t)f$. Clearly $v = u_x$ and $v^n = u^n_x$ satisfy $w_t - iw_{xx} \pm 2ww_x = 0$, with data $f_x, f^n_x$ respectively. This equation is similar to (1.1), and the results of Theorem (1.2) apply.

It follows by the continuity of $U(t)$, that $\|u^n - u\|_{H^2} \sim \|v^n - v\|_{H^1} \to 0, as \|f^n - f\|_{H^2_x} \leq C\|f^n - f\|_{H^2_x} \to 0$. This shows the continuity of $V(t) : H^2(R_1) \cap \{f : \|f\|_{B_{1/2}^2} \leq \varepsilon\} \to C^0(0, T)H^2(R_1)$.

Next, we show that $V(t)f$ is a weak solution, whenever $f \in H^2 \cap \{f : \|f\|_{B_{1/2}^2} \leq \varepsilon\}$. To that end, take $f^n$ smooth and decaying, so that $\|f^n - f\|_{H^2_x} \to 0$.

By Chihara’s theorem $u^n = V(t)f^n$ are classical solutions and thus for every test function $\psi$

$$\int u^n(x, t)\psi_t(x, t)dxdt + i\int u^n\psi_{xx}dxdt = \int (u^n_x)^2\psi dxdt = 0.$$ 

Since

$$\int (u^n)^2\psi dxdt - \int (u_x)^2\psi dxdt \leq T\|u^n_x - u_x\|_{L^2} \leq \int (u^n_x)^2\psi dxdt \to 0.$$ 

\footnote{Strictly speaking, the equation for $v$ has a factor of 2 in front of the nonlinearity, which might reduce the lifespan of the solution, but clearly the arguments in Theorem (1.2) work for it as well.}
by the continuity of $V(t)$ and $\|f^n - f\|_{L^2} \to 0$, we conclude
\[
\int u(x,t)\psi_t(x,t)dxdt + i \int u\psi_{xx}dxdt \mp \int (ux)^2\psi dxdt = 0,
\]
that is, $u$ is a weak solution to (1.2).

References

1. N. Burq, F. Planchon, personal communication.

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