

CANCELLATION AND STABLE RANK FOR DIRECT LIMITS OF RECURSIVE SUBHOMOGENEOUS ALGEBRAS

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ABSTRACT. We prove the following results for a unital simple direct limit A of recursive subhomogeneous algebras with no dimension growth:

- (1) $\text{tsr}(A) = 1$.
- (2) The projections in $M_\infty(A)$ satisfy cancellation: if $e \oplus q \sim f \oplus q$, then $e \sim f$.
- (3) A satisfies Blackadar's Second Fundamental Comparability Question: if $p, q \in M_\infty(A)$ are projections such that $\tau(p) < \tau(q)$ for all normalized traces τ on A , then $p \prec q$.
- (4) $K_0(A)$ is unperforated for the strict order: if $\eta \in K_0(A)$ and there is $n > 0$ such that $n\eta > 0$, then $\eta > 0$.

The last three of these results hold under certain weaker dimension growth conditions and without assuming simplicity. We use these results to obtain previously unknown information on the ordered K-theory of the crossed product $C^*(\mathbf{Z}, X, h)$ obtained from a minimal homeomorphism of a finite-dimensional infinite compact metric space X . Specifically, $K_0(C^*(\mathbf{Z}, X, h))$ is unperforated for the strict order, and satisfies the following K-theoretic version of Blackadar's Second Fundamental Comparability Question: if $\eta \in K_0(A)$ satisfies $\tau_*(\eta) > 0$ for all normalized traces τ on A , then there is a projection $p \in M_\infty(A)$ such that $\eta = [p]$.

0. INTRODUCTION

Recursive subhomogeneous algebras were introduced in [34]; we recall the definition below. They include finite direct sums of not necessarily trivial unital homogeneous C*-algebras, the dimension drop intervals and matrix algebras over them, and the algebras A_Y arising in Qing Lin's study [25] of the transformation group C*-algebras of minimal homeomorphisms (provided $\text{int}(Y) \neq \emptyset$). In this paper, we generalize to certain direct limits of recursive subhomogeneous algebras some of the known results on direct limits of homogeneous C*-algebras with slow dimension growth. We use these results to obtain previously unknown information on the ordered K-theory of the C*-algebras of minimal homeomorphisms. In particular, our results make it possible in many cases to compute the Elliott invariant [12] for the crossed product by a minimal homeomorphism.

We prove the following results for direct limits of recursive subhomogeneous algebras. (See later in the Introduction, and Section 1, for explanations of the dimension growth conditions.)

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Theorem 0.1. *Let $A = \varinjlim(A_i, \varphi_{ij})$ be a unital direct limit of a system of recursive subhomogeneous algebras with slow dimension growth. Then:*

- (1) *The map $U(A)/U_0(A) \rightarrow K_1(A)$ is an isomorphism.*
- (2) *If in addition the system has strict slow dimension growth, then the projections in $M_\infty(A)$ satisfy cancellation: if $e \oplus q \sim f \oplus q$, then $e \sim f$.*
- (3) *If in addition the maps φ_{ij} of the system are all injective and unital, then A satisfies Blackadar's Second Fundamental Comparability Question ([4], 1.3.1): if $p, q \in M_\infty(A)$ are projections such that $\tau(p) < \tau(q)$ for all normalized traces τ on A , then $p \precsim q$.*
- (4) *If in addition the maps φ_{ij} of the system are all injective and unital, and A is simple, then $K_0(A)$ is unperforated for the strict order. That is, if $\eta \in K_0(A)$ and there is $n > 0$ such that $n\eta > 0$, then $\eta > 0$.*
- (5) *If in addition the system has no dimension growth, and A is simple, then $\text{tsr}(A) = 1$.*

In this theorem, and throughout the paper, notation is as follows. The notation $p \sim q$ means Murray-von Neumann equivalence of projections, $p \precsim q$ means p is Murray-von Neumann equivalent to a subprojection of q , $M_\infty(A)$ is the algebraic direct limit $\varinjlim M_n(A)$ under the maps $a \mapsto a \oplus 0$ (following Definition 5.1.1 of [3]), $\text{tsr}(A)$ is the topological stable rank of A ([39]), $U(A)$ is the unitary group of a unital C^* -algebra A , and $U_0(A)$ is the identity component of $U(A)$.

Parts (2), (3), and (4) of the theorem partially generalize results of [5], [29], and [19]. Part (5) generalizes [9]. We have had to impose extra conditions in part (2) (strict slow dimension growth) and part (4) (simplicity); we do not know whether these extra conditions are really necessary. Similarly, we do not know whether the condition “no dimension growth” in part (5) can be relaxed to “slow dimension growth”, as was done for the homogeneous case in [6].

Two other results from the homogeneous case, Theorem 2 of [6] and Theorem 2.7 of [19], do not generalize to direct limits of recursive subhomogeneous algebras. Specifically, there is a simple direct limit A of separable recursive subhomogeneous algebras, with no dimension growth and in which the maps of the system are all injective and unital, such that the projections in A distinguish the traces on A but A does not have real rank zero, and such that $K_0(A)$ does not have Riesz decomposition. Examples of these sorts were known before, but we give one (Example 4.8) which arises naturally from our applications to crossed products by minimal homeomorphisms and has a simple proof. (In [35], we give criteria for exactly when a simple direct limit of separable recursive subhomogeneous algebras with no dimension growth has real rank zero, and when it has the property (SP): every nonzero hereditary subalgebra contains a nonzero projection. In particular, we will see that the combination of Riesz decomposition, (SP), and projections distinguish traces and implies real rank zero.)

Let X be an infinite compact metric space, and let h be a minimal homeomorphism of X . The crossed product $C^*(\mathbf{Z}, X, h)$ has been well studied when X is the Cantor set (see for example [17]), and when h is an irrational rotation of the circle. The C^* -algebras of minimal homeomorphisms of higher-dimensional spaces have remained somewhat mysterious. Connes has shown [8] that, unlike the cases mentioned above, the crossed products sometimes have no nontrivial projections. (See Corollary 12 in Section 6 of [14] for a generalization.) Qing Lin [25] has studied simple subalgebras of the crossed product which can be realized as direct limits of

recursive subhomogeneous algebras in which the maps of the system are all injective and unital; moreover, if X is finite dimensional, then the system has no dimension growth. Using those subalgebras and the results above, we obtain the following theorem.

Theorem 0.2. *Let X be a finite-dimensional infinite compact metric space, and let h be a minimal homeomorphism of X . Then:*

(1) *The map*

$$U(C^*(\mathbf{Z}, X, h))/U_0(C^*(\mathbf{Z}, X, h)) \longrightarrow K_1(C^*(\mathbf{Z}, X, h))$$

is surjective.

- (2) *$C^*(\mathbf{Z}, X, h)$ satisfies the following K -theoretic version of Blackadar’s Second Fundamental Comparability Question: if $\eta \in K_0(A)$ satisfies $\tau_*(\eta) > 0$ for all normalized traces τ on A , then there is a projection $p \in M_\infty(A)$ such that $\eta = [p]$.*
- (3) *$K_0(C^*(\mathbf{Z}, X, h))$ is unperforated for the strict order.*

This theorem can be used to completely determine the order on the K_0 -group of the crossed product in interesting cases. In Example 4.9, we easily obtain the description, proved in [22], of the positive cone in the K_0 -group of the crossed product by a Furstenberg transformation of the 2-torus. We also completely determine the Elliott invariant (see [12]) for the crossed product by a minimal homeomorphism h of an odd sphere S^n with $n \geq 3$. It follows from our computation that the Elliott invariant depends only on the simplex of invariant Borel probability measures for h , and in particular not on the dimension n of the sphere (as long as $n \geq 3$). The Elliott classification conjecture would therefore imply that if $n_1, n_2 \geq 3$ are odd, and h_j is a uniquely ergodic minimal homeomorphism of S^{n_j} , then $C^*(\mathbf{Z}, S^{n_1}, h_1) \cong C^*(\mathbf{Z}, S^{n_2}, h_2)$.

In [28] (see the survey article [27]), it is proved that the transformation group C^* -algebra of a minimal diffeomorphism of a compact smooth manifold is a direct limit of recursive subhomogeneous algebras with topological dimension at most the dimension of the manifold. In particular, the results of this paper apply to such algebras. In [24], the results of this paper and of [35] play a key role in a proof that if X is a finite-dimensional infinite compact metric space, $h: X \rightarrow X$ is a minimal homeomorphism, and the map from $K_0(C^*(\mathbf{Z}, X, h))$ to the affine functions on the tracial state space of $C^*(\mathbf{Z}, X, h)$ has dense range (a necessary condition for real rank zero); then $C^*(\mathbf{Z}, X, h)$ falls under the Elliott classification conjecture.

We now recall the definition of a recursive subhomogeneous algebra and some useful associated terminology. (See Definitions 1.1 and 1.2 of [34].) First recall that if A, B , and C are C^* -algebras, and $\varphi: A \rightarrow C$ and $\rho: B \rightarrow C$ are homomorphisms, then the pullback $A \oplus_C B$ is given by

$$A \oplus_C B = \{(a, b) \in A \oplus B : \varphi(a) = \rho(b)\}.$$

Definition 0.3. A recursive subhomogeneous algebra is a unital C^* -algebra of the form

$$R = \left[\cdots \left[\left[C_0 \oplus_{C_1^{(0)}} C_1 \right] \oplus_{C_2^{(0)}} C_2 \right] \cdots \right] \oplus_{C_l^{(0)}} C_l,$$

with $C_k = C(X_k, M_{n(k)})$ for compact Hausdorff spaces X_k and positive integers $n(k)$, with $C_k^{(0)} = C(X_k^{(0)}, M_{n(k)})$ for compact subsets $X_k^{(0)} \subset X_k$ (possibly

empty), and where the maps $C_k \rightarrow C_k^{(0)}$ are always the restriction maps. An expression of this type will be referred to as a *decomposition* of R (over $\coprod_{k=0}^l X_k$).

Associated with this decomposition are:

- (1) its *length* l ;
- (2) its *base spaces* X_0, X_1, \dots, X_l and *total space* $X = \coprod_{k=0}^l X_k$;
- (3) its *matrix sizes* $n(0), \dots, n(l)$, and *matrix size function* $m: X \rightarrow \mathbf{N} \cup \{0\}$, defined by $m(x) = n(k)$ when $x \in X_k$ (this is called the *matrix size of A at x*);
- (4) its *minimum matrix size* $\min_k n(k)$ and *maximum matrix size* $\max_k n(k)$;
- (5) its *topological dimension* $\dim(X)$ (the covering dimension of X [30], Definition 3.1.1; here equal to $\max_k \dim(X_k)$), and *topological dimension function* $d: X \rightarrow \mathbf{N} \cup \{0\}$, defined by $d(x) = \dim(X_k)$ when $x \in X_k$ (this is called the *topological dimension of A at x*);
- (6) its *standard representation* $\sigma = \sigma_R: R \rightarrow \bigoplus_{k=0}^l C(X_k, M_{n(k)})$, defined by forgetting the restriction to a subalgebra in each of the fibered products in the decomposition;
- (7) the associated *evaluation maps* $\text{ev}_x: R \rightarrow M_{n(k)}$ for $x \in X_k$, defined to be the restriction of the usual evaluation map to R , identified with a subalgebra of $\bigoplus_{k=0}^l C(X_k, M_{n(k)})$ via σ .

At this point, we make a few remarks on the notions of slow dimension growth. This notion was introduced and its importance recognized in [6]; here, we refer to the extensions of the concept to nonsimple C^* -algebras in [29] and [19]. For direct limits of direct sums of homogeneous C^* -algebras, it is usual to assume that the spaces associated with the summands are connected. Slow dimension growth is then defined in terms of the dimensions of these spaces and the multiplicities of the partial maps between direct summands at one level and those at later levels. Connectedness ensures that these multiplicities are well defined. In a direct system of recursive subhomogeneous algebras, because of the way the summands “overlap” in the pullbacks, the multiplicities of the partial maps between components of the recursive subhomogeneous decompositions, and even the partial maps themselves, need not be well defined. Because of this, and because of what happens in some of our proofs, it is not clear what the right definition of slow dimension growth is. A finite direct sum of algebras $C(X, M_n)$ is a single recursive subhomogeneous algebra. Therefore the right definition should include the situations of [29] and [19]. It should also enable one to prove cancellation in the general case, and stable rank one in the simple case. Slow dimension growth might be more tractable for direct systems of noncommutative CW complexes [32] and cell morphisms (Definition 11.3 of [32]).

We do not make a serious effort here to find the right definition. Rather, we give several versions which suffice for the proofs of our theorems, and which apply to the algebras we are most interested in, namely simple direct limits with no dimension growth. (These are the algebras required for the applications to the C^* -algebras of minimal homeomorphisms.) We also do not formally consider weakenings of the dimension growth conditions to “relatively large entries” in the sense of Section 3 of [29]. (One can, however, see from the proofs that conditions of that type suffice for some of our results.) The condition we call slow dimension growth is similar to the conditions used in [29] and [19], and the condition we call strict slow dimension growth includes in addition a kind of mixing condition on the summands.

This paper consists of four sections. The first defines and proves useful relations between the various forms of slow dimension growth, and compares them with what is already in the literature. Section 2 contains the proofs of the first four parts of Theorem 0.1. These proofs follow by standard methods from the work done in [34]. Section 3 contains the proof of the last part of Theorem 0.1. We were not able to follow the method of [9], and in fact our proof does not use any version of the selection theorem there. Instead, we rely on perturbation results, functional calculus, and a kind of approximate polar decomposition. Finally, in Section 4 we give the applications to the C^* -algebras of minimal homeomorphisms, and the subalgebras of them considered in [25].

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Some of the results of this paper were announced in [26].

1. DIMENSION GROWTH

The results on direct limits that we want to generalize from the homogeneous case are mostly stated for systems with slow dimension growth. (See [6], [5], [19], and [29].) We therefore discuss dimension growth in this section. For direct systems of recursive subhomogeneous algebras, it is not clear what the appropriate definition of slow dimension growth is. (See the Introduction for further discussion.) We therefore confine ourselves to giving some usable definitions, proving several easy results, and showing that our definitions are satisfied for simple direct limits with no dimension growth. These results suffice for our applications.

We state two versions of slow dimension growth. The weaker version is more closely related to the definitions in [19] and [29]. The stronger version includes a kind of mixing condition, which seems to be needed in some of our proofs in the next section.

Definition 1.1. Let $((A_i)_{i \in \mathbf{N}}, (\varphi_{ij}))$ be a direct system of recursive subhomogeneous algebras, and let each A_i be equipped with a specific decomposition of length l_i with total space X_i and topological dimension function $d_i: X_i \rightarrow \mathbf{N} \cup \{0\}$. The system is said to have *slow dimension growth* (with respect to the given collection of decompositions) if for every i , every projection $p \in M_\infty(A_i)$, and every $N \in \mathbf{N}$, there is j_0 such that for all $j \geq j_0$ and $x \in X_j$ we have

$$\text{ev}_x(\varphi_{ij}(p)) = 0 \quad \text{or} \quad \text{rank}(\text{ev}_x(\varphi_{ij}(p))) \geq Nd_j(x).$$

The system is said to have *strict slow dimension growth* if, in the above, for $p \neq 0$ it is possible to choose j_0 such that we always have $\text{rank}(\text{ev}_x(\varphi_{ij}(p))) \geq Nd_j(x)$.

We note that $\text{rank}(\text{ev}_x(\varphi_{ij}(p)))$ depends only on the ranks of the projections $\text{ev}_y(p)$ for suitable $y \in X_i$, namely those y for which ev_y occurs among the irreducible subrepresentations of the finite-dimensional representation $\text{ev}_x \circ \varphi_{ij}$ of A_i . The global topological nature of p is irrelevant. The global topology does, however, have a strong influence on the existence of projections p with specified values of $\text{rank}(\text{ev}_y(p))$. We would like to have $\text{rank}(\text{ev}_y(p)) = 1$. However, it follows from Example 4.8 below that there are recursive subhomogeneous algebras A whose minimum matrix size is arbitrarily large but which contain no nontrivial projections.

We do not even know whether, given $p \in M_\infty(A)$, there is a projection $q \in A$ such that $\text{ev}_y(q) = 0$ exactly when $\text{ev}_y(p) = 0$.

This definition is complicated in practice, and we will therefore seek simpler conditions which imply it. First, we compare it with the definitions already in the literature for the (nonsimple) homogeneous case.

Proposition 1.2. *Let $((A_i)_{i \in \mathbf{N}}, (\varphi_{ij}))$ be a direct system in which*

$$A_i = \bigoplus_{l=1}^{r(i)} C(X_{il}, M_{n(i,l)}),$$

with X_{il} compact and connected. Assume that the kernel of the map $A_i \rightarrow \varinjlim A_j$ contains no entire summand $C(X_{ik}, M_{n(i,k)})$. (That is, no summands vanish in the limit.) Regard each A_i as a recursive subhomogeneous algebra with the obvious decomposition (Example 1.4 of [34]). Then:

- (1) *Slow dimension growth in the sense of 2.1 of [19] implies slow dimension growth in the sense of Definition 3.6 of [29].*
- (2) *Slow dimension growth in the sense of 2.1 of [19] implies slow dimension growth in the sense of Definition 1.1.*
- (3) *If the direct system has slow dimension growth in the sense of Definition 3.6 of [29], then it has a subsystem which has slow dimension growth in the sense of Definition 1.1.*
- (4) *Slow dimension growth in the sense of Definition 1.1 implies slow dimension growth in the sense of Definition 3.6 of [29].*

We note that, for direct systems as in the proposition, slow dimension growth in the sense of Definition 1.1 does not imply slow dimension growth in the sense of 2.1 of [19]. Also, Definition 3.6 of [29] is formally stated only for systems with unital injective maps, but makes sense in general. The definition in [19] is stated for direct systems over general directed sets, but here we only consider direct systems over \mathbf{N} .

Proof of Proposition 1.2. Let $e_i^{(l)}$ be the identity of the summand $C(X_{il}, M_{n(i,l)})$. Let $\pi_i^{(l)}: A_i \rightarrow C(X_{il}, M_{n(i,l)})$ be the projection map. For $j \geq i$ and $1 \leq m \leq r(j)$, define the following quantities:

$$\alpha_{ij}^{(m)} = \min \left\{ \text{rank}(\pi_j^{(m)} \circ \varphi_{ij}(e_i^{(l)})) : 1 \leq l \leq r(i) \text{ and } \pi_j^{(m)} \circ \varphi_{ij}(e_i^{(l)}) \neq 0 \right\},$$

$$\delta_j = \max \left\{ \frac{\dim(X_{jm})}{n(j,m)} : 1 \leq m \leq r(j) \right\},$$

and

$$\mu_{ij} = \min \left\{ \frac{\text{rank}(\pi_j^{(m)} \circ \varphi_{ij}(e_i^{(l)}))}{n(j,m)} : 1 \leq l \leq r(i), 1 \leq m \leq r(j), \right.$$

$$\left. \text{and } \pi_j^{(m)} \circ \varphi_{ij}(e_i^{(l)}) \neq 0 \right\}.$$

Note that the ranks appearing here are constant functions, which we identify with the corresponding integers, because the spaces X_{jm} are connected. The quantity δ_j is called d_j in 2.1 of [19]. Also, φ_{ij} and μ_{ij} are written as φ_{ji} and μ_{ji} there. With

these definitions, our system has slow dimension growth in the sense of 2.1 of [19] if and only if for every $i \in \mathbf{N}$ we have

$$\lim_{j \rightarrow \infty} \frac{\delta_j}{\mu_{ij}} = 0,$$

and it has slow dimension growth in the sense of Definition 3.6 of [29] if and only if for every $i \in \mathbf{N}$ we have

$$\liminf_{j \rightarrow \infty} \max \left\{ \frac{\dim(X_{jm})}{\alpha_{ij}^{(m)}} : 1 \leq m \leq r(j) \right\} = 0.$$

(Here, and in the rest of this proof, such limits are taken over $j \geq i$.)

We now claim that a system has slow dimension growth in the sense of Definition 1.1 if and only if for every $i \in N$ we have

$$\lim_{j \rightarrow \infty} \max \left\{ \frac{\dim(X_{jm})}{\alpha_{ij}^{(m)}} : 1 \leq m \leq r(j) \right\} = 0.$$

To see this, first assume the condition of Definition 1.1 holds. Choose i and N , and apply the condition to the projections $e_i^{(l)}$ for $1 \leq l \leq r(i)$. Call the resulting numbers $j_0(l)$ and set $j_0 = \max\{j_0(l) : 1 \leq l \leq r(i)\}$. Then for all $j \geq j_0$ and all m with $1 \leq m \leq r(j)$, we have

$$\frac{\dim(X_{jm})}{\alpha_{ij}^{(m)}} \leq \frac{1}{N}.$$

Thus, the limit condition above is satisfied.

Conversely, assume the limit condition above, and let $p \in M_\infty(A_i)$ be a projection. For $1 \leq l \leq r(i)$ set $\rho_l = \text{rank}(\pi_i^{(l)}(p)) / \text{rank}(e_i^{(l)})$. Set

$$\rho = \min\{\rho_l : 1 \leq l \leq r(i) \text{ and } \rho_l \neq 0\}.$$

Note that the projection $\pi_j^{(m)} \circ \varphi_{ij}(p)$ has constant rank equal to

$$\sum_{l=1}^{r(i)} \rho_m \text{rank}(\pi_j^{(m)} \circ \varphi_{ij}(e_i^{(l)})).$$

If $\pi_j^{(m)} \circ \varphi_{ij}(p) \neq 0$, then there is l_0 with $\rho_{l_0} \neq 0$ and $\text{rank}(\pi_j^{(m)} \circ \varphi_{ij}(e_i^{(l_0)})) \neq 0$. So

$$\text{rank}(\pi_j^{(m)} \circ \varphi_{ij}(p)) \geq \rho_{l_0} \text{rank}(\pi_j^{(m)} \circ \varphi_{ij}(e_i^{(l_0)})) \geq \rho \alpha_{ij}^{(m)}.$$

According to the limit condition above, we can choose j_0 such that if $j \geq j_0$ and $1 \leq m \leq r(j)$, then

$$\frac{\dim(X_{jm})}{\alpha_{ij}^{(m)}} \leq \frac{\rho}{N}.$$

For such j , and whenever $\pi_j^{(m)} \circ \varphi_{ij}(p) \neq 0$, we have

$$\text{rank}(\pi_j^{(m)} \circ \varphi_{ij}(p)) \geq N \dim(X_{jm}).$$

This verifies slow dimension growth in the sense of Definition 1.1, and proves the claim.

Part (4) of the proposition is immediate from the claim. To get part (2), we merely observe that for $1 \leq m \leq r(j)$ we have

$$\frac{\dim(X_{jm})}{\alpha_{ij}^{(m)}} \leq \frac{\delta_j}{\mu_{ij}}.$$

Part (1) follows immediately from parts (2) and (4).

It remains to prove part (3). Given the claim, we must prove that we can replace the \liminf in Definition 3.6 of [29] by a limit by passing to a cofinal subset. We do this by a kind of diagonalization argument. So assume that for all i we have

$$\liminf_{j \rightarrow \infty} \max \left\{ \frac{\dim(X_{jm})}{\alpha_{ij}^{(m)}} : 1 \leq m \leq r(j) \right\} = 0.$$

Set $i(1) = 1$, and choose an infinite subset $I_1 \subset \mathbf{N}$, with $i(1) \in I_1$, such that

$$\lim_{j \rightarrow \infty, j \in I_1} \max \left\{ \frac{\dim(X_{jm})}{\alpha_{i(1)j}^{(m)}} : 1 \leq m \leq r(j) \right\} = 0.$$

Let $i(2)$ be the second element of I_1 , and choose an infinite subset $I_2 \subset I_1$, with $i(1), i(2) \in I_2$, such that

$$\lim_{j \rightarrow \infty, j \in I_2} \max \left\{ \frac{\dim(X_{jm})}{\alpha_{i(2)j}^{(m)}} : 1 \leq m \leq r(j) \right\} = 0.$$

Proceed inductively. Take

$$I = \bigcap_{n=1}^{\infty} I_n = \{i(1), i(2), \dots\},$$

which is cofinal in \mathbf{N} . Then

$$\lim_{j \rightarrow \infty, j \in I} \max \left\{ \frac{\dim(X_{jm})}{\alpha_{ij}^{(m)}} : 1 \leq m \leq r(j) \right\} = 0$$

for all $i \in I$, so that the corresponding subsequence has slow dimension growth in the sense of Definition 1.1. \square

We can generalize the approach of [19] and [29] somewhat.

Definition 1.3. Let $\varphi: A \rightarrow B$ be a (not necessarily unital) homomorphism of recursive subhomogeneous algebras. Let decompositions of A over $X = \coprod_{k=0}^K X_k$ and B over $Y = \coprod_{l=0}^L Y_l$ be given, with all X_k and Y_l connected. For $y \in Y$ define $\mu_{k,y}(\varphi)$, the k -th partial multiplicity of φ at y , as follows. Consider all possible direct sum decompositions $\text{ev}_y \circ \varphi \cong 0 \oplus \bigoplus_{j=1}^n \text{ev}_{x_j}$ with $x_j \in X$. (Because ev_y is a finite-dimensional representation, Lemma 2.1 of [34] implies that there is always at least one such decomposition.) Then $\mu_{k,y}(\varphi)$ is the maximum, over all such decompositions, of the number (counting multiplicity) of x_j that are in X_k . Moreover, define $\mu_{k,l}(\varphi)$, the k -th partial multiplicity of φ at Y_l , to be $\sup_{y \in Y_l} \mu_{k,y}(\varphi)$. Finally, say that φ is zero in the (k, l) -component if for all $a \in A$ there is $\tilde{a} \in A$ such that $\text{ev}_x(\tilde{a}) = \text{ev}_x(a)$ for $x \in X_k$ and $\text{ev}_y(\varphi(\tilde{a})) = 0$ for all $y \in Y_l$.

Note that, for direct sums of trivial homogeneous C^* -algebras, zero in the (k, l) -component simply means that the partial map from the k -th summand of A to the l -th summand of B is zero.

Lemma 1.4. *Let $((A_i)_{i \in \mathbf{N}}, (\varphi_{ij}))$ be a direct system of recursive subhomogeneous algebras, and let each A_i be equipped with a specific decomposition of length L_i , with connected base spaces $X_{1,0}, X_{i,1}, \dots, X_{i,L_i}$ and with total space $X_i = \coprod_{k=0}^{L_i} X_{i,k}$. Assume that for every i and every $N \in \mathbf{N}$, there is j_0 such that, for all $j \geq j_0$, all k with $0 \leq k \leq L_i$, and all l with $0 \leq l \leq L_j$, either φ_{ij} is zero in the (k, l) -component or $\mu_{k,l}(\varphi_{ij}) \geq N \dim(X_{j,k})$. Then the direct system has slow dimension growth.*

Proof. Let $p \in M_\infty(A_i)$, choose j_0 as in the hypotheses of the lemma, and let $j \geq j_0$. Write $p = (p_0, p_1, \dots, p_{L_i})$, where p_k is the restriction of the standard representation of p (a function on X_i) to $X_{i,k}$. Let $q = \varphi_{ij}(p)$, and analogously write $q = (q_0, q_1, \dots, q_{L_j})$. Since the spaces are all connected, the p_k and q_l have constant ranks.

Let $0 \leq l \leq L_j$. Suppose that there is some k with $p_k \neq 0$ and $\mu_{k,l}(\varphi_{ij}) \geq N \dim(X_{j,l})$. Choose $y \in X_{k,l}$ such that $\mu_{k,y}(\varphi_{ij}) \geq N \dim(X_{j,l})$. Then clearly

$$\text{rank}(\text{ev}_y(q)) \geq N \dim(X_{j,l}) \text{rank}(p_k) \geq N \dim(X_{j,l}).$$

Since q_l has constant rank, this inequality holds for arbitrary $y \in X_{k,l}$.

Otherwise, φ_{ij} is zero in the (k, l) -component for all k with $p_k \neq 0$. For such k , choose $a_k \in M_\infty(A_i)$ such that $\text{ev}_x(a_k) = \text{ev}_x(p)$ for $x \in X_{k,i}$ and $\text{ev}_y(\varphi_{ij}(a_k)) = 0$ for all $y \in X_{j,l}$. (This can clearly be done by considering the entries separately.) Set $a = \sum_{k: p_k \neq 0} a_k^* a_k$. Then $a \geq p$ and $\text{ev}_y(\varphi_{ij}(a)) = 0$ for all $y \in X_{j,l}$. Therefore $q_l = 0$. \square

For present applications, we are primarily interested in the simple case, and we devote the rest of this section to it. In this case, at least with no (rather than slow) dimension growth, we do not need connectedness assumptions.

The following lemma is a slight generalization of part of Proposition 2.1 of [9].

Lemma 1.5. *Let $A = \varinjlim A_i$ be a simple direct limit of recursive subhomogeneous algebras, such that all the maps $\varphi_{ij}: A_i \rightarrow A_j$ in the system are unital and injective. Let X_i be the total space of A_i . Let $a \in A_i \setminus \{0\}$ for some i . Then there exists j_0 such that, for every $j \geq j_0$ and every $x \in X_j$, we have $\text{ev}_x(\varphi_{ij}(a)) \neq 0$.*

Proof. Without loss of generality $i = 0$. Assume the conclusion fails for some a . Passing to a subsequence in the direct system, we may assume that for every j there is $x \in X_j$ such that $\text{ev}_x(\varphi_{0j}(a)) = 0$. Therefore the ideal $I_j = A_j \varphi_{0j}(a) A_j$ is nontrivial. Since $\varphi_{j,j+1}(I_j) \subset I_{j+1}$, we may form the ideal $I = \varinjlim I_j \subset A$. This ideal is nonzero since it contains the image of a . If $1 \in I$, then (using injectivity of the φ_{ij}) there is j and $b \in I_j$ with $\|b - 1\| < \frac{1}{2}$, which contradicts $I_j \neq A_j$. So I is a proper ideal in A , contradicting simplicity. \square

Lemma 1.6. *Let A be any C^* -algebra in which there do not exist $n + 1$ mutually orthogonal nonzero selfadjoint elements. Then $\dim(A) \leq n^2$.*

Proof. Without loss of generality there are n mutually orthogonal nonzero selfadjoint elements $a_1, \dots, a_n \in A$. Then there is a unique nonzero $\alpha_j \in \text{sp}(a_j)$ for each j . (If some $\text{sp}(a_j)$ has more than one nonzero element, then continuous functional calculus gives two mutually orthogonal nonzero selfadjoint elements $b, c \leq |a_j|$, and

using them in place of a_j contradicts the assumption.) The elements $p_j = \alpha_j^{-1}a_j$ are mutually orthogonal nonzero projections. Moreover, $p = \sum_{j=1}^n p_j$ must be an identity for A , since the existence of a nonzero selfadjoint element of $(1-p)A(1-p)$ contradicts the assumption.

If p_jAp_j contains a selfadjoint element b not a scalar multiple of p_j , then $\text{sp}(b)$ (taken relative to the unital algebra p_jAp_j) has at least two elements. We can then use continuous functional calculus to get a contradiction as in the previous paragraph. So $p_jAp_j = \mathbf{C} \cdot p_j$ for all j .

Now suppose $p_jAp_k \neq 0$ for some j and k . Let $c \in p_jAp_k$ be nonzero. Then there are $\lambda, \mu \in (0, \infty)$ such that $cc^* = \lambda p_j$ and $c^*c = \mu p_k$. Replacing c by a suitable scalar multiple, we can assume $\lambda = 1$. Then also $\mu = 1$. Let $d \in p_jAp_k$ be arbitrary. Then $c^*d \in p_kAp_k$, so $c^*d = \alpha p_k$ for some $\alpha \in \mathbf{C}$. It follows that

$$d = p_jd = cc^*d = c \cdot \alpha p_k = \alpha c.$$

This computation shows that $\dim(p_jAp_k) \leq 1$. Since $A = \bigoplus_{j,k} p_jAp_k$ as a Banach space, we conclude that $\dim(A) \leq n^2$. □

Lemma 1.7. *Let A be a simple C^* -algebra. Suppose A has a hereditary subalgebra B such that $B \cong M_n$. Then A is isomorphic to the algebra $K(H)$ of all compact operators on some Hilbert space H .*

Proof. Without loss of generality $B \cong \mathbf{C}$, that is, $B = \mathbf{C} \cdot p$ for some projection $p \in A$. Then there is a state ω on A such that $pap = \omega(a)p$ for all $a \in A$. With the help of this state, it is easy to make pA a \mathbf{C} - A strong Morita equivalence bimodule. (It is full as an A -module because A is simple.) In particular, $H = pA$ is a Hilbert space such that $A \cong K(H)$. □

Lemma 1.8. *Let $A = \varinjlim A_i$ be an infinite-dimensional simple direct limit of recursive subhomogeneous algebras, such that all the maps $\varphi_{ij}: A_i \rightarrow A_j$ in the system are unital and injective. Let X_i be the total space of A_i . Let $a \in A_i \setminus \{0\}$ for some i . Then for every $n \in \mathbf{N}$ there exists j_0 such that, for every $j \geq j_0$ and every $x \in X_j$, we have $\text{rank}(\text{ev}_x(\varphi_{ij}(a))) \geq n$.*

Proof. Without loss of generality $i = 0$. Moreover, since $\text{rank}(b^*b) = \text{rank}(b)$ for any $b \in M_n$, without loss of generality $a \geq 0$.

Choose l such that there are n mutually orthogonal nonzero selfadjoint elements $b_1, \dots, b_n \in \overline{\varphi_{0l}(a)A_l\varphi_{0l}(a)}$. (If this is not possible, then Lemma 1.6 implies that $\dim(\overline{\varphi_{0j}(a)A_j\varphi_{0j}(a)}) \leq (n-1)^2$ for all j . Then, with c being the image of a in A , we have $\dim(\overline{cAc}) \leq (n-1)^2$. Since A is simple and unital, Lemma 1.7 implies that $A \cong M_m$ for some m . This contradicts infinite dimensionality.) Now choose (by Lemma 1.5) $j_0 \geq l$ such that, for every $j \geq j_0$, every k , and every $x \in X_j$, we have $\text{ev}_x(\varphi_{lj}(b_k)) \neq 0$. For such j , and $x \in X_j$, the image

$$\text{ev}_x(\overline{\varphi_{0j}(a)A_j\varphi_{0j}(a)}) = \overline{\text{ev}_x(\varphi_{0j}(a))M_m\text{ev}_x(\varphi_{0j}(a))}$$

(for some suitable m) contains n mutually orthogonal nonzero selfadjoint elements $\text{ev}_x(b_1), \dots, \text{ev}_x(b_n)$. Therefore $\text{rank}(\text{ev}_x(\varphi_{0j}(a))) \geq n$. □

Corollary 1.9. *Let $A = \varinjlim A_i$ be an infinite-dimensional simple direct limit of recursive subhomogeneous algebras, such that all the maps $\varphi_{ij}: A_i \rightarrow A_j$ of the system are unital and injective. Assume the system has no dimension growth, that*

is, there is a finite d such that all A_i have topological dimension at most d . Then the system has strict slow dimension growth.

Proof. We verify Definition 1.1 for a particular projection p by applying Lemma 1.8 to the algebras $M_n(A) = \varinjlim M_n(A_i)$, taking $a = p$. \square

We now briefly consider the effect on dimension growth of forcing the maps of a system to be injective. This means that the algebras in the system must be replaced by quotients. Proposition 3.1 of [34] shows that this does not increase the topological dimension. However, (strict) slow dimension growth also depends on the details of the decompositions, and we do not know how to obtain a decomposition of a quotient which is suitably related to a given decomposition of the original algebra. For simple direct limits with no dimension growth, the special case we are most interested in, the previous corollary eliminates this difficulty.

Proposition 1.10. *Let $A = \varinjlim A_i$ be a simple direct limit of separable recursive subhomogeneous algebras which has no dimension growth in the sense of Corollary 1.9. Then A is the direct limit of a direct system $(B_i)_{i \in \mathbf{N}}$ with no dimension growth, and in which in addition all the maps $\psi_i: B_i \rightarrow B_{i+1}$ of the system are injective.*

Proof. Let d be a finite upper bound on the topological dimensions of the A_i . Let B_i be the image of A_i in A . Then also $\varinjlim B_i \cong A$. Since the A_i are separable, Proposition 3.1 of [34] implies that the B_i are separable recursive subhomogeneous algebras with topological dimension at most d . \square

2. CANCELLATION AND COMPARISON IN DIRECT LIMITS

In this section, we prove the positive results on direct limits with slow dimension growth, except for stable rank 1. The first result (essentially cancellation for unitaries) seems not to have been noticed before, but the others are all analogs of known results in the homogeneous case. We do not know whether Theorem 2.2 (cancellation) remains true without strict slow dimension growth, or whether Theorem 2.4 (weak unperforation) remains true without simplicity.

We impose injectivity on the maps of the system when convenient. Proposition 1.10 shows this condition can be eliminated for simple direct limits with no dimension growth.

Theorem 2.1. *Let $A = \varinjlim (A_i, \varphi_{ij})$ be a unital direct limit of a system of recursive subhomogeneous algebras with slow dimension growth. Then the map $U(A)/U_0(A) \rightarrow K_1(A)$ is an isomorphism.*

Proof. Without loss of generality all of the maps of the system are unital. Let $\varphi_{i\infty}: A_i \rightarrow A$ be the induced maps to the direct limit; these are also unital.

We first prove surjectivity. Let $\eta \in K_1(A)$. Choose i, n , and $u \in U(M_n(A_i))$ such that $(\varphi_{i\infty})_*([u]) = \eta$. Let q be the identity of $M_n(A_i)$, and let p be the identity of A_i , regarded as a subalgebra of $M_n(A_i)$ by identifying it with the upper left corner. Apply the definition of slow dimension growth to the projection p . This gives $j \geq i$ such that, if X is the total space of A_j and d is its topological dimension function, then $\text{rank}(\text{ev}_x(\varphi_{ij}(p))) \geq \frac{1}{2}d(x)$ for all $x \in X$. (Note that $\text{ev}_x(\varphi_{ij}(p))$ is never zero, because $\varphi_{ij}(p)$ is the identity of A_j .)

Apply Proposition 4.4(1) of [34] to $\varphi_{ij}(p)$, $\varphi_{ij}(q)$, and $\varphi_{ij}(u)$, obtaining $v \in U(A_j)$ such that $v \oplus 1$ is homotopic to $\varphi_{ij}(u)$ in $U(M_n(A_j))$. Then $\varphi_{j\infty}(v) \in U(A)$ and $[\varphi_{j\infty}(v)] = \eta$ in $K_1(A)$.

Now we prove injectivity. Let $u \in U(A)$ satisfy $[u] = 0$ in $K_1(A)$. We show $u \in U_0(A)$. We have $u \oplus 1 \in U_0(M_n(A))$ for some n . By standard methods there is i and $u_0 \in U(A_i)$ such that $\varphi_{i\infty}(u_0)$ is homotopic to u in $U(A)$ and such that $u_0 \oplus 1 \in U_0(M_n(A_i))$. It suffices to find $j \geq i$ such that $\varphi_{ij}(u) \in U_0(A_j)$.

As in the proof of surjectivity, let q be the identity of $M_n(A_i)$, and let p be the identity of A_i . For the same reason as there, we can choose $j \geq i$ such that, with X and d as there, we have $\text{rank}(\text{ev}_x(\varphi_{ij}(p))) \geq d(x)$ for all $x \in X$. Since also $\text{rank}(\text{ev}_x(\varphi_{ij}(p))) \geq 1$ for all $x \in X$, it follows that $\text{rank}(\text{ev}_x(\varphi_{ij}(p))) \geq \frac{1}{2}(d(x) + 1)$ for all $x \in X$. Using $\varphi_{ij}(u_0) \oplus 1 \in U_0(M_n(A_j))$, apply Proposition 4.4(2) of [34] to obtain $\varphi_{ij}(u_0) \in U_0(A_j)$, as desired. \square

The next theorem is the analog in our situation of Proposition 3.7(a) of [5], of one part of Theorem 3.7 of [29], and of Proposition 2.6 of [19].

Theorem 2.2. *Let $A = \varinjlim(A_i, \varphi_{ij})$ be a direct limit of a system of recursive subhomogeneous algebras with strict slow dimension growth. Then the projections in $M_\infty(A)$ satisfy cancellation: if $e \oplus q \sim f \oplus q$, then $e \sim f$.*

Proof. By standard arguments, we may assume that e, f , and q are in $M_\infty(A_i)$ for some i , and that $e \oplus q \sim f \oplus q$ in $M_\infty(A_i)$. Suppose first that $e = 0$. Let X be the total space of A_i . For every $x \in X$ we have $\text{ev}_x(q) \sim \text{ev}_x(f) \oplus \text{ev}_x(q)$, whence $f = 0$. So certainly $e \sim f$. Otherwise, use strict slow dimension growth to choose $j \geq i$ such that, if X is the total space of A_j and d is its topological dimension function, then $\text{rank}(\text{ev}_x(\varphi_{ij}(e))) \geq \frac{1}{2}d(x)$ for all $x \in X$. Under these conditions, $\varphi_{ij}(e) \sim \varphi_{ij}(f)$ by Proposition 4.3(2) of [34]. So $e \sim f$ in $M_\infty(A_i)$. \square

The next theorem is the analog in our situation of the other part of Theorem 3.7 of [29] and of one part of Proposition 3.7(b) of [5].

Theorem 2.3. *Let $A = \varinjlim(A_i, \varphi_{ij})$ be a direct limit of a system of recursive subhomogeneous algebras with slow dimension growth, and in which the maps φ_{ij} of the system are all injective and unital. Then A satisfies Blackadar’s Second Fundamental Comparability Question ([4], 1.3.1): if $p, q \in M_\infty(A)$ are projections such that $\tau(p) < \tau(q)$ for all normalized traces τ on A , then $p \prec\prec q$.*

Proof. By standard arguments we may assume that p and q are in $M_n(A_i)$ for some n and i . Dropping initial terms, without loss of generality $i = 0$. Replacing every A_i by $M_n(A_i)$ does not change any of the hypotheses (see Lemma 1.12 of [34]), so without loss of generality $p, q \in A_0$. Arguing as in the proof of Proposition 4.1 of [2], we find i such that $\tau(\varphi_{0i}(p)) < \tau(\varphi_{0i}(q))$ for all normalized traces τ on A_i . Letting X be the total space of A_i , letting $m: X \rightarrow \mathbf{N} \cup \{0\}$ be its matrix size function (as in Definition 0.3), letting Tr be the usual trace on matrices, and taking $\tau = \frac{1}{m(x)}\text{Tr} \circ \text{ev}_x$, we obtain $\text{rank}(\text{ev}_x(\varphi_{0i}(p))) < \text{rank}(\text{ev}_x(\varphi_{0i}(q)))$ for all $x \in X$. Let $N = \max_{x \in X} \text{rank}(\text{ev}_x(\varphi_{0i}(q)))$. Note that $N > 0$. The slow dimension growth hypothesis provides $j \geq i$ such that, with Y being the total space of A_j and $d: Y \rightarrow \mathbf{N} \cup \{0\}$ being its topological dimension function (as in Definition 0.3), we have

$$\text{ev}_y(\varphi_{0j}(q)) = 0 \quad \text{or} \quad \text{rank}(\text{ev}_y(\varphi_{0j}(q))) \geq Nd(y)$$

for all $y \in Y$. For y for which the second case holds, note that $\text{ev}_y \circ \varphi_{ij}$ is a finite-dimensional representation of A_i , and therefore is equivalent to a direct sum $0 \oplus \bigoplus_{r=1}^R \text{ev}_{x_r}$ with $x_1, \dots, x_R \in X$. Since

$$\text{rank}(\text{ev}_y(\varphi_{0j}(q))) = \sum_{r=1}^R \text{rank}(\text{ev}_{x_r}(\varphi_{0i}(q))) \quad \text{and} \quad \text{rank}(\text{ev}_{x_r}(\varphi_{0i}(q))) \leq N,$$

we get $R \geq d(y)$. We have $\text{rank}(\text{ev}_x(\varphi_{0i}(q))) \geq \text{rank}(\text{ev}_x(\varphi_{0i}(p))) + 1$ for all $x \in X$, so

$$\text{rank}(\text{ev}_y(\varphi_{0j}(q))) \geq \text{rank}(\text{ev}_y(\varphi_{0j}(p))) + d(y).$$

It follows from Proposition 4.3(1) of [34] that $\varphi_{0j}(p) \precsim \varphi_{0j}(q)$ in A_j . This proves the result. \square

Finally, we deal with unperforation. At least two different definitions of weak unperforation appear in the literature, namely Definition 6.7.1 of [3] and 2.1 of [10]. These agree in the simple case (2.1 of [10]), which is the one relevant below. To minimize confusion, however, we use the more descriptive term “unperforated for the strict order” for the property we prove (Definition 6.7.1 of [3]).

The following result is the analog of another part of Proposition 3.7(b) of [5], and of Proposition 2.10 of [19]. Unfortunately, we have not been able to generalize the method of [19], so we get the result only in the simple case. As observed in [5], in that case it is immediate from the Second Fundamental Comparability Question.

Theorem 2.4. *Let $A = \varinjlim(A_i, \varphi_{ij})$ be a simple direct limit of a system of recursive subhomogeneous algebras with slow dimension growth, and in which the maps φ_{ij} of the system are all injective and unital. Then $K_0(A)$ is unperforated for the strict order. That is, if $\eta \in K_0(A)$ and there is $n > 0$ such that $n\eta > 0$, then $\eta > 0$.*

Proof. Write $\eta = [p] - [q]$ for projections $p, q \in M_\infty(A)$. Since A is simple and $p \oplus \dots \oplus p \sim q \oplus \dots \oplus q \oplus r$ for some nonzero projection r (n copies each of p and q), we have $\tau(p) > \tau(q)$ for all normalized traces τ on A . So $p \succsim q$ by Theorem 2.3. \square

3. REDUCTION OF STABLE RANK

In this section, we prove that if A is a simple direct limit of a system of recursive subhomogeneous algebras with no dimension growth, then A has stable rank 1. This generalizes the result of [9]. Our proof, however, is somewhat different, being based on a notion of an “approximate polar decomposition”. Essentially, if a is an element of a recursive subhomogeneous algebra, and if each $\text{ev}_x(a)$ is small on a sufficiently large subspace of the (finite-dimensional) space on which it acts, then there is a unitary u such that $u(a^*a)^{1/2}$ is close to a .

We start with two preparatory results. The first, which provides a means of constructing continuous projection-valued functions without appealing to any selection theorems, has independent usefulness. Most of the work is contained in Lemma 3.3, which essentially does the induction step in the construction of the approximate polar decomposition by induction on the length of a recursive subhomogeneous decomposition.

Proposition 3.1. *Let X be a compact Hausdorff space, let $a \in C(X, M_n)_{\text{sa}}$, and let $\lambda \in \mathbf{R}$. For $x \in X$ define a projection $p(x)$ by $p(x) = \chi_{(-\infty, \lambda)}(a(x))$. Then there*

exist open sets U_k , for $0 \leq k \leq n$, and continuous rank k projections $p_k: U_k \rightarrow M_n$, such that:

- (1) $\bigcup_{k=0}^n U_k = X$.
- (2) If $k \leq l$ and $x \in U_k \cap U_l$, then $p_k(x) \leq p_l(x)$.
- (3) $p(x) = \sup\{p_k(x) : x \in U_k\}$ for all $x \in X$.
- (4) $p_k(x)$ commutes with $a(x)$ for all $x \in U_k$.

Proof. For $x \in X$, write the eigenvalues of $a(x)$ as

$$\alpha_1(x) \leq \alpha_2(x) \leq \dots \leq \alpha_n(x)$$

(repeated according to multiplicity). It follows from Theorem 8.1 of [1] that the α_k are continuous functions on X . Further, set $\alpha_{n+1}(x) = \infty$ for all x . Define

$$\beta_k(x) = \frac{1}{2}(\alpha_k(x) + \lambda) \quad \text{and} \quad U_k = \{x \in X : \alpha_k(x) < \beta_k(x) < \alpha_{k+1}(x)\}$$

for $1 \leq k \leq n$. Then define $p_k(x) = \chi_{(-\infty, \beta_k(x))}(a(x))$ for $x \in U_k$. Further take $U_0 = X$ and $p_0(x) = 0$.

We verify that these sets and projections satisfy the conclusion of the proposition. The U_k are open because the functions α_k and β_k are continuous. To see that p_k is continuous, rewrite $p_k(x) = f_x(a(x))$, where

$$f_x(t) = \begin{cases} 1, & t \leq \alpha_k(x), \\ \frac{\alpha_{k+1}(x) - t}{\alpha_{k+1}(x) - \alpha_k(x)}, & \alpha_k(x) \leq t \leq \alpha_{k+1}(x), \\ 0, & \alpha_{k+1}(x) \leq t. \end{cases}$$

The function $(t, x) \mapsto f_x(t)$ is jointly continuous, so $x \mapsto f_x(a(x))$ is continuous by Proposition 2.12 of [33]. Clearly, $\text{rank}(p_k(x)) = k$ for all x . It is obvious that the U_k cover X . It is also obvious that if $k \leq l$, then $p_k(x) \leq p_l(x)$ wherever both are defined. To verify part (3), we note that if $x \in U_k$, then $\frac{1}{2}(\alpha_k(x) + \lambda) > \alpha_k(x)$, whence $\alpha_k(x) < \lambda$, so that $p_k(x) \leq p(x)$. On the other hand, for $x \in X$ there is some k with $\alpha_k(x) < \lambda \leq \alpha_{k+1}(x)$. For this k , we have $x \in U_k$ and $p_k(x) = p(x)$. We have thus proved that $p(x) = \sup\{p_k(x) : x \in U_k\}$. Finally, part (4) is immediate because $p_k(x)$ is obtained from $a(x)$ using functional calculus. \square

Lemma 3.2. *Let $\varepsilon > 0$. Then there is $\delta > 0$ such that the following holds. Let $Z = Z_1 \cup Z_2$ be a compact Hausdorff space, with Z_1 and Z_2 closed subsets, and let U be a neighborhood of Z_1 . Let A be a unital C^* -algebra, and let $p \in C(Z, A)$ be a projection. Let $s_j \in C(Z_j, A)$ be a partial isometry with initial projection $s_j^*s_j = p|_{Z_j}$ for $j = 1, 2$, and suppose $\|s_1|_{Z_1 \cap Z_2} - s_2|_{Z_1 \cap Z_2}\| < \delta$. Then there is a partial isometry $s \in C(Z, A)$ with initial projection $s^*s = p$, such that*

$$s|_{Z_1} = s_1, \quad s|_{Z \setminus U} = s_2|_{Z \setminus U}, \quad \text{and} \quad \|s|_{Z_2} - s_2\| < \varepsilon.$$

Proof. The partial isometry s will be constructed as follows. Choose some continuous function $c: Z \rightarrow A$ such that $c|_{Z_1} = s_1$. Let

$$V = Z_1 \cup \{x \in Z_2 : \|c(x)p(x) - s_2(x)\| < \delta\},$$

which is a neighborhood of Z_1 . Choose a continuous function $f: Z \rightarrow [0, 1]$ which is equal to 1 on Z_1 and equal to zero on $Z \setminus (U \cap V)$. Then define

$$a(x) = f(x)c(x)p(x) + (1 - f(x))s_2(x) \quad \text{and} \quad s(x) = a(x)[a(x)^*a(x)]^{-1/2},$$

with functional calculus in $p(x)Ap(x)$, noting that $a(x)p(x) = p(x)$ for all x . It is clear that if $\|c(x)p(x) - s_2(x)\|$ is small enough for $x \notin U \cap V$ (depending only on ε), then we will get $\|s(x) - s_2(x)\| < \varepsilon$ for $x \in Z_2$. \square

The following lemma is the heart of the construction of the approximate polar decomposition. It is a relative version of the result for $C(X, M_n)$.

Lemma 3.3. *Let $\alpha, \varepsilon > 0$ and $n \in \mathbf{N}$. Then there is $\delta > 0$ such that the following holds. Let X be a compact Hausdorff space with dimension $d = \dim(X)$, and let $X^{(0)} \subset X$ be closed. Let $a \in C(X, M_n)$ satisfy $\|a\| \leq 1$, and assume that for each $x \in X$ there is a subspace E_x of \mathbf{C}^n with $\dim(E_x) \geq \frac{1}{2}d$ such that $\|a(x)\xi\| < \alpha\|\xi\|$ for $\xi \in E_x \setminus \{0\}$. Let p be the lower semicontinuous projection*

$$x \mapsto \chi_{(-\infty, \alpha)}([a(x)^*a(x)]^{1/2}).$$

Let $u^{(0)} \in U_0(C(X^{(0)}, M_n))$ be a unitary such that

$$\| [u^{(0)}(x)[a(x)^*a(x)]^{1/2} - a(x)] [1 - p(x)] \| < \delta$$

for $x \in X^{(0)}$. Then there exists a unitary $u \in U_0(C(X, M_n))$ such that $u|_{X^{(0)}} = u^{(0)}$ and

$$\| [u(x)[a(x)^*a(x)]^{1/2} - a(x)] [1 - p(x)] \| < \varepsilon$$

for all $x \in X$. Moreover, if we are given a homotopy $t \mapsto u_t^{(0)}$ from 1 to $u^{(0)}$ in $U(C(X^{(0)}, M_n))$, then u can be chosen such that there is a homotopy $t \mapsto u_t$ from 1 to u in $U(C(X, M_n))$ such that $u_t|_{X^{(0)}} = u_t^{(0)}$.

Proof. We may as well assume a homotopy $t \mapsto u_t^{(0)}$ is given.

Let r be the least integer such that $r \geq \frac{1}{2}d$.

We choose δ by an inductive process. Set $\delta_n = \varepsilon$. Given $\delta_{k+1} > 0$, choose $\delta_k > 0$ so small that the value $\delta = 2\delta_k/\alpha$ works for $\varepsilon = \frac{1}{2}\delta_{k+1}$ in Lemma 3.2, and also so small that $\delta_k < \delta_{k+1}$. Then set $\delta = \delta_{r-1}$.

Now we start the proof. First observe that $\text{rank}(p(x)) \geq r$ for all $x \in X$. Indeed, $\| [a(x)^*a(x)]^{1/2}\xi \| = \|a(x)\xi\|$ for all x and $\xi \in \mathbf{C}^n$. Therefore $\| [a(x)^*a(x)]^{1/2}\xi \| < \alpha\|\xi\|$ for $\xi \in E_x \setminus \{0\}$. Let F_x be the linear span of the eigenspaces of $[a(x)^*a(x)]^{1/2}$ for eigenvalues in $[\alpha, \infty)$. Then $\| [a(x)^*a(x)]^{1/2}\xi \| \geq \alpha\|\xi\|$ for all $\xi \in F_x$. Therefore $E_x \cap F_x = \{0\}$. It follows that $\text{rank}(p(x)) = n - \dim(F_x) \geq \dim(E_x) \geq r$.

Applying Proposition 3.1, write $p(x) = \sup(\{p_k(x) : x \in U_k\})$, where p_k has rank k , the U_k are open, $\bigcup_{k=0}^n U_k = X$, and $p_k(x) \leq p_l(x)$ for $k \leq l$ and $x \in U_k \cap U_l$. Since $\text{rank}(p(x)) \geq r$ for all $x \in X$, without loss of generality $U_k = \emptyset$ for $k < r$. We now construct, by induction on k , closed sets $Y_k \subset X$ such that

$$X^{(0)} \cup \{x \in X : \text{rank}(p(x)) \leq k\} \subset Y_k \subset X^{(0)} \cup \bigcup_{l=r}^k U_l$$

and

$$\{x \in X : \text{rank}(p(x)) \leq k\} \subset \text{int}(Y_k),$$

unitaries $v_k \in U_0(C(Y_k, M_n))$ such that $v_k|_{X^{(0)}} = u^{(0)}$ and

$$\| [v_k(x)[a(x)^*a(x)]^{1/2} - a(x)] [1 - p(x)] \| < \delta_k$$

for all $x \in Y_k$, and unitary homotopies $(t, x) \mapsto w_t^{(k)}(x)$ in $C(Y_k, M_n)$ with

$$w_0^{(k)} = 1, \quad w_1^{(k)} = v_k, \quad \text{and} \quad w_t^{(k)}|_{X^{(0)}} = u_t^{(0)}.$$

We start by taking $Y_{r-1} = X^{(0)}$, $v_{r-1} = u^{(0)}$, and $w_0^{(r-1)} = u_t^{(0)}$.
 Suppose now we are given Y_k, v_k , and $w_0^{(k)}$. Let

$$R = \{x \in X : \text{rank}(p(x)) = k + 1\} \setminus \text{int}(Y_k).$$

We need two facts about R : that it is closed, and that the union of Y_k and any neighborhood of R is a neighborhood of $\{x \in X : \text{rank}(p(x)) \leq k + 1\}$. For the first, let (x_λ) be a net in R with $x_\lambda \rightarrow x$. One easily sees that

$$\text{rank}(p(x)) \leq \liminf \text{rank}(p(x_\lambda)) \leq k + 1.$$

If now $\text{rank}(p(x)) < k + 1$, then $x \in \text{int}(Y_k)$, by the assumption on Y_k . This is a contradiction. So $\text{rank}(p(x)) = k + 1$, and $x \in R$ because $\text{int}(Y_k)$ is open. For the second, let Z be a neighborhood of R . Then $\text{int}(Z)$ contains R , and $\text{int}(Y_k)$ contains all other points $x \in X$ such that $\text{rank}(p(x)) = k + 1$. Therefore $\text{int}(Y_k \cup Z)$ contains R . The assumption on Y_k implies that $\text{int}(Y_k)$ contains $\{x \in X : \text{rank}(p(x)) \leq k\}$, so that $\text{int}(Y_k \cup Z)$ contains $\{x \in X : \text{rank}(p(x)) \leq k + 1\}$.

Clearly U_{k+1} is a neighborhood of R , and $p_{k+1}(x) = p(x)$ for $x \in R$. Therefore

$$[1 - p_{k+1}(x)] [a(x)^* a(x)]^{1/2} [1 - p_{k+1}(x)] = [1 - p(x)] [a(x)^* a(x)]^{1/2} [1 - p(x)]$$

is invertible in $[1 - p_{k+1}(x)]M_n[1 - p_{k+1}(x)]$ for x in R . So

$$c_0(x) = a(x) \left([1 - p_{k+1}(x)] [a(x)^* a(x)]^{1/2} [1 - p_{k+1}(x)] \right)^{-1}$$

(inverse taken in $[1 - p_{k+1}(x)]M_n[1 - p_{k+1}(x)]$) exists for x in some neighborhood V of R . Moreover, $\|c_0(x)[1 - p_{k+1}(x)]\xi\| = \|[1 - p_{k+1}(x)]\xi\|$ for $\xi \in \mathbf{C}^n$ and $x \in R$. Therefore, using the compactness of the closed unit ball of \mathbf{C}^n , we may reduce the size of V so that in addition the partial isometry $c(x) = c_0(x) [c_0(x)^* c_0(x)]^{-1/2}$, with initial projection $1 - p_{k+1}(x)$, is defined for $x \in V$. (Functional calculus is taken in $[1 - p_{k+1}(x)]M_n[1 - p_{k+1}(x)]$.)

For $x \in Y_k$, we have

$$\| [v_k(x)[a(x)^* a(x)]^{1/2} - a(x)] [1 - p(x)] \| < \delta_k,$$

and for $x \in R$ we have

$$[c(x)[a(x)^* a(x)]^{1/2} - a(x)] [1 - p(x)] = 0$$

(using the relation $p(x) = p_{k+1}(x)$ and the fact that $p(x)$ commutes with $[a(x)^* a(x)]^{1/2}$). Again using $p(x) = p_{k+1}(x)$ for $x \in R$, we get

$$\| [v_k(x) - c(x)][a(x)^* a(x)]^{1/2} [1 - p_{k+1}(x)] \| < \delta_k$$

for $x \in Y_k \cap R$. Now $[a(x)^* a(x)]^{1/2}$ commutes with $1 - p_{k+1}(x) = 1 - p(x)$, and

$$\| ([1 - p_{k+1}(x)][a(x)^* a(x)]^{1/2} [1 - p_{k+1}(x)])^{-1} \| < \frac{1}{\alpha}$$

(inverse taken in $[1 - p_{k+1}(x)]M_n[1 - p_{k+1}(x)]$), so

$$\| [v_k(x) - c(x)][1 - p_{k+1}(x)] \| < \frac{\delta_k}{\alpha}$$

for $x \in Y_k \cap R$. Choose a neighborhood W of $Y_k \cap R$ with $W \subset V$ such that this last norm is at most $2\delta_k/\alpha$ for $x \in W$. Then $(V \setminus Y_k) \cup W$ is a neighborhood of R .

Choose a closed neighborhood Z of R with $Z \subset (V \setminus Y_k) \cup W$, and so small that

$$\| [c(x)[a(x)^*a(x)]^{1/2} - a(x)] [1 - p_{k+1}(x)] \| < \frac{1}{2}\delta_{k+1}$$

for $x \in Z$.

Apply Lemma 3.2 to $s_1(x) = v_k(x)[1 - p_{k+1}(x)]$ on $Z_1 = Z \cap Y_k$ and $s_2(x) = c(x)$ on $Z_2 = Z$. The choice of δ_k provides a partial isometry s with initial projection $(1 - p_{k+1})|_Z$ such that

$$s|_{Z \cap Y_k} = [v_k(1 - p_{k+1})]|_{Z \cap Y_k} \quad \text{and} \quad \|s - c\| < \frac{1}{2}\delta_{k+1}.$$

Let $q \in C(Z, M_n)$ be the projection $q = 1 - ss^*$.

Apply Proposition 4.2(2) of [34] with Z in place of X , with $Z \cap Y_k$ in place of Y , with

$$p_1 = p_{k+1}, \quad p_2 = q, \quad q_1 = 1 - p_{k+1}, \quad q_2 = 1 - q, \quad v_0 = v_k p_{k+1}|_{Z \cap Y_k}, \quad \text{and} \quad u = 1,$$

with s as given, and with $w_t^{(k)}|_{Z \cap Y_k}$ in place of $w_t^{(0)}$. This gives $v \in C(Z, M_n)$ with

$$v^*v = p_{k+1}, \quad vv^* = q, \quad \text{and} \quad v|_{Z \cap Y_k} = v_0,$$

and a homotopy $(t, x) \mapsto w_t(x)$ of unitaries in $C(Z, M_n)$ such that

$$w_0 = 1, \quad w_1 = s + v, \quad \text{and} \quad w_t|_{Z \cap Y_k} = w_t^{(k)}|_{Z \cap Y_k}.$$

Define $Y_{k+1} = Z \cup Y_k$, and set

$$v_{k+1}(x) = \begin{cases} v_k(x), & x \in Y_k, \\ s(x) + v(x), & x \in Z. \end{cases}$$

Since Z contains a neighborhood of R , the facts about R discussed right after its choice imply that Y_{k+1} contains a neighborhood of $\{x \in X : \text{rank}(p(x)) \leq k+1\}$. By construction, we have $Y_{k+1} \subset X^{(0)} \cup \bigcup_{l=r}^{k+1} U_l$. Define

$$w_t^{(k+1)}(x) = \begin{cases} w_t^{(k)}(x), & x \in Y_k, \\ w_t(x), & x \in Z. \end{cases}$$

Then $w_t^{(k+1)}$ is a homotopy from 1 to v_{k+1} in $U(C(Y_{k+1}, M_n))$, such that $w_t^{(k+1)}|_{Y_{k+1}} = w_t^{(k)}$. Moreover,

$$v_{k+1}|_{X^{(0)}} = v_k|_{X^{(0)}} = u^{(0)} \quad \text{and} \quad w_t^{(k+1)}|_{X^{(0)}} = w_t^{(k)}|_{X^{(0)}} = u_t^{(0)}.$$

It remains only to show that

$$\| [v_{k+1}(x)[a(x)^*a(x)]^{1/2} - a(x)] [1 - p(x)] \| < \delta_{k+1}$$

for $x \in Y_{k+1}$. This estimate holds on Y_k because $v_{k+1}|_{Y_k} = v_k$ and $\delta_k < \delta_{k+1}$. For $x \in Z$, we have (using $p_{k+1}(x) \leq p(x)$ and $v_{k+1}|_Z = s$)

$$\begin{aligned} & \| [v_{k+1}(x)[a(x)^*a(x)]^{1/2} - a(x)] [1 - p(x)] \| \\ & \leq \| [s(x)[a(x)^*a(x)]^{1/2} - a(x)] [1 - p_{k+1}(x)] \| \\ & \leq \|s(x) - c(x)\| \|a\| + \| [c(x)[a(x)^*a(x)]^{1/2} - a(x)] [1 - p_{k+1}(x)] \| \\ & < \frac{1}{2}\delta_{k+1} + \frac{1}{2}\delta_{k+1} = \delta_{k+1}. \end{aligned}$$

This completes the induction.

The proof is now finished by setting $u = v_n$ and $u_t = w_t^{(n)}$. \square

We can now prove the result on approximate polar decomposition.

Proposition 3.4. *Let A be a recursive subhomogeneous algebra with total space X , and let m and d be its matrix size and topological dimension functions as in Definition 0.3. Let $\alpha, \varepsilon > 0$. Let $a \in A$, and suppose that for every $x \in X$ there is a subspace E_x of $\mathbf{C}^{m(x)}$ with $\dim(E_x) \geq \frac{1}{2}d(x)$ and such that $\|\text{ev}_x(a)\xi\| < \alpha\|\xi\|$ for $\xi \in E_x \setminus \{0\}$. Then there is a unitary $u \in U_0(A)$ such that $\|u(a^*a)^{1/2} - a\| < 2\alpha + \varepsilon$.*

Proof. We first reduce to the case in which, following the notation of Definition 0.3, the first space X_0 has just one point. To do this, replace $C(X_0, M_{n(0)})$ by $M_{n(0)} \oplus_{M_{n(0)}} C(X_0, M_{n(0)})$, where the map $M_{n(0)} \rightarrow M_{n(0)}$ is the identity map and the map $C(X_0, M_{n(0)}) \rightarrow M_{n(0)}$ is ev_x for some $x \in X_0$. This change increases the length of the decomposition by 1, but does not affect any of the hypotheses, or the conclusion, of the proposition.

Assuming now that all decompositions we consider start with a one point space, we prove the following by induction on the length: Let $a \in A$ satisfy the hypotheses, and let $p(x) = \chi_{(-\infty, \alpha)}(\text{ev}_x((a^*a)^{1/2}))$ for $x \in X$. Then there is a unitary $u \in U_0(A)$ such that

$$\|\text{ev}_x(u(a^*a)^{1/2} - a)[1 - p(x)]\| < \varepsilon$$

for all $x \in X$. This unitary will then be shown to satisfy the conclusion of the proposition.

The case of length zero is now trivial, since then $A = M_{n(0)}$. So assume that the result is known for length L . By scaling a, α , and ε , without loss of generality $\|a\| \leq 1$. Let $A = B \oplus_{C(X^{(0)}, M_n)} C(X, M_n)$, with $\varphi: B \rightarrow C(X^{(0)}, M_n)$ unital and $\rho: C(X, M_n) \rightarrow C(X^{(0)}, M_n)$ the restriction map, where B is a recursive subhomogeneous algebra of length L . Let Y be the total space of B , so that the total space of A is $Y \amalg X$. Let $d = \dim(X)$. Let b be the image of a in B , and let a_0 be the image of a in $C(X, M_n)$, under the obvious projection maps. Note that $\chi_{(-\infty, \alpha)}(\text{ev}_y((b^*b)^{1/2})) = p(y)$ for $y \in Y$. For the given values of α, ε , and n , let δ be as in Lemma 3.3. Choose, using the induction assumption, a unitary $v \in B$ such that

$$\|\text{ev}_y(v(b^*b)^{1/2} - b)[1 - p(y)]\| < \delta$$

for all $y \in Y$, and such that there is a homotopy $t \mapsto v^{(t)}$ in $U(B)$ with $v^{(0)} = 1$ and $v^{(1)} = v$.

For each $x \in X^{(0)}$, the map $c \rightarrow \varphi(c)(x)$ is a finite-dimensional representation of B . It follows from Lemma 2.1 of [34] that there are a unitary $w \in M_n$ and points $y_1, \dots, y_l \in Y$ such that

$$\varphi(c)(x) = w \left(\bigoplus_{j=1}^l \text{ev}_{y_j}(c) \right) w^*$$

for all $c \in B$. So

$$a_0(x) = \text{ev}_x(a) = w \left(\bigoplus_{j=1}^l \text{ev}_{y_j}(b) \right) w^* \quad \text{and} \quad \varphi(v)(x) = w \left(\bigoplus_{j=1}^l \text{ev}_{y_j}(v) \right) w^*.$$

Applying functional calculus to the first of these (with $(a^*a)^{1/2}$ and $(b^*b)^{1/2}$ in place of a and b) gives

$$p(x) = w \left(\bigoplus_{j=1}^l p(y_j) \right) w^*.$$

It now follows from the choice of v that

$$\| [\varphi(v)(x)[a_0(x)^*a_0(x)]^{1/2} - a_0(x)] [1 - p(x)] \| < \delta$$

for all $x \in X^{(0)}$. Moreover, $t \mapsto \varphi(v^{(t)})$ is a homotopy from 1 to $\varphi(v)$. According to Lemma 3.3 and the choice of δ , there exist a unitary $u_0 \in C(X, M_n)$ and a unitary homotopy $t \mapsto u_0^{(t)}$ from 1 to u_0 , such that $u_0|_{X^{(0)}} = \varphi(v)$ and

$$\| [u(x)[a_0(x)^*a_0(x)]^{1/2} - a_0(x)] [1 - p(x)] \| < \varepsilon$$

for all $x \in X$, and such that $u_0^{(t)}|_{X^{(0)}} = \varphi(v^{(t)})$. Then $u = (v, u_0)$ is a unitary in $U_0(A)$ such that

$$\| \text{ev}_x (u(a^*a)^{1/2} - a) [1 - p(x)] \| < \varepsilon$$

for all $x \in Y \amalg X$. Moreover, the homotopy $t \mapsto u^{(t)} = (v^{(t)}, u_0^{(t)})$ in $U(A)$ shows that $u \in U_0(A)$. This completes the induction, and the proof of the claim.

To get the desired estimate, write

$$\begin{aligned} & \| \text{ev}_x (u(a^*a)^{1/2} - a) \| \\ & \leq \| \text{ev}_x (u(a^*a)^{1/2} - a) [1 - p(x)] \| + \| \text{ev}_x (u(a^*a)^{1/2} - a) p(x) \|. \end{aligned}$$

The first term on the right is less than ε . For the second, we have, for all ξ ,

$$\| \text{ev}_x (u(a^*a)^{1/2}) p(x) \xi \| = \| \text{ev}_x ((a^*a)^{1/2}) p(x) \xi \| = \| \text{ev}_x(a) p(x) \xi \| \leq \alpha \| \xi \|,$$

so

$$\| \text{ev}_x (u(a^*a)^{1/2} - a) p(x) \| \leq 2\alpha \| \xi \|.$$

It follows that $\| \text{ev}_x (u(a^*a)^{1/2} - a) \| < \varepsilon + 2\alpha$. This is true for all x in the total space of A , so $\| u(a^*a)^{1/2} - a \| < \varepsilon + 2\alpha$. \square

In order to apply our approximate polar decomposition to simple direct limits, we need the following lemma.

Lemma 3.5. *Let $A = \varinjlim A_i$ be an infinite dimensional simple direct limit of recursive subhomogeneous algebras, such that all the maps $\varphi_{ij}: A_i \rightarrow A_j$ in the system are unital and injective. Let X_i be the total space of A_i , and let $m_i: X_i \rightarrow \mathbf{N}$ be the matrix size function. Let $a \in A_i$, for some i , be noninvertible. Then for every $n \in \mathbf{N}$ and $\varepsilon > 0$ there exists j_0 such that, for every $j \geq j_0$ and every $x \in X_j$ there is a subspace E_x of $\mathbf{C}^{m_j(x)}$ such that $\dim(E_x) \geq n$ and $\| \text{ev}_x(\varphi_{ij}(a)) \xi \| < \varepsilon \| \xi \|$ for $\xi \in E_x \setminus \{0\}$.*

Proof. Since $\| (b^*b)^{1/2} \xi \| = \| b \xi \|$ for any b and ξ , we may replace a by $(a^*a)^{1/2}$, and thus assume $a \geq 0$.

Let $f: [0, \infty) \rightarrow [0, 1]$ be a continuous function such that $f(0) \neq 0$ but $f(t) = 0$ for $t \geq \frac{1}{2}\varepsilon$. Then $f(a)$ is a nonzero selfadjoint element of A_i . Applying Lemma 1.8, we obtain j_0 such that, for every $j \geq j_0$ and every $x \in X_j$, the matrix $\text{ev}_x(\varphi_{ij}(f(a)))$ has rank at least n . Let $g: [0, \infty) \rightarrow [0, 1]$ be a continuous function such that $g = 1$ on $[0, \frac{1}{2}\varepsilon]$ and $g = 0$ on $[\varepsilon, \infty)$. Since $\| ag(a) \| < \varepsilon$ and $g(a)f(a) = f(a)$, it follows that $\| \text{ev}_x(\varphi_{ij}(a)) \xi \| < \varepsilon \| \xi \|$ for $\xi \in \text{ev}_x(\varphi_{ij}(f(a))) \mathbf{C}^{n_j(x)} \setminus \{0\}$. The lemma is therefore proved by taking $E_x = \text{ev}_x(\varphi_{ij}(f(a))) \mathbf{C}^{m_j(x)}$. \square

Theorem 3.6. *Let $A = \varinjlim A_i$ be a simple direct limit of separable recursive subhomogeneous algebras. Assume the system has no dimension growth, that is, there is $d \in \mathbf{N}$ such that all A_i have topological dimension at most d . Then $\text{tsr}(A) = 1$.*

Proof. If A is finite dimensional, the conclusion is obvious. So assume A is infinite dimensional.

We first consider the case in which all the maps $\varphi_{ij}: A_i \rightarrow A_j$ in the system are unital and injective. By Lemma 3.5 of [9], it suffices to let $a \in A_i$ for some i , let $\varepsilon > 0$, and find $j \geq i$ and an invertible element $c \in A_j$ such that $\|\varphi_{ij}(a) - c\| < \varepsilon$. Use Lemma 3.5 to find $j \geq i$ such that for every $x \in X_j$ there is a subspace E_x of \mathbf{C}^m (where m is the matrix size of A_j at the point x) such that $\dim(E_x) \geq \frac{1}{2}d$ and $\|\text{ev}_x(\varphi_{ij}(a))\xi\| < \frac{1}{4}\varepsilon\|\xi\|$ for $\xi \in E_x \setminus \{0\}$. Set $b = \varphi_{ij}(a)$. Using Proposition 3.4, find a unitary $u \in A_j$ such that $\|u(b^*b)^{1/2} - b\| < \frac{3}{4}\varepsilon$. Then $c = u[(b^*b)^{1/2} + \frac{1}{4}\varepsilon \cdot 1]$ is an invertible element of A_j which satisfies $\|\varphi_{ij}(a) - c\| = \|b - c\| < \varepsilon$.

Now we drop the assumption that the maps are unital. Without loss of generality $A_0 \neq 0$. Let $p_i = \varphi_{0i}(1_{A_0})$, which is a nonzero projection in A_i , and let p be the image of 1_{A_0} in A . Then $pAp \cong \varinjlim p_i A_i p_i$. By Corollary 1.11 of [34], the algebras $p_i A_i p_i$ are recursive subhomogeneous algebras with topological dimension at most d . Also pAp is simple because A is simple. Therefore the previous case shows $\text{tsr}(pAp) = 1$. Since pAp is stably isomorphic to A (Theorem 2.8 of [7]), it follows from Theorem 3.6 of [39] that $\text{tsr}(A) = 1$.

Finally, we use Proposition 1.10 to drop the assumption that the maps are injective. □

4. EXAMPLES AND APPLICATIONS

In this section, we apply our general results to crossed products by minimal homeomorphisms. In particular, we compute the Elliott invariant [12] of crossed products by minimal homeomorphisms of odd spheres of dimension at least 3, and we give a much more direct calculation of the Elliott invariant of the crossed product by a Furstenberg transformation on the 2-torus (see [22]). We also give a related example which shows the failure of the generalization to direct limits of recursive subhomogeneous algebras of results on Riesz decomposition and real rank zero. The applications and the example use the subalgebras of the crossed product studied in [25] (see Example 1.6 of [34]).

We note here that the preprint [25] will be absorbed into [28], in which a much stronger result is proved. (However, a sketch of what is needed here has been published in Section 3 of [26].) For our purposes, the main result of [25] is that if h is a minimal homeomorphism of a finite-dimensional infinite compact metric space X , then a useful “large” subalgebra ($A_{\{x\}}$, described below) of the crossed product $C^*(\mathbf{Z}, X, h)$ is a simple direct limit of recursive subhomogeneous algebras with no dimension growth.

In case X is a manifold and h is actually a diffeomorphism, it is shown in [28] that $C^*(\mathbf{Z}, X, h)$ is itself such a direct limit. The application of the theorems of this paper to $C^*(\mathbf{Z}, X, h)$ gives considerably more information than their application to the subalgebra $A_{\{x\}} \subset C^*(\mathbf{Z}, X, h)$. Moreover, in case the map from $K_0(C^*(\mathbf{Z}, X, h))$ to the affine functions on the tracial state space of $C^*(\mathbf{Z}, X, h)$ has dense range (but without assuming X is a manifold), it is shown in [24] (using the results of this paper and of [35]) that $C^*(\mathbf{Z}, X, h)$ has real rank zero and falls under the Elliott

classification conjecture. In particular, $C^*(\mathbf{Z}, X, h)$ is a direct limit of homogeneous C^* -algebras with no dimension growth.

We have three reasons for giving the weaker results here. First, it is quite straightforward to prove that $A_{\{x\}}$ is a simple direct limit of recursive subhomogeneous algebras with no dimension growth, while the proof of the corresponding theorem for the crossed product is extremely long. The result for $A_{\{x\}}$ already gives enough information to compute the Elliott invariant of the crossed product. Second, the result for $A_{\{x\}}$ is used in the proof of the theorem of [24] described above. Third, the direct limit decomposition for the crossed product has only been proved for minimal diffeomorphisms and for the cases in which the transformation group C^* -algebra is expected to have real rank zero. If X is finite-dimensional but not a manifold, or if it is a manifold but h is not smooth, and if the map from $K_0(C^*(\mathbf{Z}, X, h))$ to the affine functions on the tracial state space does not have dense range, the results presented here are the best currently known.

We begin by recalling from [25] the subalgebra $A_{\{x\}}$ of the crossed product, and stating its relation with the entire crossed product algebra.

Theorem 4.1. *Let X be an infinite compact metric space, let h be a minimal homeomorphism of X , and set $A = C^*(\mathbf{Z}, X, h)$. Let $u \in A$ be the unitary representing the generator of \mathbf{Z} . For $x \in X$, set*

$$A_{\{x\}} = C^*(C(X), uC_0(X \setminus \{x\})) \subset C^*(\mathbf{Z}, X, h),$$

as in [25] (see also Example 1.6 of [34]), and let $\iota: A_{\{x\}} \rightarrow A$ be the inclusion map. Then:

- (1) $A_{\{x\}}$ is simple.
- (2) $A_{\{x\}}$ is a direct limit of a system of recursive subhomogeneous algebras with topological dimension at most $\dim(X)$, in which the maps are all unital and injective.
- (3) $\iota_*: K_0(A_{\{x\}}) \rightarrow K_0(A)$ is an isomorphism.
- (4) There is a short exact sequence

$$0 \longrightarrow K_1(A_{\{x\}}) \xrightarrow{\iota_*} K_1(A) \xrightarrow{\gamma} \mathbf{Z} \longrightarrow 0,$$

in which $\gamma([u]) = 1$.

- (5) There is a one-to-one correspondence between normalized traces τ on A and h -invariant Borel probability measures μ on X , given by

$$\tau \left(\sum_k f_k u^k \right) = \int_X f_0 d\mu.$$

Moreover, the map $\tau \mapsto \tau \circ \iota$ is a bijection between normalized traces τ on A and normalized traces on $A_{\{x\}}$.

Proof. (1) This is Proposition 12 of [25].

(2) This is Example 1.6 of [34] (derived from Theorem 3 of [25]).

(3), (4) Example 2.6 of [37] gives an exact sequence

$$\begin{array}{ccccc} K_0(K(l^2(\mathbf{Z}))) & \longrightarrow & K_0(A_{\{x\}}) & \longrightarrow & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \longleftarrow & K_1(A_{\{x\}}) & \longleftarrow & K_1(K(l^2(\mathbf{Z}))), \end{array}$$

in which the maps $K_*(A_{\{x\}}) \rightarrow K_*(A)$ are induced by the inclusion. Moreover, we claim that the map $K_1(A) \rightarrow K_0(K(l^2(\mathbf{Z})))$ sends the class $[u]$ of the generating unitary to a generator of $K_0(K(l^2(\mathbf{Z})))$. Since $K_0(K(l^2(\mathbf{Z}))) \cong \mathbf{Z}$ and $K_1(K(l^2(\mathbf{Z}))) = 0$, the conclusions will then follow from exactness.

To prove the claim, we chase through the definitions in [37]. (The map in question is called $[L]_*$ there, and it is defined in the discussion following Lemma 3.10 of [37].) We find that this map is determined by the odd Kasparov A - $K(l^2(\mathbf{Z}))$ -bimodule $(K(l^2(\mathbf{Z})), \bar{\lambda}, 1 - 2p)$, in which the right $K(l^2(\mathbf{Z}))$ action on $K(l^2(\mathbf{Z}))$ is simply multiplication, in which $\bar{\lambda}: A \rightarrow L(l^2(\mathbf{Z})) = M(K(l^2(\mathbf{Z})))$ is the regular representation induced as in 7.7.1 of [31] by the representation ev_x of $C(X)$, and in which p is the projection from $l^2(\mathbf{Z})$ onto $l^2(\{1, 2, \dots\})$. Since $\bar{\lambda}(u)$ is the bilateral shift, this map does indeed send $[u]$ to a generator of $K_0(K(l^2(\mathbf{Z})))$.

(5) This is Proposition 16 of [25] and the well-known correspondence between traces on A and h -invariant Borel probability measures on X . \square

Theorem 4.2. *Let $X, h,$ and $A_{\{x\}}$ be as in Theorem 4.1, and assume in addition that $\dim(X) < \infty$. Then:*

- (1) $\text{tsr}(A_{\{x\}}) = 1$.
- (2) *The map $U(A_{\{x\}})/U_0(A_{\{x\}}) \rightarrow K_1(A_{\{x\}})$ is an isomorphism.*
- (3) *The projections in $M_\infty(A_{\{x\}})$ satisfy cancellation.*
- (4) *The algebra $A_{\{x\}}$ satisfies Blackadar's Second Fundamental Comparability Question: if $p, q \in M_\infty(A_{\{x\}})$ are projections such that $\tau(p) < \tau(q)$ for all normalized traces τ on $A_{\{x\}}$, then $p \preceq q$.*
- (5) *$K_0(A_{\{x\}})$ is unperforated for the strict order: if $n\eta > 0$ in $K_0(A_{\{x\}})$, with $n > 0$, then $\eta > 0$.*

Proof. Parts (1) and (2) of Theorem 4.1 show that $A_{\{x\}}$ is a simple direct limit, with no dimension growth, of a system of separable recursive subhomogeneous algebras. Therefore part (1) follows from Theorem 3.6, part (2) follows from Corollary 1.9 and Theorem 2.1, part (3) follows from Corollary 1.9 and Theorem 2.2, part (4) follows from Corollary 1.9 and Theorem 2.3, and part (5) follows from Corollary 1.9 and Theorem 2.4. \square

While not all of the properties in this theorem can be transferred to the entire crossed product A , Theorem 4.1 does give us some information.

Theorem 4.3. *Let X be a finite-dimensional infinite compact metric space, and let h be a minimal homeomorphism of X . Then the map*

$$U(C^*(\mathbf{Z}, X, h))/U_0(C^*(\mathbf{Z}, X, h)) \longrightarrow K_1(C^*(\mathbf{Z}, X, h))$$

is surjective.

Proof. Theorem 4.1(4) implies (using the notation there) that $K_1(C^*(\mathbf{Z}, X, h))$ is generated by $\iota_*(K_1(A_{\{x\}}))$ and $[u]$. The image of the map $U(C^*(\mathbf{Z}, X, h)) \rightarrow K_1(C^*(\mathbf{Z}, X, h))$ is a subgroup of $K_1(C^*(\mathbf{Z}, X, h))$ which contains $\iota_*(K_1(A_{\{x\}}))$ by Theorem 4.2(2), and which obviously contains $[u]$, so it is all of $K_1(C^*(\mathbf{Z}, X, h))$. \square

Theorem 4.4. *Let $X, h, A,$ and $A_{\{x\}}$ be as in Theorem 4.1, and assume in addition that $\dim(X) < \infty$. Then $\iota_*: K_0(A_{\{x\}}) \rightarrow K_0(A)$ is an order isomorphism.*

Proof. By Theorem 4.1(3), we need only prove that $(\iota_*)^{-1}$ is order preserving. So let $\eta \in K_0(A)$ satisfy $\eta > 0$. Write $(\iota_*)^{-1}(\eta) = [p] - [q]$ for some projections $p, q \in M_\infty(A_{\{x\}})$. Let σ be any normalized trace on $A_{\{x\}}$. By Theorem 4.1(5), there is a normalized trace τ on A such that $\tau \circ \iota = \sigma$. We have $\tau_*(\eta) > 0$ because A is simple and $\eta > 0$. Therefore $\tau(\iota(p)) > \tau(\iota(q))$, whence $\sigma(p) > \sigma(q)$. Since σ is arbitrary, Theorem 4.2(4) implies that $p \succsim q$, so that $(\iota_*)^{-1}(\eta) = [p] - [q] > 0$. \square

Theorem 4.5. *Let X be a finite-dimensional infinite compact metric space, and let h be a minimal homeomorphism of X . Then:*

- (1) $C^*(\mathbf{Z}, X, h)$ satisfies the following K -theoretic version of Blackadar's Second Fundamental Comparability Question: if $\eta \in K_0(A)$ satisfies $\tau_*(\eta) > 0$ for all normalized traces τ on A , then there is a projection $p \in M_\infty(A)$ such that $\eta = [p]$.
- (2) $K_0(C^*(\mathbf{Z}, X, h))$ is unperforated for the strict order.

Proof. This follows from Theorem 4.4 and parts (4) and (5) of Theorem 4.2. \square

We note at this point that the examples constructed by Villadsen in [44] do not satisfy (2) (see Proposition 11(ii) of [44]), and so also do not satisfy (1). Thus, the C^* -algebras of minimal homeomorphisms of finite-dimensional compact metric spaces are not as badly behaved as Villadsen's examples. (In [28], we will show that crossed products by minimal diffeomorphisms in fact have stable rank one.) On the other hand, the C^* -algebras covered by the real rank one classification theorem of [13] have Riesz decomposition in K_0 even in the real rank one case (see Theorems 2.6 and 4.8 of [23]), and we show in the next example that the C^* -algebras of minimal homeomorphisms of finite-dimensional compact metric spaces need not have Riesz decomposition in K_0 .

Example 4.6. We compute the ordered K_0 -group of Connes' example (Example 4 in Section 5 of [8]) of a simple unital stably finite C^* -algebra with no nontrivial projections. As there, let h be a minimal diffeomorphism of S^3 . Then $A = C^*(\mathbf{Z}, S^3, h)$ is simple, and Corollary 3 in Section 5 of [8] implies that $\tau_*(K_0(A)) = \mathbf{Z}$ for any normalized trace τ on A . (See also Corollary 10.10.6 of [3].) Since h has no fixed points, the Lefschetz Fixed Point Theorem (Theorem 4.7.7 of [41]) implies that h is orientation preserving. The Pimsner-Voiculescu exact sequence [36] then implies that $K_0(A) \cong \mathbf{Z}^2$.

We next note that all normalized traces on A agree on $K_0(A)$. Indeed, if τ_1 and τ_2 are normalized traces, then $t \mapsto [(1-t)\tau_1 + t\tau_2]_*$ is a homotopy, in an obvious sense, of maps from $K_0(A)$ to \mathbf{R} whose ranges are contained in \mathbf{Z} . Therefore it is constant. We let τ_* denote the map from $K_0(A)$ to \mathbf{Z} determined by any trace.

The map $n \mapsto n \cdot [1]$, from \mathbf{Z} to $K_0(A)$, is a left inverse of τ_* . It induces an isomorphism $K_0(A) \cong \ker(\tau_*) \oplus \mathbf{Z}$, with $\ker(\tau_*) \cong \mathbf{Z}$ also. We may thus identify $K_0(A)$ with \mathbf{Z}^2 in such a way that $[1] = (0, 1)$ and $\tau_*(m, n) = n$. Using Theorem 4.4(1), we find that $(m, n) \geq 0$ exactly when $n > 0$ or $(m, n) = (0, 0)$.

We note that $K_0(A)$ does not have Riesz decomposition. Indeed, write $(0, 2) = (1, 1) + (-1, 1)$, and note that $(0, 1) \leq (0, 2)$ but there do not exist $\eta, \mu \in K_0(A)$ with

$$(0, 0) \leq \eta \leq (1, 1), \quad (0, 0) \leq \mu \leq (-1, 1), \quad \text{and} \quad \eta + \mu = (0, 1).$$

In this example, S^3 could be replaced by S^n for any odd $n \geq 3$. Moreover, the outcome shows that the Elliott invariant depends only on the space of h -invariant

Borel probability measures. The following conjecture is therefore a special case of the Elliott classification conjecture.

Conjecture 4.7. Let g and h be minimal diffeomorphisms of S^m and S^n respectively, with $m, n \geq 3$ and odd. Suppose the spaces of g -invariant and h -invariant Borel probability measures are affinely homeomorphic. Then there is an isomorphism of transformation group C^* -algebras $C^*(\mathbf{Z}, S^m, g) \cong C^*(\mathbf{Z}, S^n, h)$.

See Section 5 of [26] for further discussion of what the Elliott classification conjecture might imply for crossed products by minimal homeomorphisms, and in particular the contrast between the suggested behavior of minimal diffeomorphisms of high-dimensional manifolds with what is known to happen for minimal homeomorphisms of the Cantor set [17] and of the circle.

The Connes example can also be used to show that some of the results of [6] and [19] do *not* generalize to direct limits of recursive subhomogeneous algebras. Examples of this type have been given previously. See [43] and [21] for the failure of real rank zero, and [42] (recalling from Theorem 10.17 and Proposition 2.1 of [18] that Riesz decomposition implies that the state space of the group is a Choquet simplex) for the failure of Riesz decomposition. (More general results on existence of algebras with prespecified Elliott invariants can be found in [11].) However, our example comes up naturally and has a simple proof.

Example 4.8. We give an example of a simple direct limit B of separable recursive subhomogeneous algebras, with no dimension growth, which has the following properties:

- (1) The projections distinguish the traces, but B does not have real rank zero.
- (2) $K_0(B)$ does not have Riesz decomposition.

Thus, there is no analog of Theorem 2 of [6], or of Theorem 2.7 of [19], for direct limits of recursive subhomogeneous algebras.

Let A be as in Example 4.6, using a uniquely ergodic minimal diffeomorphism h of S^3 . Such a thing exists by [15]. Then $A = C^*(\mathbf{Z}, S^3, h)$ is simple and has a unique normalized trace τ . Let $x \in S^3$, and let $B = A_{\{x\}}$ be as in Theorem 4.1. As there, it is a simple unital direct limit of separable recursive subhomogeneous algebras, with no dimension growth. Moreover, by Theorem 4.1(5), it has a unique trace, since there is a unique h -invariant Borel probability measure. In particular, the projections distinguish the traces. Corollary 3 in Section 5 of [8] implies that A has no nontrivial projections. Therefore $A_{\{x\}}$, being a subalgebra of A , also has no nontrivial projections. Consequently it does not have real rank zero. The ordered K_0 -group of $A_{\{x\}}$ is the same as for A , and was shown in the previous example not to have Riesz decomposition.

As another example, we compute the Elliott invariants of crossed products by Furstenberg transformations on the 2-torus, recovering in particular the main result of [22] (namely, the order on K_0) with much less effort. The computations of the (unordered) K -theory and the effect of the traces on K_0 were done in [20], which was never published. Moreover, there is now better machinery available [14] for the computation of the effect of the traces on K_0 . It therefore seems appropriate to start from scratch.

Example 4.9. Fix $\theta \in [0, 1] \setminus \mathbf{Q}$, a continuous function $f_0: S^1 \rightarrow \mathbf{R}$, and $n \in \mathbf{Z} \setminus \{0\}$. We define $h: S^1 \times S^1 \rightarrow S^1 \times S^1$ to be the inverse of the homeomorphism

$$(\zeta_1, \zeta_2) \mapsto (e^{2\pi i \theta} \zeta_1, e^{2\pi i f_0(\zeta_1)} \zeta_1^n \zeta_2).$$

(One sees that the given map does in fact have a continuous inverse by writing down an explicit formula for it. This homeomorphism is called $\varphi_{f_0, \theta}$ in [22].) The homeomorphism h is minimal by the remark after Theorem 2.1 in Section 2.3 of [16]. Normalized Lebesgue measure on $S^1 \times S^1$ is invariant, and when f_0 is Lipschitz, this is the only invariant measure (Theorem 2.1 of [16]). Define $\alpha: C(S^1 \times S^1) \rightarrow C(S^1 \times S^1)$ by $\alpha(f) = f \circ h^{-1}$. Let

$$A = C^*(\mathbf{Z}, S^1 \times S^1, h) = C^*(\mathbf{Z}, C(S^1 \times S^1), \alpha).$$

We compute the Elliott invariant of A , and we start by describing the ingredients of the description we intend to prove. Define $\alpha_0: C(S^1) \rightarrow C(S^1)$ by $\alpha_0(f)(\zeta) = f(e^{2\pi i \theta} \zeta)$. Then the crossed product $C^*(\mathbf{Z}, C(S^1), \alpha_0)$ is just the irrational rotation algebra A_θ . Moreover, the homomorphism $a \mapsto a \otimes 1$, from $C(S^1)$ to $C(S^1) \otimes C(S^1) \cong C(S^1 \times S^1)$, intertwines α_0 and α , thus giving a homomorphism $\varphi: A_\theta \rightarrow A$. Let $p \in A_\theta$ be a projection for which the unique trace τ on A_θ satisfies $\tau(p) = \theta$ ([38]). Let $\iota: C(S^1 \times S^1) \rightarrow A$ be the inclusion, and let $\beta \in K_0(C(S^1 \times S^1))$ be the Bott element. We prove that the Elliott invariant of A is given as follows:

$$K_0(A) \cong \mathbf{Z} \cdot [1] \oplus \mathbf{Z} \cdot \iota_*(\beta) \oplus \mathbf{Z} \cdot [\varphi(p)],$$

every trace τ on A satisfies

$$\tau_*(m_1[1] + m_2\iota_*(\beta) + m_3[\varphi(p)]) = m_1 + m_3\theta,$$

$$K_0(A)_+ = \{m_1[1] + m_2\iota_*(\beta) + m_3[\varphi(p)]: m_1 + m_3\theta > 0 \text{ or } m_1 = m_2 = m_3 = 0\},$$

and

$$K_1(A) \cong \mathbf{Z}^3 \oplus \mathbf{Z}/n\mathbf{Z}.$$

We first compute $K_*(A)$ using the Pimsner-Voiculescu exact sequence [36], which here takes the form

$$\begin{array}{ccccc} K_0(C(S^1 \times S^1)) & \xrightarrow{\text{id} - \alpha_*^{-1}} & K_0(C(S^1 \times S^1)) & \longrightarrow & K_0(A) \\ \exp \uparrow & & & & \downarrow \partial \\ K_1(A) & \longleftarrow & K_1(C(S^1 \times S^1)) & \xleftarrow{\text{id} - \alpha_*^{-1}} & K_1(C(S^1 \times S^1)). \end{array}$$

Let $z \in U(C(S^1))$ be $z(\zeta) = \zeta$. Then we identify $K_1(C(S^1 \times S^1)) \cong \mathbf{Z}^2$ as the free abelian group on the generators $[z \otimes 1]$ and $[1 \otimes z]$ (in that order). We also identify $K_0(C(S^1 \times S^1)) \cong \mathbf{Z}^2$ as the free abelian group on the generators $[1]$ and the Bott element β , which we take to be the image of $[z] \otimes [z]$ under the homomorphism

$$K_1(C(S^1)) \otimes K_1(C(S^1)) \rightarrow K_0(C(S^1) \otimes C(S^1)).$$

(See the beginning of Section 2 of [40].) Again, for the purpose of writing group homomorphisms as matrices, we take these generators in the order given.

To compute α_* , we use the homotopic map given by $f \mapsto f \circ h_0^{-1}$ with

$$h_0^{-1}(\zeta_1, \zeta_2) = (\zeta_1, \zeta_1^n \zeta_2).$$

It is then clear that

$$\alpha_*([1]) = [1], \quad \alpha_*([z \otimes 1]) = [z \otimes 1], \quad \text{and} \quad \alpha_*([1 \otimes z]) = n[z \otimes 1] + [1 \otimes z].$$

To calculate $\alpha_*(\beta)$, we shift to the topological K-theory $K^*(S^1 \times S^1)$, and use its ring structure. Note that $[z \otimes 1]^2 = 0$, because all elements of $K^1(S^1)$ have square zero. Therefore

$$\alpha_*(\beta) = (h^{-1})^*([z \otimes 1] \cdot [1 \otimes z]) = [z \otimes 1] \cdot (n[z \otimes 1] + [1 \otimes z]) = \beta.$$

It follows that $\alpha_*: K_0(C(S^1 \times S^1)) \rightarrow K_0(C(S^1 \times S^1))$ is the identity, and that $\alpha_*: K_1(C(S^1 \times S^1)) \rightarrow K_1(C(S^1 \times S^1))$ is given by the matrix $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

We now know that the upper right horizontal map in the Pimsner-Voiculescu sequence is zero, and that the lower left horizontal map is $\begin{pmatrix} 0 & -n \\ 0 & 0 \end{pmatrix}$. Therefore

$$K_1(A) \cong \mathbf{Z}^3 \oplus \mathbf{Z}/n\mathbf{Z} \quad \text{and} \quad K_0(A) \cong \mathbf{Z}^3.$$

We identify $[1]$ and β with their images in $K_0(A)$, and choose any $\eta_0 \in K_0(A)$ such that $\partial(\eta_0) = (-1, 0) \in K_1(C(S^1 \times S^1))$. (Note that $(-1, 0)$ generates the kernel of $\text{id} - \alpha_*^{-1}$ on $K_1(C(S^1 \times S^1))$.) Then $K_0(A) = \mathbf{Z} \cdot [1] \oplus \mathbf{Z} \cdot \beta \oplus \mathbf{Z} \cdot \eta_0$. (Later we will replace η_0 by a more carefully chosen generator.)

Now let τ be an arbitrary normalized trace on A . Then τ is induced by an h -invariant measure μ on $S^1 \times S^1$. We compute $\tau_*: K_0(A) \rightarrow \mathbf{R}$. Trivially $\tau(1) = 1$, and $\tau_*(\beta) = 0$ because $\beta \in K^0(S^1 \times S^1)$ is represented as the difference of two vector bundles of the same rank (namely 1). To calculate $\tau_*(\eta_0)$, we combine Definition VI.8 and Theorems V.12 and VI.11 of [14] to get (notation explained afterwards)

$$\exp(2\pi i \tau_*(\eta_0)) = R_\alpha^\mu([z^{-1} \otimes 1]).$$

Here $[z^{-1} \otimes 1]$ now represents the homotopy class of the function $(\zeta_1, \zeta_2) \mapsto \zeta_1^{-1}$ (an element of $[S^1 \times S^1, S^1]$). Following Definitions VI.3 and VI.5 of [14], $R_\alpha^\mu([v])$ is computed by finding a continuous function $f: S^1 \times S^1 \rightarrow \mathbf{R}$ such that

$$v(h^{-1}(x))^* v(x) = e^{if(x)}$$

for all $x \in S^1 \times S^1$, and setting

$$R_\alpha^\mu([v]) = \exp\left(i \int_X f d\mu\right).$$

(By comparing Definition VI.2 with the proof of Proposition VI.10 in [14], one sees that the automorphism $C(S^1 \times S^1)$ given by h really is $\alpha(f) = f \circ h^{-1}$.) With $v = z \otimes 1$, one checks that we may choose the function $f(x) = 2\pi\theta$ for all x , whence $\exp(2\pi i \tau_*(\eta_0)) = \exp(2\pi i\theta)$. Therefore there is $k \in \mathbf{Z}$ such that $\tau_*(\eta_0) = \theta + k$.

A priori k depends on τ . However, the space of normalized traces is connected, and $\tau \mapsto \tau_*(\eta_0)$ is continuous, so in fact k is independent of τ . Replacing η_0 by $\eta_0 - k[1]$, we may therefore assume that $\tau_*(\eta_0) = \theta$ for all traces τ . It follows that $\tau_*: \mathbf{Z}^3 \rightarrow \mathbf{R}$ is given, for every τ , by $\tau_*(m_1, m_2, m_3) = m_1 + m_3\theta$.

Every normalized trace τ on A must restrict to the unique trace on the image $\varphi(A_\theta) \subset A$ of the irrational rotation algebra A_θ . Therefore, with the projection $p \in A_\theta$ being as at the beginning, we have $\tau_*([\varphi(p)] - \eta_0) = 0$. Consequently there is $l \in \mathbf{Z}$ such that $\eta_0 = [\varphi(p)] + l\beta$. We then also have

$$K_0(A) = \mathbf{Z} \cdot [1] \oplus \mathbf{Z} \cdot \beta \oplus \mathbf{Z} \cdot [\varphi(p)],$$

and the formula for every τ_* with respect to the new identification of $K_0(A)$ with

\mathbf{Z}^3 is still $\tau_*(m_1, m_2, m_3) = m_1 + m_3\theta$. The identification of $K_0(A)_+$ with

$$\{(m_1, m_2, m_3) \in \mathbf{Z}^3 : m_1 + m_3\theta > 0 \text{ or } m_1 = m_2 = m_3 = 0\}$$

(the main result of [22]) is now immediate from Theorem 4.5(2).

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