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MIXED MULTIPLICITIES OF IDEALS VERSUS MIXED VOLUMES OF POLYTOPES

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ABSTRACT. The main results of this paper interpret mixed volumes of lattice polytopes as mixed multiplicities of ideals and mixed multiplicities of ideals as Samuel's multiplicities. In particular, we can give a purely algebraic proof of Bernstein's theorem which asserts that the number of common zeros of a system of Laurent polynomial equations in the torus is bounded above by the mixed volume of their Newton polytopes.

INTRODUCTION

Let us first recall the definition of mixed volumes. Given two polytopes P, Q in $\mathbb{R}^n$ (which need not be different), their Minkowski sum is defined as the polytope

$$P + Q := \{a + b \mid \ a \in P, \ b \in Q\}.$$

The n-dimensional *mixed volume* of a collection of n polytopes $Q_1, ..., Q_n$ in $\mathbb{R}^n$ is the value

$$MV_n(Q_1, \dots, Q_n) := \sum_{h=1}^{s} \sum_{1 \le i_1 < \dots < i_h \le n} (-1)^{n-h} V_n(Q_{i_1} + \cdots + Q_{i_h}).$$

Here V_n denotes the n-dimensional Euclidean volume. Mixed volumes play an important role in convex geometry (see [BF], [Ew]) and elimination theory (see [GKZ], [CLO], [Stu]).

Our interest in mixed volumes arises from the following result of Bernstein [Be] which relates the number of solutions of a system of polynomial equations to the mixed volume of their Newton polytopes (see also [Kh], [Ku]).

Bernstein's Theorem. *Let $f_1, ..., f_n$ be Laurent polynomials in $\mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ with finitely many common zeros in the torus $(\mathbb{C}^*)^n$. Then the number of common zeros of $f_1, ..., f_n$ in $(\mathbb{C}^*)^n$ is bounded above by the mixed volume $MV_n(Q_1, ..., Q_n)$, where Q_i denotes the Newton polytope of f_i. Moreover, this bound is attained for a generic choice of coefficients in $f_1, ..., f_n$.*

Bernstein's theorem is a generalization of the classical Bezout's theorem. It is a beautiful example of the interaction between algebra and combinatorics. However, the original proof in [Be] has more or less a combinatorial flavor. A geometric proof using intersection theory was given by Teissier [Te3] (see also the expositions

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[Fu2], [GKZ]). This paper grew out of our attempt to find an algebraic proof of Bernstein's theorem by using Samuel's multiplicity as it is usually done in a proof of Bezout's theorem. The relationship between toric varieties and multigraded rings used in the geometric proof suggests that mixed multiplicities of ideals may be the link between mixed volume of Newton polytopes of Laurent polynomials and the number of their common zeros. To produce this link we encountered two problems which are of independent interest:

- Can one interpret the number of common zeros of Laurent polynomials in the torus as mixed multiplicity of ideals?
- Does there exist any relationship between mixed multiplicities of ideals and mixed volume of polytopes?

We will solve these problems and we will thereby obtain a proof for Bernstein's theorem which uses mixed multiplicities of ideals in a similar way as Samuel's multiplicity for Bezout's theorem. In fact, the number of common zeros of general polynomials in the torus counted with multiplicities and the mixed volume of their Newton polytopes can be interpreted as the same mixed multiplicity of ideals.

Now we are going to give a brief introduction of mixed multiplicities. Let $J_1, ..., J_n$ be a collection of ideals in a local ring $(A, \mathfrak{m})$ and I an $\mathfrak{m}$-primary ideal. Then the length function

$$\ell(I^{u_0} J_1^{u_1} \cdots J_n^{u_n} / I^{u_0+1} J_1^{u_1} \cdots J_n^{u_n})$$

is a polynomial $P(u)$ for $u_0, u_1, ..., u_n$ large enough [Ba], [R2], [Te1]. If we write this polynomial in the form

$$P(u) = \sum_{\alpha \in \mathbb{N}^{n+1},\ |\alpha| = r} \frac{1}{\alpha!} e_\alpha u^\alpha + \{\text{terms of total degree} < r\},$$

where $r = \deg P(u)$ and $\alpha = (\alpha_0, \alpha_1, ..., \alpha_n)$ of weight

$$|\alpha| := \alpha_0 + \alpha_1 + \cdots + \alpha_n = r, \quad \alpha! := \alpha_0!\alpha_1!...\alpha_n!, \quad u^\alpha := u_0^{\alpha_0} u_1^{\alpha_1} ... u_n^{\alpha_n},$$

then the coefficients e_α are non-negative integers. One calls e_α the *mixed multiplicities* of the ideals $I, J_1, \ldots, J_n$ [Te1]. We will denote e_α by $e_\alpha(I|J_1, ..., J_n)$. This notion can also be defined for homogeneous ideals in a standard multi-graded algebra over a field. Applications of mixed multiplicities can be found in [KaV], [Ro], [Te1], [Te2], [Tr2], [Ve1] and [Ve2].

If the ideals $J_1, ..., J_n$ are $\mathfrak{m}$-primary ideals, one can interpret $e_\alpha(I|J_1, ..., J_n)$ as Samuel's multiplicity of general elements ([Te1], [R2], [Sw]). However, the techniques used in the $\mathfrak{m}$-primary case are not applicable for non-$\mathfrak{m}$-primary ideals. For instance, mixed multiplicities of $\mathfrak{m}$-primary ideals are always positive, whereas they may be zero in the general case. We will develop new techniques to prove the following general result which allows us to test the positivity of mixed multiplicities and to compute them by means of Samuel's multiplicity.

Corollary 1.6. *Assume that the local ring A has an infinite residue field. Let Q be an ideal generated by α_i general elements in J_i, for $i = 1, ..., n$, and $J := J_1 \cdots J_n$. Then $e_\alpha(I|J_1, ..., J_n) > 0$ if and only if* $\dim A/(Q : J^\infty) = \alpha_0 + 1$. *In this case,*

$$e_\alpha(I|J_1, ..., J_n) = e(I, A/(Q : J^\infty)).$$

More generally, we can specify a class of concrete ideals Q that can be used to compute $e_\alpha(I|J_1, ..., J_n)$ (Theorem 1.4). Such a result was already obtained for two ideals in [Tr2]. The novelties here are the use of diagonal subalgebras and the introduction of superficial sequences for a set of ideals which provide a simpler way to study mixed multiplicities. As consequences, we will show that the positivity of $e_\alpha(I|J_1, ..., J_n)$ does not depend on the ideal I and is rigid with respect to a certain order of the indices α (Corollary 1.8).

There is already a close relationship between mixed multiplicities of multigraded rings and mixed volumes. First, the multiplicity of a graded toric ring can be expressed in terms of the volume of a convex polytope (which is a consequence of Ehrhart's theory on the number of lattice points in convex polytopes). Second, mixed volume can be defined as a coefficient of the multivariate polynomial representing the volumes of linear combinations of the polytopes (Minkowski formula). Using these facts we find the following interpretation of mixed volumes as mixed multiplicities of ideals.

Corollary 2.5. *Let $Q_1, ..., Q_n$ be an arbitrary collection of lattice convex polytopes in $\mathbb{R}^n$. Let $A = k[x_0, x_1, ..., x_n]$ and $\mathfrak{m}$ be the maximal graded ideal of A. Let M_i be any set of monomials of the same degree in A such that Q_i is the convex hull of the lattice points of their dehomogenized monomials in $k[x_1, ..., x_n]$. Let J_i be the ideal of A generated by the monomials of M_i. Then*

$$MV_n(Q_1, ..., Q_n) = e_{(0,1,...,1)}(\mathfrak{m}|J_1, ..., J_n).$$

This interpretation has interesting consequences. For instance, one can deduce properties of mixed volumes from those of mixed multiplicities. Conversely, properties of mixed volumes may predict unknown properties of mixed multiplicities. For instance, well-known inequalities for mixed volumes such as the Alexandroff-Fenchel inequality (see e.g. [Kh], [Te3]) lead us to raise the question whether similar inequalities are valid for mixed multiplicities of ideals (Question 2.7). To give an answer to this question turns out to be a challenging problem.

To prove Bernstein's theorem we first reformulate it for a system of homogeneous polynomial equations. In this case, the number of common zeros of general polynomials $f_1, \ldots, f_n$ can be seen as Samuel's multiplicity of certain graded algebra. It turn out that this Samuel's multiplicity and the mixed volume of their Newton polytopes are the same mixed multiplicity $e_{(0,1,...,1)}(\mathfrak{m}|J_1, ..., J_n)$, where $J_1, ..., J_n$ are the ideals generated by the supporting monomials of $f_1, \ldots, f_n$. By the principle of conservation of number, this implies the bound in Bernstein's theorem for any algebraically closed field (Theorem 3.1).

Finally, we would like to point out that computing mixed volumes is a hard enumerative problem (see e.g. [EC], [HS1], [HS2]) and that the above relationships between mixed volumes, mixed multiplicities and Samuel multiplicity provide an alternative method for the computation of mixed volumes since many computer algebra programs can compute the Samuel multiplicity or the Hilbert polynomial of multigraded algebras.

This paper is organized as follows. Section 1 will deal with the characterization of mixed multiplicities as Samuel's multiplicities. In Section 2 we will interpret mixed volumes as mixed multiplicities. The algebraic proof of Bernstein's theorem will be given in Section 3.

1. Mixed multiplicities of ideals

We begin with some general observations on Hilbert polynomials of multigraded algebras.

Let s be any non-negative integer. Let $R = \bigoplus_{u\in\mathbb{N}^{s+1}} R_u$ be a finitely generated standard $\mathbb{N}^{s+1}$-graded algebra over an Artin local ring R_0. We say R is *standard* if it is generated by homogeneous elements of degrees $(0,...,1,...,0)$, where 1 occurs only as the ith component, $i = 0,1,...,s$. The *Hilbert function* of R is defined by $H_R(u) := \ell(R_u)$, where ℓ denotes the length. If we view u as a set of $s+1$ variables $u_0,...,u_s$, then there exists a polynomial $P_R(u)$ and integers $n_0,n_1,...,n_s$ such that $H_R(u) = P_R(u)$ for $u_i \geq n_i$, $i = 0,1,...,s$ (abbr. for $u \gg 0$) [Wa]. One calls $P_R(u)$ the *Hilbert polynomial* of R. If $P_R(u) \neq 0$, we write $P_R(u)$ in the form

$$P_R(u) = \sum_{\alpha\in\mathbb{N}^{s+1}, |\alpha|=r} \frac{1}{\alpha!} e_\alpha(R) u^\alpha + \{\text{terms of degree} < r\},$$

where $r = \deg P_R(u)$ and $\alpha = (\alpha_0,\alpha_1,...,\alpha s)$ with

$$|\alpha| := \alpha_0 + \alpha_1 + \cdots + \alpha_s = r, \quad \alpha! := \alpha_0!\alpha_1!...\alpha_s!, \quad \text{and} \quad u^\alpha := u_0^{\alpha_0} u_1^{\alpha_1} ... u_s^{\alpha_s}.$$

One calls the coefficients $e_\alpha(R)$ the *mixed multiplicities* of the multigraded algebra R. If $s = 0$, i.e. R is an $\mathbb{N}$-graded algebra, then R has only one mixed multiplicity. It is the usual multiplicity of R, and we will denote it by $e(R)$.

The mixed multiplicities of R can be studied by means of certain $\mathbb{N}$-graded subalgebras. Let $\lambda = (\lambda_0,\lambda_1,...,\lambda_s)$ be any sequence of non-negative integers. Set

$$R^\lambda := \bigoplus_{n\geq 0} R_{n\lambda}.$$

Then R^λ is a finitely generated $\mathbb{N}$-graded algebra over R_0. One calls R^λ the λ-*diagonal subalgebra* of R. This notion plays an important role in the study of embeddings of blowups of projective schemes [CHTV].

Lemma 1.1. *Let $r = \deg P_R(u) \geq 0$ and let all components of λ be positive. Then $\dim R^\lambda = r + 1$ and*

$$e(R^\lambda) = r! \sum_{\alpha\in\mathbb{N}^{s+1},\ |\alpha|=r} \frac{1}{\alpha!} e_\alpha(R) \lambda^\alpha.$$

Proof. Since all components of λ are positive, we have

$$P_{R^\lambda}(n) = H_{R^\lambda}(n) = H_R(n\lambda) = \sum_{\alpha\in\mathbb{N}^{s+1},\ |\alpha|=r} \frac{1}{\alpha!} e_\alpha(R)\lambda^\alpha n^r + \{\text{terms of degree} < r\}$$

for $n \gg 0$. This implies the conclusion because $\dim R^\lambda = \deg P_{R^\lambda}(n) + 1$. □

Let $(A,\mathfrak{m})$ be a local ring (or a standard graded algebra over a field, where $\mathfrak{m}$ is the maximal graded ideal). Let I be an $\mathfrak{m}$-primary ideal and $J_1,\dots,J_s$ a sequence of ideals of A. One can define the $\mathbb{N}^{s+1}$-graded algebra

$$R(I|J_1,\dots,J_s) := \bigoplus_{(u_0,u_1,...,u_s)\in\mathbb{N}^{s+1}} I^{u_0} J_1^{u_1} ... J_s^{u_s} / I^{u_0+1} J_1^{u_1} ... J_s^{u_s}.$$

This algebra can be viewed as the associated graded ring of the Rees algebra $A[J_1t_1,...,J_st_s]$ with respect to the ideal generated by the elements of I.

For short, set $R = R(I|J_1,\dots,J_s)$. Then R is a standard $\mathbb{N}^{s+1}$-graded algebra. Hence it has a Hilbert polynomial $P_R(u)$. For any $\alpha \in \mathbb{N}^{s+1}$ with $|\alpha| = \deg P_R(u)$ we will set

$$e_\alpha(I|J_1,\dots,J_s) := e_\alpha(R).$$

The mixed multiplicities $e_\alpha(I|J_1,\dots,J_s)$ were studied first for $\mathfrak{m}$-primary ideals in [Ba], [R1] [R2], [Te1] and then for arbitrary ideals in [KaMV], [KaV], [Tr2], [Vi].

Throughout this section let

$$J := J_1 ... J_s,$$
$$d := \dim A/(0 : J^\infty),$$

where for any ideal $Q \subset A$ we set $Q : J^\infty := \bigcup_{m\ge 0}(Q : J^m)$. Moreover, for any finitely generated A-module E we will denote by $e(I,E)$ the Samuel multiplicity of E with respect to I.

Theorem 1.2. *Let $R = R(I|J_1,\dots,J_s)$. Assume that $d = \dim A/(0 : J^\infty) \ge 1$. Then*

(a) $\deg P_R(u) = d-1$,

(b) $e_{(d-1,0,\dots,0)}(I|J_1,\dots,J_s) = e(I, A/(0 : J^\infty))$.

Proof. Let $I', J_1',...,J_s'$ be the sequence of ideals generated by $I, J_1,...,J_s$ in the quotient ring $A/(0 : J^\infty)$ and put $R' = R(I'|J_1',...,J_s')$. Then

$$\begin{aligned} R_u' &= (I^{u_0}J_1^{u_1}...J_s^{u_s} + (0 : J^\infty))/(I^{u_0+1}J_1^{u_1}...J_s^{u_s} + (0 : J^\infty)) \\ &= I^{u_0}J_1^{u_1}...J_s^{u_s}/(I^{u_0+1}J_1^{u_1}...J_s^{u_s} + I^{u_0}J_1^{u_1}...J_s^{u_s} \cap (0 : J^\infty)). \end{aligned}$$

Since $I^{u_0}J_1^{u_1}...J_s^{u_s} \cap (0 : J^\infty) = 0$ for $u \gg 0$, we get $R_u = R_u'$ for $u \gg 0$. Hence

$$P_R(u) = P_{R'}(u).$$

So we may replace A by $A/(0 : J^\infty)$. If we do so, we may assume that $0 : J^\infty = 0$ and $d = \dim A \ge 1$. Then $\operatorname{ht} J \ge 1$. For $\lambda = (1,...,1)$ we have

$$R^\lambda = \bigoplus_{n\ge 0} I^nJ^n/I^{n+1}J^n \cong A[IJt]/(I),$$

where $A[IJt]$ is the Rees algebra of the ideal IJ. Since $\operatorname{ht}(IJ) \ge 1$, we have $\dim A[IJt] = d+1$ [Va, Corollary 1.6]. Hence $\dim R^\lambda \le d$. By Lemma 1.1, this implies $\deg P_R(u) \le d-1$.

On the other hand, $\dim A/J^m < d$ for any $m \ge 1$. Therefore,

$$e(I,A) = e(I,J^m) = \lim_{n\to\infty} \frac{\ell(I^nJ^m/I^{n+1}J^m)}{n^{d-1}/(d-1)!} = \lim_{n\to\infty} \frac{P_R(n,m,...,m)}{n^{d-1}/(d-1)!}$$

for $m \gg 0$. Since $e(I,A) > 0$, this implies $\dim P_R(u) \ge d-1$. So we can conclude that $\deg P_R(u) = d-1$ and that $e_{(d-1,0,\dots,0)}(R) = e(I,A)$. □

The computation of mixed multiplicities can be passed to the case $e_{(d-1,0,\dots,0)}(R)$. For this we shall need the following notation.

Given a standard $\mathbb{Z}^{s+1}$-graded algebra S, we will denote by S_+ the ideal of S generated by the homogeneous elements of degrees with positive components. A sequence of homogeneous elements $z_1,...,z_m$ in S is called *filter-regular* if

$$[(z_1,...,z_{i-1}) : z_i]_u = (z_1,...,z_{i-1})_u$$

for $u \gg 0$, $i = 1,...,m$. It is easy to see that this is equivalent to the condition $z_i \notin P$ for any associated prime $P \not\supseteq S_+$ of $S/(z_1,...,z_{i-1})$.

Remark. Filter-regular sequences have their origin in the theory of Buchsbaum rings [SV, Appendix]. It can be shown that if S is a standard graded algebra over a field, then $\mathrm{Proj}(S)$ is an equidimensional Cohen-Macaulay scheme if and only if every homogeneous system of parameters of S is filter-regular.

We will now work in the $\mathbb{Z}^{s+1}$-graded algebra

$$S := \bigoplus_{u\in\mathbb{Z}^{s+1}} I^{u_0}J_1^{u_1}...J_s^{u_s}/I^{u_0+1}J_1^{u_1+1}...J_s^{u_s+1}.$$

Let $\varepsilon_1,...,\varepsilon_m$ be any non-decreasing sequence of indices with $1 \le \varepsilon_i \le s$. Let $x_1,...,x_m$ be a sequence of elements of A with $x_i \in J_{\varepsilon_i}$, $i = 1,...,m$. We denote by x_i^* the residue class of x_i in $J_{\varepsilon_i}/IJ_1...J_{\varepsilon_i-1}J_{\varepsilon_i}^2J_{\varepsilon_i+1}...J_s$. We call $x_1,...,x_m$ an $(\varepsilon_1,...,\varepsilon_m)$-*superficial sequence* for the ideals $J_1,...,J_s$ (with respect to I) if $x_1^*,...,x_m^*$ is a filter-regular sequence in S.

The above notion can be considered as a generalization of the classical notion of a superficial element of an ideal, which plays an important role in the theory of multiplicity. Recall that an element x is called superficial with respect to an ideal $\mathfrak{a}$ if there is an integer c such that

$$(\mathfrak{a}^n : x) \cap \mathfrak{a}^c = \mathfrak{a}^{n-1}$$

for $n \gg 0$. A sequence of elements $x_1,...,x_m \in \mathfrak{a}$ is called a superficial sequence of $\mathfrak{a}$ if the residue class of x_i in $A/(x_1,...,x_{i-1})$ is a superficial element of the ideal $\mathfrak{a}/(x_1,...,x_{i-1})$, $i = 1,...,m$. It is known that this is equivalent to the condition that the initial forms of $x_1,...,x_m$ in $\mathfrak{a}/\mathfrak{a}^2$ form a filter-regular sequence in the associated graded ring $\bigoplus_{n\ge 0}\mathfrak{a}^n/\mathfrak{a}^{n+1}$ (see e.g. [Tr1, Lemma 6.2]).

We may use superficial sequences to reduce the dimension of the base ring.

Lemma 1.3. *Let Q be an ideal of A generated by an $(\varepsilon_1,...,\varepsilon_m)$-superficial sequence of $J_1,...,J_s$. Let $\bar I, \bar J_1,...,\bar J_s$ be the sequence of ideals generated by $I, J_1,...,J_s$ in the quotient ring A/Q and put $\bar R = R(\bar I|\bar J_1,...,\bar J_s)$. Let α_j be the number of the indices i such that $\varepsilon_i = j$, $j = 1,...,s$. Let $\Delta^{(0,\alpha_1,...,\alpha_s)}P_R(u)$ denote the $(0,\alpha_1,...,\alpha_s)$-difference of the polynomial $P_R(u)$. Then*

$$P_{\bar R}(u) = \Delta^{(0,\alpha_1,...,\alpha_s)}P_R(u).$$

Proof. If $m = 1$, we may assume that $(\alpha_1,...,\alpha_s) = (1,0,...,0)$. Then $Q = (x)$, where $x \in J_1$ such that $(0 : x^*)_u = 0$ for $u \gg 0$. This means

$$(1) \qquad (I^{u_0+1}J_1^{u_1+2}J_2^{u_2+1}...J_s^{u_s+1} : x) \cap I^{u_0}J_1^{u_1}...J_s^{u_s} = I^{u_0+1}J_1^{u_1+1}...J_s^{u_s+1}.$$

As a consequence we get

$$(I^{u_0+1}J_1^{u_1+2}J_2^{u_2+1}...J_s^{u_s+1} : x) \cap I^{u_0}J_1^{u_1+1}...J_s^{u_s+1} = I^{u_0+1}J_1^{u_1+1}...J_s^{u_s+1}$$

for $u \gg 0$. Consider R as a quotient ring of S. The above formula shows that $(0_R : x^*)_u = 0$ for $u \gg 0$. Hence $P_{R/(0_R:x^*)}(u) = P_R(u)$. Now, from the exact sequence

$$0 \longrightarrow R/(0 :_R x^*) \xrightarrow{x^*} R \longrightarrow R/(x^*) \longrightarrow 0$$

we can deduce that

$$(2) \qquad P_{R/(x^*)}(u) = \Delta^{(0,1,0,...,0)}P_R(u).$$

On the other hand, (1) implies

$$I^{u_0+1}J_1^{u_1+2}...J_s^{u_s+1} \cap xI^{v_0}J_1^{v_1}...J_s^{v_s} = xI^{u_0+1}J_1^{u_1+1}...J_s^{u_s+1}$$

for $(u_0, u_1, ..., u_s) \gg 0$ and $v_i = u_i, u_i + 1$, $i = 0, 1, ..., s$. Using this formula we can easily show that

$$I^{u_0}J_1^{u_1}...J_s^{u_s} \cap xI^{v_0}J_1^{v_1}...J_s^{v_s} = xI^{\max\{u_0,v_0\}}J_1^{\max\{u_1-1,v_1\}}...J_s^{\max\{u_s,v_s\}}$$

for $u \gg 0$, $v \gg 0$. By the Artin-Rees lemma, there exists $(c_0, c_1, ..., c_s) \in \mathbb{N}^{s+1}$ with $c_1 > 0$ such that

$$I^{u_0}J_1^{u_1}...J_s^{u_s} \cap (x) \subseteq xI^{u_0-c_0}J_1^{u_1-c_1}...J_s^{u_s-c_s}$$

for $u_i \geq c_i$, $i = 0, 1, ..., s$. Therefore,

$$\begin{aligned} I^{u_0}J_1^{u_1}...J_s^{u_s} \cap (x) &= I^{u_0}J_1^{u_1}...J_s^{u_s} \cap xI^{u_0-c_0}J_1^{u_1-c_1}...J_s^{u_s-c_s} \\ &= xI^{u_0}J_1^{u_1-1}J_2^{u_2}...J_s^{u_s} \end{aligned}$$

for $u \gg 0$. This implies

$$\begin{aligned} \bar{R}_u &= (I^{u_0}J_1^{u_1}...J_s^{u_s}, x)/(I^{u_0+1}J_1^{u_1}...J_s^{u_s}, x) \\ &= I^{u_0}J_1^{u_1}...J_s^{u_s}/(I^{u_0+1}J_1^{u_1}...J_s^{u_s} + I^{u_0}J_1^{u_1}...J_s^{u_s} \cap (x)) \\ &= I^{u_0}J_1^{u_1}...J_s^{u_s}/(I^{u_0+1}J_1^{u_1} \dots J_s^{u_s} + xI^{u_0}J_1^{u_1-1}J_2^{u_2}...J_s^{u_s}) \\ &= (R/(x^*))_u. \end{aligned}$$

Thus, $P_{\bar{R}}(u) = P_{R/(x^*)}(u)$. Combining this with (2) we get $P_{\bar{R}}(u) = \Delta^{(0,..,1,..,0)}P_R(u)$ which proves the case $m = 1$.

If $m > 1$, we may assume that $\alpha_1 > 0$. Then $x_1 \in J_1$. Let $I^*, J_1^*, ..., J_s^*$ denote the sequence of the ideals generated by $I, J_1, ..., J_s$ in the quotient ring $A/(x_1)$. Put $R^* = R(I^*|J_1^*, ..., J_s^*)$. As shown above, we have

$$P_{R^*}(u) = \Delta^{(0,1,0,...,0)}P_R(u). \tag{3}$$

Let $S^* := \bigoplus_{u\in\mathbb{Z}^{s+1}}(I^*)^{u_0}(J_1^*)^{u_1} \dots (J_s^*)^{u_s}/(I^*)^{u_0+1}(J_1^*)^{u_1+1}...(J_s^*)^{u_s+1}$. For $u \gg 0$ we have

$$\begin{aligned} [S/(x_1^*)]_u &= I^{u_0}J_1^{u_1} \dots J_s^{u_s}/(I^{u_0+1}J_1^{u_1+1} \dots J_s^{u_s+1} + x_1I^{u_0}J_1^{u_1-1}J_2^{u_2}...J_s^{u_s}) \\ &= I^{u_0}J_1^{u_1} \dots J_s^{u_s}/(I^{u_0+1}J_1^{u_1+1} \dots J_s^{u_s+1} + (x_1) \cap I^{u_0}J_1^{u_1} \dots J_s^{u_s}) \\ &= (I^{u_0}J_1^{u_1} \dots J_s^{u_s}, x_1)/(I^{u_0+1}J_1^{u_1+1} \dots J_s^{u_s+1}, x_1) \\ &= S_u^*. \end{aligned}$$

Since $[(x_1^*, ..., x_{i-1}^*) : x_i]_u = (x_1^*, ..., x_{i-1}^*)_u$ for $u \gg 0$, $i = 2, ..., m$, we also have

$$[(x_2^*, ..., x_{i-1}^*)S^* : x_i]_u = (x_2^*, ..., x_{i-1}^*)S_u^*$$

for $u \gg 0$, $i = 2, ..., m$. Therefore, $x_2^*, ..., x_m^*$ is an $(\varepsilon_1 - 1, \varepsilon_2, ..., \varepsilon_s)$-superficial sequence of the ideals $J_1^*, ..., J_s^*$. Now, we may use induction on m to assume that

$$P_{\bar{R}}(u) = \Delta^{(0,\alpha_1-1,\alpha 2,...,\alpha s)}P_{R^*}(u).$$

Combining this with (3) we get $P_{\bar{R}}(u) = \Delta^{(0,\alpha_1,\alpha 2,...,\alpha s)}P_R(u)$. □

Using Theorem 1.2 and Lemma 1.3 we obtain the following criterion for the positivity of mixed multiplicities.

Theorem 1.4. *Let $\alpha = (\alpha_0, \alpha_1, ..., \alpha s)$ be any sequence of non-negative integers with $|\alpha| = d-1$. Let Q be any ideal generated by an $(\alpha_1, ..., \alpha_s)$-superficial sequence* *of the ideals $I, J_1, ..., J_s$. Then $e_\alpha(I|J_1, ..., J_s) > 0$ if and only if $\dim A/(Q : J^\infty) = \alpha_0 + 1$. In this case,*

$$e_\alpha(I|J_1, ..., J_s) = e(I, A/(Q : J^\infty)).$$

Proof. If $\alpha = (d-1, 0, ..., 0)$, the conclusion follows from Theorem 1.2. If $\alpha \neq (d-1, 0, ..., 0)$, then $d \geq 2$. Let $\bar{R}, \bar{I}, \bar{J}_1, ..., \bar{J}_s$ be as in Lemma 1.3. Then $\deg P_{\bar{R}}(u) \leq d - 1 - m = \alpha_0$ where $m = \alpha_1 + \cdots + \alpha_s$. Write

$$P_{\bar{R}}(u) = \sum_{\beta \in \mathbb{N}^{s+1}, |\beta| = \alpha_0} \frac{e_\beta(\bar{I}|\bar{J}_1, ..., \bar{J}_s)}{\beta!} u^\beta + \{\text{terms of degree} < \alpha_0\}.$$

Then

$$e_{(\alpha_0, \alpha_1, ..., \alpha_s)}(I|J_1, ..., J_s) = e_{(\alpha_0, 0, ..., 0)}(\bar{I}|\bar{J}_1, ..., \bar{J}_s).$$

If $e_\alpha(I|J_1, ..., J_s) > 0$, then $e_{(\alpha_0, 0, ..., 0)}(\bar{I}|\bar{J}_1, ..., \bar{J}_s) > 0$. Therefore, $\deg P_{\bar{R}}(u) = \alpha_0$. By Theorem 1.2(a), this implies $\dim A/(Q : J^\infty) = \alpha_0 + 1$.

Conversely, if $\dim A/(Q : J^\infty) = \alpha_0 + 1$ and if we put $\bar{J} = \bar{J}_1 ... \bar{J}_s$, then

$$e_{(\alpha_0, 0, ..., 0)}(\bar{I}|\bar{J}_1, ..., \bar{J}_s) = e(\bar{I}, \bar{A}/(0 : \bar{J}^\infty)) = e(I, A/(Q : J^\infty))$$

by Theorem 1.2(b). Since the Samuel multiplicity is always positive, this implies $e_{(\alpha_0, 0, ..., 0)}(\bar{I}|\bar{J}_1, ..., \bar{J}_s) > 0$. So we can conclude that $e_\alpha(I|J_1, ..., J_s) > 0$ if and only if $\dim A/(Q : J^\infty) = \alpha_0 + 1$. □

Let k be the residue field of A. Using the prime avoidance characterization of a superficial element we can easily see that superficial sequences exist if k is infinite. In fact, general elements of $J_1, ..., J_s$ always form a superficial sequence. Recall that a property holds for a *general element* x of an ideal $Q = (y_1, ..., y_m)$ if there exists a non-empty Zariski-open subset $U \subseteq k^m$ such that whenever $x = \sum_{j=1}^m c_j x_j$ and the image of $(c_1, ..., c_m)$ in k^m belongs to U, then the property holds for x.

Lemma 1.5. *Assume that k is infinite. Any sequence which consists of α_1 general elements in J_1, ... , α_s elements in J_s forms an $(\alpha_1, ..., \alpha_s)$-superficial sequence for the ideals $J_1, ..., J_s$.*

Proof. Let $x_1, ..., x_m$ be a sequence of such general elements, $m = \alpha_1 + \cdots + \alpha_s$. Assume that $x_i \in J_{\varepsilon_i}$. Since x_i is a general element of J_{ε_i}, we have $x_i^* \notin P$ for any associated prime P of $(x_1^*, ..., x_{i-1}^*)$ with $P \not\supseteq J_{\varepsilon_i}/IJ_1...J_{\varepsilon_i - 1}J_{\varepsilon_i}^2 J_{\varepsilon_i + 1}...J_s$. Since S_+ is contained in the ideal generated by the elements of $J_{\varepsilon_i}/IJ_1...J_{\varepsilon_i - 1}J_{\varepsilon_i}^2 J_{\varepsilon_i + 1}...J_s$, this implies $x_i^* \notin P$ for any associated prime P of $(x_1^*, ..., x_{i-1}^*)$ with $P \not\supseteq S_+$. Hence $x_1^*, ..., x_m^*$ is a filter-regular sequence in S. □

Corollary 1.6. *Assume that the local ring A has infinite residue field. Let Q be an ideal generated by α_1 general elements in J_1, ... , α_n elements in J_n. Then $e_\alpha(I|J_1, ..., J_n) > 0$ if and only if $\dim A/(Q : J^\infty) = \alpha_0 + 1$. In this case,*

$$e_\alpha(I|J_1, ..., J_n) = e(I, A/(Q : J^\infty)).$$

Now we shall see that the characterization of mixed multiplicities of $\mathfrak{m}$-primary ideals given in [Te1] is a special case of Corollary 1.6.

Corollary 1.7 ([Te1, Ch. 0, Proposition 2.1])**.** *Assume that the local ring A has infinite residue field. Let $I, J_1, \ldots, J_s$ be $\mathfrak{m}$-primary ideals. Let $\alpha = (\alpha_0, \alpha_1, ..., \alpha_s)$ be any sequence of non-negative integers with $|\alpha| = \dim A - 1$. Let P be an ideal* *of A generated by $\alpha_0 + 1$ general elements in I, α_1 general elements in J_1, ... , α_s elements in J_s. Then*

$$e_\alpha(I|J_1, ..., J_s) = e(P, A).$$

Proof. Let Q be the subideal of P generated by α_1 general elements in J_1, ... , α_s elements in J_s. By Lemma 1.5, these elements form a superficial sequence of the ideals $J_1, ..., J_s$. Since $J_1, ..., J_s$ are $\mathfrak{m}$-primary ideals, Q is generated by a subsystem of parameters of A and J is an $\mathfrak{m}$-primary ideal. Therefore,

$$\dim A/(Q : J^\infty) = \dim A/Q = \dim A - (\alpha_1 + \cdots + \alpha_s) = \alpha_0 + 1.$$

By Theorem 1.4 and the above equation, we get

$$e_\alpha(I|J_1, ..., J_s) = e(I, A/(Q : J^\infty)) = e(I, A/Q).$$

But $e(I, A/Q) = e(P, A/Q)$ because P generates a minimal reduction of I in A/Q. So we can conclude that

$$e_\alpha(I|J_1, ..., J_s) = e(P, A/Q) = e(P, A).$$

□

Using Corollary 1.6 we obtain interesting properties of mixed multiplicities.

Corollary 1.8. *Let $\alpha = (\alpha_0, \alpha_1, ..., \alpha_s)$ be any sequence of non-negative integers with $|\alpha| = d - 1$. Assume that $e_\alpha(I|J_1, \dots, J_n) > 0$. Then*

(a) *$e_\alpha(I'|J_1, \dots, J_n) > 0$ for any $\mathfrak{m}$-primary ideal I',*

(b) *$e_\beta(I|J_1, \dots, J_n) > 0$ for all $\beta = (\beta_0, \dots, \beta_n)$ with $|\beta| = d - 1$ and $\beta_i \le \alpha_i$, $i = 1, \dots, n$.*

Proof. Without loss of generality, we may assume that the residue field of A is infinite. Let Q be an ideal generated by α_1 general elements in J_1, ... , α_n elements in J_n.

(a) By Corollary 1.6, the assumption implies $\dim A/(Q : J^\infty) = \alpha_0 + 1$. Since this condition does not depend on I, we also have $e_\alpha(I'|J_1, ..., J_n) > 0$.

(b) Let Q' denote the subideal of Q generated by β_i general elements in J_i, $i = 1, ..., n$. Put $A^* = A/Q'$, $I^* = IA^*$ and $J_i = J_iA^*$. Let $R^* = R(I^*|J_1^*, \dots, J_n^*)$. By Lemma 1.3 we have

$$P_{R^*}(u) = \Delta^{(0,\beta_1,\dots,\beta_n)} P_R(u).$$

From this it follows that

$$e_{(\alpha_0,\alpha_1-\beta_1,\dots,\alpha_n-\beta_n)}(I^*|J_1^*, \dots, J_n^*) = e_\alpha(I|J_1, \dots, J_n) > 0.$$

Hence $\deg P_{R^*}(u) = (d - 1) - (\beta_1 + \cdots + \beta_n)$. By Theorem 1.2(a), this implies

$$\dim A/(Q' : J^\infty) = \deg P_{R^*}(u) + 1 = \beta_0 + 1.$$

Therefore, $e_\beta(I|J_1, \dots, J_n) > 0$ by Corollary 1.6. □

2. Mixed volumes and toric rings

The aim of this section is to interpret mixed volumes as mixed multiplicities.

Usually, a mixed volume is defined for a collection of n convex polytopes in $\mathbb{R}^n$ (see e.g. [CLO]). But it is obvious that it may also be defined for any collection of convex polytopes in $\mathbb{R}^n$ as follows. Let $Q_1, ..., Q_r$ be convex polytopes in $\mathbb{R}^n$ with $\dim(Q_1 + \cdots + Q_r) \le r$. We call the value

$$MV_r(Q_1, \dots, Q_r) := \sum_{h=1}^{r} \sum_{1 \le i_1 < \dots < i_h \le r} (-1)^{r-h} V_r(Q_{i_1} + \cdots + Q_{i_h})$$

the *mixed volume* of $Q_1, ..., Q_r$. Here V_r denotes the r-dimensional Euclidean volume.

Let $\mathbf{Q} = (Q_1, ..., Q_s)$ be a sequence of convex polytopes in $\mathbb{R}^n$. Let $\lambda = (\lambda_1, ..., \lambda_s)$ be any sequence of non-negative integers. We denote by $\lambda\mathbf{Q}$ the Minkowski sum $\lambda_1 Q_1 + \cdots + \lambda_s Q_s$ and by $\mathbf{Q}_\lambda$ the multiset of λ_1 polytopes Q_1,...,λ_s polytopes Q_s. Minkowski showed that the volume of the polytope $\lambda\mathbf{Q}$ is a homogeneous polynomial in λ whose coefficients are mixed volumes up to constants (see e.g. [CLO, Ch. 7, Proposition 4.9]).

Proposition 2.1 (Minkowski formula)**.** *Let $r = \dim(Q_1 + \cdots + Q_s)$. Then*

$$V_r(\lambda\mathbf{Q}) = \sum_{\alpha \in \mathbb{N}^s, |\alpha| = r} \frac{1}{\alpha!} MV_r(\mathbf{Q}_\alpha)\lambda^\alpha.$$

We will use the Minkowski formula to establish the relationship between mixed volumes and mixed multiplicities. For that we need to work with graded toric rings.

Let $A = k[x_1, ..., x_n]$ be a polynomial ring over a field k. Let M be a finite set of monomials in A. The subalgebra $k[M]$ of A generated by the monomials of M is called the *toric ring* (or affine semigroup ring) of M. We associate with every monomial $x_1^{a_1} ... x_n^{a_n} \in A$ the lattice point $a = (a_1, ..., a_n) \in \mathbb{N}^n$. Many ring-theoretic properties of $k[M]$ can be described by means of the lattice points of M (see e.g. [BH, Section 6] or [Sta, Chap. I]). For instance,

$$\dim k[M] = \operatorname{rank} \mathbb{Z}(M),$$

where $\mathbb{Z}(M)$ denotes the subgroup of $\mathbb{Z}^n$ generated by the lattice points of M.

Assume furthermore that the lattice points of M lie on an affine hyperlane of $\mathbb{R}^n$. This is for example the case when M consists of monomials of the same degree. Then $k[M]$ has a natural $\mathbb{N}$-graded structure. The multiplicity $e(k[M])$ can be expressed in terms of the lattice points of M as follows.

Let Q_M denote the convex hull of the lattice points of M in $\mathbb{R}^n$. Then Q_M is a convex polytope with

$$\dim Q_M = \operatorname{rank} \mathbb{Z}(M) - 1.$$

Proposition 2.2. *Let $r = \operatorname{rank} \mathbb{Z}(M) - 1$. Let E be any subset of M such that its lattice points form a basis of $\mathbb{Z}(M)$. Then*

$$e(k[M]) = \frac{V_r(Q_M)}{V_r(Q_E)}.$$

This multiplicity formula is a consequence of Ehrhart's theory for the number of lattice points in lattice polytopes (see e.g. [BH, Theorem 6.3.12] or [Sta, Chap. I, Theorem 10.3]). The number $V_r(Q_M)/V_r(Q_E)$ is often called the *normalized volume* of the polytope Q_M with respect to the lattice $\mathbb{Z}(M)$.

In the following we will be concerned with products of finite sets of monomials, which is the counterpart of Minkowski sums of convex polytopes.

Let $M_1, ..., M_s$ be sets of monomials in A such that each M_i consists of monomials of the same degree. For any sequence $\lambda = (\lambda_1, ..., \lambda_s)$ of positive integers we denote by M^λ the set of all products of λ_1 monomials of M_1,..., λ_s monomials of M_s. Using the above propositions we can express the multiplicity of the toric ring $k[M^\lambda]$ in terms of mixed volumes.

Corollary 2.3. *Let $r = \operatorname{rank}\mathbb{Z}(M^{(1,\dots,1)}) - 1$. Let E be any subset of $M^{(1,\dots,1)}$ such that its lattice points form a basis of $\mathbb{Z}(M^{(1,\dots,1)})$. Let $\mathbf{Q}$ be the sequence of polytopes $Q_{M_1},...,Q_{M_s}$. Then*

$$e(k[M^\lambda]) = \frac{1}{V_r(Q_E)} \sum_{\alpha\in\mathbb{N}^s, |\alpha|=r} \frac{1}{\alpha!} MV_r(\mathbf{Q}_\alpha)\lambda^\alpha.$$

Proof. Every lattice vector of M^λ is a sum of λ_1 lattice points of M_1,...,λ_s lattice points of M_s. Therefore,

$$Q_{M^\lambda} = \lambda_1 Q_1 + \cdots + \lambda_s Q_s = \lambda\mathbf{Q}.$$

Since $\mathbb{Z}(M^\lambda) = \mathbb{Z}(M^{(1,\dots,1)})$, we have $\operatorname{rank}\mathbb{Z}(M^\lambda) = r+1$. Using Proposition 2.2 we obtain

$$e(k[M^\lambda]) = \frac{V_r(\lambda\mathbf{Q})}{V_r(Q_E)}.$$

Hence the conclusion follows from Proposition 2.1. □

This formula for the multiplicity of the toric rings $k[M^\lambda]$ resembles the formula for the multiplicity of diagonal subalgebras in Lemma 1.1. Therefore, if we can find a standard multigraded algebra such that the toric rings $k[M^\lambda]$ are its diagonal subalgebras, then a comparison of these formulas will imply a relationship between mixed volumes and mixed multiplicities.

Theorem 2.4. *Let $A = k[x_1,...,x_n]$ and $M_0, M_1,..., M_s$ be a sequence of sets of monomials such that $M_0 = \{x_1,...,x_n\}$ and each M_i consists of monomials of the same degree d_i for $i = 0, 1, \dots, s$. Let $\mathfrak{m}$ be the maximal graded ideal of A and J_i the ideal generated by the monomials of M_i. Let $R = R(\mathfrak{m}|J_1,...,J_s)$ and let $\mathbf{Q}$ be the sequence of polytopes $Q_{M_0}, Q_{M_1},...,Q_{M_s}$. Then $\deg P_R(u) = n-1$ and for any $\alpha\in\mathbb{N}^{s+1}$ with $|\alpha| = n-1$,*

$$e_\alpha(\mathfrak{m}|J_1,...,J_s) = \frac{MV_{n-1}(\mathbf{Q}_\alpha)}{\sqrt{n}}.$$

Proof. Let S denote the subalgebra of the polynomial ring $A[t_0, t_1,...,t_s]$ generated by all monomials of the form $f_i t_i$ with $f_i \in M_i$. Then S is a standard $\mathbb{N}^{s+1}$-graded algebra over k. We shall see that $R \cong S$ as $\mathbb{N}^{s+1}$-graded algebras. Let $u = (u_0, u_1,...,u_s)$ be any sequence of non-negative integers. The vector space R_u has a basis consisting of the monomials of $\mathfrak{m}^{u_0} J_1^{u_1}...J_s^{u_s}$ which are not contained in $\mathfrak{m}^{u_0+1} J_1^{u_1}...J_s^{u_s}$. Since each J_i is generated by M_i and since M_i consists of monomials of the same degree, these monomials are of the form $f_0 f_1...f_s$, where each f_i is a product of u_i monomials of M_i, $i = 0, 1,...,s$. By mapping the elements $f_0 f_1...f_s \in R_u$ to the elements $(f_0 t_0^{u_0})(f_1 t_1^{u_1})...(f_n t_n^{u_n}) \in S_u$ we obtain an $\mathbb{N}^{s+1}$-graded isomorphism of R and S.

Let $\lambda = (\lambda_0, \lambda_1,...,\lambda_s)$ be any sequence of $s+1$ positive integers. The above isomorphism induces an $\mathbb{N}$-graded isomorphism of diagonal subalgebras $R^\lambda \cong S^\lambda$. Let M^λ denote the set of all products of λ_0 monomials of M_0,..., λ_s monomials in M_s. Then

$$S^\lambda \cong k[M^\lambda].$$

Let f be a product of s monomials of $M_1, ..., M_s$. Put $E = \{x_1 f, ..., x_n f\} \subseteq M^{(1,\dots,1)}$. Then $\mathbb{Z}(E)$ contains all lattice points of the form $e_i - e_j$, where $e_1, ..., e_n$ denote the basic vectors of $\mathbb{R}^n$. Therefore, $\mathbb{Z}(E)$ contains all lattice points of the hyperplane $x_1 + \cdots + x_n = \deg f + 1$. Since all monomials of $M^{(1,\dots,1)}$ have the degree $\deg f + 1$, the lattice points of E form a basis for $\mathbb{Z}(M^{(1,\dots,1)})$. Hence $\operatorname{rank} \mathbb{Z}(M^{(1,\dots,1)}) = n$. Since Q_E is congruent to the convex polytope spanned by the points e_i,

$$V_{n-1}(Q_E) = \frac{\sqrt{n}}{(n-1)!}.$$

Applying Corollary 2.3 we get

$$e(S^\lambda) = \frac{(n-1)!}{\sqrt{n}} \sum_{\alpha \in \mathbb{N}^{s+1}, |\alpha| = n-1} \frac{1}{\alpha!} MV_{n-1}(\mathbf{Q}_\alpha) \lambda^\alpha.$$

On the other hand, since $\dim S^\lambda = \operatorname{rank} \mathbb{Z}(M^\lambda) = n$, using Lemma 1.1 we get $\deg P_S(u) = n-1$ and

$$e(S^\lambda) = (n-1)! \sum_{\alpha \in \mathbb{N}^{s+1}, |\alpha| = n-1} \frac{1}{\alpha!} e_\alpha(S) \lambda^\alpha.$$

Since the above two formulas for $e(S^\lambda)$ hold for all sequences λ of positive integers, we can conclude that their corresponding terms are equal. This means

$$e_\alpha(S) = \frac{MV_{n-1}(\mathbf{Q}_\alpha)}{\sqrt{n}}$$

for any $\alpha \in \mathbb{N}^{s+1}$ with $|\alpha| = n-1$. □

It is now easy to interpret mixed volumes as mixed multiplicities of ideals.

Corollary 2.5. *Let $Q_1, ..., Q_n$ be an arbitrary collection of lattice convex polytopes in $\mathbb{R}^n$. Let $A = k[x_0, x_1, ..., x_n]$ and let $\mathfrak{m}$ be the maximal graded ideal of A. Let M_i be any set of monomials of the same degree in A such that Q_i is the convex hull of the lattice points of their dehomogenized monomials in $k[x_1, ..., x_n]$. Let J_i be the ideal of A generated by the monomials of M_i. Then*

$$MV_n(Q_1, ..., Q_n) = e_{(0,1,\dots,1)}(\mathfrak{m}|J_1, ..., J_n).$$

Proof. By definition, the projection of the lattice point of a monomial on the hyperplane $x_0 = 0$ is the lattice point of its dehomogenized monomial. Therefore, the convex hull Q_{M_i} of the lattice points of M_i is the projection of the polytope Q_i on the hyperplane $x_0 = 0$. As a consequence, the volume $V_n(Q_{M_i})$ is proportional to $V_n(Q_i)$. This proportion can be computed as the volume of the convex hull Q_E of the basic vectors $e_0, ..., e_n$ of $\mathbb{R}^{n+1}$. Since $V_n(Q_E) = \sqrt{n+1}$, we obtain

$$V_n(Q_i) = \frac{V_n(Q_{M_i})}{\sqrt{n+1}}.$$

From this it follows that the corresponding mixed volumes are also proportional:

$$MV_n(Q_1, ..., Q_n) = \frac{MV_n(Q_{M_1}, ..., Q_{M_n})}{\sqrt{n+1}}.$$

On the other hand, applying Theorem 2.4 to the sequence $M_0, M_1, ..., M_n$ of monomials in $n+1$ variables we obtain

$$e_{(0,1,...,1)}(\mathfrak{m}|J_1, ..., J_n) = \frac{MV_n(Q_{M_1}, ..., Q_{M_n})}{\sqrt{n+1}}.$$

Therefore, we can conclude that $MV_n(Q_1, ..., Q_n) = e_{(0,1,...,1)}(\mathfrak{m}|J_1, ..., J_n)$. □

An immediate consequence of the interpretation of mixed volumes as mixed multiplicities is the non-trivial fact that mixed volumes are always non-negative numbers. In fact, we can reprove the following result given in [Fu2, p. 117].

Corollary 2.6. *Let $P_1, ..., P_n$ and $Q_1, ..., Q_n$ be two sequences of convex lattice polytopes in $\mathbb{R}^n$ with $P_i \supseteq Q_i$. Then*

$$MV_n(P_1, ..., P_n) \geq MV_n(Q_1, ..., Q_n).$$

Proof. By Corollary 2.5 we have

$$MV_n(P_1, ..., P_n) = e_{(0,1,...,1)}(\mathfrak{m}|I_1, ..., I_n),$$
$$MV_n(Q_1, ..., Q_n) = e_{(0,1,...,1)}(\mathfrak{m}|J_1, ..., J_n),$$

where I_i and J_i are ideals generated by monomial ideals with the same degree and $I_i \supseteq J_i$. Note that the vector space $\mathfrak{m}^{u_0} I_1^{u_1} ... I_n^{u_n} / \mathfrak{m}^{u_0+1} I_1^{u_1} ... I_n^{u_n}$ contains the vector space $\mathfrak{m}^{u_0} J_1^{u_1} ... J_n^{u_n} / \mathfrak{m}^{u_0+1} J_1^{u_1} ... J_n^{u_n}$ for all $u = (u_0, u_1, ..., u_n) \in \mathbb{N}^{n+1}$. Then

$$H_{R(\mathfrak{m}|I_1,...,I_n)}(u) \geq H_{R(\mathfrak{m}|J_1,...,J_n)}(u).$$

Since $e_{(0,1,...,1)}(\mathfrak{m}|I_1, ..., I_n)$ and $e_{(0,1,...,1)}(\mathfrak{m}|J_1, ..., J_n)$ are the coefficients of one of the leading terms of the corresponding Hilbert polynomials, we obtain

$$e_{(0,1,...,1)}(\mathfrak{m}|I_1, ..., I_n) \geq e_{(0,1,...,1)}(\mathfrak{m}|J_1, ..., J_n),$$

which implies the conclusion. □

Remark. Relations among mixed volumes of lattice polytopes always hold for arbitrary convex polytopes by approximating them with rational convex polytopes and then using finer lattices [Te3].

Now we come to the famous Alexandroff-Fenchel inequality between mixed volumes:

$$MV_n(Q_1, ..., Q_n)^2 \geq MV_n(Q_1, Q_1, Q_3, ..., Q_n) MV_n(Q_2, Q_2, Q_3, ..., Q_n).$$

Khovanski [Kh] and Teissier [Te3] used the Hodge index theorem in intersection theory to prove this inequality. This leads us to believe that a similar inequality should hold between mixed multiplicities.

Question 2.7. Let $(A, \mathfrak{m})$ be a local (or standard graded) ring with $\dim A = n+1 \geq 3$. Let I be an $\mathfrak{m}$-primary ideal and $J_1, ..., J_n$ ideals of height n. Put $\alpha = (0, 1, \ldots, 1)$. Is it true that

$$e_\alpha(I|J_1, ..., J_n)^2 \geq e_\alpha(I|J_1, J_1, J_3, ..., J_n) e_\alpha(I|J_2, J_2, J_3, ..., J_n)\ ?$$

Using Theorem 1.4 we can reduce this theorem to the case $\dim A = 3$. In this case, we have to prove the simpler formula:

$$e_{(0,1,1)}(I|J_1, J_2)^2 \geq e_{(0,1,1)}(I|J_1, J_1) e_{(0,1,1)}(I|J_2, J_2).$$

Unfortunately, we were unable to give an answer to the above question. The difficulty can be seen from the following observation.

Remark. The above inequality does not hold if $J_1, ..., J_n$ are $\mathfrak{m}$-primary ideals. In this case, we can even show that the inverse inequality holds, namely,

$$e_\alpha(I|J_1, ..., J_n)^2 \leq e_\alpha(I|J_1, J_1, J_3, ..., J_n)e_\alpha(I|J_2, J_2, J_3, ..., J_n)$$

where $\alpha = (0, 1, 1, \dots, 1)$. Using Corollary 1.7 we can translate it to the inequality

$$e_{(1,1)}(J_1|J_2)^2 \leq e(J_1, A)e(J_2, A)$$

for a two-dimensional ring A. This inequality was proved first by Teissier [Te2] for reduced Cohen-Macaulay rings over an algebraically closed field of characteristic zero and then by Rees and Sharp [RS] in general.

It is known that computing mixed volumes is a hard enumerative problem (see [EC], [HS1], [HS2] for algorithms and software for doing these computations). Instead of that we can now compute mixed multiplicities of the associated graded ring of the multigraded Rees algebra $A[J_1t_1, ..., J_nt_n]$ with respect to the ideal $\mathfrak{m}$. By Corollary 1.6, these mixed multiplicities can be interpreted as Samuel multiplicities. The computation of these multiplicities can be carried out by computer algebra systems such as *Cocoa, Macaulay 2* and *Singular.*

3. Bernstein's theorem

Let $k[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ be a Laurent polynomial ring over a field k. For any Laurent polynomial

$$f = \sum_{a \in \mathbb{Z}^n} c_a x^a \ (c_a \in k)$$

we will denote by $M(f)$ the set of monomials x^a with $c_a \neq 0$. Let Q_f denote the convex hull of the lattice points a with $c_a \neq 0$ in $\mathbb{R}^n$, i.e. $Q_f = Q_{M(f)}$. One calls Q_f the *Newton polytope* of f.

Bernstein's theorem says that the mixed volume of the associated Newton polytopes of n Laurent polynomials is a sharp bound for the number of common zeros in the torus $(\mathbb{C}^*)^n$ [Be, Theorem A]. Here we will prove Bernstein's theorem by purely algebraic means for any algebraically closed field k.

Theorem 3.1. *Let k be an algebraically closed field. Let $f_1, ..., f_n$ be Laurent polynomials in $k[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ with finitely many common zeros in $(k^*)^n$. Then the number of common zeros of $f_1, ..., f_n$ in $(k^*)^n$ is bounded above by $MV_n(Q_{f_1}, ..., Q_{f_n})$. Moreover, this bound is attained for a generic choice of coefficients in $f_1, ..., f_n$ if k has characteristic zero.*

Here, a generic choice of coefficients in $f_1, ..., f_n$ means that the supporting monomials of $f_1, ..., f_n$ remain the same while their coefficients vary in a non-empty open parameter space.

Now we are going to give a homogeneous version of Bernstein's theorem.

Let f^h denote the homogenization of a Laurent polynomial f in $k[x_0^{\pm 1}, x_1^{\pm 1}, ..., x_n^{\pm 1}]$. Then Q_{f^h} is a polytope in $\mathbb{R}^{n+1}$. Its projection to the hyperplane $x_0 = 0$ is a polytope canonically identified with Q_f. We have $V_n(Q_{f^h}) = \sqrt{n+1}\ V_n(Q_f)$. Hence

$$MV_n(Q_{f_1^h}, ..., Q_{f_n^h}) = \sqrt{n+1}\ MV_n(Q_{f_1}, ..., Q_{f_n}).$$

It is also obvious that the number of common zeros of $f_1, ..., f_n$ in $(k^*)^n$ is equal to the number of common zeros of $f_1^h, ..., f_n^h$ in $\mathbb{P}^n_{k^*}$, where $\mathbb{P}^n_{k^*}$ denotes the set of all points of the projective space $\mathbb{P}^n_k$ with non-zero components. Thus, Theorem 3.1 can be translated as follows.

Theorem 3.2. *Let k be an algebraically closed field. Let $g_1, ..., g_n$ be homogeneous Laurent polynomials in $k[x_0^{\pm 1}, x_1^{\pm 1}, ..., x_n^{\pm 1}]$ with finitely many common zeros in $\mathbb{P}^n_{k^*}$. Then*

$$|\{\alpha \in \mathbb{P}^n_{k^*} \mid g_i(\alpha) = 0,\ i = 1, 2, \ldots, n\}| \leq \frac{MV_n(Q_{g_1}, ..., Q_{g_n})}{\sqrt{n+1}}.$$

Moreover, this bound is attained for a generic choice of coefficients in $g_1, ..., g_n$ if k has characteristic zero.

We may reduce the above theorems to the case of polynomials. In fact, if we multiply the given Laurent polynomials with an appropriate monomial, then we will obtain a new system of polynomials. Obviously, the new polynomials in $(k^*)^n$ or in $\mathbb{P}^n_{k^*}$ have the same common zeros. Since their Newton polytopes are translations of the old ones, their mixed volumes do not change, too.

Now assume that $g_1, ..., g_n$ are homogeneous polynomials in $A = k[x_0, x_1, ..., x_n]$. Let M_i be the set of monomials occuring in g_i. Let $\mathfrak{m}$ be the maximal graded ideal of A and J_i the ideals of A generated by M_i. Put

$$R = R(\mathfrak{m}|J_1, ..., J_n).$$

We know by Theorem 2.4 that $\deg P_R(u) = n + 1$ and

$$e_{(0,1,...,1)}(R) = \frac{MV_n(Q_{g_1}, ..., Q_{g_n})}{\sqrt{n+1}}.$$

Therefore, Theorem 3.2 follows from the following result.

Theorem 3.3. *Let k be an algebraically closed field. Let $g_1, ..., g_n$ be homogeneous polynomials in $k[x_0, x_1, ..., x_n]$ with finitely many common zeros in $\mathbb{P}^n_{k^*}$. Then*

$$|\{\alpha \in \mathbb{P}^n_{k^*} \mid g_i(\alpha) = 0,\ i = 1, 2, \ldots, n\}| \leq e_{(0,1,...,1)}(R).$$

Moreover, this bound is attained for a generic choice of coefficients in $g_1, ..., g_n$ if k has characteristic zero.

Proof. Let Q be the ideal $(g_1, ..., g_n)$. Then there is a one-to-one correspondence between common zeros of $g_1, ..., g_n$ in $\mathbb{P}^n_{k^*}$ and the one-dimensional homogeneous primes of A which contain $Q : (x_0...x_n)^\infty$. As a consequence, the assumption on $g_1, ..., g_n$ implies that $Q : (x_0...x_n)^\infty$ is a one-dimensional ideal. Therefore, the number of common zeros of $g_1, ..., g_n$ in $\mathbb{P}^n_{k^*}$ is equal to the number of minimal associated prime ideals of $Q : (x_0...x_n)^\infty$ which is bounded above by the multiplicity $e(A/(Q : (x_0...x_n)^\infty))$ in view of the associativity formula for multiplicities. By the principle of conservation of number (see e.g. Fulton [Fu1, Section 10.2]), we only need to show that for a generic choice of the coefficients of $g_1, ..., g_n$, $Q : (x_0...x_n)^\infty$ there is a radical ideal with

$$e(A/(Q : (x_0...x_n)^\infty)) = e_{(0,1,...,1)}(R).$$

Let $J := J_1...J_n$. We may multiply $g_1, ..., g_n$ with $x_0...x_n$ to obtain a new system of equations with $J \subseteq (x_0...x_n)$. Since $(x_0...x_n)^m \in J$ for $m \gg 0$,

$$Q : (x_0...x_n)^\infty = Q : J^\infty.$$

By Corollary 1.6 we have for a generic choice of the coefficients of $g_1, ..., g_n$,

$$e(A/(Q : J^\infty)) = e_{(0,1,...,1)}(R).$$

Thus, the number of common zeros of $g_1, ..., g_n$ in $\mathbb{P}^n_{k^*}$ is bounded by the mixed multiplicity $e_{(0,1,...,1)}(R)$. It remains to show that $Q : J^\infty$ is a radical ideal for a generic choice of the coefficients of $g_1, ..., g_n$ if k has characteristic zero. But this follows from Bertini theorem [Fl, Satz 5.4(e)]. □

Finally, we would like to remark that the last statement of the above theorems does not hold if the ground field has positive characteristic.

Example. Let k be an algebraically closed field with $\text{char}(k) = p$. Let $f(x) = ax^p + b$ be a polynomial in one variable, $a, b \in k$. For $a, b \neq 0$ we choose $c \in k$ such that $c^p = b/a$. Then $f(x) = a(x+c)^p$ has only one zero in k^*, whereas the Newton polygon of f has volume p.

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