EQUIVALENCE OF DOMAINS ARISING FROM DUALITY OF ORBITS ON FLAG MANIFOLDS III

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ABSTRACT. In Gindikin and Matsuki 2003, we defined a $G_K$-invariant subset $C(S)$ of $G_C$ for each $K_C$-orbit $S$ on every flag manifold $G_C/P$ and conjectured that the connected component $C(S)_0$ of the identity would be equal to the Akhiezer-Gindikin domain $D$ if $S$ is of nonholomorphic type. This conjecture was proved for closed $S$ in Wolf and Zierau 2000 and 2003, Fels and Huckleberry 2005, and Matsuki 2006 and for open $S$ in Matsuki 2006. It was proved for the other orbits in Matsuki 2006, when $G_R$ is of non-Hermitian type. In this paper, we prove the conjecture for an arbitrary non-closed $K_C$-orbit when $G_R$ is of Hermitian type. Thus the conjecture is completely solved affirmatively.

1. INTRODUCTION

Let $G_C$ be a connected complex semisimple Lie group and $G_R$ a connected real form of $G_C$. Let $K_C$ be the complexification in $G_C$ of a maximal compact subgroup $K$ of $G_R$. Let $X = G_C/P$ be a flag manifold of $G_C$, where $P$ is an arbitrary parabolic subgroup of $G_C$. Then there exists a natural one-to-one correspondence between the set of $K_C$-orbits $S$ and the set of $G_R$-orbits $S'$ on $X$ given by the condition:

$$(1.1) \quad S \leftrightarrow S' \iff S \cap S' \text{ is non-empty and compact}$$

(M2). For each $K_C$-orbit $S$ we defined in [GM1] a subset $C(S)$ of $G_C$ by

$C(S) = \{ x \in G_C \mid xS \cap S' \text{ is non-empty and compact} \}$

where $S'$ is the $G_R$-orbit on $X$ given by (1.1).

Akhiezer and Gindikin defined a domain $D/K_C$ in $G_C/K_C$ as follows ([AG]). Let $\mathfrak{g}_R = \mathfrak{k} \oplus \mathfrak{m}$ denote the Cartan decomposition of $\mathfrak{g}_R = \text{Lie}(G_R)$ with respect to $K$. Let $t$ be a maximal abelian subspace in $i\mathfrak{m}$. Put

$$t^+ = \{ Y \in t \mid |\alpha(Y)| < \frac{\pi}{2} \text{ for all } \alpha \in \Sigma \}$$

where $\Sigma$ is the restricted root system of $\mathfrak{g}_C$ with respect to $t$. Then $D$ is defined by

$$D = G_R(\exp t^+)K_C.$$
Remark 1.2. When $G_\mathbb{R}$ is of Hermitian type, there exist two special closed $K_C$-orbits $S_1 = K_C B / B = Q / B$ and $S_2 = K_C w_0 B / B = Q w_0 / B$ on the full flag manifold $G_C / B$, where $Q = K_C B$ is the usual maximal parabolic subgroup of $G_C$ defined by a nontrivial central element in $i \mathfrak{r}$ and $w_0$ is the longest element in the Weyl group. For each parabolic subgroup $P$ containing the Borel subgroup $B$, two closed $K_C$-orbits $S_1 P$ and $S_2 P$ on $G_C / P$ are called of holomorphic type and all the other $K_C$-orbits are called of nonholomorphic type. Especially all the non-closed $K_C$-orbits are defined to be of nonholomorphic type.

When $G_\mathbb{R}$ is of non-Hermitian type, we define that all the $K_C$-orbits are of nonholomorphic type.

Let $S_{op}$ denote the unique open dense $K_C B$ double coset in $G_C$. Then $S_{op}'$ is the unique closed $G_\mathbb{R} B$ double coset in $G_C$. In this case we see that

$$C(S_{op}) = \{ x \in G_C \mid x S_{op} \supset S_{op}' \}.$$  

It follows easily that $C(S_{op})$ is a Stein manifold (cf. [GM1], [H]). The connected component $C(S_{op})_0$ is often called the Iwasawa domain.

The inclusion

$$D \subset C(S_{op})_0$$

was proved in [H]. (Later [M3] gave a proof without complex analysis.) On the other hand, it was proved in [GM1], Proposition 8.1 and Proposition 8.3, that $C(S_{op})_0 \subset C(S)_0$ for all $K_C P$ double cosets $S$ for any $P$. So we have the inclusion

$$(1.2) \ D \subset C(S)_0.$$  

Hence we have only to prove the converse inclusion

$$(1.3) \ C(S)_0 \subset D$$

for $K_C$-orbits $S$ of nonholomorphic type in Conjecture 1.1.

If $S$ is closed in $G_C$, then we can write

$$C(S) = \{ x \in G_C \mid x S \subset S' \}.$$  

So the connected component $C(S)_0$ is essentially equal to the cycle space introduced in [WW]. For Hermitian cases the inclusion (1.3) for closed $S$ was proved in [WZ2] and [WZ3]. For non-Hermitian cases it was proved in [FH] and [M4].

When $S$ is the open $K_C P$ double coset in $G_C$, the inclusion (1.3) was proved in [M4] for an arbitrary $P$ generalizing the result in [B].

Recently the inclusion (1.3) was proved in [M5] for an arbitrary orbit $S$ when $G_\mathbb{R}$ is of non-Hermitian type. So the remaining problem was to prove (1.3) for non-closed and non-open orbits when $G_\mathbb{R}$ is of Hermitian type.

In this paper we solve this problem.

In the next section we prove the following theorem.

Theorem 1.3. Suppose that $G_\mathbb{R}$ is of Hermitian type and let $S$ be a non-closed $K_C P$ double coset in $G_C$. Then there exist $K_C B$ double cosets $\bar{S}_1$ and $\bar{S}_2$ contained in the boundary $\partial S = S^c - S$ of $S$ such that

$$x(\bar{S}_1 \cup \bar{S}_2)^c \cap S_0^c \neq \emptyset$$

for all the elements $x$ in the boundary of $D$. Here $S_0$ denotes the dense $K_C B$ double coset in $S$. 
Remark 1.4. It seems that \( \tilde{S}_1 \) and \( \tilde{S}_2 \) are always distinct \( K_C \)-orbits. But we do not need this distinctness.

Corollary 1.5. Suppose that \( G_R \) is of Hermitian type and let \( S \) be a non-closed \( K_C \cdot P \) double coset in \( G_C \). Then \( C(S)_0 = D \).

Proof. Let \( S_0 \) be as in Theorem 1.3. Let \( \Psi \) denote the set of the simple roots in the positive root system for \( B \). For each \( \alpha \in \Psi \) we define a parabolic subgroup

\[ P_\alpha = B \sqcup Bw_\alpha B \]

of \( G_C \). By [GM2], Lemma 2, we can take a sequence \( \{\alpha_1, \ldots, \alpha_m\} \) of simple roots such that

\[ \dim_G S_0 P_{\alpha_1} \cdots P_{\alpha_k} = \dim_G S_0 + k \]

for \( k = 0, \ldots, m = \text{codim}_G S_0 \). Then it is shown in [M5], Theorem 1.4, that

\[ x \in C(S) \cap D^{\text{cl}} \implies xS^{\text{cl}} \cap S_0^{\text{op}} P_{\alpha_m} \cdots P_{\alpha_1} = xS \cap S_0. \tag{1.4} \]

Let \( x \) be an element in the boundary of \( D \). Then it follows from Theorem 1.3 that

\[ x(\partial S) \cap S_0^{\text{cl}} \neq \phi. \]

If \( x \) is also contained in \( C(S) \), then it follows from (1.4) that

\[ x(\partial S) \cap S_0^{\text{cl}} P_{\alpha_m} \cdots P_{\alpha_1} = \phi. \]

Since \( S_0^{\text{cl}} \) is contained in the closed set \( S_0^{\text{op}} P_{\alpha_m} \cdots P_{\alpha_1} \), we have

\[ x(\partial S) \cap S_0^{\text{cl}} = \phi, \]

a contradiction. Hence \( x \notin C(S) \). Thus we have proved \( C(S)_0 \subset D \). \( \square \)

Section 3 is devoted to the explicit computation of the case where \( G_R = Sp(2, \mathbb{R}) \). We use Proposition 3.2 in the proof of Lemma 2.4 in Section 2. Another simple example of the \( SU(2,1) \)-case is explicitly computed in [M4] Example 1.5.

2. Proof of Theorem 1.3

Let \( j \) be a maximal abelian subspace of \( i\mathbb{R} \). Let \( \Delta \) denote the root system of the pair \( (g_C, j) \). Since \( G_R \) is a group of Hermitian type, there exists a nontrivial central element \( Z \) of \( i\mathbb{R} \) and we can write

\[ g_C = f_C \oplus n \oplus p \]

where \( \Delta^+_n = \{ \alpha \in \Delta \mid \alpha(Z) > 0 \} \), \( n = \bigoplus_{\alpha \in \Delta^+_n} g_C(1, \alpha) \) and * \( \rightarrow \overline{\tau} \) denotes the conjugation in \( g_C \) with respect to \( g_R \). Let \( Q \) be the maximal parabolic subgroup of \( G_C \) defined by \( Q = K_C \exp n \). Let \( \Delta^+ \) be a positive system of \( \Delta \) containing \( \Delta^+_n \). Then it defines a Borel subgroup \( B = B(j, \Delta^+) \) of \( G_C \) contained in \( Q \).

Let \( P \) be a parabolic subgroup of \( G_C \) containing \( B \). Let \( S \) be a non-closed \( K_C \cdot P \) double coset in \( G_C \) and let \( S_0 \) denote the dense \( K_C \cdot B \) double coset in \( S \). By [M1], Theorem 2, we can write

\[ S_0 = K_C c_{\gamma_1} \cdots c_{\gamma_k} wB \]

with some \( w \in W \) and a strongly orthogonal system \( \{\gamma_1, \ldots, \gamma_k\} \) of roots in \( \Delta^+_n \). Here \( W \) is the Weyl group of \( \Delta \) and

\[ c_{\gamma_j} = \exp(X - X) \]

with some \( X \in g_C(j, \gamma_j) \) such that \( c_{\gamma_j}^2 \) is the reflection with respect to \( \gamma_j \).
Let $\Theta$ denote the subset of $\Psi$ such that $P = BW_\Theta B$ where $W_\Theta$ is the subgroup of $W$ generated by $\{w_\alpha \mid \alpha \in \Theta\}$. Let $\Delta_{\Theta}$ denote the subset of $\Delta$ defined by

$$\Delta_{\Theta} = \{\beta \in \Delta \mid \beta = \sum_{\alpha \in \Theta} n_\alpha \alpha \text{ for some } n_\alpha \in \mathbb{Z}\}.$$ 

If $\gamma_j \in w\Delta_{\Theta}$ for all $j = 1, \ldots, k$, then it follows that $c_{\gamma_j} \in wPw^{-1}$ for all $j = 1, \ldots, k$ and therefore

$$Sw^{-1} = S_0 Pw^{-1} = K_{c_{\gamma_1}} \cdots c_{\gamma_k} wPw^{-1} = K_{c_{\gamma_1}}wPw^{-1}$$

becomes closed in $G_C$, contradicting the assumption. Hence there exists a $j$ such that $\gamma_j \not\in w\Delta_{\Theta}$. Replacing the order of $\gamma_1, \ldots, \gamma_k$, we may assume that

$$\gamma_1 \not\in w\Delta_{\Theta}.$$ 

Let $I$ denote the complex Lie subalgebra of $\mathfrak{g}_C$ generated by $\mathfrak{g}_C(1, \gamma_1) \oplus \mathfrak{g}_C(1, -\gamma_1)$ which is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ and let $L$ be the analytic subgroup of $G_C$ for $I$. Then we have $(L \cap K_C)c_{\gamma_1}(L \cap wBw^{-1}) = (L \cap K_C)c_{\gamma_1}^{-1}(L \cap wBw^{-1})$ since both of the double cosets are open dense in $L$. Hence we have

$$S_0 = K_{c_{\gamma_1}} \cdots c_{\gamma_k} wB = K_{c_{\gamma_1}}^{-1} c_{\gamma_2} \cdots c_{\gamma_k} wB = K_{c_{\gamma_1}} \cdots c_{\gamma_k} w_{\gamma_1} wB.$$ 

If $\gamma_1 \not\in w\Delta^+$, then $\gamma_1 \in w_{\gamma_1} w\Delta^+$. So we may assume

$$\gamma_1 \in w\Delta^+,$$

replacing $w$ with $w_{\gamma_1} w$ if necessary. Let $\ell$ denote the real rank of $G_\mathbb{R}$.

**Lemma 2.1.** There exists a maximal strongly orthogonal system $\{\beta_1, \ldots, \beta_\ell\}$ of roots in $\Delta_+^\mathbb{R}$ satisfying the following conditions:

(i) If $\gamma_1$ is a long root of $\Delta$, then $\beta_1 = \gamma_1$ and $\gamma_2, \ldots, \gamma_k \in \mathbb{R}\beta_2 \oplus \cdots \oplus \mathbb{R}\beta_\ell$. (If the roots in $\Delta$ have the same length, then we define that all the roots are long roots.)

(ii) If $\gamma_1$ is a short root of $\Delta$, then $\gamma_1 \in \mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2$ and $\gamma_2, \ldots, \gamma_k \in \mathbb{R}\beta_3 \oplus \cdots \oplus \mathbb{R}\beta_\ell$.

**Proof.** First suppose that $\mathfrak{g}_\mathbb{R}$ is of type AIII, DIII, EIII, EVII or DI (of real rank 2). Then the roots in $\Delta$ have the same length. So we have only to take $\beta_j = \gamma_j$ for $j = 1, \ldots, k$ and choose an orthogonal system $\{\beta_1, \ldots, \beta_\ell\}$ of roots in $\Delta_+^\mathbb{R}$ containing $\{\beta_1, \ldots, \beta_k\}$.

Next suppose that $\mathfrak{g}_\mathbb{R} \cong \mathfrak{sp}(\ell, \mathbb{R})$. Write

$$\Delta = \{\pm e_r \pm e_s \mid 1 \leq r < s \leq \ell\} \cup \{\pm 2e_r \mid 1 \leq r \leq \ell\}$$

and

$$\Delta_+^\mathbb{R} = \{e_r + e_s \mid 1 \leq r < s \leq \ell\} \cup \{2e_r \mid 1 \leq r \leq \ell\}$$

as usual using an orthonormal basis $\{e_1, \ldots, e_\ell\}$ of $\mathfrak{g}_\mathbb{R}$. If $\gamma_1 = 2e_r$, then $\{\beta_2, \ldots, \beta_\ell\} = \{2e_s \mid s \neq r\}$ satisfies condition (i). If $\gamma_1 = e_r + e_s$ with $r \neq s$, then we put $\beta_1 = 2e_r$ and $\beta_2 = 2e_s$. Assertion (ii) is clear if we put $\{\beta_3, \ldots, \beta_\ell\} = \{2e_p \mid p \neq r, s\}$.

Finally suppose that $\mathfrak{g}_\mathbb{R} = \mathfrak{so}(2, 2p - 1)$ with $p \geq 2$. Then the real rank of $\mathfrak{g}_\mathbb{R}$ is two, and we can write

$$\Delta = \{\pm e_r \pm e_s \mid 1 \leq r < s \leq p\} \cup \{\pm e_r \mid 1 \leq r \leq p\}$$

and

$$\Delta_+^\mathbb{R} = \{e_1 \pm e_s \mid 2 \leq s \leq p\} \cup \{e_1\}$$

where
with an orthonormal basis \( \{e_1, \ldots, e_n\} \) of \( j^* \). If \( k = 2 \), then we have \( \gamma_1 = \beta_1 = e_1 \pm e_s \)
and \( \gamma_2 = \beta_2 = e_1 \mp e_s \) with some \( s \). If \( k = 1 \) and \( \gamma_1 = e_1 \pm e_s \), then \( \beta_1 = \gamma_1 \)
and \( \beta_2 = e_1 \mp e_s \). If \( k = 1 \) and \( \gamma_1 = e_1 \), then we may put \( \beta_1 = e_1 + e_2 \)
and \( \beta_2 = e_1 - e_2 \).

**Definition 2.2.** (i) Define a subroot system \( \Delta_1 \) of \( \Delta \) as follows.

If \( \gamma_1 \) is a long root of \( \Delta \), then we put

\[
\Delta_1 = \{ \pm \beta_1 \} = \{ \pm \gamma_1 \}.
\]

On the other hand if \( \gamma_1 \) is a short root of \( \Delta \), then we put

\[
\Delta_1 = \Delta \cap (\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2)
\]

(which is of type \( C_2 \)).

(ii) Put \( \Delta_2 = \{ \alpha \in \Delta \mid \alpha \) is orthogonal to \( \Delta_1 \} \).

(iii) Let \( t_j \) denote the complex Lie subalgebras of \( \mathfrak{g}_C \) generated by \( \bigoplus_{\alpha \in \Delta_j} \mathfrak{g}_C(j, \alpha) \)
for \( j = 1, 2 \).

(iv) Let \( L_1 \) and \( L_2 \) denote the analytic subgroups of \( G_C \) for \( t_1 \) and \( t_2 \), respectively.

It follows from Lemma 2.1 that

\[
c_{\gamma_1} \in L_1 \quad \text{and that} \quad c_{\gamma_2} \cdots c_{\gamma_k} \in L_2.
\]

Let \( X_j \) be nonzero root vectors in \( \mathfrak{g}_C(j, \beta_j) \) for \( j = 1, \ldots, \ell \). Then we can define a maximal abelian subspace

\[
t = \mathbb{R}(X_1 - X_1) + \cdots + \mathbb{R}(X_\ell - X_\ell)
\]
in \( \mathfrak{m} \) and a maximal abelian subspace

\[
a = \mathbb{R}(X_1 + X_1) + \cdots + \mathbb{R}(X_\ell + X_\ell)
\]
in \( \mathfrak{m} \) as in [GMI, Section 2]. Since the restricted root system \( \Sigma(t) \) is of type \( BC_\ell \)
or \( C_\ell \), the set \( t^+ \) is defined by the long roots in \( \Sigma(t) \). Hence it is of the form

\[
t^+ = \{ Y_1 + \cdots + Y_\ell \mid Y_j \in t_j^+ \}
\]

where \( t_j^+ = \{ s(X_j - X_j) \mid -(\pi/4) < s < \pi/4 \} \) by a suitable normalization of \( X_j \)
for \( j = 1, \ldots, \ell \).

Put \( T^+ = \exp t^+ \) and \( A = \exp a \). Then it is shown in [GMI, Lemma 2.1], that
\( AQ = T^+ Q \) and hence that

\[
G_R Q = K A Q = K T^+ Q
\]

by the Cartan decomposition \( G_R = K A K \). The closure of \( G_R Q \) in \( G_C \) is written as
\[
(G_R Q)^c = G_R Q \sqcup G_R c_{\beta_1} Q \sqcup G_R c_{\beta_1} c_{\beta_2} Q \sqcup \cdots \sqcup G_R c_{\beta_1} \cdots c_{\beta_k} Q
\]
where \( c_{\beta_j} = \exp(\pi/4)(X_j - X_j) \) for \( j = 1, \ldots, \ell \) ([WZI, Theorem 3.8]). We also see that

\[
(2.1) \quad G_R c_{\beta_1} \cdots c_{\beta_k} Q = K c_{\beta_1} \cdots c_{\beta_k} T_{k+1}^+ \cdots T_\ell^+ Q
\]

where \( T_j^+ = \exp t_j^+ \) since we can consider the action of the Weyl group \( W_K(T) \) on \( T \) which is of type \( BC_\ell \).

By the map

\[
\iota : xK_C \mapsto (xQ, xQ)
\]
the complex symmetric space $G_C/K_C$ is embedded in $G_C/Q \times G_C/Q$ (WZ2). It is shown in [BHH], Section 3, and [GMI], Proposition 2.2, that
\[ \iota(D/K_C) = G_R Q / Q \times G_R Q / Q. \]

**Lemma 2.3.** Suppose that
\[ \iota(x K_C) \in G_R c_\beta Q / Q \times G_R Q / Q \]
and that $\gamma_1$ is a long root of $\Delta^+_k$. (If the roots in $\Delta$ have the same length, then we define that all the roots are long roots.) Define a $K_C$-$B$ double coset $\bar{S}_1$ by
\[ \bar{S}_1 = K_C c_{\gamma_1} \cdots c_{\gamma_k} w B. \]
Then $\bar{S}_1$ is contained in $\partial S = S^d - S$ and
\[ x \bar{S}_1 \cap S'_0 \neq \emptyset. \]

**Proof.** It is clear that we may replace $x$ by any elements in the double coset $G_R x K_C$. By the left $G_R$-action we may assume that $x \in Q$. By the right $K_C$-action we may moreover assume that $x \in \mathcal{N}$ since $Q = \mathcal{N} K_C$. Since $K = K_C \cap G_R$ normalizes $\mathcal{N}$, we may assume by (2.1) that
\[ x Q = c_{\beta_1} t_2 \cdots t_\ell Q \]
with some $t_j \in T^+_j$ for $j = 2, \ldots, \ell$. As in [WZ2], we write
\[ c_{\beta_1} = c_\gamma = c^c \quad \text{and} \quad t_j = t_j^- t_j^+ \quad \text{for} \quad j = 2, \ldots, \ell \]
with $c^c, t_j^- \in \mathcal{N}$ and $c^c$, $t_j^+ \in Q$. Then we have
\[ x = c^c t_2^- \cdots t_\ell^- \]
It follows from Lemma 2.1 and Definition 2.2 that $c_{\gamma_2} \cdots c_{\gamma_k} \in L_2$. Since $\text{Ad}(c_{\gamma_2} \cdots c_{\gamma_k})$ is $\theta$-stable, the double cosets
\[ S_{L_2} = (L_2 \cap K_C) c_{\gamma_2} \cdots c_{\gamma_k} (L_2 \cap w B w^{-1}) \]
and
\[ S'_{L_2} = (L_2 \cap G_R) c_{\gamma_2} \cdots c_{\gamma_k} (L_2 \cap w B w^{-1}) \]
correspond by the duality ([M1], Theorem 2).

It follows from Lemma 2.1 (i) and Definition 2.2 that
\[ c^c \in L_1 \quad \text{and} \quad t_2^+, \ldots, t_\ell^+ \in L_2. \]

It follows moreover from Definition 2.2 (i) that $t_1 \cong \mathfrak{s}(2, \mathbb{C})$.

Write $y = t_2^- \cdots t_\ell^-$. Then we have
\[ y Q = t_2 \cdots t_\ell Q \subset T^+ Q \subset G_R Q \]
and
\[ y Q \subset \overline{G_R Q}. \]
Hence we have
\[ y \in L_2 \cap (C(S_1) \cap C(S_2)) = L_2 \cap D \]
by [GMI], (1.3). By the inclusion (1.2) this implies that the set $y S_{L_2} \cap S'_{L_2}$ is nonempty and closed in $L_2$. Take an element $z$ of $y S_{L_2} \cap S'_{L_2}$.
Since \( \gamma_1 \in w\Delta^+ \), we have \( c^+ \in wBw^{-1} \). Since \( c^+ \in L_1 \) commutes with elements in \( L_2 \), we have
\[
\begin{align*}
  cz \in cS_{L_2} &= c^{-1}c^+y(L_2 \cap K_C)c_{\gamma_2} \cdots c_{\gamma_k}(L_2 \cap wBw^{-1}) \\
  &= c^+y(L_2 \cap K_C)c_{\gamma_2} \cdots c_{\gamma_k}(L_2 \cap wBw^{-1}) \\
  &\subset c^+yK_Cc_{\gamma_2} \cdots c_{\gamma_k}wBw^{-1} = xS_1w^{-1}.
\end{align*}
\]

On the other hand we have
\[
\begin{align*}
  cz \in cS_{L_2} &= c(L_2 \cap G_R)c_{\gamma_2} \cdots c_{\gamma_k}(L_2 \cap wBw^{-1}) \\
  &= (L_2 \cap G_R)c_{\gamma_1}c_{\gamma_2} \cdots c_{\gamma_k}(L_2 \cap wBw^{-1}) \subset S_0w^{-1}.
\end{align*}
\]

Hence \( x\bar{S}_1 \cap S_0^\prime \neq \emptyset \). It is clear that \( \bar{S}_1 \subset S_0^\prime = S^d \) because
\[
(L_1 \cap K_C)(L_1 \cap wBw^{-1}) \subset ((L_1 \cap K_C)c(L_1 \cap wBw^{-1}))^d = L_1.
\]

Now we will prove \( \bar{S}_1 \not\subset S \). Consider the map
\[
\varphi : K_C^\prime \backslash G_C/B \ni K_C\theta \rightarrow B\theta(g)^{-1}gB \in B \backslash G_C/B
\]
introduced in \( \text{Sp} \) where \( \theta \) is the holomorphic involution in \( G_C \) defining \( K_C \). We have
\[
\varphi(\bar{S}_1) = Bw^{-1}w_{\gamma_2} \cdots w_{\gamma_k}wB
\]
and
\[
\varphi(S) = \varphi(S_0P) \subset Pw^{-1}w_{\gamma_1} \cdots w_{\gamma_k}wP = BW_\phi w^{-1}w_{\gamma_1} \cdots w_{\gamma_k}wW_\phi B.
\]
So we have only to show
\[
(2.2) \quad w^{-1}w_{\gamma_2} \cdots w_{\gamma_k}w \not\in W_\phi w^{-1}w_{\gamma_1} \cdots w_{\gamma_k}wW_\phi.
\]

Let \( Z \) be an element in \( j \) defining \( P \). This implies that \( Z \) is dominant for \( \Delta^+ \) and that \( \{ \alpha \in \Psi \mid \alpha(Z) = 0 \} = \Theta \). Let \( w_1 \) and \( w_2 \) be elements in \( W_\phi \). Let \( B(\ , \ ) \) denote the Killing form on \( g \) and let \( Y_{\gamma_1} \) denote the element in \( j \) such that
\[
\gamma_1(Y) = B(Y, Y_{\gamma_1}) \quad \text{for all} \ Y \in j.
\]

Then we have
\[
\begin{align*}
  &B(Z, w^{-1}w_{\gamma_2} \cdots w_{\gamma_k}wZ) - B(Z, w_1w^{-1}w_{\gamma_1}w_{\gamma_2} \cdots w_{\gamma_k}wwZ) \\
  &= B(wZ - w_{\gamma_1}wZ, w_{\gamma_2} \cdots w_{\gamma_k}wZ) \\
  &= \frac{2B(Y_{\gamma_1}, wZ)}{B(Y_{\gamma_1}, Y_{\gamma_1})}B(Y_{\gamma_1}, w_{\gamma_2} \cdots w_{\gamma_k}wZ) \\
  &= \frac{2B(Y_{\gamma_1}, wZ)^2}{B(Y_{\gamma_1}, Y_{\gamma_1})} > 0
\end{align*}
\]

since \( \gamma_1 \not\in w\Delta_\phi \). Thus we have proved (2.2). \( \square \)

**Lemma 2.4.** Suppose that
\[
i(xK_C) \in G_Rc_\beta Q/Q \times G_Rc_\beta Q/Q
\]
and that \( \gamma_1 \) is a short root of \( \Delta^+_1 \). (We assume that \( g_R \cong \mathfrak{sp}(l, R) \) or \( \mathfrak{so}(2, 2p - 1) \) with \( p \geq 2 \).) Define a \( K_C \)-B double coset \( \bar{S}_1 \) by \( \bar{S}_1 = K_Cc_{\gamma_2} \cdots c_{\gamma_k}wB \) where
\[
g = \begin{cases} 
  e & \text{if } \gamma_1 \text{ is the simple short root of } \Delta^+_1, \\
  c_\beta & \text{if } \gamma_1 \text{ is the non-simple short root of } \Delta^+_1.
\end{cases}
\]

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Here $\Delta_1^+ = \Delta_1 \cap w\Delta^+$ and $\beta$ is the long simple root of $\Delta_1^+$. Then $\tilde{S}_1$ is contained in $\partial S = S^{cl} - S$ and

$$x\tilde{S}_1 \cap S_0^{cl} \neq \emptyset.$$  

**Proof.** It follows from Lemma 2.1 (ii) and Definition 2.2 that

$$c_{\beta_1}^+, t_1^+ \in L_1 \quad \text{and} \quad t_2^+, \ldots, t_\ell^+ \in L_2.$$  

It follows moreover from Definition 2.2 (i) that $L_1 \cong \mathfrak{sp}(2, \mathbb{C})$.

Write $y = t_3 \cdots t_\ell$. Then by the same argument as in the proof of Lemma 2.3 we see that the set $yS_{L_2} \cap S'_{L_2}$ is nonempty and closed in $L_2$. Take an element $z$ of $yS_{L_2} \cap S'_{L_2}$.

The positive system $\Delta_1^+$ of $\Delta_1$ consists of two long roots and two short roots. Since $\gamma_1 \in \Delta_1^+$, $\gamma_1$ is either of these two short roots. Write $x_1 = c_{\beta_1}t_2^+$.

First assume that $\gamma_1$ is the simple short root of $\Delta_1^+$. Then it follows from Proposition 3.2 (i) in the next section that

$$x_1(L_1 \cap K_C)(L_1 \cap wBw^{-1}) \cap ((L_1 \cap G_R)c_{\gamma_1}(L_1 \cap wBw^{-1}))^{cl}$$  

is nonempty. Note that $L_1 \cap wBw^{-1}$ and $\gamma_1$ correspond to $w_{\beta_2}Bw_{\beta_2}^{-1}$ and $\delta$ in the next section, respectively. Let $z_1$ be an element of (2.3). Then we have

$$z_1z \in x_1(L_1 \cap K_C)(L_1 \cap wBw^{-1})yS_{L_2}$$  

$$= x_1(L_1 \cap K_C)(L_1 \cap wBw^{-1})y(L_2 \cap K_C)c_{\gamma_2} \cdots c_{\gamma_\ell}(L_2 \cap wBw^{-1})$$  

$$= x_1y(L_1 \cap K_C)(L_2 \cap K_C)c_{\gamma_2} \cdots c_{\gamma_\ell}(L_1 \cap wBw^{-1})(L_2 \cap wBw^{-1})$$  

$$\subset xK_Cc_{\gamma_2} \cdots c_{\gamma_\ell}wBw^{-1} = x\tilde{S}_1w^{-1}$$  

and

$$z_1z \in ((L_1 \cap G_R)c_{\gamma_1}(L_1 \cap wBw^{-1}))^{cl}S'_{L_2}$$  

$$= ((L_1 \cap G_R)c_{\gamma_1}(L_1 \cap wBw^{-1}))^{cl}(L_2 \cap G_R)c_{\gamma_2} \cdots c_{\gamma_\ell}(L_2 \cap wBw^{-1})$$  

$$\subset (G_Rc_{\gamma_1}c_{\gamma_2} \cdots c_{\gamma_\ell}wBw^{-1})^{cl} = S_0^{cl}w^{-1}.$$  

So we have $x\tilde{S}_1 \cap S_0^{cl} \neq \emptyset$. We can prove $\tilde{S}_1 \subset S^{cl} - S$ by the same arguments as in the proof of Lemma 2.3.

Next assume that $\gamma_1$ is the non-simple short root of $\Delta_1^+$. Then it follows from Proposition 3.2 (ii) in the next section that

$$x_1(L_1 \cap K_C)c_{\beta}(L_1 \cap wBw^{-1}) \cap ((L_1 \cap G_R)c_{\gamma_1}(L_1 \cap wBw^{-1}))^{cl}$$  

is nonempty. Note that $L_1 \cap wBw^{-1}$, $\gamma_1$ and $\beta$ correspond to $B$, $\delta$ and $\beta_2$ in the next section, respectively. By the same argument as above we can prove

$$x\tilde{S}_1 \cap S_0^{cl} \neq \emptyset.$$  

It follows from Remark 3.3 that $\tilde{S}_1 \subset S^{cl}$. Finally we will prove that $\tilde{S}_1 \not\subset S$. Using the same argument as in the proof of Lemma 2.3, we have only to show

$$w^{-1}w_\beta w_{\gamma_2} \cdots w_{\gamma_\ell}w \not\in W_\beta w^{-1}w_{\gamma_2} \cdots w_{\gamma_\ell}wW_\beta.$$  

Let $Z$ and $Y_{\gamma_1}$ be as in the proof of Lemma 2.3. Define $Y_\beta \in j$ so that

$$\beta(Y) = B(Y, Y_\beta) \quad \text{for all} \ Y \in j.$$
Then we have
\[ B(Z, w^{-1}w_{\beta}w_{\gamma_2} \cdots w_{\gamma_k} wZ) - B(Z, w_1 w^{-1}w_{\gamma_1} w_{\gamma_2} \cdots w_{\gamma_k} w_2 wZ) \]
\[ = B(w_{\beta} wZ - w_{\gamma_1} wZ, w_{\gamma_2} \cdots w_{\gamma_k} wZ) \]
\[ = B(wZ - w_{\gamma_1} wZ, w_{\gamma_2} \cdots w_{\gamma_k} wZ) - B(wZ - w_{\beta} wZ, w_{\gamma_2} \cdots w_{\gamma_k} wZ) \]
\[ = \frac{2B(Y_{\gamma_1}, wZ)}{B(Y_{\gamma_1}, Y_{\gamma_1})} B(Y_{\gamma_1}, w_{\gamma_2} \cdots w_{\gamma_k} wZ) - \frac{2B(Y_{\beta}, wZ)}{B(Y_{\beta}, Y_{\beta})} B(Y_{\beta}, w_{\gamma_2} \cdots w_{\gamma_k} wZ) \]
\[ = \frac{2B(Y_{\gamma_1}, wZ)^2}{B(Y_{\gamma_1}, Y_{\gamma_1})} - \frac{2B(Y_{\beta}, wZ)^2}{B(Y_{\beta}, Y_{\beta})} > 0 \]
for \( w_1, w_2 \in W_{\Theta} \) since
\[ B(Y_{\gamma_1}, wZ) > 0, \quad 0 < B(Y_{\beta}, wZ) \leq B(Y_{\gamma_1}, wZ) \quad \text{and} \quad B(Y_{\beta}, Y_{\beta}) = 2B(Y_{\gamma_1}, Y_{\gamma_1}). \]
Thus we have proved (2.4). \( \square \)

Using the conjugation on \( G_C \) with respect to the real form \( G_R \), the following follows from Lemma 2.3 and Lemma 2.4.

**Corollary 2.5.** Suppose that
\[ \iota(xK_C) \in G_R Q/Q \times G_R \overline{c_{\beta} Q} \overline{Q}/\overline{Q}. \]
Then there exists a \( K_C \)-B double coset \( \tilde{S}_2 \) contained in \( \partial S \) such that
\[ x\tilde{S}_2 \cap S_0^{cl} \neq \phi. \]

**Proof of Theorem 1.3.** Let \( S \) be a non-closed \( K_C \)-\( P \) double coset in \( G_C \). Then it follows from Lemma 2.3, Lemma 2.4 and Corollary 2.5 that there exist \( K_C \)-B double cosets \( \tilde{S}_1 \) and \( \tilde{S}_2 \) contained in \( \partial S \) such that
\[ x(\tilde{S}_1 \cup \tilde{S}_2) \cap S_0^{cl} \neq \phi \]
for all \( x \in \partial D \) satisfying
\[ xK_C \in \iota^{-1}((G_R c_{\beta} Q/Q \times G_R \overline{Q} \overline{Q}) \cup (G_R Q/Q \times G_R \overline{c_{\beta} Q} \overline{Q} \overline{Q})). \]
Suppose that
\[ y(\tilde{S}_1 \cup \tilde{S}_2)^{cl} \cap S_0^{cl} = \phi \]
for some \( y \in \partial D \). Then there exists a neighborhood \( U \) of \( y \) in \( G_C \) such that
\[ x(\tilde{S}_1 \cup \tilde{S}_2)^{cl} \cap S_0^{cl} = \phi \]
for all \( x \in U \). But this contradicts (2.5) because the right hand side of (2.6) is dense in \( \partial(D/K_C) \). \( \square \)

3. \( Sp(2, \mathbb{R}) \)-case

Let \( G_C = Sp(2, \mathbb{C}) = \{ g \in GL(4, \mathbb{C}) | \iota' g J g = J \} \) where
\[ J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}. \]
Let
\[ K_C = \left\{ \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \middle| g \in GL(2, \mathbb{C}) \right\} \quad \text{and} \quad G_R = G_C \cap U(2, 2) \cong Sp(2, \mathbb{R}). \]
Put $U_+ = C_{e_1} \oplus C_{e_2}$ and $U_- = C_{e_3} \oplus C_{e_4}$ by using the canonical basis $\{e_1, e_2, e_3, e_4\}$ of $C^4$. Then we have

$$K_C = Q \cap \overline{Q}$$

where $Q = \{g \in G_C \mid gU_+ = U_+\}$ and $\overline{Q} = \{g \in G_C \mid gU_- = U_-\}$.

The full flag manifold $X$ of $G_C$ consists of the flags $(V_1, V_2)$ in $C^4$ where dim $V_j = j$, $V_1 \subset V_2$ and $\langle uJv \rangle = 0$ for all $u, v \in V_2$. Let $B$ denote the Borel subgroup of $G_C$ defined by

$$B = \{g \in G_C \mid gC e_1 = C e_1 \text{ and } gU_+ = U_+\}.$$ 

Then the full flag manifold $X$ is identified with $G_C/B$ by the map

$$gB \mapsto (V_1, V_2) = (gC e_1, gU_+).$$

There are eleven $K_C$-orbits

$$S_1 = \{(V_1, V_2) \mid V_2 = U_+\},$$
$$S_2 = \{(V_1, V_2) \mid V_2 = U_-\},$$
$$S_3 = \{(V_1, V_2) \mid V_1 \subset U_+, \text{ dim}(V_2 \cap U_-) = 1\},$$
$$S_4 = \{(V_1, V_2) \mid V_1 \subset U_-, \text{ dim}(V_2 \cap U_+) = 1\},$$
$$S_5 = \{(V_1, V_2) \mid V_1 \subset U_+ \} - (S_1 \cup S_3),$$
$$S_6 = \{(V_1, V_2) \mid V_1 \subset U_- \} - (S_2 \cup S_4),$$
$$S_7 = \{(V_1, V_2) \mid \text{dim}(V_2 \cap U_+) = \text{dim}(V_2 \cap U_-) = 1\} - (S_3 \cup S_4),$$
$$S_8 = \{(V_1, V_2) \mid V_i \subset U_+ = \{0\}, \text{dim}(V_2 \cap U_-) = 1, V_2 \cap U_+ = \{0\}\},$$
$$S_9 = \{(V_1, V_2) \mid V_i \subset U_- = \{0\}, \text{dim}(V_2 \cap U_-) = 1, V_2 \cap U_+ = \{0\}\},$$
$$S_{10} = \{(V_1, V_2) \mid V_2 \cap U_+ = \{0\}, \langle uJv \rangle = 0 \text{ for } v \in V_1\},$$
$$S_{op} = \{(V_1, V_2) \mid V_2 \cap U_+ = \{0\}, \langle uJv \rangle \neq 0 \text{ for } v \in V_1 \}$$

on $X$ where

$$\tau(v) = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} v$$

for $v \in C^4$. These orbits are related as follows (MO, Fig. 12):
Let $P_1$ and $P_2$ be the parabolic subgroups of $G_C$ defined by

$$P_1 = Q \quad \text{and} \quad P_2 = \{ g \in G_C \mid gC e_1 = C e_1 \},$$

respectively. Then the above diagram implies, for example, that

$$S_1 P_2 = S_3 P_2 \quad \text{and that} \quad \dim S_1 = \dim S_5 - 1$$

by the arrow attached with the number 2 joining $S_1$ and $S_5$.

On the other hand define subsets

$$C_+ = \{ z \in C^4 \mid (z, z) > 0 \}, \quad C_- = \{ z \in C^4 \mid (z, z) < 0 \}$$

and $C_0 = \{ z \in C^4 \mid (z, z) = 0 \}$ of $C^4$ using the Hermitian form $(w, z) = \overline{w_1 z_1} + \overline{w_2 z_2} - \overline{w_3 z_3} - \overline{w_4 z_4}$ defining $U(2, 2)$. For $v \in C^4$ define subspaces

$$v' = \{ u \in C^4 \mid (vJu) = 0 \} \quad \text{and} \quad v^\perp = \{ u \in C^4 \mid (v, u) = 0 \}$$

of $C^4$. Then $C_0$ is devided as $C_0 = C_0^s \sqcup C_0^r$ where

$$C_0^s = \{ v \in C_0 \mid v' = v^\perp \} \quad \text{and} \quad C_0^r = \{ v \in C_0 \mid v' \neq v^\perp \}.$$  

The $G_R$-orbits on $X$ are

$$S_1' = \{(V_1, V_2) \mid V_2 - \{0\} \subset C_+\},$$

$$S_2' = \{(V_1, V_2) \mid V_2 - \{0\} \subset C_-\},$$

$$S_3' = \{(V_1, V_2) \mid V_1 - \{0\} \subset C_+, \ V_2 \cap C_\neq \phi\},$$

$$S_4' = \{(V_1, V_2) \mid V_1 - \{0\} \subset C_-, \ V_2 \cap C_\neq \phi\},$$

$$S_5' = \{(V_1, V_2) \mid V_1 - \{0\} \subset C_+, \ V_2 \cap C_0^s \neq \{0\}\},$$

$$S_6' = \{(V_1, V_2) \mid V_1 - \{0\} \subset C_-, \ V_2 \cap C_0^s \neq \{0\}\},$$

$$S_7' = \{(V_1, V_2) \mid V_1 - \{0\} \subset C_0^r, \ V_2 \not\subset C_0\},$$

$$S_8' = \{(V_1, V_2) \mid V_1 \subset C_0^s, \ V_2 \cap C_\neq \phi\},$$

$$S_9' = \{(V_1, V_2) \mid V_1 \subset C_0^s, \ V_2 \cap C_- \neq \phi\},$$

$$S_{10}' = \{(V_1, V_2) \mid V_1 - \{0\} \subset C_0^r, \ V_2 \subset C_0\},$$

$$S_{10}'_{{\text{op}}} = \{(V_1, V_2) \mid V_1 \subset C_0^r, \ V_2 \subset C_0\}.$$  

Here the $K_C$-orbit $S_j$ and the $G_R$-orbit $S'_j$ correspond by the duality for each $j = 1, \ldots, 10, \text{op}$.

Take a maximal abelian subspace

$$j = \left\{ Y(a_1, a_2) = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & -a_1 & 0 \\ 0 & 0 & 0 & -a_2 \end{pmatrix} \bigg| a_1, a_2 \in \mathbb{R} \right\}$$

of $im$. Using the linear forms $e_j : Y(a_1, a_2) \mapsto a_j$ for $j = 1, 2$, we can write

$$\Delta = \{ \pm 2 e_1, \pm 2 e_2, \pm e_1 \pm e_2 \} \quad \text{and} \quad \Delta^+_n = \{ 2 e_1, 2 e_2, e_1 + e_2 \}.$$  

Write $\beta_1 = 2 e_1$, $\beta_2 = 2 e_2$ and $\delta = e_1 + e_2$. Take root vectors $X_1 = -E_{13}$ of $g_C(j, \beta_1)$ and $X_2 = -E_{24}$ of $g_C(j, \beta_2)$ where $E_{ij}$ $(i, j = 1, \ldots, 4)$ denotes the matrix units.
Define
\[
t_1(s) = \exp s(X_1 - X_1) = \exp s(E_{31} - E_{13}) = \begin{pmatrix}
\cos s & 0 & -\sin s & 0 \\
0 & 1 & 0 & 0 \\
\sin s & 0 & \cos s & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
and
\[
t_2(s) = \exp s(X_2 - X_2) = \exp s(E_{42} - E_{24}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos s & 0 & -\sin s \\
0 & 0 & 1 & 0 \\
0 & \sin s & 0 & \cos s
\end{pmatrix}
\]
for \( s \in \mathbb{R} \). Then we can write the Akhiezer-Gindikin domain \( D \) as
\[
D = G_{\mathbb{R}} T^+ K_C
\]
where \( T^+ = \{ t_1(s_1)t_2(s_2) \mid |s_1| < \pi/4, |s_2| < \pi/4 \} \). Write \( c_{\beta_j} = t_j(\pi/4) \) and \( w_{\beta_j} = t_j(\pi/2) \) for \( j = 1, 2 \). Then we can write
\[
S_j = K_C g_{\mathbb{R}} B \quad \text{and} \quad S'_j = G_{\mathbb{R}} g_B
\]
for \( j = 1, \ldots, 10, \text{op} \) with the following representatives \( g \) ([Mi], Theorem 2):

<table>
<thead>
<tr>
<th>( j )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>( 6 )</th>
<th>( 7 )</th>
<th>( 8 )</th>
<th>( 9 )</th>
<th>( 10 )</th>
<th>( \text{op} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g )</td>
<td>( e )</td>
<td>( w_{\beta_1}w_{\beta_2} )</td>
<td>( w_{\beta_1} )</td>
<td>( c_{\beta_2} )</td>
<td>( c_{\beta_2}w_{\beta_1} )</td>
<td>( c_{\delta}w_{\beta_2} )</td>
<td>( c_{\beta_1} )</td>
<td>( c_{\beta_1}w_{\beta_2} )</td>
<td>( c_{\delta} )</td>
<td>( c_{\beta_1}c_{\beta_2} )</td>
<td></td>
</tr>
</tbody>
</table>

Here
\[
c_{\delta} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix} = \exp \frac{\pi}{4} (X_\delta - X_\delta)
\]
with \( X_\delta = -(E_{14} + E_{23}) \in g_C(1, \delta) \).

The standard maximal flag manifold \( G_C/Q \) is identified with the space \( Y \) of two dimensional subspaces \( V_+ \) of \( \mathbb{C}^4 \) such that \( \imath u^* v = 0 \) for all \( u, v \in V_+ \) by the map
\[
G_C/Q \ni gQ \mapsto V_+ = gU_+ \in Y.
\]
Similarly we also identify \( G_C/\bar{Q} \) with \( Y \) by the map
\[
G_C/\bar{Q} \ni g\bar{Q} \mapsto V_- = gU_- \in Y.
\]
As in Section 2 the complex symmetric space \( G_C/K_C \) is naturally identified with the open subset
\[
\{(V_+, V_-) \in G_C/Q \times G_C/\bar{Q} \mid V_+ \cap V_- = \{0\}\}
\]
of \( G_C/Q \times G_C/\bar{Q} \cong Y \times \bar{Y} \) by the map
\[
\imath : gK_C \ni (V_+, V_-) = (gU_+, gU_-).
\]
Then the Akhiezer-Gindikin domain \( D/K_C \) is identified with
\[
G_{\mathbb{R}} Q/Q \times G_{\mathbb{R}} \bar{Q}/\bar{Q} = \{(V_+, V_-) \in Y \times \bar{Y} \mid V_+ - \{0\} \subset C_+ \text{ and } V_- - \{0\} \subset C_-\}.
\]
Let \( xK_C \) be an element of \( \partial(D/K_C) \) such that \( \imath(xK_C) \in G_{\mathbb{R}} c_{\beta_1} Q/Q \times G_{\mathbb{R}} \bar{Q}/\bar{Q} \). Then it follows from Lemma 2.3 that
\[
xK_C gB \cap G_{\mathbb{R}} c_{\beta_1} gB \neq \phi
\]
for \( g = e, w_{\beta_2} \) and \( c_{\beta_2} \). This implies that

\[
\begin{align*}
(3.1) & \quad xS_1 \cap S'_8 \neq \phi, \\
(3.2) & \quad xS_3 \cap S'_9 \neq \phi
\end{align*}
\]

and that

\[
(3.3) \quad xS_5 \cap S'_{op} \neq \phi.
\]

Since \( S'_7^{cl} = \{(V_1, V_2) \mid V_1 \subset C_0 \} \supset S'_9 \), it follows from (3.2) that

\[
(3.4) \quad xS_3 \cap S'_7^{cl} \neq \phi.
\]

On the other hand, since \( S'_1^{cl} \supset S'_9^{op} \), it follows from (3.3) that

\[
(3.5) \quad xS_5 \cap S'_1^{cl} \neq \phi.
\]

**Remark 3.1.** (i) If \( \iota(xK_C) \in G_{\mathbb{R}}Q/Q \times G_{\mathbb{R}}c_{\beta_2}Q/Q \), then we can prove

\[
\begin{align*}
xS_2 \cap S'_9 \neq \phi, & \quad xS_4 \cap S'_8 \neq \phi, & \quad xS_6 \cap S'_{op} \neq \phi, \\
xS_4 \cap S'_7^{cl} \neq \phi \quad \text{and} \quad xS_6 \cap S'_1^{cl} \neq \phi
\end{align*}
\]

in the same way.

(ii) If we apply [M4], Theorem 1.3, to this case, then we have

\[
x \in \partial D \Rightarrow x(S_5 \sqcup S_6)^{cl} \cap S'_{op} \neq \phi.
\]

So we see that the results in this paper are refinements of this theorem for Hermitian cases.

By (3.4) and (3.5) we proved the following.

**Proposition 3.2.** If \( \iota(xK_C) \in G_{\mathbb{R}}c_{\beta_1}Q/Q \times G_{\mathbb{R}}c_{\beta_2}Q/Q \), then we have:

(i) \( xK_Cw_{\beta_2}B \cap (G_{\mathbb{R}}c_{\beta_2}B)^{cl} \neq \phi \).

(ii) \( xK_Cc_{\beta_2}B \cap (G_{\mathbb{R}}c_{\beta_2}B)^{cl} \neq \phi \).

**Remark 3.3.** It is clear that \( K_Cw_{\beta_2}B = S_3 \subset S'_7^{cl} = (K_Cc_{\beta_2}B)^{cl} \) and that \( K_Cc_{\beta_2}B = S_5 \subset S'_1^{cl} = (K_Cc_{\beta_4}B)^{cl} \).

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**References**


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