BRAUER ALGEBRAS, SYMPLECTIC SCHUR ALGEBRAS
AND SCHUR-WEYL DUALITY

RICHARD DIPPER, STEPHEN DOTY, AND JUN HU

Abstract. In this paper we prove the Schur-Weyl duality between the symplectic group and the Brauer algebra over an arbitrary infinite field $K$. We show that the natural homomorphism from the Brauer algebra $B_n(-2m)$ to the endomorphism algebra of the tensor space $(K^{2m})^\otimes n$ as a module over the symplectic similitude group $GSp_{2m}(K)$ (or equivalently, as a module over the symplectic group $Sp_{2m}(K)$) is always surjective. Another surjectivity, that of the natural homomorphism from the group algebra for $GSp_{2m}(K)$ to the endomorphism algebra of $(K^{2m})^\otimes n$ as a module over $B_n(-2m)$, is derived as an easy consequence of S. Oehms’s results [S. Oehms, J. Algebra (1) 244 (2001), 19–44].

1. Introduction

Let $K$ be an infinite field. Let $m, n \in \mathbb{N}$. Let $U$ be an $m$-dimensional $K$-vector space. The natural left action of the general linear group $GL(U)$ on $U^\otimes n$ commutes with the right permutation action of the symmetric group $\mathfrak{S}_n$. Let $\varphi, \psi$ be the natural representations

$$
\varphi : (K\mathfrak{S}_n)^{\text{op}} \to \text{End}_K(U^\otimes n), \quad \psi : KGL(U) \to \text{End}_K(U^\otimes n),
$$

respectively. The well-known Schur-Weyl duality (see [Sc], [W], [CC], [CL]) says that

(a) $\varphi(K\mathfrak{S}_n) = \text{End}_{KGL(U)}(U^\otimes n)$, and if $m \geq n$, then $\varphi$ is injective, and hence an isomorphism onto $\text{End}_{KGL(U)}(U^\otimes n)$,

(b) $\psi(KGL(U)) = \text{End}_{K\mathfrak{S}_n}(U^\otimes n)$,

(c) if $\text{char } K = 0$, then there is an irreducible $(KGL(U), (K\mathfrak{S}_n)^{\text{op}})$-bimodule decomposition

$$
U^\otimes n = \bigoplus_{\lambda=(\lambda_1,\lambda_2,\ldots) \vdash n, \ell(\lambda) \leq m} \Delta_\lambda \otimes S^\lambda,
$$

where $\Delta_\lambda$ (resp., $S^\lambda$) denotes the irreducible $KGL(U)$-module (resp., irreducible $K\mathfrak{S}_n$-module) associated to $\lambda$, and $\ell(\lambda)$ denotes the largest integer $i$ such that $\lambda_i \neq 0$.

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Let $\tau$ be the automorphism of $K\mathfrak{S}_n$ which is defined on generators by $\tau(s_i) = -s_i$ for each $1 \leq i \leq n - 1$. Then (by using this automorphism) it is easy to see that the same Schur-Weyl duality still holds if one replaces the right permutation action of $\mathfrak{S}_n$ by the right sign permutation action, i.e.,

$$(v_{i_1} \otimes \cdots \otimes v_{i_n})s_j := -(v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes v_{i_j} \otimes v_{i_{j+1}} \otimes \cdots \otimes v_{i_n}),$$

for any $1 \leq j \leq n - 1$ and any $v_{i_1}, \ldots, v_{i_n} \in U$.

In the case of $K = \mathbb{C}$, there are also Schur-Weyl dualities for other classical groups—symplectic groups and orthogonal groups. In this paper, we shall consider only the symplectic case.\(^1\) Recall that symplectic groups are defined by certain bilinear forms $(\cdot, \cdot)$ on vector spaces. Let $V$ be a $2n$-dimensional $K$-vector space equipped with a nondegenerate skew-symmetric bilinear form $(\cdot, \cdot)$. Then (see [Gri], [Dt, Section 4]) the symplectic similitude group (resp., the symplectic group) relative to $(\cdot, \cdot)$ is

$$GSp(V) := \left\{ g \in GL(V) \mid 0 \neq d \in K, \text{ such that } (gv, gw) = d(v, w), \forall v, w \in V \right\}$$

(resp., $Sp(V) := \left\{ g \in GL(V) \mid (gv, gw) = (v, w), \forall v, w \in V \right\}$).

By restriction from $GL(V)$, we get natural left actions of $GSp(V)$ and $Sp(V)$ on $V^{\otimes n}$. Again we denote by $\psi$ the natural $K$-algebra homomorphism

$$\psi : KGSp(V) \rightarrow \text{End}_K(V^{\otimes n}),$$

$$\psi : KSp(V) \rightarrow \text{End}_K(V^{\otimes n}).$$

Note that if $0 \neq d \in K$ is such that $(gv, gw) = d(v, w)$ for any $v, w \in V$, then $(\sqrt{d^{-1}}g)v, (\sqrt{d^{-1}}g)w = (v, w)$ for any $v, w \in V$. Therefore, if $K$ is large enough such that $\sqrt{d} \in K$ for any $d \in K$, then $g \in GSp(V)$ implies that $(a \text{id}_V)g \in Sp(V)$ for some $0 \neq a \in K$. In that case,

$$\psi(g) = \psi((a^{-1} \text{id}_V)(a \text{id}_V)g) = \psi(a^{-1} \text{id}_V)\psi((a \text{id}_V)g) = (a^{-n} \text{id}_{V^{\otimes n}})\psi((a \text{id}_V)g) = a^{-n}\psi((a \text{id}_V)g).$$

It follows that

$$\psi(KSp(V)) = \psi(KGSp(V))$$

provided $K$ is large enough.

In the setting of Schur-Weyl duality for the symplectic group, the symmetric group $\mathfrak{S}_n$ should be replaced by Brauer algebras (introduced in [B]). Recall that the Brauer algebra $B_n(x)$ over a Noetherian integral domain $R$ (with parameter $x \in R$) is a unital $R$-algebra with generators $s_1, \cdots, s_{n-1}, e_1, \cdots, e_{n-1}$ and relations (see [E]):

$$s_i^2 = 1, \quad e_i^2 = xe_i, \quad e_i s_i = s_i = e_i s_i, \quad \forall 1 \leq i \leq n - 1,$$

$$s_i s_j = s_j s_i, \quad s_i e_j = e_j s_i, \quad s_i e_j = e_j s_i, \quad \forall 1 \leq i < j \leq n - 1,$$

$$s_is_{i+1}s_i = s_i s_{i+1}s_i, \quad e_i s_i s_{i+1}s_i = e_{i+1}s_i, \quad \forall 1 \leq i < n - 1,$$

$$s_i e_{i+1}e_i = s_{i+1}e_i, \quad e_{i+1}e_i s_{i+1}s_i = e_{i+1}s_i, \quad \forall 1 \leq i < n - 1.$$

Note that $B_n(x)$ was originally defined as the linear space with basis the set of all Brauer $n$-diagrams, graphs on $2n$ vertices and $n$ edges with the property that

\(^1\)In this paper, we will use the results by Oehms and also by Donkin on symplectic Schur algebras. To deal with the orthogonal case, one needs analogous results for orthogonal Schur algebras, which are not presently available.
every vertex is incident to precisely one edge. One usually thinks of the vertices as arranged in two rows of \( n \) each, the top and bottom rows. Label the vertices in each row of an \( n \)-diagram by the indices \( 1, 2, \cdots , n \) from left to right. Then \( s_i \) corresponds to the \( n \)-diagram with edges connecting vertices \( i \) (resp., \( i + 1 \)) on the top row with \( i + 1 \) (resp., \( i \)) on the bottom row, and all other edges are vertical, connecting vertex \( k \) on the top and bottom rows for all \( k \neq i, i + 1 \). \( e_i \) corresponds to the \( n \)-diagram with horizontal edges connecting vertices \( i, i + 1 \) on the top and bottom rows, and all other edges are vertical, connecting vertex \( k \) on the top and bottom rows for all \( k \neq i, i + 1 \). The multiplication is given by the linear extension of a product defined on diagrams. For more details, see [B], [GW].

There are right actions of Brauer algebras (with certain parameters) on tensor space. The definition of the actions depends on the choice of an orthogonal basis with respect to the defining bilinear form. Let \( \delta_{ij} \) denote the value of the usual Kronecker delta. For any \( 1 \leq i \leq 2m \), set \( i' := 2m + 1 - i \). We fix an ordered basis \( \{ v_1, v_2, \cdots , v_{2m} \} \) of \( V \) such that

\[
(v_i, v_j) = 0 = (v_i, v_{j'}), \quad (v_i, v_{j'}) = \delta_{ij} = -(v_{j'}, v_i), \quad \forall \ 1 \leq i, j \leq m.
\]

For any \( i, j \in \{ 1, 2, \cdots , 2m \} \), let

\[
\epsilon_{ij} := \begin{cases} 
1 & \text{if } j = i' \text{ and } i < j, \\
-1 & \text{if } j = i' \text{ and } i > j, \\
0 & \text{otherwise.}
\end{cases}
\]

The right action of \( B_n(-2m) \) on \( V^\otimes n \) is defined on generators by

\[
(v_{i_1} \otimes \cdots \otimes v_{i_n}) s_j := -(v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes v_{i_{j+1}} \otimes v_{i_j} \otimes v_{i_{j+2}} \otimes \cdots \otimes v_{i_n}),
\]

\[
(v_{i_1} \otimes \cdots \otimes v_{i_n}) e_j := \epsilon_{i, i_{j+1}} v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes \left( \sum_{k=1}^{m} (v_{k'} \otimes v_k - v_k \otimes v_{k'}) \right) \otimes v_{i_{j+2}} \cdots \otimes v_{i_n}.
\]

Let \( \varphi \) be the natural \( K \)-algebra homomorphism

\[
\varphi : (B_n(-2m))^\text{op} \to \text{End}_K(V^\otimes n).
\]

The following results are well known.

**Theorem 1.2 ([B], [B1], [B2]).** 1) The natural left action of \( GSp(V) \) on \( V^\otimes n \) commutes with the right action of \( B_n(-2m) \). Moreover, if \( K = \mathbb{C} \), then

\[
\varphi(B_n(-2m)) = \text{End}_{\mathbb{C}GSp(V)}(V^\otimes n) = \text{End}_{\mathbb{C}Sp(V)}(V^\otimes n),
\]

\[
\psi(\mathbb{C}Sp(V)) = \psi(\mathbb{C}GSp(V)) = \text{End}_{B_n(-2m)}(V^\otimes n).
\]

2) If \( K = \mathbb{C} \) and \( m \geq n \), then \( \varphi \) is injective, and hence an isomorphism onto \( \text{End}_{\mathbb{C}GSp(V)}(V^\otimes n) \).

3) If \( K = \mathbb{C} \), then there is an irreducible \( \mathbb{C}GSp(V) \), \( (B_n(-2m))^\text{op} \)-bimodule decomposition

\[
V^\otimes n = \bigoplus_{f=0}^{[n/2]} \bigoplus_{\lambda \vdash n-2f, \ell(\lambda) \leq m} \Delta(\lambda) \otimes D(\lambda'),
\]

where \( \Delta(\lambda) \) (resp., \( D(\lambda') \)) denotes the irreducible \( \mathbb{C}GSp(V) \)-module (resp., the irreducible \( B_n(-2m) \)-module) corresponding to \( \lambda \) (resp., corresponding to \( \lambda' \), and \( \lambda' = (\lambda_1', \lambda_2', \cdots) \) denotes the conjugate partition of \( \lambda \).
The aim of this work is to remove the restriction on \( K \) in part 1) and part 2) of the above theorem. We shall see that the following holds for any infinite field \( K \).

**Proposition 1.3.** For any infinite field \( K \), \( \psi(KGSp(V)) = \text{End}_{B_n(-2m)}(V^\otimes n) \).

In fact, this is an easy consequence of [Oe, (6.1), (6.2), (6.3)] and [Dt, (3.2(b))]. The proof is given in Section 2. The main result of this paper is

**Theorem 1.4.** Let \( K \) be an arbitrary infinite field. Then
\[
\varphi(B_n(-2m)) = \text{End}_{KGSp(V)}(V^\otimes n) = \text{End}_{KSp(V)}(V^\otimes n),
\]
and if \( m \geq n \), then \( \varphi \) is also injective, and hence an isomorphism onto
\[
\text{End}_{KGSp(V)}(V^\otimes n).
\]

**Remark 1.5.**
1) Note that when \( m < n \), \( \varphi \) is in general not injective. For example, let \( m = 2, n = 3, U = K^2, G = Sp_4(K) \). Then it is easy to check that the element \( \alpha := (1 + s_1)(1 + s_2 + s_2s_1) + (1 + s_2 + s_1s_2)e_1(1 + s_2 + s_2s_1) \) lies in the kernel of \( \varphi : B_3(-4) \to \text{End}_{Sp_4(K)}(V^\otimes 3) \). In fact, \( \ker(\varphi) = K\alpha \).

2) It would be interesting to know if the quantized versions of Proposition 1.3 and Theorem 1.4 hold (see [BW], [CP] and [M]).

3) Although the paper [CC] should apply in the symplectic case (see [T]), our results provide an alternative approach (one which is rather more detailed).

### 2. The algebra \( A_R^s(m) \)

In this section, we shall show how Proposition 1.3 follows from results of [Oe, (6.1), (6.2), (6.3)] and [Dt, (3.2(b))].

We shall first introduce (following [Oe, Section 6]) a \( \mathbb{Z} \)-graded \( R \)-algebra \( A_R^s(m) \) for any Noetherian integral domain \( R \). Over an algebraically closed field, this algebra is isomorphic to the coordinate algebra of the symplectic monoid, and the dual of its \( n \)-th homogeneous summand is isomorphic to the symplectic Schur algebra introduced by S. Donkin ([Do2]).

Let \( R \) be a Noetherian integral domain. Let \( x_{i,j}, 1 \leq i, j \leq 2m \) be \( 4m^2 \) commuting indeterminates over \( R \). Let \( A_R(2m) \) be the free commutative \( R \)-algebra (i.e., polynomial algebra) in these \( x_{i,j}, 1 \leq i, j \leq 2m \). Let \( I_R \) be the ideal of \( A_R(2m) \) generated by elements of the form

\[
\begin{aligned}
\sum_{k=1}^{2m} \epsilon_k x_{i,k} x_{j,k'}, & \quad 1 \leq i \neq j' \leq 2m; \\
\sum_{k=1}^{2m} \epsilon_k x_{i,k} x_{k',j}, & \quad 1 \leq i \neq j' \leq 2m; \\
\sum_{k=1}^{2m} \epsilon_k (x_{i,k} x_{k',j} - x_{k,j} x_{k',j'}), & \quad 1 \leq i, j \leq m.
\end{aligned}
\]

(2.1)

The \( R \)-algebra \( A_R(2m)/I_R \) will be denoted by \( A_R^s(m) \). Write \( c_{i,j} \) for the canonical image \( x_{i,j} + I_R \) of \( x_{i,j} \) in \( A_R^s(m) \) (1 \leq i, j \leq 2m). Then in \( A_R^s(m) \) we have the relations

\[
\begin{aligned}
\sum_{k=1}^{2m} \epsilon_k c_{i,k} c_{j,k'} & = 0, \quad 1 \leq i \neq j' \leq 2m; \\
\sum_{k=1}^{2m} \epsilon_k c_{i,k} c_{k',j} & = 0, \quad 1 \leq i \neq j' \leq 2m; \\
\sum_{k=1}^{2m} \epsilon_k (c_{i,k} c_{k',j} - c_{k,j} c_{k',j'}) & = 0, \quad 1 \leq i, j \leq m.
\end{aligned}
\]

(2.2)

Note that \( A_R(2m) \) is a graded algebra, \( A_R(2m) = \bigoplus_{n \geq 0} A_R(2m,n) \), where the \( A_R(2m,n) \) is the subspace spanned by the monomials of the form \( x_{i,j} \) for \((\hat{i}, \hat{j}) \in \)
\[ I^2(2m, n) \], where
\[ I(2m, n) := \{ \mathbf{i} = (i_1, \cdots, i_n) \mid 1 \leq i_j \leq 2m, \forall j \}, \]
\[ I^2(2m, n) = I(2m, n) \times I(2m, n), \quad x_{\mathbf{i}, \mathbf{j}} := x_{i_1, j_1} \cdots x_{i_n, j_n}. \]
Since \( I_R \) is a homogeneous ideal, \( A_R^+(m) \) is graded too and \( A_R^+(m) = \bigoplus_{n \geq 0} A_R^+(m, n) \), where \( A_R^+(m, n) \) is the subspace spanned by the monomials of the form \( c_{\mathbf{i}, \mathbf{j}} \) for \( (\mathbf{i}, \mathbf{j}) \in I^2(2m, n) \), where
\[ c_{\mathbf{i}, \mathbf{j}} := c_{i_1, j_1} \cdots c_{i_n, j_n}. \]

By convention, throughout this paper, we identify the symmetric group \( \mathfrak{S}_n \) with the set of maps acting on their arguments on the right. In other words, if \( \sigma \in \mathfrak{S}_n \) and \( a \in \{1, \ldots, n\} \) we write \((a)\sigma\) for the value of \( a \) under \( \sigma \). This convention carries the consequence that, when considering the composition of two symmetric group elements, the leftmost map is the first to act on its argument. For example, we have \((1, 2, 3)(2, 3) = (1, 3)\) in the usual cycle notation.

Note that the symmetric group \( \mathfrak{S}_n \) acts on the right on the set \( I(2m, n) \) by the rule\(^2\)
\[ i\sigma := (i(1)\sigma^{-1}, \cdots, i(n)\sigma^{-1}), \quad \sigma \in \mathfrak{S}_n. \]
It is clear (see [Dt]) that \( A_R^+(m, n) \cong A_R(2m, n)/I_R(n) \), where \( I_R(1) = 0 \), and for \( n \geq 2 \), \( I_R(n) \) is the \( R \)-submodule of \( A_R(2m, n) \) generated by elements of the form
\begin{equation}
(2.3)
\begin{cases}
\sum_{k=1}^{2m} \epsilon_k x_{(i_1, \cdots, i_n), (k, k', k_3, \cdots, k_n)}, \\
\sum_{k=1}^{2m} \epsilon_k x_{(k, k', i_3, \cdots, i_n), (j_1, \cdots, j_n)}, \\
\sum_{k=1}^{2m} \epsilon_k (x_{(i, i', i_3, \cdots, i_n), (k, k', j_3, \cdots, j_n)} - x_{(k, k', i_3, \cdots, i_n), (j_1, j'_3, j_3, \cdots, j_n)}),
\end{cases}
\end{equation}
where \( 1 \leq i, j \leq m, i, j \in I(2m, n) \) such that \( i_1 \neq i'_2, j_1 \neq j'_2 \).

Furthermore, if one defines
\[ \Delta(x_{\mathbf{i}, \mathbf{j}}) = \sum_{k \in I(2m, n)} x_{\mathbf{i}, \mathbf{k}} \otimes x_{\mathbf{k}, \mathbf{j}}, \quad \varepsilon(x_{\mathbf{i}, \mathbf{j}}) = \delta_{\mathbf{i}, \mathbf{j}}, \quad \forall \mathbf{i}, \mathbf{j} \in I(2m, n), \forall n, \]
then the algebra \( A_R(2m) \) becomes a graded bialgebra, and each \( A_R(2m, n) \) is a subcoalgebra of \( A_R(2m) \). Its linear dual \( S_R(2m, n) := \text{Hom}_R(A_R(2m, n), R) \) is the so-called Schur algebra over \( R \) (see [Gr]). Let \( S_R^+(m, n) := \text{Hom}_R(A_R^+(m, n), R) \). By [Oe, Section 6], \( A_R^+(m, n) \) is in fact a quotient coalgebra of \( A_R(2m, n) \); hence \( S_R^+(m, n) \) is a subalgebra of \( S_R(2m, n) \).

We define \( (\mathbf{i}, \mathbf{j}) \sim (\mathbf{u}, \mathbf{v}) \) if there exists some \( \sigma \in \mathfrak{S}_n \) with \( i\sigma = u, j\sigma = v \). Let \( I^2(2m, n) / \sim \) be the set of orbits for the action of \( \mathfrak{S}_n \) on \( I^2(2m, n) \). For each \( (\mathbf{i}, \mathbf{j}) \in I^2(2m, n) / \sim \), we define \( \xi_{\mathbf{i}, \mathbf{j}} \in S_R(2m, n) \) by
\[ \xi_{\mathbf{i}, \mathbf{j}}(x_{\mathbf{u}, \mathbf{v}}) = \begin{cases} 1, & \text{if } (\mathbf{i}, \mathbf{j}) \sim (\mathbf{u}, \mathbf{v}), \\
0, & \text{otherwise}, \end{cases} \quad \forall (\mathbf{u}, \mathbf{v}) \in I^2(2m, n) / \sim. \]

The set \( \{ \xi_{\mathbf{i}, \mathbf{j}} \mid (\mathbf{i}, \mathbf{j}) \in I^2(2m, n) / \sim \} \) forms an \( R \)-basis of \( S_R(2m, n) \). The natural action of \( S_R(2m, n) \) on \( V^\otimes n \) is given as follows:
\[ \xi_{\mathbf{i}, \mathbf{j}} : V^\otimes n \rightarrow V^\otimes n \]
\[ v_{\mathbf{a}} := v_{a_1} \otimes \cdots \otimes v_{a_n} \mapsto \sum_{\mathbf{b} \in I(2m, n), (\mathbf{a}, \mathbf{b}) \sim (\mathbf{i}, \mathbf{j})} v_{\mathbf{b}}. \quad \forall \mathbf{a} := (a_1, \cdots, a_n) \in I(2m, n). \]

\(^2\)This action is the so-called right place permutation action.
Let \( \xi = \sum_{(k,j) \in I^2(2m,n)} a_{k,j} \xi_{k,j} \in S_R(2m,n) \). By (2.3), it is easy to see that \( \xi \in S_R^+(m,n) \) if and only if

\[
(2.4) \quad \begin{align*}
\sum_{k=1}^{2m} \xi_k a_{(i_1,\ldots,i_n),(k,k',\ldots,k_n)} &= 0, \\
\sum_{k=1}^{2m} \xi_k a_{(k,k',i_3,\ldots,i_n),(j_1,\ldots,j_n)} &= 0, \\
\sum_{k=1}^{2m} \xi_k a_{(i_1,i_2,\ldots,i_n),(k,k',j_3,\ldots,j_n)} - a_{(k,k',i_3,\ldots,i_n),(j_1,\ldots,j_n)} &= 0,
\end{align*}
\]

where \( 1 \leq i, j \leq m \), \( i \neq j \), \( i \neq j' \).

Let \( R = K \) be an infinite field. Recall the ordered basis \( \{ v_1, v_2, \ldots, v_{2m} \} \) of \( V \). Let \( \langle , \rangle \) be the unique (nondegenerate) skew-symmetric bilinear form on \( V \) such that

\[
(v_i, v_j) = 0 = (v_i', v_j'), \quad (v_i, v_j') = -\delta_{ij} = -(v_j', v_i), \quad \forall 1 \leq i, j \leq m.
\]

This form is given (relative to the above ordered basis) by the block matrix

\[
J := \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix},
\]

where \( J_m \) is the unique anti-diagonal \( m \times m \) permutation matrix. With respect to the above ordered basis of \( V \), the group \( GSp(V) \) may be identified with the group \( GSp_{2m}(K) \) given by

\[
GSp_{2m}(K) := \left\{ A \in GL_{2m}(K) \mid \exists 0 \neq d(A) \in K, \text{ such that } A^T J A = d(A) J \right\}.
\]

Let \( M_{2m}(K) \) denote the affine algebraic monoid of \( n \times n \) matrices over \( K \). With respect to the above basis of \( V \), the symplectic monoid \( SpM(V) \), which by definition consists of the linear endomorphisms of \( V \) preserving the bilinear form up to any scalar (see [Dt, Section 4.2]), may be identified with

\[
SpM_{2m}(K) := \left\{ A \in M_{2m}(K) \mid \exists d(A) \in K, \text{ such that } A^T J A = d(A) J \right\}.
\]

Let \( \overline{K} \) be the algebraic closure of \( K \). The coordinate algebra \( \overline{K}[M_{2m}(\overline{K})] \) is isomorphic to \( A_{\overline{K}}(2m) := A_K(2m) \otimes \overline{K} \). The coordinate algebra of \( GL_{2m}(\overline{K}) \) is isomorphic to \( \overline{K}[\text{det}^{-1}(x_{i,j})_{m \times 2m}; x_{i,j}]_{1 \leq i,j \leq 2m} \). The embedding \( GSp_{2m}(K) \hookrightarrow GL_{2m}(\overline{K}) \) induces a surjective map \( \overline{K}[GL_{2m}(\overline{K})] \twoheadrightarrow \overline{K}[GSp_{2m}(\overline{K})] \). Denote by \( A_{\overline{K}}^{sy}(m,n) \) (resp., \( A_{\overline{K}}^{sp}(m,n) \)) the image of \( A_{\overline{K}}(2m) \) (resp., of \( A_{\overline{K}}(2m, n) \)) under this map. Then, by [Do2],

1. \( A_{\overline{K}}^{sy}(2m) \) is isomorphic to the coordinate algebra of \( SpM_{2m}(\overline{K}) \),
2. \( A_{\overline{K}}^{sy}(2m) = \bigoplus_{0 \leq n \in \mathbb{Z}} A_{\overline{K}}^{sy}(m,n) \), and the dimension of \( A_{\overline{K}}^{sy}(m,n) \) is independent of the field \( K \),
3. the linear dual of \( A_{\overline{K}}^{sy}(m,n) \), say, \( S_{\overline{K}}^{sy}(m,n) \) is a generalized Schur algebra in the sense of [Do1].

The algebra \( S_{\overline{K}}^{sy}(m,n) \) is called by S. Donkin the \emph{symplectic Schur algebra}.

We define \( A_{\overline{K}}^{sy}(m,n) \) (resp., \( A_{\overline{K}}^{sy}(m,n) \)) to be the image of \( A_K(2m) \) (resp., of \( A_K(2m, n) \)) under the surjective map \( \overline{K}[GL_{2m}(\overline{K})] \twoheadrightarrow \overline{K}[GSp_{2m}(\overline{K})] \). It is clear that

\[
A_{\overline{K}}^{sy}(m,n) \otimes \overline{K} = A_{\overline{K}}^{sy}(m,n), \quad A_{\overline{K}}^{sy}(m,n) \otimes \overline{K} = A_{\overline{K}}^{sy}(m,n),
\]

and hence \( A_{\overline{K}}^{sy}(2m) = \bigoplus_{0 \leq n \in \mathbb{Z}} A_{\overline{K}}^{sy}(m,n) \).

On the other hand, by definition of \( SpM_{2m}(K) \), it is easy to check that the defining relations (2.1) vanish on every matrix in \( SpM_{2m}(K) \). It follows that there is an epimorphism of graded bialgebras from \( A_{\overline{K}}^{sy}(m,n) \) onto \( A_{\overline{K}}^{sy}(m,n) \). Note that for each
0 ≤ n ∈ ℤ, the dimensions of both $A^s_K(m, n)$ (see [Oe, (6.1)]) and $A^{sy}_K(m, n)$ are independent of the field $K$. By [Dt, (9.5)], $A^s_K(m, n) \cong A^{sy}_K(m, n)$. So the two coalgebras always have the same dimension. It follows that $A^s_K(m, n) \cong A^{sy}_K(m, n)$ and $A^s_K(m) \cong A^{sy}_K(m)$. In particular, we have that $S^s_K(m, n) \cong S^{sy}_K(m, n)$. Therefore we have

**Theorem 2.5** ([Oe, (6.2)]). For any infinite field $K$, there is an isomorphism of graded bialgebras from $A^s_K(m)$ onto $A^{sy}_K(m)$. In particular, $A^s_K(m, n) \cong A^{sy}_K(m, n)$ and $S^s_K(m, n) \cong S^{sy}_K(m, n)$ for each $n \in \mathbb{N}$.

As a $\mathbb{Z}$-submodule of $\text{End}_\mathbb{Z}(V^{\otimes n})$, the algebra $\text{End}_{B_n(-2m)}(V^\otimes n)$ is a free $\mathbb{Z}$-module of finite rank. Oehms proved in [Oe, (6.3)] that the symplectic Schur algebra $S^s(m, n)$ is isomorphic to the centralizer algebra $\text{End}_{B_n(-2m)}(V^\otimes n)$ over any Noetherian integral domain. The following two results follow directly from the construction of his isomorphism.

**Theorem 2.6** ([Oe, (6.3)]). For any field $K$, under the natural homomorphism $S_K(2m, n) \to \text{End}(V^\otimes n)$, the subalgebra $S^s_K(m, n)$ is mapped isomorphically onto the subalgebra $\text{End}_{B_n(-2m)}(V^\otimes n)$.

**Corollary 2.7** ([Oe, (6.3)]). For any field $K$, the map which sends $f \otimes a$ to $af$ naturally extends to a $K$-algebra isomorphism

$$\text{End}_{B_n(-2m)}(V^\otimes n) \otimes_\mathbb{Z} K \cong \text{End}_{B_n(-2m)}(V^\otimes n).$$

Now we can prove Proposition 1.3. By Theorem 2.6 and the canonical isomorphism $S^{sy}_K(m, n) \cong S^s_K(m, n)$ from Theorem 2.5, we know that the natural homomorphism from $S^{sy}_K(m, n)$ to $\text{End}(V^\otimes n)$ maps $S^{sy}_K(m, n)$ isomorphically onto $\text{End}_{B_n(-2m)}(V^\otimes n)$. Now the second author showed in [Dt, (3.2(b))] that the images of $KGSp(V)$ and of $S^s_K(m, n)$ (which is denoted by $S_d(G)$ in [Dt, (3.2(b))]) in $\text{End}(V^\otimes n)$ are the same when $K$ is infinite. It follows that for any infinite field $K$,

$$\psi(KGSp(V)) = \text{End}_{B_n(-2m)}(V^\otimes n).$$

Note that this is also equivalent to the fact that the natural evaluation map

$$\text{(2.8) } KGSp(V) \to S^{sy}_K(m, n) \cong S^s_K(m, n)$$

is surjective. This completes the proof of Proposition 1.3.

3. The action of $B_n(-2m)$ on $V^\otimes n$ for $m ≥ n$

In this section, we shall give the proof of Theorem 1.4 in the case where $m ≥ n$. Let $R$ be a Noetherian integral domain with $q ∈ R$ a fixed invertible element. It is well known that the Hecke algebra $H_{R,q}(S_n)$ associated with the symmetric group $S_n$, and hence the group algebra of the symmetric group $S_n$ itself, are cellular algebras. An important cellular basis of $H_{R,q}(S_n)$ is the Murphy basis, introduced in [Mu]. Another cellular basis is the Kazhdan-Lusztig basis [KL]. The latter one was extended by Graham and Lehrer to a cellular basis of the Brauer algebra. Xi extended this in [Xi] to the Birman-Murakami-Wenzl algebra, a quantization of the Brauer algebra; this algebra is also cellular. It is known that ([GL], [Xi], [E])

3Note that though Oehms assumed in [Oe, (6.2)] that $K$ is an algebraically closed field, the validity over an arbitrary infinite field is an immediate consequence (as shown in our previous discussion).
any cellular basis of the Hecke algebra $\mathcal{H}_{R,q}(S_k)$ ($k \in \mathbb{N}$) can be extended to a cellular basis of the Birman-Murakami-Wenzl algebra. We shall follow Enyang’s formulation in [E], which describes the basis explicitly in terms of the generators. We will use the Murphy basis of $RS_k$ ($k \in \mathbb{N}$), extended to a cellular basis of $B_n(-2m)$. We now describe this basis.

For a composition $\lambda = (\lambda_1, \cdots, \lambda_s)$ of $k$ (i.e., $\lambda_i \in \mathbb{Z}_{\geq 0}$, $\sum \lambda_i = k$), let

$$G_\lambda = G_{\{1, \cdots, \lambda_1\}} \times G_{\{\lambda_1+1, \cdots, \lambda_1+\lambda_2\}} \times \cdots$$

be the corresponding Young subgroup of $G_k$, and set $x_\lambda = \sum_{w \in G_\lambda} w \in R\mathbb{C}$, where $\mathbb{C}$ is a row standard tableau with $\lambda_i$ many nodes in row $i$. A $\lambda$-tableau $t$ is such a diagram in which the nodes are replaced by the numbers $1, \cdots, k$, in some order. The initial $\lambda$-tableau $t^\lambda$ is the one obtained by filling in the numbers $1, \cdots, k$ in order along successive rows. For example,

$$
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 \\
\end{array}
$$

is the initial $(3,2)$-tableau. The symmetric group $G_k$ acts naturally on the set of $\lambda$-tableaux (on the right), and for any $\lambda$-tableau $t$ we define $d(t)$ to be the unique element of $G_k$ with $t^\lambda d(t) = t$. A $\lambda$-tableau $t$ is called row standard if the numbers increase along rows. If $\lambda_1 \geq \cdots \geq \lambda_s$, i.e., $\lambda$ is a partition of $k$, then $t$ is called column standard if the numbers increase down columns, and standard if it is both row and column standard. The set $D_\lambda = \{ d(t) \mid t \text{ is a row standard } \lambda\text{-tableau} \}$ is a set of right coset representatives of $G_\lambda$ in $G_k$; its elements are known as distinguished coset representatives. For any standard $\lambda$-tableaux $s, t$, we define $m_{st} = d(s)^{-1}x_\lambda d(t)$. Murphy [Mu] showed

**Theorem 3.1** ([Mu]). $\{ m_{st} \mid \lambda \vdash k, s, t \text{ are standard } \lambda\text{-tableaux} \}$ is a cellular basis of $R\mathbb{C}$ for any Noetherian integral domain $R$.

To describe Enyang’s cellular basis of the Brauer algebra $B_n(x)$, we need some more notation. First we fix certain bipartitions of $n$, namely $\nu = \nu_f := ((2^f), (n-2f))$, where $(2^f):=(2,2,\cdots,2)$ and $(n-2f)$ are considered as partitions of

$2f$ and $n-2f$ respectively, and $0 \leq f \leq [n/2]$. Here $[n/2]$ is the largest non-negative integer not greater than $n/2$. In general, a bipartition of $n$ is a pair $(\lambda^{(1)}, \lambda^{(2)})$ of partitions of numbers $n_1$ and $n_2$ with $n_1 + n_2 = n$. The notions of Young diagram, bitableaux, etc., carry over easily. Let $t^\nu$ be the standard $\nu$-bitableau in which the numbers $1, 2, \cdots, n$ appear in order along successive rows of the first component tableau, and then in order along successive rows of the second component tableau. We define

$$D_\nu := \left\{ d \in \mathfrak{S}_n \left| \begin{array}{c}
(t^{(1)}, t^{(2)}) = t^\nu d \text{is row standard and the first column of } t^{(1)} \text{ is an increasing sequence when read from top to bottom} \\
\end{array} \right. \right\}.$$ 

For each partition $\lambda$ of $n-2f$, we denote by Std($\lambda$) the set of all the standard $\lambda$-tableaux with entries in $\{2f+1, \cdots, n\}$. The initial tableau $t^\lambda$ in this case has the numbers $2f+1, \cdots, n$ in order along successive rows. Again, for each $t \in \text{Std}(\lambda)$, let $d(t)$ be the unique element in $\mathfrak{S}_{\{2f+1, \cdots, n\}} \subseteq \mathfrak{S}_n$ with $t^\lambda d(t) = t$.

For each integer $f$ with $0 \leq f \leq [n/2]$, we denote by $B(f)$ the two-sided ideal of $B_n(-2m)$ generated by $e_1e_2\cdots e_{2f-1}$. Note that $B(f)$ is spanned by all Brauer
diagrams with at least $2f$ horizontal edges ($f$ edges in each of the top and the bottom rows in the diagram).

Let $f$ be an integer with $0 \leq f \leq [n/2]$. Let $\sigma \in \mathcal{S}_{2f+1,\ldots,n}$ and $d_1, d_2 \in D_\nu$, where again $\nu$ is the bipartition $((2^f), (n - 2f))$ of $n$. Then $d_1^{-1}e_1 e_3 \cdots e_{2f-1} \sigma d_2$ corresponds to the Brauer diagram where the top horizontal edges connect $(2i-1)d_1$ and $(2i)d_1$, the bottom horizontal edges connect $(2i-1)d_2$ and $(2i)d_2$, for $i = 1, 2, \ldots, f$, and the vertical edges are determined by $d_1^{-1}\sigma d_2$. By [Xi, (3.5)], every Brauer diagram $d$ can be written in this way.

**Theorem 3.2** ([E]). Let $R$ be a Noetherian integral domain with $x \in R$. Let $B_n(x)_R$ be the Brauer algebra with parameter $x$ over $R$. Then the set

$$\left\{ d_1^{-1}e_1 e_3 \cdots e_{2f-1} m s d_2 \left| \begin{array}{c} 0 \leq f \leq [n/2], \lambda \vdash n - 2f, s, t \in \text{Std}(\lambda), \\ d_1, d_2 \in D_\nu, \text{ where } \nu := ((2^f), (n - 2f)) \end{array} \right. \right\}$$

is a cellular basis of the Brauer algebra $B_n(x)_R$.

As a consequence, by combining Theorems 3.1 and 3.2, we get that

**Corollary 3.3.** With the above notation, the set

$$\left\{ d_1^{-1}e_1 e_3 \cdots e_{2f-1} \sigma d_2 \left| \begin{array}{c} 0 \leq f \leq [n/2], \sigma \in \mathcal{S}_{2f+1,\ldots,n}, d_1, d_2 \in D_\nu, \\ \text{where } \nu := ((2^f), (n - 2f)) \end{array} \right. \right\}$$

is a basis of the Brauer algebra $B_n(x)_R$, which coincides with the natural basis given by Brauer n-diagrams.

We now specialize $R$ to be a field $K$, assume $m \geq n$, $V = K^n$ and consider the special Brauer algebra $B_n(-2m) = B_n(-2m \cdot 1_K)_K$. As pointed out in Section 1, this algebra acts on the tensor space $V^{\otimes n}$, centralizing the action of the symplectic similitude group $GSp(V)$ and hence that of the symplectic group $Sp(V)$ as well.

The proof of the next result will be given at the end of the section, after a series of preparatory lemmas.

**Theorem 3.4.** Let $K$ be a field. If $m \geq n$, then the natural homomorphism $\varphi : B_n(-2m) \to \text{End}_K(V^{\otimes n})$ is injective; if furthermore $K$ is infinite, then it is in fact an isomorphism onto $\text{End}_{KSp(V)}(V^{\otimes n})$.

Suppose that $m \geq n$. Our first goal here is to show that the action of $B_n(-2m)$ on $V^{\otimes n}$ is faithful; that is, the annihilator $\text{ann}_{B_n(-2m)}(V^{\otimes n})$ is $(0)$. Note that

$$\text{ann}_{B_n(-2m)}(V^{\otimes n}) = \bigcap_{v \in V^{\otimes n}} \text{ann}_{B_n(-2m)}(v).$$

Thus it is enough to calculate $\text{ann}_{B_n(-2m)}(v)$ for some set of chosen vectors $v \in V^{\otimes n}$ such that the intersection of annihilators is $(0)$. We write

$$\text{ann}(v) = \text{ann}_{B_n(-2m)}(v) := \{ x \in B_n(-2m) \mid vx = 0 \}.$$

Recall that $(v_1, \ldots, v_{2m})$ denotes an ordered basis of $V$, and $I(2m, n)$ denotes the set of multi-indices $\underline{i} := (i_1, \ldots, i_n)$ with $i_j \in \{1, \ldots, 2m\}$ for $j = 1, \ldots, n$. We write $v_{\underline{i}} = \prod_{j=1}^n v_{i_j}$ for $\underline{i} := (i_1, \ldots, i_n) \in I(2m, n)$. Thus $\{ v_{\underline{i}} \mid \underline{i} \in I(2m, n) \}$ is a $K$-basis of $V^{\otimes n}$. Consider the action of the symmetric group $S_n$ on $I(2m, n)$ given by

$$\pi := (i_1, \pi^{-1}, \ldots, i_n, \pi^{-1})$$

for $\underline{i} := (i_1, \ldots, i_n) \in I(2m, n)$ and $\pi \in S_n$. Thus, in particular, by definition, $v_{\underline{i}} \pi = (-1)^{\ell(\pi)} v_{\underline{i}}$. For $\underline{i} \in I(2m, n)$, an ordered pair $(s, t)$ ($1 \leq s < t \leq n$) is called a symplectic pair in $\underline{i}$ if $i_s = i_t'$. Two ordered pairs
(s, t) and (u, v) are called disjoint if \{s, t\} \cap \{u, v\} = \emptyset. We define the symplectic length \(\ell_s(v_i)\) to be the maximal number of disjoint symplectic pairs \(s, t\) in \(i\). For \(\sigma, \pi \in \mathcal{S}_n\) and \(1 \leq j \leq n - 1\), it is easy to see that \(v_j \sigma e_j \pi\) is zero or a linear combination of tensors \(v_i\) with \(\ell_s(v_i) = \ell_s(v_i')\). Moreover, for \(f > \ell_s(v_i)\) we have \(B^{(f)} \subseteq \text{ann}(v_i)\). Note that \(\pi \mapsto (-1)^{\ell(\pi)}\pi\) for \(\pi \in \mathcal{S}_n\) defines an automorphism \(\tau\) of the group algebra \(K\mathcal{S}_n\), and that our action of \(\mathcal{S}_n\) on tensor space is precisely the standard place-permutation action (see Section 2) twisted by this automorphism. In particular, this shows that \(K\mathcal{S}_n\) acts faithfully on \(V^{\otimes n}\) for \(m \geq n\). Moreover, for \(\pi \in \mathcal{S}_n\) and \(i \in I(2m, n)\), \(\text{ann}(v_i\pi) = \text{ann}(v_i)\pi = \pi^{-1}\text{ann}(v_i)\).

Now suppose again that \(m \geq n\). We shall prove by induction on \(f\) that \(B^{(f)} \supseteq \text{ann}_{B_n(-2m)}(V^{\otimes n})\) for all \(f\). Since \(B^{(f)} = 0\) for \(f > [n/2]\), this shows the main result of this section, that is, \(B_n(-2m)\) acts faithfully on \(V^{\otimes n}\) if \(m \geq n\). The start of the induction is the following.

**Lemma 3.5.** \(\text{ann}_{B_n(-2m)}(V^{\otimes n}) \subseteq B^{(1)}\).

**Proof.** Since \(m \geq n\), the tensor \(v := v_1 \otimes v_2 \otimes \cdots \otimes v_n\) is defined. Then \(v\pi = (-1)^{\ell(\pi)}v(1)\pi^{-1} \otimes \cdots \otimes v(n)\pi^{-1}\) for \(\pi \in \mathcal{S}_n\). Now \(B^{(1)}\) is contained in the annihilator of \(v\pi\), hence is contained in the intersection of all annihilators of \(v\pi\), as \(\pi\) ranges over \(\mathcal{S}_n\). Hence \(B^{(1)}\) annihilates the subspace \(S\) spanned by the \(v\pi\), where \(\pi\) runs through \(\mathcal{S}_n\). Then the subspace \(S\) becomes a \(B_n(-2m)\)-submodule of tensor space (since \(B^{(1)}\) acts as zero).

On the other hand, since \(S\), as a module for the symmetric group part, which is isomorphic with \(B_n(-2m)\) modulo the ideal \(B^{(1)}\), is faithful, it follows that the annihilator of \(S\) must be in \(B^{(1)}\). Hence \(\text{ann}_{B_n(-2m)}(V^{\otimes n}) \subseteq B^{(1)}\). \(\Box\)

Suppose that we have already shown \(\text{ann}_{B_n(-2m)}(V^{\otimes n}) \subseteq B^{(f)}\) for some natural number \(f \geq 1\). We want to show \(\text{ann}_{B_n(-2m)}(V^{\otimes n}) \subseteq B^{(f+1)}\). Suppose \(f > [n/2]\), we are done already. Thus we may assume \(f \leq [n/2]\).

For \(i := (i_1, \cdots, i_n) \in I(2m, n)\), we define the weight \(\lambda(v_i) = \lambda\) to be the composition \(\lambda = (\lambda_1, \cdots, \lambda_{2m})\) of \(n\) into \(2m\) parts, where \(\lambda_j\) is the number of times \(v_j\) occurs as a tensor factor in \(v_i\), \(j = 1, \cdots, 2m\). Note that the tensors of weight \(\lambda\) for a given composition \(\lambda\) of \(n\) span a \(K\mathcal{S}_n\)-submodule \(M^\lambda\) of \(V^{\otimes n}\); thus

\[
V^{\otimes n} = \bigoplus_{\lambda \in \Lambda(2m, n)} M^\lambda
\]

as a \(K\mathcal{S}_n\)-module, where \(\Lambda(2m, n)\) denotes the set of compositions of \(n\) into \(2m\) parts. It is well known that \(M^\lambda\) is isomorphic to the sign permutation representation of \(\mathcal{S}_n\) on the cosets of the Young subgroup \(\mathcal{S}_\lambda\) of \(\mathcal{S}_n\).

As a consequence, each element \(v \in V^{\otimes n}\) can be written as a sum

\[
v = \sum_{\lambda \in \Lambda(2m, n)} v_\lambda
\]

for uniquely determined \(v_\lambda \in M^\lambda\).

Fix an index \(i \in I(2m, 2f)\) of the form \((i_1, i'_1, i_2, i'_2, \cdots, i_f, i'_f)\) with \(1 \leq i_s \leq 2m\) for \(1 \leq s \leq f\); for example, \(i = (1, 1', 2, 2', \cdots, f, f')\). Since \(e_1 e_3 \cdots e_{2f-1}\) acts only on the first \(2f\) parts of any simple tensor \(v_i\), \(i \in I(2m, n)\), we may consider these operators as acting on \(V^{\otimes 2f}\).

Let \(v = v_i := ((2f), (n - 2f))\). Consider the subgroup \(\Pi\) of \(\mathcal{S}_{1, \ldots, 2f} \leq \mathcal{S}_n\) permuting the rows of \(t^{(i)}\) but keeping the entries in the rows fixed. Obviously, \(\Pi\)
normalizes the stabilizer $\mathfrak{S}_{(2')}$ of $t^v(1)$ in $\mathfrak{S}_{2f}$, where $\mathfrak{S}_{2f} := \mathfrak{S}_{\{1,2,\ldots,2f\}}$. In fact, it is well known that the semi-direct product $\Psi := \mathfrak{S}_{(2')} \times \Pi$ is the normalizer of $\mathfrak{S}_{(2')}$. If

Let $\lambda^{(1)} \in \Lambda(2m,2f)$ be the weight of $v_\mathfrak{S}$ with $\mathfrak{S} = (f+1, (f+1)', \ldots, 2f, (2f)') \in I(2m,2f)$. Note, if $j = (j_1, \ldots, j_{n-2f}) \in I(2m, n-2f)$ satisfies $2f + 1 \leq j_s \leq m$ for $s = 1, \ldots, n - 2f$, and if $\lambda^{(2)} \in \Lambda((2m, n-2f))$ denotes the weight of $v_\mathfrak{S}$ in $V^\otimes n - 2f$, then we obtain the weight $\lambda \in \Lambda(2m, n)$ of $v_\mathfrak{S} \otimes v_\mathfrak{S}$ by adding $\lambda^{(1)}$ to $\lambda^{(2)}$ componentwise. Note that $\{ s \mid \lambda^s_1 \neq 0 \} \cap \{ s \mid \lambda^s_2 \neq 0 \} = \emptyset$. We write for this weight $\lambda = \lambda^{(1)} \otimes \lambda^{(2)}$. We define $E_\lambda \in B_{2n}(-2m)$ to be $e_1 e_3 \cdots e_{2f-1}$.

**Lemma 3.6.** The weight component of $v_\mathfrak{S} e_1 e_3 \cdots e_{2f-1}$ to weight $\lambda^{(1)}$ is

$$
(v_\mathfrak{S} E_{\lambda^{(1)}}) = (-1)^f \sum_{\psi \in \Psi} v_\mathfrak{S}^{(1)} \psi = (-1)^f \sum_{\psi \in \Psi} (-1)^{f(\psi)} v_\mathfrak{S}^{(1)} \psi.
$$

**Proof.** By definition,

$$
v_\mathfrak{S} E_{\lambda^{(1)}} = \left( \sum_{j=1}^m (v_{j'} \otimes v_j - v_j \otimes v_{j'}) \right)^\otimes = (-1)^f \left( \sum_{j=1}^m (v_j \otimes v_{j'} - v_{j'} \otimes v_j) \right)^\otimes.
$$

To obtain the components in the weight space $(V^\otimes 2f)^{\lambda^{(1)}}$, we have to consider all occurring simple tensors which are obtained from $v_\mathfrak{S} = w_1 \otimes \cdots \otimes w_f$ with $w_i = v_{f+i} \otimes v_{f+i}'$ by first permuting the tensors $w_i$, which is done by a permutation $\pi \in \Pi$, and then replacing (for some $i \in \{1, \ldots, f\}$) $w_i$ by $w_{\pi'} = v_{(f+i)'} \otimes v_{f+i}$, which amounts to applying a permutation $\sigma \in \mathfrak{S}_{(2f)}$. On the other hand, each such tensor occurs exactly once, and the sign $(-1)^{\psi(\lambda^{(1)})}$ is calculated taking in account that if we factor out $(-1)^f$, the $w_i$ carry a positive sign and the $w_{\pi'}$ a negative sign, the elements of $\Pi$ all have even length and the action of $\mathfrak{S}_n$ on $V^\otimes n$ considered here carries a sign as well. This proves the lemma. $\Box$

Recall $\nu = v_\nu := ((2f), (n - 2f)) = (\nu^{(1)}, \nu^{(2)})$ and the definition of the set $\mathcal{D}_\nu$ in the beginning of this section. We set $\mathcal{D}_f = \mathcal{D}_\nu \cap \mathfrak{S}_\mu$, where $\mu = ((2f), (n - 2f)) \in \Lambda(2, n)$. Since $t^\nu(2) d$ is row standard for any $d \in \mathcal{D}_\nu$, thus $\mathcal{D}_f$ consists of all $d \in \mathcal{D}_\nu$ which fix every element in the set $\{2f + 1, \ldots, n\}$. That is, $\mathcal{D}_f = \mathcal{D}_\nu \cap \mathfrak{S}_{2f}$.

**Lemma 3.7.** We have the equality

$$
\mathfrak{S}_{2f} = \bigcup_{d \in \mathcal{D}_f} \Psi d,
$$

where "\bigcup" means a disjoint union.

**Proof.** Let $t = t^v(1) w$, where $w \in \mathfrak{S}_{2f}$, be a $\nu^{(1)}$-tableau. Then $w^{-1} \mathfrak{S}_{(2')} w$ is its row stabilizer and $w^{-1} \Pi w$ is the subgroup of $\mathfrak{S}_{2f}$ permuting the rows of $t$. We therefore find a $\rho \in \mathfrak{S}_{(2')}$ such that $t w^{-1} \rho w$ is row standard, and then a $\pi \in \Pi$ such that $t w^{-1} \rho w^{-1} \pi w = t^\nu(1) \rho \pi w$ is row standard and has increasing first column. Thus $t^\nu(1) \rho \pi w = t^\nu(1) d$ for some $d \in \mathcal{D}_\nu \cap \mathfrak{S}_{2f} = \mathcal{D}_f$. Thus we have shown $\psi = d$ with $\psi = \rho \pi \in \Psi$, and hence $w \in \Psi d$. To show that the union is disjoint, let $d_1, d_2 \in \mathcal{D}_f$ and suppose $d_1 = \psi d_2$ for some $\psi \in \Psi$. Consider $t_i = t^v(1) d_i$, $i = 1, 2$. We see from $d_1 = \psi d_2$ that $t_1$ and $t_2$ have the same numbers in their rows, in fact up to
a permutation the same rows, since they are row standard. But the first column has to be increasing, by definition of $D_{\nu_f^t}$; hence the orders of the rows in $t_1$ and $t_2$ have to be the same as well. This proves $d_1 = d_2$ and the union is disjoint. \qed

We now turn to the full set $D_{\nu_f^t}$. Fix $d \in D_{\nu_f^t}$ and let $t = (t^{(1)}, t^{(2)})$ be the corresponding $\nu_f^t$-bitableau. Since $t^{(2)}$ consists of a single row with increasing entries, it is completely determined by those entries. On the other hand, taking an arbitrary set partition $\{1, \cdots, n\} = \{i_1, \cdots, i_{2f}\} \sqcup \{i_{2f+1}, \cdots, i_{2n}\}$, and inserting the entries of the first set in increasing order along successive rows in $t^{(1)}$, and the numbers in the second set in increasing order into $t^{(2)}$, we obtain a $\nu_f^t$-bitableau $t = (t^{(1)}, t^{(2)})$ such that obviously $d(t) \in D_{\nu_f^t}$. Thus we may index those elements of $D_{\nu_f^t}$ by the set $\mathcal{P}_f$ of subsets of $\{1, \cdots, n\}$ of size $2f$. Writing $d_J$ for $J \in \mathcal{P}_f$, for an arbitrary $d \in D_{\nu_f^t}$ with $t^{(r)}d = t = (t^{(1)}, t^{(2)})$, the subset $J$ of $\{1, \cdots, n\}$ of entries of $t^{(1)}$ is an element of $\mathcal{P}_f$, and one sees by direct inspection that there is an element $d_1 \in D_f = D_{\nu_f^t} \cap \mathcal{S}_{2f}$ such that $t = t^{(r)}d_{1}d_{1}d_J = d_1d_J$. That is, $d = d_{1}d_J$. Note also that each element $d_J$ is a distinguished right coset representative of $\mathcal{S}_{(2f,n-2f)}$ in $\mathcal{S}_{n}$. Thus we have shown

**Lemma 3.8.**

$$D_{\nu_f^t} = \bigcup_{J \in \mathcal{P}_f} D_f d_J.$$ 

We define $I_f$ to be the set of multi-indices $(i_{2f+1}, \cdots, i_m)$ of length $n-2f$ with $2f+1 \leq i_\rho \leq m$ for $\rho = 2f+1, \cdots, n$, where we choose the position index $\rho$ to run from $2f+1$ to $n$ in order to keep the notation straight, when we act by an element of $\mathcal{S}_n$. Note that for $2f+1 \leq i \leq m$, we have $i' > m$; hence $\ell_s(v_k) = 0$ for all $k \in I_f$.

For an arbitrary element $v \in V^{\otimes n}$, we say the simple tensor $v_k = v_{i_1} \otimes \cdots \otimes v_{i_m}$ is involved in $v$ if $v_k$ has a nonzero coefficient in writing $v$ as a linear combination $\sum_{J \in I(2m,n)} k_J v_J$ of the basis $\{v_J \mid J \in I(2m,n)\}$ of $V^{\otimes n}$.

**Lemma 3.9.** Let $k \in I_f$, $v = v_k \otimes v_k \in V^{\otimes n}$. Let $1 \neq d \in \mathcal{S}_n$. If either $d \notin \mathcal{S}_{(2f,n-2f)}$ or $d \in D_f = D_{\nu_f^t} \cap \mathcal{S}_{(2f,n-2f)}$, then $d^{-1}zE_f \in \text{ann}(v)$ for any $z \in \Psi$.

**Proof.** Write $v_k = w_1 \otimes \cdots \otimes w_f$ with $w_j = v_{j} \otimes v_{j'}$, $j = 1, \cdots, f$. If $d \notin \mathcal{S}_{(2f,n-2f)}$, then $d^{-1}$ is not contained in $\mathcal{S}_{(2f,n-2f)}$ either. In particular, there is some $j$, $2f+1 \leq j \leq n$, such that $1 \leq jd^{-1} \leq 2f$, and hence the basis vector $v_{k_j}$ with $2f+1 \leq k_j \leq m$ appears at position $jd^{-1}$ in $vd^{-1}$. However, $m < k_j' \leq 2m - 2f$; hence $v_{k_j'}$ does not occur as a factor in $vd^{-1}$ at all and hence for any $z \in \Psi$, $0 = vd^{-1}zv_{jd^{-1}-1}$ if $jd^{-1}$ is even, $0 = vd^{-1}zv_{jd^{-1}}$ if $jd^{-1}$ is odd. As the $e_i$’s in $E_f = e_1 e_3 \cdots e_{2f-1}$ commute we have $vd^{-1}zE_f = 0$ in this case. If $d \in D_f = D_{\nu_f^t} \cap \mathcal{S}_{2f}$, then $d$ and hence $d^{-1}$ as well is not contained in the subgroup $\Psi$ of $\mathcal{S}_{2f}$ defined above. Therefore there exists $j \in \{1, 3, \cdots, 2f-1\}$ such that $jd^{-1}, (j+1)d^{-1}$ are not in the same row of $(t^{(2)})d^{-1}$. Now we see similarly as above that $zE_f$ annihilates $vd^{-1}$ for any $z \in \Psi$. \qed

We are now ready to prove the key lemma from which our main result in this section will follow easily.
Lemma 3.10. Let $S$ be the subset

$$\left\{ d_1^{-1}E_f \sigma d_2 \middle| d_1, d_2 \in \mathcal{D}_{\nu_f}, d_1 \neq 1, \sigma \in \mathfrak{S}_{t+1, \ldots, n} \right\}$$

of the basis (3.3) of $B_n(-2m)$, and let $U$ be the subspace spanned by $S$. Then

$$B^{(f)} \cap \left( \bigcap_{k \in I_f} \text{ann}(v_k \otimes v_k) \right) = B^{(f+1)} \oplus U.$$

Proof. Since $\ell_s(v_k) = 0$, by definition of $I_f$, hence $\ell_s(v_k \otimes v_k) = f$, it follows that $B^{(f+1)} \subseteq \text{ann}(v_k \otimes v_k)$. This, together with Lemma 3.9, shows that the right-hand side is contained in the left-hand side.

Now let $x \in B^{(f)} \cap \left( \bigcap_{k \in I_f} \text{ann}(v_k \otimes v_k) \right)$. Using Lemma 3.9 and the basis (3.3) of $B_n(-2m)$, we may assume that $x = E_f \sum_{d \in \mathcal{D}_\nu} z_d d$, where $\nu = \nu_f = ((2f), (n-2f))$ and the coefficients $z_d$, $d \in \mathcal{D}_\nu$ are taken from $K\mathcal{S}_{t+1, \ldots, n} \subseteq K\mathcal{S}_n$. We then have to show $x = 0$.

Fix $k \in I_f$ and write $x = v_k \otimes v_k$. As in Lemma 3.6, choose the weight $\lambda(1) \in \Lambda(2m, 2f)$ to be the weight of $v_k = w_1 \otimes \cdots \otimes w_f$, where $w_i = v_{f+i} \otimes v_{(f+i)', i = 1, \ldots, f}$, and let $\lambda(2)$ be the weight of $v_k$; thus $\lambda = \lambda(1) \otimes \lambda(2)$ is the weight of $v_k \otimes v_k$. Since $V^\otimes n$ is the direct sum of its weight spaces $M^\lambda$, we conclude $(vx)_\mu = 0$ for all $\mu \in \Lambda(2m, n)$. In particular,

$$0 = (vx)\lambda = (v_k \otimes v_k x)_\lambda = \sum_{d \in \mathcal{D}_\nu} (v_k E_f \otimes v_k)_{\lambda} z_d d = \sum_{d \in \mathcal{D}_\nu} (v_k E_f)_{\lambda(1)} \otimes v_k z_d d.$$

The latter equality holds, since the action of $\mathcal{S}_n$ preserves weight spaces.

By Lemma 3.6 we have $(v_k E_f)_{\lambda(1)} = (-1)^f \sum_{\psi \in \Psi} \nu_\psi v_k \Psi = \hat{v}$, where again $\Psi$ is the normalizer of the Young subgroup $\mathcal{S}_{2f}$ in $\mathcal{S}_{2f}$. Thus we have to investigate

$$\sum_{d \in \mathcal{D}_\nu} (\hat{v} \otimes v_k) z_d d = 0$$

for the unknown element $z_d \in K\mathcal{S}_{t+1, \ldots, n}$. Note that

$$(\hat{v} \otimes v_k) z_d = \hat{v} \otimes (v_k z_d).$$

We fix $d \in \mathcal{D}_{\nu_f}$. By Lemma 3.8 we find a $2f$-element subset $J$ of $\{1, \ldots, n\}$ and $d_1 \in \mathcal{D}_f \subseteq \mathcal{S}_{2f}$ such that $d = d_1 d_J$. Thus

$$(\hat{v} \otimes v_k) z_d d = (\hat{v} \otimes v_k z_d) d = (\hat{v} \otimes v_k z_d) d_1 d_J = (\hat{v} d_1 \otimes v_k z_{d_1, d_J}) d_J,$$

since $d_1 \in \mathcal{S}_{2f}$ and $z_d \in K\mathcal{S}_{t+1, \ldots, n}$.

If $J, L \in \mathcal{P}_f, J \neq L$, choose $1 \leq l \leq n$ with $l \in J$ but $l \not\in L$. Thus there exists a $j \in \{1, 2, \ldots, 2f\}$ which is mapped by $d_J$ to $l$, but $(l)d_J^{-1} > 2f$. Note that for any $d \in \mathcal{D}_f$ all basis vectors $v_i$ occurring in $\hat{v} d$ as factors have index in the set $\{f+1, f+2, \ldots, 2f, (2f)', \ldots, (f+2)', (f+1)\}'$, and all those $v_i$ occurring in $v_k z_{d_1, d_J}$, respectively in $v_k z_{d_1, d_J}$, have index $i$ between $2f+1$ and $m$. Let $v_{i_1} \cdots v_{i_n}$ be a simple tensor involved in $(\hat{v} d_1 \otimes v_k z_{d_1, d_J}) d_J$ and $v_{j_1} \cdots v_{j_n}$ be a simple tensor involved in $(\hat{v} d_2 \otimes v_k z_{d_2, d_J}) d_J$ for $d_1, d_2 \in \mathcal{D}_f$. Then, by the above, we have that $2f+1 \leq j_l \leq m$, and either $v_{i_l} = v_k$ or $v_{i_l} = v_{k'}$ for some $f+1 \leq l \leq 2f$. Consequently the simple tensors $v_{i_l} \in I(2m, n)$ involved in $(\hat{v} d_1 \otimes v_k z_{d_1, d_J}) d_J$ and in $(\hat{v} d_2 \otimes v_k z_{d_2, d_J}) d_J$ are disjoint; hence both sets are linearly independent.

We conclude that $\sum_{d \in \mathcal{D}_f} (\hat{v} \otimes v_k z_{dd_J}) d_J = 0$ for each $J \in \mathcal{P}_f$; hence $\sum_{d \in \mathcal{D}_f} \hat{v} d_1 \otimes v_k z_{d_1, d_J} = 0$, since $d_J$ is invertible.
Lemma 3.7 says in particular that \( \hat{d}_1 \) is a linear combination of basis tensors \( v_i = v_{i_1} \otimes \cdots \otimes v_{i_f} \), with \( i \in 2\Psi d_1 \), and that we obtain by varying \( d_1 \) through \( D_f \) precisely the partition of \( \mathcal{G}_{2f} \) into \( \Psi \)-cosets. These are mutually disjoint. This is because \( \mathcal{G}_{2f} \) acts faithfully on the \( K \)-span of \( \{ v_{\sigma} \mid \sigma \in \mathcal{G}_{2f} \} \), and hence all the basis vectors \( v_i, 1 \leq i \leq n \) appearing as factors in \( \hat{d}_1 \) are pairwise distinct. Consequently the cosets of \( \Psi d_1, d_1 \in D_f \), partition the basis vectors in this set into mutually disjoint subsets, and we conclude that the basic tensors involved in \( \hat{d}_1 \) are disjoint for different choices of \( d_1 \in D_f \). Therefore, the equality \( \sum_{d_1 \in D_f} \hat{d}_1 \otimes v_{\bar{k}} z_{d_1 d_j} = 0 \) implies that \( \hat{d}_1 \otimes v_{\bar{k}} z_{d_1 d_j} = 0 \) for each fixed \( d_1 \in D_f \). Now we vary \( \bar{k} \in I_f \). The \( K \)-span of \( \{ v_{\bar{k}} \mid \bar{k} \in I_f \} \) is isomorphic to the tensor space \( V^\otimes n-2f \) for the symmetric group \( \mathcal{G}_{(2f+1,\ldots,n)} \cong \mathcal{G}_{n-2f} \). Since \( m-2f \geq n-2f \), hence \( \mathcal{G}_{(2f+1,\ldots,n)} \) acts faithfully on it. This implies \( z_{d_1 d_j} = 0 \) for all \( d_1 \in D_f, J \in \mathcal{P} f \). Thus \( x = 0 \) and the lemma is proved.

**Corollary 3.11.** Let \( d \in D_\nu, \nu = \nu_f \). Then

\[
B^{(f)} \cap \left( \bigcap_{\bar{k} \in I_f} \text{ann}(v_{\bar{k}} \otimes v_{\bar{k}})d \right) = B^{(f+1)} \oplus \left( \bigoplus_{\bar{d} \notin \bar{d}_1, \bar{d}_2 \in D_\nu, \sigma \in \mathcal{G}_{(2f,\ldots,n-2f)}} K \hat{d}_1^{-1} E_\sigma \bar{d}_2 \right).
\]

Hence \( B^{(f)} \cap \left( \bigcap_{d \in D_\nu} \bigcap_{\bar{k} \in I_f} \text{ann}(v_{\bar{k}} \otimes v_{\bar{k}})d \right) = B^{(f+1)} \).

**Proof.** First, we claim that for any \( \bar{d}, \bar{d}_2 \in D_\nu \) with \( \bar{d} \neq \bar{d}_2 \),

\[
\left( (v_{\bar{k}} \otimes v_{\bar{k}})d \right) \bar{d}^{-1} E_\sigma = 0,
\]

and thus the right-hand side of the above equality is contained in the left-hand side.

By Lemma 3.9, it suffices to consider the case where \( \bar{d} \bar{d}^{-1} = w \in \mathcal{G}_{(2f,\ldots,n-2f)} \). By Lemma 3.8, we can write \( d = d_1 d_J, \bar{d} = \bar{d}_1 d_L \), where \( d_1, \bar{d}_1 \in D_f, J, L \in \mathcal{P} f \). Since \( d_J, d_L \) are distinguished right coset representatives of \( \mathcal{G}_{(2f,\ldots,n-2f)} \) in \( \mathcal{G}_n \), we deduce that \( J = L \). Hence \( d_1 = w \bar{d}_1 \) and hence \( w \in \mathcal{G}_{2f} \). By Lemma 3.7, we can write \( w = z d_3 \), where \( z \in \Psi, d_3 \in D_f \). Since \( d \neq \bar{d}_2 \), it follows that \( d_1 \neq \bar{d}_1 \). Therefore, by the decomposition given in Lemma 3.7, we know that \( d_1 = w \bar{d}_1 = z d_3 \bar{d}_1 \) implies that \( d_3 \neq 1 \). Therefore, by Lemma 3.9, \( v_{\bar{k}} \bar{d}^{-1} d_3 z \bar{d}_1^{-1} E_\sigma = 0 \). This proves our first claim.

Second, note that the annihilator of \( (v_{\bar{k}} \otimes v_{\bar{k}})d \) \( (\bar{k} \in I_f, d \in D_\nu) \) in \( B^{(f)} \) is precisely \( d^{-1} \text{ann}(v_{\bar{k}} \otimes v_{\bar{k}}) \cap B^{(f)} \). By Lemma 3.10, to complete the proof of the corollary, it suffices to show that each \( d^{-1} d_1 F_{\sigma} \bar{d}_2 \), where \( \bar{d} \neq \bar{d}_1, \bar{d}_2 \in D_\nu, \sigma \in \mathcal{G}_{(2f,\ldots,n-2f)} \), can be written in the form \( d_1^{-1} E_{\bar{d}} \bar{d}_2 \), where \( \bar{d} \neq \bar{d}_1, \bar{d}_2 \in D_\nu, \sigma \in \mathcal{G}_{(2f,\ldots,n-2f)} \). In fact, assume that \( d_1 = z d_3 d_J \), where \( z \in \Psi, d_3 \in D_f, J \in \mathcal{P} f \). Then by the decomposition given in Lemma 3.7, \( d_1 \neq 1 \) implies that \( d_3 d_J \neq d \). Note that \( \Psi \) is generated by \( s_1, s_2, s_{2i+1}, s_{2i-1}, i = 1, \ldots, f \). Using the fact that \( s_1 e_1 = e_1 \) and

\[
s_{2i}s_{2i+1}s_{2i-1}s_{2i-1}s_{2i-1}e_{2i+1}e_{2i-1}e_{2i+1} = e_{2i+1}e_{2i+1}e_{2i-1}e_{2i+1} = e_{2i+1}e_{2i+1}e_{2i-1}e_{2i+1} = e_{2i-1}e_{2i+1},
\]

it is easy to see that \( z E_\sigma = E_\sigma \) for any \( z \in \Psi \). It follows that

\[
d^{-1} d_1^{-1} E_{\bar{d}} \bar{d}_2 = (d_3 d_J)^{-1} z^{-1} E_\sigma \bar{d}_2 = (d_3 d_J)^{-1} E_\sigma \bar{d}_2,
\]

as required. This completes the proof of the corollary. \( \square \)
Proof of Theorem 3.4. We have seen in Lemma 3.5 that \( \text{ann}_{B_n(-2m)}(V^\otimes n) \subseteq B^{(1)} \), and Corollary 3.11 implies that \( \text{ann}_{B_n(-2m)}(V^\otimes n) \subseteq B^{(f+1)} \) provided that \( \text{ann}_{B_n(-2m)}(V^\otimes n) \subseteq B^{(f)} \) for all natural numbers \( f \). Thus by induction on \( f \) we have \( \text{ann}_{B_n(-2m)}(V^\otimes n) \subseteq B^{(f)} \). Since \( B^{(f+1)} = 0 \) for \( f > [n/2] \) it follows that \( \text{ann}_{B_n(-2m)}(V^\otimes n) = 0 \). In other words, \( \varphi \) is injective if \( m \geq n \).

Suppose furthermore \( K \) is an infinite field. By (2.8) the natural homomorphism from the group algebra \( K^GSp(V) \) to the symplectic Schur algebra \( S^{sy}_{K}(m, n) \) is surjective. Note that since \( S^{sy}_{K}(m, n) \) is a quasi-hereditary algebra and \( V \cong L(\varepsilon_1) \cong \Delta(\varepsilon_1) \cong \vee(\varepsilon_1) \), it follows that \( V^\otimes n \) is also a tilting module over \( S^{sy}_{K}(m, n) \). By the general theory from tilting modules (e.g. \([DPS, \text{Lemma 4.4 (c)}] \)),

\[
\text{End}_{K^GSp(V)}(V^\otimes n) \otimes_K \overline{K} = \text{End}_{S^{sy}_{K}(m, n)}(V^\otimes n) \otimes_K \overline{K} = \text{End}_{K^GSp(V)}(V^\otimes n) \overline{K},
\]

where \( V_{\overline{K}} := V \otimes_K \overline{K} \), and \( \dim \text{End}_{S^{sy}_{K}(m, n)}(V^\otimes n) = \dim \text{End}_{S^{sy}_{K}(m, n)}(V^\otimes n) \).

Therefore

\[
\dim \text{End}_{K^GSp(V)}(V^\otimes n) = \sum_{0 \leq f \leq [n/2]} (\dim \overline{S}^\lambda)^2
\]

(by the fact that \( m \geq n \) and \([GW, (10.3.3)]\))

\[
= \dim B_n(-2m),
\]

where \( \overline{S}^\lambda \) is the cell module for \( B_n(-2m) \) associated to \( \lambda \). By comparing dimensions, we see that \( \varphi \) is in fact an isomorphism. This completes the proof of Theorem 3.4 and hence the proof of Theorem 1.4 in the case \( m \geq n \).

\[ \square \]

4. The case \( m < n \)

We shall now embark on the case where \( m < n \). Our proof will use the result for \( m \geq n \), which was done in the previous section.

Recall that for \( m < n \) the algebra \( B_n(-2m) \) does not in general act faithfully on \( V^\otimes n \). To prove Theorem 1.4, it suffices to show that the dimension of \( \text{im}(\varphi) \) is independent of the choice of the infinite field \( K \). From now on, unless otherwise stated, we assume that \( K \) is algebraically closed. In particular, by (1.1) we can work with \( Sp(V) \) instead of \( GSp(V) \).

We fix \( m_0 \in \mathbb{N} \) such that \( m_0 \geq m \) and \( m_0 - m \) is even. Let \( \tilde{V} \) be an \( m_0 \)-dimensional symplectic \( K \)-vector space with ordered basis \( \tilde{v}_1, \ldots, \tilde{v}_{m_0}, \tilde{v}_{m'_0}, \ldots, \tilde{v}_{1'} \) and the symplectic form given by \( (\tilde{v}_i, \tilde{v}_j) = \tilde{\epsilon}_{ij}, \forall 1 \leq i, j \leq 1' \), where

\[
\tilde{\epsilon}_{ij} := \begin{cases} 
1 & \text{if } j = i' \text{ and } i < j, \\
-1 & \text{if } j = i' \text{ and } i > j, \\
0 & \text{otherwise}.
\end{cases}
\]

We make the convention that \( 1 < 2 < \cdots < m_0 < m'_0 < \cdots < 2' < 1' \). Identifying \( v_i \) with \( \tilde{v}_i \) and \( v_{i'} \) with \( \tilde{v}_{i'} \) for each \( 1 \leq i \leq m \), we embed \( V \) into \( \tilde{V} \) as a \( K \)-subspace. In the following we shall construct objects and maps with respect to \( \tilde{V} \) and \( V \), which will without further notice carry a symbol “~” if they are constructed
with respect to $\tilde{V}$ and without this symbol for $V$. The notion of the signs $\tilde{\epsilon}_{ij}$ for $i, j \in \{1, \cdots, m_0, m'_0, \cdots, 1'\}$ extends the $\epsilon_{ij}$ defined in the beginning for $V$.

We have a natural embedding of $Sp(V)$ into $Sp(\tilde{V})$, that is,

\[(4.1) \quad Sp(V) = \left\{ g \in Sp(\tilde{V}) \mid g\tilde{v}_j = \tilde{v}_j, \text{for each } m + 1 \leq j \leq (m + 1)' \right\}.\]

The tensor space $V^\otimes n$ is a direct summand of $\tilde{V}^\otimes n$; let $\pi_K : \tilde{V}^\otimes n \rightarrow V^\otimes n$ be the corresponding projection. That is, $\pi_K$ sends all simple tensors which contain a tensor factor $\tilde{v}_i$ or $\tilde{v}_i'$ for $m + 1 \leq i \leq m_0$ to zero.

The symplectic form defines a $KS\!p(V)$-isomorphism $\iota$ from $V$ onto $V^* := \text{Hom}_K(V, K)$, taking $v \in V$ to $\iota(v) := (v, -) \in V^*$; thus $V$ and hence $V^\otimes n$ are self-dual $KS\!p(V)$-modules. The analogous statement holds for $\tilde{V}$ and $K\!Sp(\tilde{V})$.

We identify $\text{End}_K(V)$ with $V \otimes V^*$ in the standard way. If we represent a $K$-endomorphism of $V$ as a matrix $(d_{ij})$ ($i, j \in \{1, \cdots, m, m' \cdots, 1'\}$), relative to a basis $(v_i)$, then the corresponding vector of $V \otimes V^*$ is

\[\sum_{i,j} d_{ij} (v_i \otimes \iota(v_{j'})).\]

Note that $\iota(v_{j'})(v_s) = \delta_{s,j}$ for any $1 \leq s \leq 1'$. This construction extends easily to a tensor product by

\[\text{End}_K(V^\otimes n) \cong V^\otimes n \otimes (V^*)^\otimes n \cong V^\otimes n \otimes (V^* \otimes n)\]

and works similarly for $\tilde{V}$. If $g \in Sp(V)$, $\rho : KS\!p(V) \rightarrow \text{End}_K(V^\otimes n)$ is the representation afforded by the tensor space, then $\rho(g)$ acts on $\text{End}_K(V^\otimes n)$ by conjugation. Hence $\text{End}_K(V^\otimes n)$ is naturally a $KS\!p(V)$-module and the isomorphisms above are $KS\!p(V)$-module maps. In particular

\[\text{End}_{KS\!p(V)}(V^\otimes n) \cong (V^\otimes n \otimes (V^* \otimes n))^{Sp(V)},\]

where the latter denotes the invariants of $V^\otimes n \otimes (V^* \otimes n)$ under the left diagonal action of $KS\!p(V)$. Using the fact that $V \cong V^*$ as a $KS\!p(V)$-module, we obtain $\text{End}_K(V^\otimes n) \cong V^\otimes 2n$, and hence

\[\text{End}_{KS\!p(V)}(V^\otimes n) \cong (V^\otimes 2n)^{Sp(V)}.\]

Note that the above isomorphism sends a homomorphism represented by the matrix $(d_{i,j})$ ($i, j \in I(2m, n)$), relative to a basis $(v_i)$, to the vector

\[(4.2) \quad \sum_{i,j \in I(2m, n)} d_{i,j} (v_i \otimes v_{j'}),\]

where $i = (i_1, \cdots, i_n), j = (j_1, \cdots, j_n), j' = (j'_1, \cdots, j'_n)$. Therefore, we can express our problem in terms of invariants. A similar construction works for $\tilde{V}$ and $Sp(\tilde{V})$.

Since $Sp(V) \leq Sp(\tilde{V})$ we may restrict $\tilde{V}^\otimes 2n$ to $Sp(V)$, and it is easy to see that the projection $\pi_K : \tilde{V}^\otimes 2n \rightarrow V^\otimes 2n$ is $KS\!p(V)$-linear. In particular, $\pi_K(\tilde{V}^\otimes 2n)^{Sp(V)} \subseteq (V^\otimes 2n)^{Sp(V)}$.

**Lemma 4.3.** Let $\theta : B_n(-2m) \rightarrow B_n(-2m)$ be the $K$-linear isomorphism which is defined on the common basis of these algebras, consisting of Brauer diagrams, as
the identity. Then the following diagram

\[
\begin{array}{ccc}
B_n(-2m_0) & \xrightarrow{\tilde{\varphi}} & \text{End}_{KSp(V)}(\tilde{V}^\otimes n) \\
\theta & & \xrightarrow{\pi_K} \\
B_n(-2m) & \xrightarrow{\varphi} & \text{End}_{KSp(V)}(V^\otimes n)
\end{array}
\xrightarrow{\sim} \left(\tilde{V}^\otimes 2n\right)^{Sp(\tilde{V})} \xrightarrow{\pi_K} \left(V^\otimes 2n\right)^{Sp(V)}
\]

is commutative, where \(\pi_K'\) maps an endomorphism of \(\tilde{V}^\otimes n\) to its restriction to \(V^\otimes n \subseteq \tilde{V}^\otimes n\) followed by the projection \(\pi_K\).

**Proof.** We use the same symbols to denote the standard generators for the two Brauer algebras \(B_n(-2m_0), B_n(-2m)\). By definition, 

\[
\theta(d^{-1}_1e_1e_3\cdots e_{2f-1}\sigma d_2) = d^{-1}_1e_1e_3\cdots e_{2f-1}\sigma d_2,
\]

for any \(0 \leq f \leq [n/2]\), \(\sigma \in S_{2f+1,\ldots,n}\), \(d_1, d_2 \in D_{\nu}\), where \(\nu := ((2f), (n-2f))\). Note that \(\theta\) is a \(K\)-linear map, but does not respect multiplication, since \(\theta(e_1e_1) = -2m_0e_2 \neq -2me_1 = \theta(e_1)\theta(e_1)\). The same is true for \(\pi_K'\).

Let \(\tilde{I} = \{m+1, \ldots, m_0, m_n, \ldots, (m+1)\}\). We identify endomorphisms of \(\tilde{V}^\otimes n\) (resp., of \(V^\otimes n\)) with their matrices relative to the basis \((\tilde{v}_i)\) (resp., the basis \((v_i)\)). The map \(\pi_K'\) just sends a matrix \((d_{i,j})\) to its submatrix obtained by deleting those rows and columns indexed by elements in \(\tilde{I}\), while \(\pi_K\) sends all simple tensors which contain a tensor factor \(\tilde{v}_i\) for \(i \in \tilde{I}\) to zero. Using (4.2), one sees easily that the right square diagram is commutative. It remains to show that \(\pi_K'\tilde{\varphi} = \varphi\theta\).

We identify \(\pi_K'\) with \(\pi_K\). We have to show that for any \(0 \leq f \leq [n/2]\), \(\sigma \in S_{2f+1,\ldots,n}\), \(d_1, d_2 \in D_{\nu}\), where \(\nu := ((2f), (n-2f))\),

\[
\pi_K'\tilde{\varphi}(d^{-1}_1e_1e_3\cdots e_{2f-1}\sigma d_2) = \varphi\theta(d^{-1}_1e_1e_3\cdots e_{2f-1}\sigma d_2)
\]

or equivalently,

\[
\pi_K'\left(\tilde{\varphi}(d^{-1}_1)\tilde{\varphi}(e_1e_3\cdots e_{2f-1})\tilde{\varphi}(\sigma d_2)\right) = \varphi(d^{-1}_1)\varphi(e_1e_3\cdots e_{2f-1})\varphi(\sigma d_2).
\]

Note that for any \(w \in S_n\) where both \(\tilde{\varphi}(w)\) and \(\varphi(w)\) are given by right place permutation, it is trivial that \(\pi_K'\tilde{\varphi}(w) = \varphi(w)\). Now

\[
\tilde{\varphi}(e_i) = \sum_{j \in \tilde{I}(2m,n)} \epsilon_{j,j_i+1} \tilde{v}_{j_1} \otimes \cdots \otimes \tilde{v}_{j_{i-1}} \otimes \left(\sum_{k=1}^{m_0} (\tilde{v}_k \otimes \tilde{v}_k - \tilde{v}_k \otimes \tilde{v}_k)\right) \otimes \tilde{v}_{j_{i+2}} \otimes \\
\cdots \otimes \tilde{v}_{j_n} \otimes \tilde{v}_{j'_1} \otimes \cdots \otimes \tilde{v}_{j'_{i-1}} \otimes \cdots \otimes \tilde{v}_{j'_{n}};
\]

\[
\varphi(e_i) = \sum_{j \in \tilde{I}(2m,n)} \epsilon_{j,j_i+1} v_{j_1} \otimes \cdots \otimes v_{j_{i-1}} \otimes \left(\sum_{k=1}^{m} (v_k \otimes v_k - v_k \otimes v_k)\right) \otimes v_{j_{i+2}} \otimes \\
\cdots \otimes v_{j_n} \otimes v_{j'_1} \otimes \cdots \otimes v_{j'_{i-1}} \otimes \cdots \otimes v_{j'_{n}}.
\]

It is also easy to see that \(\pi_K\tilde{\varphi}(e_1e_3\cdots e_{2f-1}) = \varphi(e_1e_3\cdots e_{2f-1})\).

Therefore, to prove (4.4), it suffices to show that for any \(x, y \in S_n\),

\[
\pi_K'\left(\tilde{\varphi}(x)\tilde{\varphi}(e_1e_3\cdots e_{2f-1})\tilde{\varphi}(y)\right) = \pi_K'\tilde{\varphi}(x)\pi_K'\tilde{\varphi}(e_1e_3\cdots e_{2f-1})\pi_K'\tilde{\varphi}(y).
\]

But this follows from direct verification (although \(\pi_K'\) is in general not an algebra homomorphism). This completes the proof of the lemma. \(\square\)
Henceforth we assume that \( m_0 \geq n \). By Theorem 3.4, \( \overline{\varphi} \) is an isomorphism; hence \( \varphi \) is surjective if and only if \( \pi_K'\left( \text{End}_{KSp(V)}(\overline{V}^{\otimes n}) \right) = \text{End}_{KSp(V)}(V^{\otimes n}) \), or equivalently, \( \pi_K'\left( (\overline{V}^{\otimes 2n})^{Sp(V)} \right) = (V^{\otimes 2n})^{Sp(V)} \). This means that every \( KSp(V) \)-endomorphism \( f \) of \( V^{\otimes n} \) can be extended to a \( KSp(\overline{V}) \)-endomorphism \( \widetilde{f} \) of \( \overline{V}^{\otimes n} \) such that \( \pi_K'(\widetilde{f}) = f \). It also means that every \( Sp(V) \)-invariant \( v \) of \( V^{\otimes 2n} \) can be extended to a \( Sp(\overline{V}) \)-invariant \( \overline{v} \) of \( \overline{V}^{\otimes 2n} \) such that \( \pi_K'\overline{v} = v \).

To accomplish this we replace the groups \( Sp(V) \) and \( Sp(\overline{V}) \) by their Lie algebras \( g = sp_{2m} \) and \( \overline{g} = sp_{2m_0} \). Let \( A := \mathbb{Z}[v, v^{-1}] \), where \( v \) is an indeterminate over \( \mathbb{Z} \), and let \( Q(v) \) be its quotient field. Let \( U_A \) respectively \( \overline{U}_A \) be Lusztig’s \( A \)-form (see [Lu3]) in the quantized enveloping algebra of \( g \) respectively \( \overline{g} \). For any commutative integral domain \( R \) and any invertible \( q \in R \) we write \( U_R := U_A \otimes_A R \), where we consider \( R \) as an \( A \)-module by the specialization \( v \mapsto q \). Furthermore, taking \( q = 1 \in \mathbb{Z} \) and taking the quotient by the ideal generated by the \( K_i - 1 \) for \( i = 1, \cdots, m \), one gets the Kostant’s \( Z \)-form (see [Ko], [Lu2, (8.15)] and the proof of [Lu1, (6.7)(c), (6.7)(d)])

\[
U_Z \cong (U_A \otimes_A \mathbb{Z})/(K_1 - 1, \cdots, K_m - 1) \cong U_Z/(K_1 - 1, \cdots, K_m - 1)
\]

\[
\cong \left( U_A/(K_1 - 1, \cdots, K_m - 1) \right) \otimes_A \mathbb{Z}
\]

in the ordinary enveloping algebra of the complex Lie algebra \( sp_{2m}(\mathbb{C}) \), and the hyperalgebra

\[
U_K \cong U_Z \otimes_Z K \cong (U_A \otimes_A \mathbb{Z})/(K_1 - 1, \cdots, K_m - 1) \otimes Z K
\]

\[
\cong U_K/(K_1 - 1, \cdots, K_m - 1)
\]

of the simply connected simple algebraic group \( Sp_{2m}(K) \). Similarly we define \( \overline{U}_R \), \( \overline{U}_Z \) and \( \overline{U}_K \).

It is well known that (see [Ja]) there is an equivalence of categories between \{rational \( Sp_{2m}(K) \)-modules\} and \{locally finite \( U_K \)-modules\} such that the trivial \( Sp_{2m}(K) \)-module corresponds to the trivial \( U_K \)-module, where the trivial \( U_K \)-module is the one dimensional module which affords the counit map of the Hopf algebra \( U_K \). The \( Sp_{2m}(K) \)-action on tensor space gives rise to a locally finite \( U_K \)-action on tensor space. Therefore

\[
\text{End}_{KSp(V)}(V^{\otimes n}) = \text{End}_{U_K}(V^{\otimes n}) \cong (V^{\otimes 2n})^{U_K} = (V^{\otimes 2n})^{Sp(V)}.
\]

This works in the same way for \( \overline{V} \). Hence \( \pi_K \) is a \( U_K \)-linear map which maps the invariants \( (\overline{V}^{\otimes 2n})^{U_K} \) into \( (V^{\otimes 2n})^{U_K} \).

Our goal is to show that \( \pi_K'\left( (\overline{V}^{\otimes 2n})^{U_K} \right) = (V^{\otimes 2n})^{U_K} \). For this purpose, we have to investigate certain nice bases of \( (V^{\otimes 2n})^{U_K} \) respectively \( (\overline{V}^{\otimes 2n})^{U_K} \). Let \( V_A \) (resp., \( \overline{V}_A \)) be the free \( A \)-module generated by \( v_1, \cdots, v_{m_0}, v_{m_0}', \cdots, v_{1}' \) (resp., by \( v_1, \cdots, v_m, v_{m'}, \cdots, v_{1'} \)). Recall that there is an action of \( U_Q(v) \) on \( V_Q(v) := \)
\( \bar{V}_A \otimes_A \mathbb{Q}(v) \) which is defined on generators as follows.

\[
E_i \bar{v}_j := \begin{cases} 
\bar{v}_i, & \text{if } j = i + 1, \\
\bar{v}_{(i+1)'}, & \text{if } j = i', \\
0, & \text{otherwise;}
\end{cases} \\
E_{m_0} \bar{v}_j := \begin{cases} 
\bar{v}_{m_0}, & \text{if } j = m_0', \\
0, & \text{otherwise},
\end{cases}
\]

\[
F_i \bar{v}_j := \begin{cases} 
\bar{v}_{i+1}, & \text{if } j = i, \\
\bar{v}_i, & \text{if } j = (i + 1)', \\
0, & \text{otherwise;}
\end{cases} \\
F_{m_0} \bar{v}_j := \begin{cases} 
\bar{v}_{m_0'}, & \text{if } j = m_0, \\
0, & \text{otherwise},
\end{cases}
\]

\[
K_i \bar{v}_j := \begin{cases} 
v \bar{v}_j, & \text{if } j = i \text{ or } j = (i + 1)', \\
v^{-1} \bar{v}_j, & \text{if } j = i + 1 \text{ or } j = i', \\
\bar{v}_j, & \text{otherwise;}
\end{cases} \\
K_{m_0} \bar{v}_j := \begin{cases} 
v^2 \bar{v}_j, & \text{if } j = m_0, \\
v^{-2} \bar{v}_j, & \text{if } j = m_0', \\
\bar{v}_j, & \text{otherwise},
\end{cases}
\]

where \( 1 \leq i < m_0, 1 \leq j \leq l' \), and we replace \( \bar{v}_l \) in the usual natural representation of \( U_{Q(v)} \) with \( (-1)^{m_0 - 1} \bar{v}_l \) for each \( 1 \leq i \leq m_0 \). This works in the same way for \( U_{Q(v)} \) and \( V_A \). That is, we replace \( v_{l'} \) in the usual natural representation of \( U_{Q(v)} \) with \( (-1)^{m - 1} v_{l'} \) for each \( 1 \leq i \leq m \). The action of the generators of \( U_{Q(v)} \) on \( V_{Q(v)} := \bar{V}_A \otimes_A \mathbb{Q}(v) \) is as follows.

\[
E_i v_j := \begin{cases} 
v_i, & \text{if } j = i + 1, \\
v_{(i+1)'}, & \text{if } j = i', \\
0, & \text{otherwise;}
\end{cases} \\
E_m v_j := \begin{cases} 
v_{m'}, & \text{if } j = m, \\
0, & \text{otherwise},
\end{cases}
\]

\[
F_i v_j := \begin{cases} 
v_{i+1}, & \text{if } j = i, \\
v_i, & \text{if } j = (i + 1)', \\
0, & \text{otherwise;}
\end{cases} \\
F_m v_j := \begin{cases} 
v_{m'}, & \text{if } j = m, \\
0, & \text{otherwise},
\end{cases}
\]

\[
K_i v_j := \begin{cases} 
v v_j, & \text{if } j = i \text{ or } j = (i + 1)', \\
v^{-1} v_j, & \text{if } j = i + 1 \text{ or } j = i', \\
v_j, & \text{otherwise;}
\end{cases} \\
K_m v_j := \begin{cases} 
v^2 v_j, & \text{if } j = m, \\
v^{-2} v_j, & \text{if } j = m', \\
v_j, & \text{otherwise},
\end{cases}
\]

where \( 1 \leq i < m, j \in \{1, \cdots, m\} \cup \{m', \cdots, 1'\} \). Our hypothesis that \( m_0 - m \) is even ensures that the new basis of \( V_A \) is still a part of the new basis of \( \bar{V}_A \). By [Lu3, (19.3.5)], our new basis \( \{\bar{v}_i, \bar{v}_l\}_{1 \leq i \leq m_0} \) (resp., \( \{v_i, v_{l'}\}_{1 \leq i \leq m} \) is a canonical basis of \( \bar{V}_{Q(v)} \) (resp., of \( V_{Q(v)} \)) in the sense of [Lu3].

For any field \( k \) and any specialization \( v \mapsto q \in k^\times, V_k \cong L_k(\varepsilon_1) \cong \triangle_k(\varepsilon_1) \cong \bigwedge_k(\varepsilon_1), \) it follows that \( V_k \), hence \( V_k^\otimes n \), is a tilting module over \( U_k \). By [DPS, (4.4)], we have that \( \text{End}_{U_k}(V_k^\otimes n) \cong \text{End}_{U_A}(V_A^\otimes n) \otimes_A k \), and the dimension of \( \text{End}_{U_k}(V_k^\otimes n) \) is independent of \( k \). The same is true for \( \bar{V}_k \) and \( \bar{U}_k \).
For each $1 \leq i \leq l'$, $\iota(\tilde{v}_i) = (\tilde{v}_i, -) \in \tilde{V}_A^* := \text{Hom}_A(\tilde{V}_A, A)$. Then $\iota(\tilde{v}_1)$ is a highest weight vector of weight $\varepsilon_1$. The map $\tilde{v}_1 \mapsto \iota(\tilde{v}_1)$ extends naturally to a $\tilde{U}_A$-module isomorphism $\iota': \tilde{V}_A \cong \tilde{V}_A^* = \tilde{U}_A \iota(\tilde{v}_1)$. One checks easily that
\[
\iota'(\tilde{v}_1) = v^{i-1} \iota(\tilde{v}_1), \quad \iota'(\tilde{v}_i') = v^{2m_0 + 1 - i} \iota(\tilde{v}_i'), \quad \forall 1 \leq i \leq m_0.
\]
Using the isomorphism $\iota'$, we get that
\[
\text{End}_{\tilde{U}_A}(\tilde{V}_A^{\otimes n}) \cong \left(\text{End}(\tilde{V}_A^{\otimes n})\right)_{\tilde{U}_A} \cong \left(\tilde{V}_A^{\otimes n} \otimes (\tilde{V}_A^{\otimes n})^*\right)_{\tilde{U}_A} \cong \left(\tilde{V}_A^{\otimes n} \otimes (\tilde{V}_A^{\otimes n})^*\right)_{\tilde{U}_A} \cong (\tilde{V}_A^{\otimes 2n})_{\tilde{U}_A}.
\]
Similarly, $\text{End}_{\tilde{U}_A}(\tilde{V}_A^{\otimes n}) \cong (\tilde{V}_A^{\otimes n})_{\tilde{U}_A}$. Consequently, for any field $k$ and any specialization $v \mapsto q \in k^\times$,
\[
(\tilde{V}_k^{\otimes 2n})_{\tilde{U}_k} \cong \text{End}_{\tilde{U}_k}(\tilde{V}_k^{\otimes n}) \cong \text{End}_{\tilde{U}_A}(\tilde{V}_A^{\otimes n}) \otimes_A k \cong (\tilde{V}_A^{\otimes 2n})_{\tilde{U}_A} \otimes_A k.
\]
Similarly, $(\tilde{V}_k^{\otimes 2n})_{\tilde{U}_k} \cong \text{End}_{U_k}(V_k^{\otimes n}) \cong \text{End}_{U_A}(V_A^{\otimes n}) \otimes_A k \cong (V_A^{\otimes 2n})_{U_A} \otimes_A k$. Note that when specializing $q$ to 1, each $K_i$ acts as the identity on the tensor space $V^{\otimes 2n}$. It follows that
\[
(V_Z^{\otimes 2n})_{U_Z} \cong \text{End}_{U_Z}(V_Z^{\otimes n}) \cong \text{End}_{U_Z}(V_Z^{\otimes n}) \cong \text{End}_{U_A}(V_A^{\otimes n}) \otimes_A Z \cong (V_A^{\otimes 2n})_{U_A} \otimes_A Z
\]
and
\[
(V_K^{\otimes 2n})_{U_K} \cong \text{End}_{U_K}(V_K^{\otimes n}) \cong \text{End}_{U_K}(V_K^{\otimes n}) \cong \text{End}_{U_A}(V_A^{\otimes n}) \otimes_A K \cong (V_A^{\otimes 2n})_{U_A} \otimes_A Z \otimes Z \cong (V_Z^{\otimes 2n})_{U_Z} \otimes Z \otimes Z
\]
Similar results hold for $\tilde{V}$, $\tilde{U}$ and $\tilde{U}$.

In [Lu3, (27.1.2)], Lusztig introduced the notion of a based module and by [Lu3, (27.3)], the $\tilde{U}_{\mathbb{Q}(v)}$-module $\tilde{M} := (V_{\mathbb{Q}(v)}^{\otimes 2n})$ is a based module. That is, there is a canonical basis $\tilde{B}$ of $\tilde{M}$, in Lusztig’s notation ([Lu3, (27.3.2)]), each element in $\tilde{B}$ is of the form $\tilde{v}_{i_1} \circ \tilde{v}_{i_2} \circ \cdots \circ \tilde{v}_{i_{2n}}$, and $\tilde{v}_{i_1} \circ \cdots \circ \tilde{v}_{i_{2n}}$ is equal to $\tilde{v}_{i_1} \otimes \cdots \otimes \tilde{v}_{i_{2n}}$ plus a linear combination of elements $\tilde{v}_{j_1} \otimes \cdots \otimes \tilde{v}_{j_{2n}}$ with $(\tilde{v}_{j_1}, \cdots, \tilde{v}_{j_{2n}}) < (\tilde{v}_{i_1}, \cdots, \tilde{v}_{i_{2n}})$ and with coefficients in $v^{-1} \mathbb{Z}[v^{-1}]$, where “$<$” is a partial order defined in [Lu3, (27.3.1)]. In particular, $\tilde{B}$ is an $A$-basis of $V_A^{\otimes 2n}$. Similarly, we define $M := (V_{\mathbb{Q}(v)}^{\otimes 2n})$ as a module over $U_{\mathbb{Q}(v)}$, and we have a canonical basis $B$ of $M$. Each element of $B$ is of the form $v_{i_1} \circ v_{i_2} \circ \cdots \circ v_{i_{2n}}$.

Let $X_+$ be the set of all the dominant weights of $\tilde{g}$. For $\lambda \in X_+$, we denote by $\Delta_{\mathbb{Q}(v)}(\lambda)$ the irreducible $U_{\mathbb{Q}(v)}$-module of highest weight $\lambda$. We define
\[
\tilde{M}[\lambda] := \sum_{\tilde{N} \subseteq \tilde{M}} \tilde{N}.
\]
Then
\[
\tilde{M} = \bigoplus_{\lambda \in X_+} \tilde{M}[\lambda].
\]
For each $\lambda \in X_+$, let $\tilde{M}[> \lambda] := \bigoplus_{\mu < \lambda \in X_+} \tilde{M}[\mu]$ and define $\tilde{B}[> \lambda] := \tilde{B} \cap \tilde{M}[> \lambda]$. By [Lu3, (27.1.8)(b)], $\tilde{B}[> \lambda]$ is a $\mathbb{Q}(v)$-basis of $\tilde{M}[> \lambda]$. We define $\tilde{M}[> \lambda] :=$
In particular, $\bar{B}[\neq 0] := \bigcup_{\lambda \neq 0} \bar{B}[\lambda]$ forms a $\mathbb{Q}(v)$-basis of $\bar{M}[\neq 0]$. We define $\bar{M}[\neq 0]_A := \sum_{b \in \bar{B}[\neq 0]} Ab$. Then $\bar{M}[\neq 0]_A$ is stable under $\bar{U}_A$. Hence for any specialization $v \mapsto q \neq 0$ in some field $K$, $\bar{M}[\neq \lambda]_K := \sum_{b \in \bar{B}[\neq \lambda]} Kb$ is $\bar{U}_K$-stable and the set $\{ b \mid b \in \bar{B}[\neq \lambda] \}$ forms a $K$-basis of $\bar{M}[\neq \lambda]_K$. Let $\bar{M}[\neq 0] := \bigoplus_{\lambda \neq 0} \bar{M}[\lambda]$. By [Lu3, (27.2.5)],

$$\bar{M}[\neq 0] = \sum_{\mu \in \bar{X}_+ - \{0\}} \bar{M}[\geq \mu].$$

In particular, $\bar{B}[\neq 0] := \bigcup_{\lambda \neq 0} \bar{B}[\lambda]$ forms a $\mathbb{Q}(v)$-basis of $\bar{M}[\neq 0]$. We define $\bar{M}[\neq 0]_A := \sum_{b \in \bar{B}[\neq 0]} Ab$. Then $\bar{M}[\neq 0]_A$ is stable under $\bar{U}_A$. Hence for any specialization $v \mapsto q \neq 0$ in some field $K$, $\bar{M}[\neq 0]_K := \sum_{b \in \bar{B}[\neq 0]} Kb$ is $\bar{U}_K$-stable and the set $\{ b \mid b \in \bar{B}[\neq 0] \}$ forms a $K$-basis of $\bar{M}[\neq 0]_K$. The isomorphism $\iota'$ induces a natural isomorphism $\bar{V}_A^{\otimes 2n} \cong (\bar{V}_A^{\otimes 2n})^*$, which we still denote by $\iota'$. It is clear that $\iota'$ maps $(\bar{V}_A^{\otimes 2n})_{\bar{U}(v)}$ isomorphically onto $(\bar{V}_A^{\otimes 2n}/\bar{M}[\neq 0])^*$. In particular, $\iota'(a)$ vanishes on $\bar{B}[\neq 0]$ for every $a \in (\bar{V}_A^{\otimes 2n})_{\bar{U}A}$. Therefore, $\iota'$ maps $(\bar{V}_A^{\otimes 2n})_{\bar{U}A}$ into $(\bar{V}_A^{\otimes 2n}/\bar{M}[\neq 0])^*$. By comparing dimensions, we conclude that for each field $K$ which is an $\mathcal{A}$-algebra, $\iota'$ maps $(\bar{V}_K^{\otimes 2n})_{\bar{U}K}$ isomorphically onto $(\bar{V}_K^{\otimes 2n}/\bar{M}[\neq 0]_K)^*$. As a consequence, $\iota'$ also maps $(\bar{V}_A^{\otimes 2n})_{\bar{U}A}$ isomorphically onto $(\bar{V}_A^{\otimes 2n}/\bar{M}[\neq 0]_A)^*$. Similarly, one can define $\bar{X}_+$ (the set of all the dominant weights of $\mathfrak{g}$), and for each $\lambda \in \bar{X}_+$, one can define $M[\lambda]$, $M[\lambda]$, $B[\lambda]$, $M[\neq 0]$, and $B[\neq 0]$. One has $M = \bigoplus_{\lambda \in \bar{X}_+} M[\lambda]$, and $(\bar{V}_A^{\otimes 2n})_{\bar{U}A}$ is canonically isomorphic to $(\bar{V}_A^{\otimes 2n}/\bar{M}[\neq 0]_A)^*$.

Recall that (see [Lu3, (27.2.1)]),

$$B = \bigcup_{\lambda \in \bar{X}_+} B[\lambda], \quad \bar{B} = \bigcup_{\lambda \in \bar{X}_+} \bar{B}[\lambda].$$

By [Lu3, (27.2.5)], the image of $B[0]$ (resp., $B[0]$) in $\bar{V}_A^{\otimes 2n}/\bar{M}[\neq 0]_A$ (resp., in $\bar{V}_A^{\otimes 2n}/\bar{M}[\neq 0]_A$) forms an $\mathcal{A}$-basis of $\bar{V}_A^{\otimes 2n}/\bar{M}[\neq 0]_A$ (resp., of $\bar{V}_A^{\otimes 2n}/\bar{M}[\neq 0]_A$). Let

$$J_0 := \{ (i_1, \ldots, i_{2n}) \in I(2m, 2n) \mid v_{i_1} \circ \cdots \circ v_{i_{2n}} \in B[0] \},$$

$$\bar{J}_0 := \{ (i_1, \ldots, i_{2n}) \in I(2m, 2n) \mid \bar{v}_{i_1} \bar{\circ} \cdots \bar{\circ} \bar{v}_{i_{2n}} \in \bar{B}[0] \}.$$

**Corollary 4.5.** With the above notation the set

$$\{ v_{i_1} \circ \cdots \circ v_{i_{2n}} + M[> 0]_A \mid (i_1, \ldots, i_{2n}) \in J_0 \}$$

forms an $\mathcal{A}$-basis of $\bar{V}_A^{\otimes 2n}/M[> 0]_A$.

**Proof.** This is clear, by the fact that the image of $B[0]$ in $\bar{V}_A^{\otimes 2n}/M[> 0]_A$ is an $\mathcal{A}$-basis and each $v_{i_1} \circ \cdots \circ v_{i_{2n}}$ is equal to $v_{i_1} \circ \cdots \circ v_{i_{2n}}$ plus a linear combination of elements $v_{j_1} \circ \cdots \circ v_{j_{2n}}$ with $(v_{j_1}, \ldots, v_{j_{2n}}) < (v_{i_1}, \ldots, v_{i_{2n}})$ and with coefficients in $v^{-1}\mathbb{Z}[v^{-1}]$. □
Similarly, the set
\[(4.6) \quad \{\tilde{v}_{i_1} \otimes \cdots \otimes \tilde{v}_{i_{2n}} + \tilde{M}[> 0]_{\mathcal{A}} \mid (i_1, \ldots, i_{2n}) \in \tilde{J}_0\}\]
forms an $\mathcal{A}$-basis of $\tilde{V}_n^{\otimes 2n}/\tilde{M}[> 0]_{\mathcal{A}}$.

**Theorem 4.7.** With the above notation, $J_0 \subseteq \tilde{J}_0$.

**Proof.** For each $1 \leq i \leq m_0$, let $\tilde{e}_i, \tilde{f}_i$ (resp., $e_i, f_i$) be the Kashiwara operators of $\tilde{U}_{Q(v)}$ (resp., of $U_{Q(v)}$). The $\tilde{U}_{Q(v)}(sp_{2m_0})$-crystal structure on $\tilde{V}_{Q(v)}$ is given below:

\[
1 \xrightarrow{1} 2 \xrightarrow{2} \cdots m_0^{-1} \xrightarrow{m_0} m_0 \xrightarrow{m_0} \cdots m_0^{-1} \xrightarrow{2} 2' \xrightarrow{1} 1',
\]

where

\[
\begin{align*}
1 \xrightarrow{i} k & \iff \tilde{f}_i \tilde{v}_k \equiv \tilde{v}_k \pmod{v^{-1}\tilde{M}} \iff \tilde{e}_i \tilde{v}_k \equiv \tilde{v}_k \pmod{v^{-1}\tilde{M}},
\end{align*}
\]

Similarly, the $U_{Q(v)}$-crystal structure on $V_{Q(v)}$ is as below:

\[
1 \xrightarrow{1} 2 \xrightarrow{2} \cdots m^{-1} \xrightarrow{m} m \xrightarrow{m} \cdots m^{-1} \xrightarrow{2} 2' \xrightarrow{1} 1'.
\]

Comparing with the two crystal graphs, it is easy to see that for each $1 \leq i \leq m$ and each $j \in \{1, \ldots, m\} \cup \{m', \ldots, 1'\}$,

\[
\begin{align*}
\max\{k \geq 0 \mid \tilde{e}_k \tilde{v}_j \notin v^{-1}\tilde{M}\} = \max\{k \geq 0 \mid e_k v_j \notin v^{-1}M\}, \\
\max\{k \geq 0 \mid \tilde{f}_k \tilde{v}_j \notin v^{-1}\tilde{M}\} = \max\{k \geq 0 \mid f_k v_j \notin v^{-1}M\}.
\end{align*}
\]

Moreover, for each $m + 1 \leq i \leq m_0$ and each $j \in \{1, \ldots, m\} \cup \{m', \ldots, 1'\}$,

\[
\tilde{e}_i \tilde{v}_j \in v^{-1}\tilde{M}, \quad \tilde{f}_i \tilde{v}_j \in v^{-1}\tilde{M}.
\]

Let $B'$ (resp., $\tilde{B}'$) be the canonical basis of $V^{\otimes n}$ (resp., of $\tilde{V}^{\otimes n}$) constructed from the canonical basis of $V$ (resp., of $\tilde{V}$); see [Lu3, (27.3.1)]. For each $\lambda \in \lambda^+_X$ (resp., $\lambda \in \tilde{\lambda}^+_X$), let $B'[\lambda]^{lo}, B'[\lambda]^{hi}$ (resp., $\tilde{B}'[\lambda]^{lo}, \tilde{B}'[\lambda]^{hi}$) be as defined in [Lu3, (27.2.3)].

Now [Lu3, (17.2.4)] gave the rules for the action of the Kashiwara operators $\tilde{e}_i, \tilde{f}_i, e_j, f_j$ on tensor products. As a consequence, our previous discussion shows that for any $1 \leq i_1, \ldots, i_n \leq 2m$,

\[
v_{i_1} \cdots v_{i_n} \in B'[\lambda]^{hi} \iff \tilde{v}_{i_1} \cdots \tilde{v}_{i_n} \in \tilde{B}'[\lambda]^{hi},
\]

\[
v_{i_1} \cdots v_{i_n} \in B'[\lambda]^{lo} \iff \tilde{v}_{i_1} \cdots \tilde{v}_{i_n} \in \tilde{B}'[\lambda]^{lo}.
\]

Now applying [Lu3, (27.3.8)], which said that

\[
\tilde{B}[0] = \bigcup_{\lambda \in \lambda^+_X} \left\{b \tilde{b} \middle| b \in \tilde{B}'[-w_0(\lambda)]^{lo}, b' \in \tilde{B}'[\lambda]^{hi}\right\},
\]

\[
B[0] = \bigcup_{\lambda \in \lambda^+_X} \left\{b \circ b' \middle| b \in B'[-w_0(\lambda)]^{lo}, b' \in B'[\lambda]^{hi}\right\},
\]

our theorem follows immediately. \(\square\)

**Proof of Theorem 1.4.** We regard $\mathbb{Z}$ as an $\mathcal{A}$-algebra by specializing $v$ to $1 \in \mathbb{Z}$, and regard $K$ as a $\mathbb{Z}$-algebra as usual. Then it is easy to see that $v' \otimes A \mathbb{K}$ coincides with the canonical $Sp_{2m_0}$-module isomorphism $V \to V^*$, $v \mapsto \iota(v) := (v, -)$ for any $v \in V$. Let $\tilde{V}_Z := \tilde{V}_A \otimes A \mathbb{Z}$, $\tilde{M}[\neq 0]_Z := \tilde{M}[\neq 0]_{\mathcal{A}} \otimes A \mathbb{Z}$. We have similar notation for
\(V^n, M \neq 0\). We claim that the natural projection map \((V \otimes 2n)^n U_K \to (V \otimes 2n)^n U_K\) is surjective.

In fact, we have the following commutative diagram:

\[
\begin{array}{ccc}
(V \otimes 2n)^n U_K & \longrightarrow & (V \otimes 2n)^n U_K \\
\pi_K \downarrow & & \downarrow j_K \\
(V \otimes 2n)^n U_K & \longrightarrow & (V \otimes 2n)^n U_K \\
\end{array}
\]

where the rightmost vertical homomorphism is induced from the canonical homomorphism \(j_Z : V^n_{Z, M} \to V^n_{Z, M}\). Note that \(j_Z\) is well defined as \(M_{Z, M} \subseteq \tilde{M}_{Z, M}\) (which follows from the fact that for each \(\lambda \in X_+\) with \(\lambda \neq 0\), \(M_{C, \lambda}\) should be contained in \(\tilde{M}_{C, \lambda}\)).

By (4.6) and Theorem 4.7, the image of \(\left\{v_1 \otimes \cdots \otimes v_{i_2n} + M(> 0)_{K} \mid (i_1, \ldots, i_{2n}) \in J_0\right\}\) under \(j_K := j_Z \otimes 1_{K}\) is always linearly independent, which shows that \(j_K\) is injective. Hence \(j_K^n := j_Z^n \otimes 1_{K}\) is surjective. It follows that

\[
\pi_K((V \otimes 2n)^n U_K) = (V \otimes 2n)^n U_K,
\]

as required. Now using Lemma 4.3 and Theorem 3.4, we complete the proof of Theorem 1.4 when \(K\) is algebraically closed.

Now suppose that \(K\) is an arbitrary infinite field. Let \(\overline{K}\) denote the algebraic closure of \(K\). Note that the image of \(\varphi\) is generated (as an algebra) by

\[
\{\varphi(e_1), \ldots, \varphi(e_{n-1}), \varphi(s_1), \ldots, \varphi(s_{n-1})\},
\]

and the canonical homomorphism

\[
\text{End}_{KSp(V^n)}(V^n_{K}) \otimes_K \overline{K} = \text{End}_{U_K}(V^n_{K}) \otimes_K \overline{K} \\
\rightarrow \text{End}_{U_K^n}(V^n_{K}) = \text{End}_{KSp(V^n)}(V^n_{K})
\]

is an isomorphism, where \(U_K = U_Z \otimes K\), \(U_{\overline{K}} = U_Z \otimes \overline{K} \cong U_K \otimes K\). It follows that the dimension of \(\text{im}(\varphi)\) is constant under field extensions \(K \subseteq \overline{K}\). The proof is completed.

Remark 4.8. The argument above in the proof of Theorem 1.4 actually shows that

\[
\pi_Z((V \otimes 2n)^n U_Z) = (V \otimes 2n)^n U_Z,
\]

or equivalently, \(\pi_Z(\text{End}_{U_Z}(V^n_{Z})) = \text{End}_{U_Z}(V^n_{Z})\).

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Mathematisches Institut B, Universität Stuttgart, Pfaffenwaldring 57, Stuttgart, 70569, Germany
E-mail address: Richard.Dipper@mathematik.uni-stuttgart.de

Department of Mathematics and Statistics, Loyola University Chicago, 6525 North Sheridan Road, Chicago, Illinois 60626
E-mail address: doty@math.luc.edu

Department of Applied Mathematics, Beijing Institute of Technology, Beijing, 100081, People’s Republic of China
E-mail address: junhu303@yahoo.com.cn