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LOCALLY QUASICONVEX SMALL-CANCELLATION GROUPS

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ABSTRACT. In this article we prove several results about the local quasiconvexity behavior of small-cancellation groups. In addition to strengthening our previously obtained positive results, we also describe several families of negative examples. Also, as the strength of the assumed small-cancellation conditions increases, the gap between our positive results and our counterexamples narrows. Finally, as an additional application of these techniques, we include similar results and counterexamples for Coxeter groups.

It has been known for some time that the class of small-cancellation groups contains groups which are coherent, groups which are incoherent, groups which are locally quasiconvex and groups which are not locally quasiconvex [2, 13, 14, 16]. However, there remains a large gap between the hypotheses necessary to obtain positive results and available counterexamples. In this article, we begin closing this gap by combining the perimeter technique we introduced in [14] with the concept of a fan we developed in [15]. On the positive side we derive a number of new results based on the following theorem, which is a combination of the main theorem of [15] with one of the main theorems in [14].

Theorem 3.10. *Let X be a compact weighted $C(p)$-$T(q)$ complex, where p, q, and k satisfy the Euclidean restrictions. If every minimal fan of type k in X is both spread out and perimeter reducing, then $\pi_1 X$ is locally quasiconvex.*

All of the undefined terms (*Euclidean restrictions*, *minimal fan of type k*, *spread out*, *perimeter reducing*, etc.) are defined in the course of the article. In order to obtain theorems which are easier to apply, we show how residual finiteness can be used to replace the spread-out assumption and we introduce a *local* and a *global ratio* which facilitate verifying whether the fans in a collection are perimeter reducing. Our main positive results assert that *weighted small-cancellation* 2*-complexes* whose ratios are bounded above by a certain constant are coherent and locally quasiconvex. The constant itself is a function of the strength of the small-cancellation condition satisfied by the complex. For example, the following theorems are our strongest results using local ratios and fans of types 2 and 3.

Theorem 7.6. *Let X be a compact weighted $C(p)$-$T(q)$ complex in which all minimal fans of type 2 are spread out. Then $\pi_1 X$ is locally quasiconvex if either $p = 3$, $q \geq 6$ and $\mathsf{Local}(X) < \frac{q-4}{q-3}$, or $p \geq 4$, $q \geq 4$ and $\mathsf{Local}(X) < \frac{q-3}{q-2}$.*

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Theorem 7.7. *Let X be a compact weighted $C(p)$-$T(q)$ complex in which all minimal fans of type 3 are spread out. Then $\pi_1 X$ is locally quasiconvex if either $p \geq 6$, $q = 3$ and $\mathsf{Local}(X) < \frac{p-5}{2p-9}$, or $p \geq 4$, $q \geq 4$ and $\mathsf{Local}(X) < \frac{(q-2)(p-3)-1}{(q-1)(p-3)-1}$.*

On the negative side we give examples of incoherent and non-locally quasiconvex groups whose ratios are as close to these constants as we have been able to achieve. As the strength of the small-cancellation conditions improves, the gap between the positive results and the ratios of the counterexamples narrows. In one case, at least, the ratios of the counterexamples converge to the ratios in the theorems showing that in this direction our positive results are asymptotically sharp.

Finally, in the last two sections we explore the extra benefits to be gained from using fans of type k where k is large, and we examine the boundary between locally quasiconvex Coxeter groups and non-locally quasiconvex Coxeter groups. For example, we prove the following.

Theorem 12.2. *The Coxeter group $\langle a_1, \ldots, a_r \mid a_i^2, (a_i a_j)^{m_{ij}} \ (i \neq j) \rangle$ is coherent provided $m_{ij} \geq r$ for all $i \neq j$. Similarly, if $m_{ij} > r$ for all $i \neq j$, then the group is locally quasiconvex.*

A pair of counterexamples suggests that the connection between the smallest exponents and the number of generators may not be arbitrary.

Structure of the article. Sections 1, 2 and 3 review the necessary definitions and results from [15] and [14]. Section 4 is a slight digression which shows how the hypothesis that the fans are spread out can be removed when the fundamental group is residually finite. In Section 5 we define two invariants that compare the weights and perimeters in a weighted 2-complex which we call its local and global ratio. All of the positive and negative results in the remainder of the article are framed in these terms. Section 6 and Section 7 contain our main positive results, and Sections 8, 9, and 10 contain our families of counterexamples. We present, in particular, several families of groups which are incoherent and/or not locally quasiconvex. The counterexamples come quite close to violating the statements of the theorems, particularly as the strength of the small-cancellation conditions increases. In Section 11 we consider fans of type k where k is large. For comparison, all of the results up to this point are derived using fans of types 2 and 3. Finally, in Section 12 our techniques are applied to Coxeter groups.

1. Diagrams and small-cancellation theory

In this section we review some basic definitions. For a rigorous development of these notions (that is consistent with their use here), the interested reader should consult [15].

Definition 1.1 (Piece)**.** Let X be a combinatorial 2-complex. Intuitively, a piece of X is a path which is contained in the boundaries of the 2-cells of X in at least two distinct ways. More precisely, a nontrivial path $P \to X$ is a *piece* of X if there are 2-cells R_1 and R_2 such that $P \to X$ factors as $P \to R_1 \to X$ and as $P \to R_2 \to X$ but there does not exist a homeomorphism $\partial R_1 \to \partial R_2$ such that there is a commutative diagram

$$\begin{array}{ccc} P & \to & \partial R_2 \\ \downarrow & \nearrow & \downarrow \\ \partial R_1 & \to & X. \end{array}$$

Notice that the 2-cells R_1 and R_2 are not necessarily distinct. Excluding commutative diagrams of this form ensures that P occurs in ∂R_1 and ∂R_2 in essentially distinct ways.

Definition 1.2 (Disc diagram)**.** A *disc diagram* is a nonempty, contractible finite 2-complex with a specific planar embedding. Although a disc diagram is a deformation retraction of a topological disc, it need not be homeomorphic to one. The *area* of a disc diagram is its number of 2-cells.

One way a disc diagram can fail to be homeomorphic to a topological disc is if it contains a spur.

Definition 1.3 (Spur)**.** A spur in the disc diagram D is a valence 1 0-cell on a 1-cell in its boundary. Note that spurs correspond with the "backtracks" in its boundary cycle. The leftmost illustration in Figure 2 is a disc diagram with a spur.

Definition 1.4 ($C(p)$-$T(q)$-complex)**.** Roughly speaking, a 2-complex X is a $C(p)$-$T(q)$ *complex* if for every immersed path $P \to X$ and for every minimal area disc diagram $D \to X$ which has P as a boundary cycle, the internal 2-cells of D share edges with at least p other 2-cells and the internal vertices of D have valence at least q. Although this rough definition is not quite technically correct, it should give the reader unfamiliar with small-cancellation complexes an approximate idea of their properties. See [15] or [13] for precise definitions. Recall that an *arc* in a diagram is a maximal path in which all of its internal vertices have valence 2. If D has minimal area, then the arcs in the interior of D will be pieces in the sense of Definition 1.1.

As is usual with arguments about 2-complexes, 2-cells which are proper powers cause technical difficulties which need to be addressed. In particular, we need to digress for a moment to discuss exponents, packets, and packed maps.

Definition 1.5 (Exponent of a 2-cell)**.** Let X be a 2-complex, and let $R \to X$ be one of its 2-cells. Let n be the largest number such that the map $\partial R \to X$ can be expressed as a path W^n in X, where W is a closed path in X. This number n, which measures the periodicity of the map of $\partial R \to X$, is the *exponent* of R, and a path such as W is a *period* for ∂R. Notice that any other closed path which determines the same cycle as W will also be a period of ∂R. If the exponent n is greater than 1, then the $\partial R \to X$ is called a *proper power*.

Definition 1.6 (Packet)**.** Let R be a 2-cell in X of exponent n and let W be a period of ∂R. The attaching map $\partial R \to X$ can be expressed as a path $W^n \to X$. Consider a circle subdivided into $|W|$ 1-cells, and attach a copy of R by wrapping ∂R around the circle n times. We call the resulting 2-complex $\bar{R}$. Note that there is a map $\bar{R} \to X$ such that $R \to X$ factors as $R \to \bar{R} \to X$. Observe that $\pi_1 \bar{R} \cong \mathbb{Z}/n\mathbb{Z}$ and that the universal cover of $\bar{R}$ has a 1-skeleton which is identical to that of R together with n distinct copies of R attached by embeddings. The universal cover of $\bar{R}$ is the *packet of* R and is denoted by $\widetilde{R}$. Technically we should write $\widetilde{\bar{R}}$, but we will use the notation of $\widetilde{R}$ since R is its own universal cover and thus there is no danger of confusion. Notice that if the exponent of R is 1, then the packet $\widetilde{R}$ is the same as R itself. Notice also that the map $\widetilde{R} \to X$ can be viewed as an extension of the map $R \to X$.

Definition 1.7 (Packed map)**.** Let $\phi\colon Y \to X$ be a fixed map. The map ϕ will be called *packed* if whenever there is a lift of a 2-cell $R \to X$ to a 2-cell $R \to Y$, there is also a lift of $\widetilde{R} \to X$ to a map $\widetilde{R} \to Y$ which extends the map $R \to Y$. Since we will treat the packets $\widetilde{R}$ as the basic building blocks of our 2-complexes, almost all of the maps under discussion will be packed. Given a map $Y \to X$ there is a unique extension to a packed map where the 1-skeleton of the domain is unchanged. We will call this extended domain $\widetilde{Y}$ in analogy with $\widetilde{R}$.

2. Fans and ladders

We are now ready to describe the notion of a fan as introduced in [15].

Definition 2.1 (Fan)**.** A *fan* F is a 2-complex homeomorphic to a closed disc, which consists of a linearly ordered sequence of 2-cells such that the 2-cells which are not adjacent in the ordering intersect in at most a vertex. Figure 1 shows a typical fan. In addition, a portion of ∂F is designated as its *outer path* Q and Q must be a concatenation $Q = Q_1 Q_2 \dots Q_n$ where each Q_i is a subpath of ∂R_i. The unique path S such that QS^{-1} is the boundary cycle of F will be called the *inner path* of F, and the portion of the 1-skeleton of F which is not in its outer path will be the *inner portion* of its 1-skeleton.

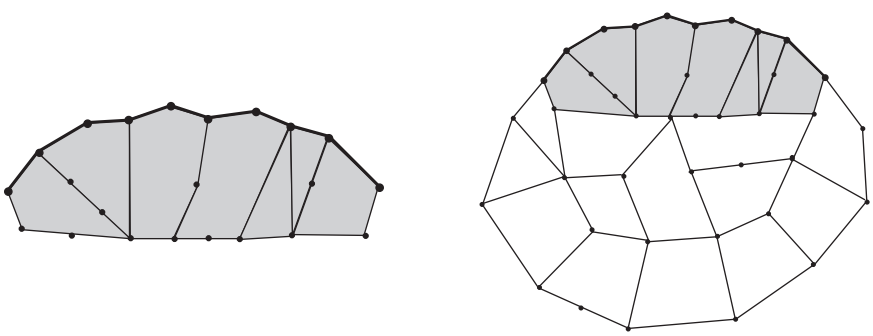

Figure 1. On the left is a fan F whose outer path Q is the bold path on its boundary. The disc diagram on the right contains the fan F as a subcomplex. Note that Q is a subpath of the boundary path of D.

We wil primarily be interested in fans equipped with a near-immersion into a 2-complex X. Recall that an *immersion* is a map which is locally injective and a *near-immersion* is a map which is locally injective except at 0-cells in the domain.

Definition 2.2 (Fan in a diagram)**.** If F is a fan and $F \to \widetilde{F} \to X$ is a near immersion, then this mapped fan is a *fan in* X. A disc diagram $D \to X$ *contains the fan* $F \to X$, provided that $F \to X$ factors as $F \to D \to X$, where the outer path Q of F maps to ∂D. In this case, we will also regard the outer path $Q \to F$ of F as a path $Q \to X$.

Three types of fans which deserve special attention are i-shells, pointed fans, and broad fans.

Definition 2.3 (i-shell)**.** Let D be a diagram. An *i-shell* of D is a 2-cell $R \hookrightarrow D$ whose boundary cycle is the concatenation of a maximal arc in ∂D and a path in the interior of D which is the concatenation of i maximal arcs. More specifically, the boundary cycle $\partial R \to D$ is the concatenation of subpaths $P_0, P_1, \ldots, P_i$ where $P_j \to D$ is an interior maximal arc of D for all $j > 0$ and $P_0 \to D$ is the preimage of ∂D in ∂R. An i-shell is a fan in D with the preimage of $P_0 \to D$ as its outer path.

Illustrated from left to right in Figure 2 are disc diagrams containing a spur, a 0-shell, a 1-shell, a 2-shell, and a 3-shell. In each case, the 2-cell R is shaded, and the maximal boundary arc P_0 is $\partial R \cap \partial D$.

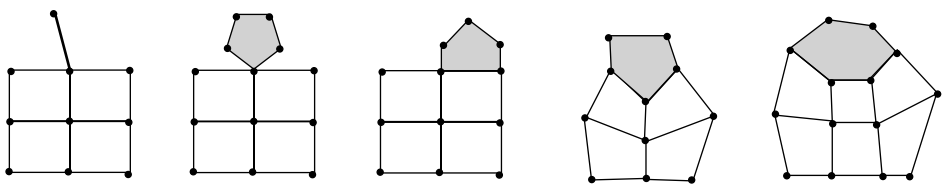

FIGURE 2. Spurs and i-shells

Definition 2.4 (Pointed fan)**.** A *pointed fan* $F \hookrightarrow D$ is a fan whose 2-cells are 2-shells. In addition, if F has i distinct 2-cells, then F contains $i+1$ pieces in D which have a common initial point and which terminate at various points on the outer path of F. The inner path $S \to D$ is the concatenation of the outer two of these maximal internal arcs, and the outer path $Q \to D$ is the concatenation of i maximal boundary arcs which are complements of two of these pieces. See the left side of Figure 3.

Definition 2.5 (Broad fan)**.** A *broad fan* $F \hookrightarrow D$ is a fan whose 2-cells are 2-shells and 3-shells. If F contains $i+j$ 2-cells, where i of them are 2-shells and j of them are 3-shells, then the outer path of F is the concatenation of $i+j$ maximal boundary arcs each of which is the complement of two or three interior pieces, and the rest of the 1-skeleton of F is the union of $i+2j+1$ pieces. The inner path S consists of $j+2$ of these pieces, and the remaining $i+j-1$ of them are internal maximal arcs of the diagram F. A broad fan F is *k-separated* if from left to right, it has at least k 2-shells at the beginning, at least k 2-shells between every pair of 3-shells, and at least k 2-shells at the end.

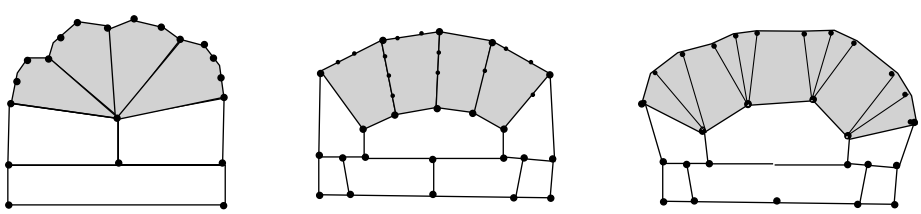

FIGURE 3. A pointed fan, a 0-separated broad fan, and a 2-separated broad fan

Pointed fans, broad fans and i-shells can be used to describe fans of types 1, 2, and 3 in a $C(p)$-$T(q)$ complex. As might be guessed from the notation, there are also collections of fans of type k for each $k \geq 1$. See [15] for a precise definition. We will return to fans of type k for $k > 3$ in Section 11. For now we list the fans of types 1, 2, and 3 as they were explicitly described in Example 7.10 of [15].

Example 2.6 (Fans of types 1, 2 and 3)**.** The fans of type 1 in a $C(p)$-$T(q)$ complex are described by the following lists:

$p \geq 4$, $q \geq 4$: {spurs, 0-shells, 1-shells, 2-shells},

$p \geq 6$, $q = 3$: {spurs, 0-shells, 1-shells, 2-shells, 3-shells}.

Fans of type 2 are described by one of the following lists:

$p \geq 4$, $q \geq 4$: {spurs, 0-shells, 1-shells, pointed fans with $i \geq q-3$},

$p = 3$, $q \geq 6$: {spurs, 0-shells, 1-shells, pointed fans with $i \geq q-4$}.

Fans of type 3 are described by the following lists:

$p \geq 4$, $q \geq 4$: {spurs, 0-shells, 1-shells, pointed fans with $i \geq q-2$, $(q-3)$-separated broad fans with $j \geq p-4$ and $i \geq (j+1)(q-3)$},

$p \geq 6$, $q = 3$: {spurs, 0-shells, 1-shells, pointed fans with $i \geq q-2$, broad fans with $j \geq p-5$ and $i \geq 0$}.

Since the restrictions given above on p and q recur frequently, we establish the following conventions.

Convention 2.7 (Restrictions)**.** Let D be a $C(p)$-$T(q)$ diagram and let k be a nonnegative integer. By *Euclidean restrictions* we mean that p, q, and k satisfy one of the following sets of conditions:

(1) $p \geq 6$, $q = 3$, and k is odd,
(2) $p \geq 4$, $q \geq 4$, and k is arbitrary,
(3) $p = 3$, $q \geq 6$, and k is even.

By the *hyperbolic restrictions* we mean that p, q, and k satisfy one of the following sets of conditions:

(1) $p \geq 7$, $q = 3$, and k is odd,
(2) $p \geq 5$, $q \geq 4$, and k is odd,
(3) $p \geq 4$, $q \geq 5$, and k is even,
(4) $p = 3$, $q \geq 7$, and k is even.

The fans of type k were introduced in [15] in order to concisely state the following generalization of the main theorem of small-cancellation theory [15, Theorem 9.4].

Theorem 2.8. *If D is a $C(p)$-$T(q)$ disc diagram and p, q, and k satisfy the Euclidean restrictions, then one of the following holds:*

(1) *D contains at least 3 separate fans of type k.*
(2) *D is a ladder of width $\leq k$.*
(3) *D is a wheel of width $\leq k$.*

Without going into the details, a ladder is a long thin diagram which looks like a regular neighborhood of a line segment, and a wheel looks like a regular

neighborhood of a single point (Lemma 8.7 and Lemma 8.8 in [15]). The size of the regular neighborhood roughly correlates to its width k. The precise definitions are stated in terms of iterated duals of the diagrams involved, but we will only need an easy corollary, which we state in terms of J-thin diagrams.

Definition 2.9 (J-thin)**.** A disc diagram D with boundary cycle PQ^{-1} is *J-thin* for some $J \in \mathbb{N}$ if every 0-cell in P is contained in a J-neighborhood of Q and vice-versa.

The following corollary is a consequence of Theorem 2.8, and Lemmas 8.7 and 8.8 in [15]. The rough idea is that ladders of width $\leq k$ and wheels of width $\leq k$ are J-thin. On the other hand, when there exist three separate fans of type k, at least one of them must avoid the endpoints of P and Q.

Corollary 2.10. *Let X be a compact $C(p)$-$T(q)$ complex and let p, q, and k satisfy the Euclidean restrictions. There is a constant J which depends on X, p, q, and k, such that every minimal area disc diagram D over X with boundary cycle PQ^{-1} is either J-thin, or there exists a fan of type k in D whose outer path is a subpath of either P or Q.*

There is one major drawback to the collection of fans of type k as described above. Even if a small-cancellation 2-complex X is compact, there might still exist an infinite number of distinct fans of type k which occur in diagrams over X. On the other hand, there are only a finite number of such fans which are minimal in a certain technical sense. Minimal fans are discussed in Section 12 of [15]. For our purposes, a minimal fan of type k in X is one that does not contain a proper subfan of type k, where the technical details are buried in what is meant by the word "contain". The three facts about minimal fans that we need are the following. They correspond to Lemma 12.4, Theorem 12.7, and Theorem 13.8 in [15], respectively.

Lemma 2.11. *Let p, q, and k satisfy the Euclidean restrictions. If F is a fan of type k, then there are only a finite number of subfans of F. Consequently, every fan of type k has a minimal subfan of type k.*

Theorem 2.12. *Let X be a compact $C(p)$-$T(q)$ complex and let p, q, and k satisfy the Euclidean restrictions. The set of all minimal fans of type k in X is finite.*

Theorem 2.13. *Let X be a $C(p)$-$T(q)$ complex, and let $F \to X$ be a minimal fan of type k. If p, q, and k satisfy the hyperbolic restrictions, then its outer path $Q \to X$ lifts to a simple path in $\widetilde{X}$.*

The first two results (combined with Corollary 2.10) ensure that every disc diagram which isn't J-thin contains a fan from a *finite* list, and the third is a more technical result that we will need in Section 4.

3. Perimeter and local quasiconvexity

In this section we recall some definitions and results from [14]. For a more detailed development, see [14].

Definition 3.1 (Side of a 2-cell)**.** Let x be a 1-cell in a combinatorial 2-complex X and let p be a point in its interior. A small open neighborhood of p looks like the spine of a book with several "pages" coming out. Each of these pages will be referred to as a *side of a 2-cell (at x).* One way to specify a side is to select a 1-cell

r in a 2-cell R in X. Both of these selections are best done via maps $r \to R \to X$. We denote this side by (R, r). A 2-cell R with characteristic map $R \to X$ has exactly $|\partial R|$-sides, even though some of these may appear along the same 1-cell in X. Continuing with the book metaphor, the pages along the spine x are called the *sides at x in X*, and the set of these sides is denoted $\mathsf{Sides}_X(x)$.

Definition 3.2 (Weighted 2-complex)**.** A 2-complex X is a *weighted 2-complex* if each of the sides in X has been assigned a nonnegative integer weight, the perimeter of each 1-cell is finite, and the weight of each 2-cell is positive. The *perimeter of a 1-cell x* in X is the sum of the weights assigned to the sides in $\mathsf{Sides}_X(x)$, and the *weight of a 2-cell R* in X is the sum of the weights assigned to its various sides. Notice that the perimeter of each 1-cell is automatically finite when X is compact, and the restriction on weights merely requires that every 2-cell have at least one side with a positive weight. In symbols,

$$\mathbf{P}(x) = \sum_{(R,r)\in\mathsf{Sides}_X(x)} \mathbf{Wt}((R,r)), \tag{1}$$

$$\mathbf{Wt}(R) = \sum_{r\in\mathrm{Edges}(\partial R)} \mathbf{Wt}((R,r)). \tag{2}$$

If X is a weighted 2-complex, and $\phi\colon Y \to X$ is a map, then there is an invariant called the *perimeter of* $Y \to X$, which we defined in [14]. In this article we restrict ourselves to the case where Y is compact and $\phi\colon Y \to X$ is a near-immersion. In this case, as was shown in [14, Lemma 2.18], the perimeter of $Y \to X$ can be defined as follows:

Definition 3.3 (Perimeter of a map)**.** If X is a weighted 2-complex, Y is compact, and the map $\phi\colon Y \to X$ is a near-immersion, then

$$\mathbf{P}(Y) = \mathbf{P}(\phi : Y \to X) = \sum_{y\in\mathsf{Edges}(Y)} \mathbf{P}(\phi(y)) - \sum_{S\in\mathsf{Cells}(Y)} \mathbf{Wt}(\phi(S)). \tag{3}$$

In [14], we used perimeter calculations to prove that various groups were coherent and locally quasiconvex. Recall that a group is *coherent* if every finitely generated subgroup is finitely presented. The best known examples of coherent groups are free groups, surface groups, polycyclic groups, and 3-manifold groups. The definition of local quasiconvexity is slightly more involved.

Definition 3.4 (Quasiconvexity)**.** A subspace Y of a geodesic metric space X is *quasiconvex* if there is an ϵ neighborhood of Y which contains all of the geodesics in X which start and end in Y. In group theory, a subgroup H of a group G generated by A is *quasiconvex* if the 0-cells corresponding to H form a quasiconvex subspace of the Cayley graph $\Gamma(G, A)$. If every finitely generated subgroup of G is quasiconvex, then G (generated by A) is *locally quasiconvex*. In general, specifying the generating set A is a crucial aspect of the definition, but when G is word-hyperbolic (as is the case for all of the groups considered in this article), then G is locally quasiconvex with respect to one finite generating set if and only if it is locally quasiconvex with respect to every finite generating set.

As we noted in [14], the notion of perimeter can be applied to fans in X.

Definition 3.5 (Perimeter-reducing fan)**.** Let X be a weighted 2-complex. A fan $F \to X$ is *perimeter-reducing* provided that the perimeter of $\widetilde{F} \to X$ is less

than the perimeter of its outer path $Q \to X$. In other words, $\mathbf{P}(\widetilde{F}) < \mathbf{P}(Q)$. If $\mathbf{P}(\widetilde{F}) \leq \mathbf{P}(Q)$, then we call F *weakly perimeter-reducing.* Rearranging the terms in Definition 3.3, we see that a fan is (weakly) perimeter-reducing if and only if the perimeter of its inner portion is less than (or equal to) the sum of the weights of the 2-cells in $\widetilde{F}$.

Another property of fans which is closely related to perimeter calculations is the property of being spread-out.

Definition 3.6 (Spread out)**.** A fan $F \to X$ is *spread out* provided that the sides of 2-cells of $\widetilde{F}$ along 1-cells in the outer path $Q \to F$ project to distinct sides of 2-cells along 1-cells in X. This condition is certainly satisfied when the outer path $Q \to \widetilde{F}$ projects to a path $Q \to X$ which does not pass through any 1-cell of X more than once. For instance, $F \to X$ is spread out when $\widetilde{F} \to X$ is an embedding, and it is spread out when $Q \to X$ is a simple path.

We are now able to quote the main theorems from [14] which use fans to establish coherence and local quasiconvexity results. These correspond to Theorems 10.11 and 12.5 in [14], respectively.

Theorem 3.7 (Fan coherence)**.** *Let X be a compact weighted 2-complex. Let $\mathcal{T}$ be a collection of spread-out weakly perimeter-reducing fans, and suppose that for each fan $F \in \mathcal{T}$, we have $\mathbf{P}(\widetilde{F}) < \mathbf{P}(\partial F)$. If every nontrivial minimal area disc diagram contains a spur or a fan in $\mathcal{T}$, then $\pi_1 X$ is coherent.*

Theorem 3.8 (Fan local quasiconvexity)**.** *Let $\mathcal{T}$ be a finite collection of perimeter reducing spread-out fans in the compact weighted 2-complex X, and let $J \in \mathbb{N}$. Suppose that for every minimal area disc diagram $D \to X$ with boundary cycle PQ^{-1}, either D is J-thin or D contains a spur or fan in $\mathcal{T}$ whose outer path is a subpath of either P or Q. Then $\pi_1 X$ is a locally quasiconvex word-hyperbolic group.*

By combining Theorems 3.7 and 3.8 with Theorem 2.8 and Corollary 2.10 we obtain the following theorems whose scope is the subject of the remainder of this paper.

Theorem 3.9 (Small-cancellation coherence)**.** *Let X be a compact weighted $C(p)$-$T(q)$ complex, where p, q, and k satisfy the Euclidean restrictions. If every minimal fan of type k in X is both spread out, weakly perimeter reducing and satisfies $\mathbf{P}(\widetilde{F}) < \mathbf{P}(\partial F)$, then $\pi_1 X$ is coherent.*

Theorem 3.10 (Small-cancellation local quasiconvexity)**.** *Let X be a compact weighted $C(p)$-$T(q)$ complex, where p, q, and k satisfy the Euclidean restrictions. If every minimal fan of type k in X is both spread out and perimeter reducing, then $\pi_1 X$ is locally quasiconvex.*

Throughout this article we focus on local quasiconvexity more than on coherence since in almost every instance where we can prove coherence, similar methods, pushed a bit more, allow us to prove the stronger property of local quasiconvexity. See, for instance, the discussion in Section 11.

There are two main difficulties in applying Theorem 3.10. While for each k, there are only finitely many minimal fans of type k in the compact 2-complex X, it is a tedious task to enumerate them and verify that they are perimeter reducing and spread out. Furthermore, for large values of k, these fans will not all be spread

out in X. In Section 4 we show that when $\pi_1 X$ is residually finite, it is no longer necessary to assume that all minimal fans of type k are spread out in X since they are spread out in one of its finite covers. Then, in Section 5, we describe two easily computed invariants of X which will enable us to readily verify that various fans are perimeter reducing. The finer invariant will be applied in Section 7 to obtain rather precise results for $k = 2$ and $k = 3$. The courser invariant will be applied in Section 11 where it will enable us to apply Theorem 3.10 for arbitrarily large k.

4. Spreading out fans

As mentioned above, one limitation of Theorems 3.7 and 3.8 and hence Theorem 3.10 is that the requirement that all fans of type $\mathcal{T}$ be spread out in X is often too restrictive for natural applications. In this section we explain how the assumption that the fans are spread out in X can be replaced with an assumption that they are spread out in the universal cover of X when $\pi_1 X$ is residually finite. The main idea is to use the residual finiteness of $\pi_1 X$ to choose a finite cover $\widehat{X}$ of X in which all of the appropriate fans are spread out. These theorems can then be applied to the finite cover, and the coherence or local quasiconvexity of $\pi_1 \widehat{X}$ will imply the coherence or local quasiconvexity of $\pi_1 X$. We begin by recalling the definition of a residually finite group.

Definition 4.1 (Residually finite)**.** A group G is *residually finite* provided that for each nontrivial element $g \in G$, there is a finite quotient $G \to \overline{G}$ such that $\bar{g}$ is nontrivial. By considering quotients $G \to \overline{G}_1 \times \overline{G}_2 \times \cdots \times \overline{G}_k$, it is easy to see that if G is residually finite, then for any finite set of nontrivial elements $g_1, g_2, \ldots, g_k$ there is a finite quotient $G \to \overline{G}$ such that each g_i is not mapped to the identity.

Lemma 4.2 (Spreading out fans)**.** *If X is a complex with a residually finite fundamental group and $\mathcal{T}$ is a finite collection of fans whose outer paths lift to simple paths in $\widetilde{X}$, then there is a finite cover $\widehat{X}$ of X such that each lift of a fan of type $\mathcal{T}$ to $\widehat{X}$ is spread out.*

Proof. Let $F \to X$ be a fan in $\mathcal{T}$ which is not spread out in X but which is spread out in $\widetilde{X}$. By definition, there are two sides of 2-cells along two 1-cells in Q which project to the same side in X. In particular, there is a subpath P of the outer path Q which projects to a closed path in X but which lifts to an open path in $\widetilde{X}$. Since $\mathcal{T}$ is finite, there are only finitely many such paths $P \to Q \to F$ where F is in $\mathcal{T}$. Each of these closed paths in X represents a nontrivial conjugacy class in $\pi_1 X$. In order to apply Definition 4.1, we can modify these paths to represent specific elements in $\pi_1 X$. Let v be a fixed basepoint for X. For each path P, choose a path P_v from v to the basepoint of P and let P' denote the closed path $P_v P P_v^{-1}$ based at v. This creates a finite list of paths based at v which represent elements of $\pi_1 X$. Moreover, notice that a lift of P' to a cover of X is closed if and only if the lift of the subpath P is closed. Consider the resulting finite set of paths based at v. By Definition 4.1, there is a finite quotient of $\pi_1 X$ in which the elements they represent map to nontrivial elements. The corresponding finite regular cover $\widehat{X}$ has the property that none of the essential closed paths P have closed lifts to $\widehat{X}$. The cover $\widehat{X}$ has the desired property. Indeed, if there was a fan $F \to \widehat{X} \to X$ in $\mathcal{T}$ which was not spread out in $\widehat{X}$, then there would exist a subpath $P \to Q \to F \to \widehat{X}$ which was a closed path in $\widehat{X}$ even though it lifts to an open path in $\widetilde{X}$. This is impossible by the way $\widehat{X}$ was constructed. $\square$

Notice that the argument actually shows that the outer paths of fans in $\mathcal{T}$ are simple paths in $\widehat{X}$. The following lemma allows us to conclude that $\pi_1 X$ is coherent (or word hyperbolic and locally quasiconvex) by proving that $\pi_1 \widehat{X}$ is coherent (word-hyperbolic and locally quasiconvex) instead.

Lemma 4.3 (Commensurability)**.** *If K is coherent and K is a finite index subgroup of G, then G is coherent. Similarly, if K is word-hyperbolic and locally-quasiconvex and K is a finite index subgroup of G, then G is word-hyperbolic and locally-quasiconvex.*

Proof. It is easy to show that a finite index subgroup of a group is finitely generated (or finitely presented) if and only if the group itself is finitely generated (finitely presented) [13, Proposition 4.2]. Let H be a finitely generated subgroup of G. Since $H \cap K$ is of finite index in H, $H \cap K$ is also finitely generated. But $H \cap K$ is now a finitely generated subgroup of the coherent group K and therefore it is finitely presented. Finally H must be finitely presented since $H \cap K$ is finitely presented. The second assertion is an easy consequence of the definitions and will be omitted. □

Using Lemma 4.2 and Lemma 4.3 we have the following version of Theorem 3.8, which eliminates the hypothesis that the minimal fans be spread-out.

Theorem 4.4. *Let X be a compact weighted 2-complex with a residually finite fundamental group and let $\mathcal{T}$ be a finite collection of perimeter reducing fans in X whose outer paths lift to simple paths in $\widetilde{X}$. Suppose there is a $J \in \mathbb{N}$ such that for every minimal area disc diagram $D \to X$ with boundary cycle PQ^{-1}, either D is J-thin or D contains a perimeter reducing fan in $\mathcal{T}$ whose outer path is a subpath of either P or Q. Then $\pi_1 X$ is word-hyperbolic and locally quasiconvex.*

Proof. Since each fan in $\mathcal{T}$ lifts to a spread-out fan in $\widetilde{X}$, Lemma 4.2 provides a finite cover $\widehat{X}$ in which the lift of each of these fans is spread out. Let $\widehat{\mathcal{T}}$ denote the set of fans $F \to \widehat{X}$ which are lifts of fans $F \to X$ in $\mathcal{T}$. Note that the complex $\widehat{X}$ inherits a weighting from X, and that each fan in $\widehat{\mathcal{T}}$ is perimeter-reducing. Any minimal area disc diagram $D \to \widehat{X}$ projects to a minimal area disc diagram $D \to X$. By hypothesis, $D \to X$ contains a fan $F \to X$ from $\mathcal{T}$, and consequently the lift of $D \to X$ to $D \to \widehat{X}$ contains the lift of $F \to X$ which belongs to $\widehat{\mathcal{T}}$. By Theorem 3.8, $\pi_1 \widehat{X}$ is word-hyperbolic and locally quasiconvex. Since $\pi_1 \widehat{X}$ is a finite index subgroup of $\pi_1 X$, Lemma 4.3 shows that $\pi_1 X$ is a locally quasiconvex word hyperbolic group as well. □

Restricting to the small-cancellation context, we have the following result. Notice that we needed to switch from the Euclidean to the hyperbolic restrictions in order to be able to use Theorem 2.13.

Theorem 4.5. *Let X be a compact weighted $C(p)$-$T(q)$ complex with a residually finite fundamental group, where p, q, and k satisfy the hyperbolic restrictions. If every minimal fan of type k in X is perimeter reducing, then $\pi_1 X$ is locally quasiconvex.*

Proof. Let $\mathcal{T}$ denote the minimal fans of type k in the $C(p)$-$T(q)$ complex X, and note that by Theorem 2.12, $\mathcal{T}$ is finite. By Theorem 4.4, any disc diagram which is not J-thin contains a fan of type k, and hence by Lemma 2.11, it contains a fan

in $\mathcal{T}$. Since p, q, k satisfy the hyperbolic restrictions, Theorem 2.13 states that the outer path of each fan in $\mathcal{T}$ lifts to a simple path in $\tilde{X}$. The result now follows from Theorem 4.4. □

A similar result is clearly possible for coherence with essentially the same proof, which we omit.

Theorem 4.6. *Let X be a compact weighted $C(p)$-$T(q)$ complex with a residually finite fundamental group, where p, q, and k satisfy the hyperbolic restrictions. If every minimal fan of type k is weakly perimeter reducing and satisfies $\mathbf{P}(\widetilde{F}) < \mathbf{P}(\partial F)$, then $\pi_1 X$ is coherent.*

5. Local and global ratios

In this section we define two invariants of a weighted 2-complex which compare the weights of its 2-cells with the perimeters of its pieces. These "ratio" invariants will enable us to quickly conclude that entire collections of fans are perimeter-reducing without performing a case-by-case calculation. These ratios appear to be the most easily computed invariants of a weighted 2-complex that give a qualitative idea of the coherence and local-quasiconvexity properties of its fundamental group. In Sections 8, 9, and 10, we will devote a considerable effort towards constructing examples of weighted 2-complexes with incoherent fundamental groups, whose ratios are as small as possible.

Definition 5.1 (Local and global ratio)**.** Let X be a weighted 2-complex, let E denote the maximum perimeter of a piece in X, and let W denote the minimum value of $n \cdot \mathbf{Wt}(R)$ where R is a 2-cell in X and n is its exponent. We define the *global ratio of* X, denoted $\mathsf{Global}(X)$, to be the fraction $\frac{E}{W}$.

Our second invariant is more precise since it pays attention to where the pieces occur among the boundaries of 2-cells. The *local ratio of* X, denoted $\mathsf{Local}(X)$, is the supremum of $\frac{\mathbf{P}(P)}{n \cdot \mathbf{Wt}(R)}$ where P varies over the pieces in ∂R, R varies over the 2-cells of X, and n is the exponent of R. It is possible for these ratio invariants to equal infinity, but this is not a favorable possibility.

Finally, an even more precise invariant, which we will rarely use, would look at sequences of two or three consecutive pieces in the boundary of R (since these are the precise paths which are involved in the perimeter calculations). Let $\mathsf{Local}_i(X)$ denote the supremum of $\frac{\mathbf{P}(S)}{i \cdot n \cdot \mathbf{Wt}(R)}$ where S is the concatenation of i consecutive pieces in ∂R, R is a 2-cell in X and n is its exponent. Using this notation, $\mathsf{Local}(X) = \mathsf{Local}_1(X)$, and $\mathsf{Local}(X) \geq \mathsf{Local}_i(X)$ for all $i \geq 1$.

Definition 5.2 (Homogeneous)**.** We say X is *homogeneous* if $n \cdot \mathbf{Wt}(R)$ is a constant independent of the 2-cell R in X. Notice that $\mathsf{Global}(X) \geq \mathsf{Local}(X)$ and that $\mathsf{Global}(X) = \mathsf{Local}(X)$ whenever X is homogeneous.

Since fans of type 1 are single i-shells or spurs, they are trivially spread out. Therefore the case $k = 1$ of Theorem 3.10 can be stated as follows:

Theorem 5.3 (Local quasiconvexity)**.** *Let X be a compact weighted $C(p)$-$T(q)$-complex. If $p = 6$ and $q = 3$ [$p = q = 4$] and for each 2-cell $R \to X$ the perimeter of any three [two] consecutive pieces in the boundary of R is strictly less than $n \cdot \mathbf{Wt}(R)$ where n is its exponent, then $\pi_1 X$ is locally quasiconvex.*

We can restate Theorem 5.3 using the local ratio notation.

Theorem 5.4. *Let X be a weighted $C(p)$-$T(q)$-complex. If either $p = 6$, $q = 3$, and $\mathsf{Local}_3(X) < \frac{1}{3}$, or $p = q = 4$ and $\mathsf{Local}_2(X) < \frac{1}{2}$, then $\pi_1 X$ is locally quasiconvex.*

Sections 6 and 7 are devoted to showing that under stronger small-cancellation assumptions (using $k = 2$ or 3), even higher local and global ratios suffice to guarantee local quasiconvexity. Section 11 will investigate global ratios using larger values of k.

6. Global ratio results

In order to apply Theorem 3.10 or Theorem 4.5, we must ensure that all of the minimal fans of type k are perimeter-reducing. In this section we give a simple inequality involving $\mathsf{Global}(X)$ and the number of 2-shells, 3-shells, and 4-shells in a fan $F \to X$. Whenever this inequality is satisfied the fan F is perimeter-reducing. We then apply this result to find upper bounds on $\mathsf{Global}(X)$ that imply that various pointed fans and broad fans are perimeter reducing. The same statements will be proven for $\mathsf{Local}(X)$ in Section 7, but the proofs are more delicate.

Lemma 6.1 (Global perimeter calculation)**.** *Let X be a weighted 2-complex, let $F \to X$ be a fan in which every 2-cell is either a 2-shell, 3-shell or 4-shell. If b, c, and d denote the numbers of 2-shells, 3-shells, and 4-shells in F, and $\mathsf{Global}(X) < \frac{b+c+d}{b+2c+3d+1}$, then $F \to X$ is a perimeter-reducing fan.*

Proof. The total number of 2-cells in F is $b + c + d$, it has $b + c + d - 1$ maximal internal arcs, and the inner path of F is the concatenation of $2+0b+1c+2d$ pieces. Thus the inner portion of F can be decomposed into $b + 2c + 3d + 1$ pieces. Let E denote the maximum perimeter of a piece, let W denote the minimum value of $n \cdot \mathbf{Wt}(R)$ and observe that

$$\sum \mathbf{P}(P) \leq E(b + 2c + 3d + 1) \quad \text{and} \quad W(b + c + d) \leq \sum_{R \in \mathsf{Cells}(\widetilde{F})} \mathbf{Wt}(R)$$

where the first sum is taken over all pieces in the inner portion of F. Therefore if $E(b+2c+3d+1) < W(b+c+d)$, then $F \to X$ is perimeter-reducing. Rearranging the terms we find that $\mathsf{Global}(X) = \frac{E}{W} < \frac{b+c+d}{b+2c+3d+1}$ is sufficient. □

Notice that when X is homogeneous and the inequality in Lemma 6.1 fails to hold, then the fan $F \to X$ is definitely not perimeter-reducing. To illustrate Lemma 6.1 we record the special cases of pointed fans and q-separated broad fans below. The first one is immediate.

Corollary 6.2 (Pointed fans)**.** *Let X be a weighted 2-complex and let $F \to X$ be a pointed fan with i 2-shells. If $\mathsf{Global}(X) < \frac{i}{i+1}$, then $F \to X$ is a perimeter-reducing fan.*

Corollary 6.3 (Broad fans)**.** *Let X be a weighted 2-complex and let $F \to X$ be a k-separated broad fan with j 3-shells. If $\mathsf{Global}(X) < \frac{(k+1)(j+1)-1}{(k+2)(j+1)-1}$, then $F \to X$ is a perimeter-reducing fan.*

Proof. Since F is k-separated it contains at least $k(j + 1)$ 2-shells. Thus if b is the number of 2-shells in F, $b = k(j + 1) + \ell$ for some $\ell \geq 0$. By Lemma 6.1 a global

ratio bounded above by $\frac{b+j}{b+2j+1}$ is sufficient, and it is easy to check that

$$\frac{b+j}{b+2j+1} = \frac{k(j+1)+\ell+j}{k(j+1)+\ell+2j+1} \geq \frac{k(j+1)+j}{k(j+1)+2j+1} = \frac{(k+1)(j+1)-1}{(k+2)(j+1)-1}.$$

□

Using these corollaries, it is straightforward to derive perimeter reduction results for all fans of types 2 and 3.

Lemma 6.4. *Let X be a weighted $C(p)$-$T(q)$ complex. Every fan $F \to X$ of type 2 is perimeter-reducing when either $p = 3$, $q \geq 6$ and $\mathsf{Global}(X) < \frac{q-4}{q-3}$, or $p \geq 4$, $q \geq 4$ and $\mathsf{Global}(X) < \frac{q-3}{q-2}$.*

Proof. This is obvious when F is a spur, 0-shell, or 1-shell, and follows from Corollary 6.2 when F is a pointed fan with $i \geq q-4$ [$i \geq q-3$]. □

Lemma 6.5. *Let X be a weighted $C(p)$-$T(q)$ complex. Every fan $F \to X$ of type 3 is perimeter-reducing when either $p \geq 6$, $q = 3$ and $\mathsf{Global}(X) < \frac{p-5}{2p-9}$, or $p \geq 4$, $q \geq 4$ and $\mathsf{Global}(X) < \frac{(q-2)(p-3)-1}{(q-1)(p-3)-1}$.*

Proof. All of the fans of type 3 are either obviously perimeter-reducing given these restrictions, or they are perimeter-reducing by Corollary 6.2 or Corollary 6.3. □

Combining these with Theorem 3.10 gives new results on local quasiconvexity.

Theorem 6.6. *Let X be a compact weighted $C(p)$-$T(q)$ complex in which all minimal fans of type 2 are spread out. Then $\pi_1 X$ is locally quasiconvex when either $p = 3$, $q \geq 6$ and $\mathsf{Global}(X) < \frac{q-4}{q-3}$, or $p \geq 4$, $q \geq 4$ and $\mathsf{Global}(X) < \frac{q-3}{q-2}$.*

Proof. The restrictions on p and q are the Euclidean restrictions with $k = 2$ and by Lemma 6.4 all fans of type 2 are perimeter reducing. Thus the result follows from Theorem 3.10. □

Theorem 6.7. *Let X be a compact weighted $C(p)$-$T(q)$ complex in which all minimal fans of type 3 are spread out. Then $\pi_1 X$ is locally quasiconvex when either $p \geq 6$, $q = 3$ and $\mathsf{Global}(X) < \frac{p-5}{2p-9}$, or $p \geq 4$, $q \geq 4$ and $\mathsf{Global}(X) < \frac{(q-2)(p-3)-1}{(q-1)(p-3)-1}$.*

Proof. The restrictions on p and q are the Euclidean restrictions with $k = 3$ and by Lemma 6.5 all fans of type 3 are perimeter reducing. Thus the result follows from Theorem 3.10. □

As usual, the assumption that various fans are spread out can be replaced with the assumption that $\pi_1 X$ is residually finite. The proofs are the same but with Theorem 4.5 used in place of Theorem 3.10. Also note that this change requires us to switch from the Euclidean to the hyperbolic restrictions.

Theorem 6.8. *Let X be a compact weighted $C(p)$-$T(q)$ complex with a residually finite fundamental group. Then $\pi_1 X$ is locally quasiconvex when either $p = 3$, $q > 6$ and $\mathsf{Global}(X) < \frac{q-4}{q-3}$, or $p \geq 4$, $q > 4$ and $\mathsf{Global}(X) < \frac{q-3}{q-2}$.*

Theorem 6.9. *Let X be a compact weighted $C(p)$-$T(q)$ complex with a residually finite fundamental group. Then $\pi_1 X$ is locally quasiconvex when either $p > 6$, $q = 3$ and $\mathsf{Global}(X) < \frac{p-5}{2p-9}$, or $p > 4$, $q \geq 4$ and $\mathsf{Global}(X) < \frac{(q-2)(p-3)-1}{(q-1)(p-3)-1}$.*

Finally, we conclude this section with a remark about limiting values.

Remark 6.10 (Limiting values). Consider the fractions $\frac{p-5}{2p-9} = \frac{1}{2} - \frac{1}{4p-18}$ and $\frac{(q-2)(p-3)-1}{(q-1)(p-3)-1} = 1 - \frac{1}{q-1} - \frac{1}{q-1} \cdot \frac{1}{(q-1)(p-3)-1}$. As p increases, the first fraction converges to $\frac{1}{2}$ and the second to $1 - \frac{1}{q-1}$. Similarly, as q increases, the first is unchanged and the second approaches 1. We will return to these limiting values in Section 11.

7. Local ratio results

In this section we show how Corollary 6.2 and Corollary 6.3 remain true when the global ratio of X is replaced by the local ratio of X. Since the weights of the 2-cells and the perimeters of the pieces can vary tremendously, an averaging process using "coefficients" must be employed to transition from the local estimate given by the local ratio to the global estimate required by a perimeter reduction. We begin with a definition.

Definition 7.1 (Sides along arcs)**.** Let D be a weighted disc diagram and let R be a 2-cell in D. If P is a subpath of ∂R and $P \to R \to D$ is a maximal arc in D, then (R, P) is a *side of R along the arc P*. The weight of this side is the sum of the weights of the sides (R, r) where r is a 1-cell in P. Notice that to assign a particular weight to the side (R, P), it is sufficient to assign weights to the sides (R, r) arbitrarily so long as the total is the desired one.

Lemma 7.2 (Pointed fan coefficients)**.** *Let $F \hookrightarrow D$ be a pointed fan in a disc diagram, and suppose that F contains exactly i 2-shells. Then we can assign non-negative coefficients to the sides of 2-cells of F along arcs in the inner portion of F such that*

1. *there is a coefficient of* 1 *along each arc in the inner path of F;*
2. *the sum of the coefficients on the two sides of each maximal internal arc is* 1*;*
3. *for each 2-cell of F, the sum of its two coefficients is $\frac{i+1}{i}$.*

Proof. Assume that the fan F is oriented so that the 2-cells of F are arranged from left to right with the outer path of F on top. Assign a coefficient of 1 to the sides along arcs in the inner path of F. Next, the left and right sides of the j-th 2-cell counting from the left will be assigned coefficients of $\frac{i+1-j}{i}$ and $\frac{j}{i}$, respectively. It is easy to check that these coefficients satisfy the conditions. On the left of Figure 4 is a pointed fan (outside its diagram) with its corresponding coefficients. (The right side of the figure will be used below in the proof of Lemma 7.5.) □

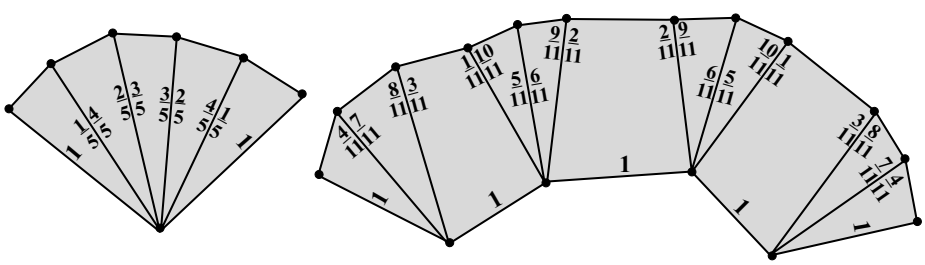

Figure 4. Coefficients on pointed and broad fans

Lemma 7.3 (Pointed fan reductions)**.** *Let X be a weighted 2-complex and let $F \to X$ be a pointed fan with i 2-cells. If $\mathsf{Local}(X) < \frac{i}{i+1}$, then F is perimeter-reducing.*

Proof. Let P be a maximal internal arc which lies between the 2-cells R_1 and R_2 of F with exponents n_1 and n_2, and let c be the constant $\frac{i+1}{i}$. By hypothesis, $n_1 \cdot \mathbf{Wt}(R_1) > c \cdot \mathbf{P}(P)$ and $n_2 \cdot \mathbf{Wt}(R_2) > c \cdot \mathbf{P}(P)$. It follows that if $a, b \geq 0$ and $a + b = 1$, then

$$a \cdot n_1 \cdot \mathbf{Wt}(R_1) + b \cdot n_2 \cdot \mathbf{Wt}(R_2) > c \cdot \mathbf{P}(P). \tag{4}$$

For each maximal internal arc we choose the constants a and b to be the coefficients which satisfy the conclusion of Lemma 7.2. If we add up all of the resulting inequalities for each maximal internal arc and the inequality $1 \cdot n \cdot \mathbf{Wt}(R) > c \cdot \mathbf{P}(P)$ for each piece in the inner path, we find that

$$c \cdot \sum_{R \in \mathrm{Cells}(F)} n \cdot \mathbf{Wt}(R) > c \cdot \sum \mathbf{P}(P)$$

where the second sum is taken over the disjoint pieces P in the inner portion of the fan F. Note that the second statement in Lemma 7.2 is used to guarantee that the sum of the two coefficients appearing with each $n \cdot \mathbf{Wt}(R)$ is c. Cancelling the constant c completes the proof. □

Similar results hold for broad fans.

Lemma 7.4 (Broad fan coefficients)**.** *Let $F \hookrightarrow D$ be a k-separated broad fan in a disc diagram, and suppose that F contains exactly j 3-shells. Then we can assign nonnegative coefficients to the 'sides' of 2-cells of F along arcs in the inner portion of F such that*

1. *there is a coefficient of 1 along each arc in the inner path of F;*
2. *the sum of the coefficients on the two sides of each maximal internal arc is 1;*
3. *for each 2-shell [3-shell], the sum of its two [three] coefficients is bounded by $\frac{(k+2)(j+1)-1}{(k+1)(j+1)-1}$.*

Proof. We assign a coefficient of 1 along each arc in the inner path of F so that Condition 1 is satisfied. Suppose that F is oriented so that the 2-cells of F are arranged from left to right with the outer path of F on top. The left and right sides of the i-th 3-shell in F will be assigned coefficients of $\frac{j+1-i}{(k+1)(j+1)-1}$ and $\frac{i}{(k+1)(j+1)-1}$, respectively. Choose coefficients on the other sides of the maximal internal arcs so that the total is 1. Notice that the sum of the coefficients on the 3-shells is exactly the bound in Condition 3 since $1 + \frac{j+1-i}{(k+1)(j+1)-1} + \frac{i}{(k+1)(j+1)-1} = \frac{(k+2)(j+1)-1}{(k+1)(j+1)-1}$.

We will now assign coefficients to the sides of 2-shells along maximal internal arcs of F. Consider a sequence of $k+1$ maximal internal arcs bounding k consecutive 2-shells, and suppose that the right side of the leftmost maximal internal arc has a coefficient of c and the right side of the rightmost maximal internal arc has a coefficient of d. Then the coefficients of the sides to the right of each of the intermediate maximal internal arcs will be chosen so that they are the intermediate terms of an arithmetic progression from c to d, and the coefficients of the sides to the left of each maximal internal arc will then be chosen so that the sum of the two coefficients at each maximal internal arc is 1. Note that using the coefficients

to the left to interpolate would have yielded the same set of coefficients. Since the interpolation is between positive numbers, the coefficients are all positive, and Condition 2 is true by definition. It is now easy to check that when there are exactly k 2-shells between adjacent 3-shells, the bound of Condition 3 is exact, and that additional 2-shells make the bound easier to achieve. Since F is k-separated, the sum of the coefficients in each 2-shell stays within the bound of Condition 3. This completes the proof. □

On the right of Figure 4 is a broad fan (outside its diagram) with its corresponding coefficients. This broad fan is 2-separated and has three 3-shells. We can use Lemma 7.4 to prove a result similar to Lemma 7.3. Since the proof is nearly identical, it will be omitted.

Lemma 7.5 (Broad fan reductions)**.** *Let X be a weighted 2-complex and let $F \to X$ be a k-separated broad fan with j 3-shells. If $\mathsf{Local}(X) < \frac{(k+1)(j+1)-1}{(k+2)(j+1)-1}$, then F is perimeter-reducing.*

Now that Lemma 7.3 and Lemma 7.5 have been established, all of the results in Section 6 which followed from Corollary 6.2 and Corollary 6.3 (namely, Lemma 6.4 through Theorem 7.9) remain true as written when each occurence of $\mathsf{Global}(X)$ in their statements is replaced with $\mathsf{Local}(X)$. Thus we have shown the following.

Theorem 7.6. *Let X be a compact weighted $C(p)$-$T(q)$ complex in which all minimal fans of type 2 are spread out. Then $\pi_1 X$ is locally quasiconvex when either $p = 3$, $q \geq 6$ and $\mathsf{Local}(X) < \frac{q-4}{q-3}$, or $p \geq 4$, $q \geq 4$ and $\mathsf{Local}(X) < \frac{q-3}{q-2}$.*

Theorem 7.7. *Let X be a compact weighted $C(p)$-$T(q)$ complex in which all minimal fans of type 3 are spread out. Then $\pi_1 X$ is locally quasiconvex when either $p \geq 6$, $q = 3$ and $\mathsf{Local}(X) < \frac{p-5}{2p-9}$, or $p \geq 4$, $q \geq 4$ and $\mathsf{Local}(X) < \frac{(q-2)(p-3)-1}{(q-1)(p-3)-1}$.*

Theorem 7.8. *Let X be a compact weighted $C(p)$-$T(q)$ complex with a residually finite fundamental group. Then $\pi_1 X$ is locally quasiconvex when either $p = 3$, $q > 6$ and $\mathsf{Local}(X) < \frac{q-4}{q-3}$, or $p \geq 4$, $q > 4$ and $\mathsf{Local}(X) < \frac{q-3}{q-2}$.*

Theorem 7.9. *Let X be a compact weighted $C(p)$-$T(q)$ complex with a residually finite fundamental group. Then $\pi_1 X$ is locally quasiconvex when either $p > 6$, $q = 3$ and $\mathsf{Local}(X) < \frac{p-5}{2p-9}$, or $p > 4$, $q \geq 4$ and $\mathsf{Local}(X) < \frac{(q-2)(p-3)-1}{(q-1)(p-3)-1}$.*

8. $C(p)$-$T(3)$ COUNTEREXAMPLES

In the next three sections we present several families of groups which are incoherent and/or not locally quasiconvex. The examples presented in this section will be derived from $C(p)$-$T(3)$ complexes, those in the next section will be derived from $C(p)$-$T(4)$ complexes, and those in Section 10 will be derived from $C(4)$-$T(q)$ complexes. These examples demonstrate the limits to which results about the behavior of small-cancellation groups can be pushed. We begin with the standard example of an incoherent finitely presented group. Finally, we note that all of the weighted 2-complexes constructed in these sections are homogeneous (Definition 5.1) and thus the global ratio is the same as the local ratio throughout.

Example 8.1 ($F_2 \times F_2$)**.** An early example of an incoherent group due to Baumslag, Boone, and Neumann (see [2]) is the direct product $F_2 \times F_2$ of two free groups. Its

usual presentation

$$\langle a, b, x, y \mid [a, x], [a, y], [b, x], [b, y] \rangle$$

is readily seen to satisfy the $C(4)$-$T(4)$ conditions. If we let $\phi\colon F_2 \times F_2 \to \mathbb{Z}$ be the homomorphism induced by mapping each of the generators of $F_2 \times F_2$ to $1 \in \mathbb{Z}$, then $\mathsf{kernel}(\phi)$ is finitely generated but does not have a finite presentation.

The unit perimeter on the standard 2-complex X of the presentation above (i.e. all sides have weight 1) gives a weight of 4 for each 2-cell, and the perimeter of each 1-cell is 4. Thus $\mathsf{Global}(X) = \mathsf{Local}(X) = \mathsf{Local}_2(X) = 1$ and the hypothesis of Theorem 5.4 fails. This is because the sum of the unit perimeters of two consecutive pieces is 8, which is greater than the weight of each 2-cell.

There is another $C(4)$-$T(4)$ 2-complex whose fundamental group is $F_2 \times F_2$ which comes closer to satisfying the conditions of the theorem. Let θ be the graph with two 0-cells and three 1-cells in which each 1-cell connects one 0-cell to the other. Let Y be the complex $\theta \times \theta$. Then $\pi_1 Y \cong F_2 \times F_2$, and it is easy to check that Y is a $C(4)$-$T(4)$-complex. The unit perimeter on Y assigns a perimeter of 3 to each 1-cell and a weight of 4 to each 2-cell. Again, since $4 \not\geq 6$, this 2-complex fails our hypothesis, even though the weight of each 2-cell is strictly greater than the perimeter of each piece and $\mathsf{Global}(Y) = \mathsf{Local}(Y) = \mathsf{Local}_2(Y) = 3/4$.

The first examples of incoherent small-cancellation groups satisfying stronger small-cancellation conditions were constructed by Rips in [16]. We will not discuss Rips's examples here except to note that they fail to satisfy the hypotheses of our coherence theorems, and that they fail these criteria more severely than the various families of examples presented below.

Definition 8.2. Let $\{a_1, \ldots, a_n\}$ be a set of distinct letters. We define a_{ij} to be the unique letter a_k with $k \equiv \binom{i}{2} + j \pmod{n}$, and for $j = 1, 2, \ldots, n$, we define the words

$$W_j = a_{1j} a_{2j} \ldots a_{nj}, \qquad W'_j = a_{2j} a_{3j} \ldots a_{nj}. \tag{5}$$

The notation of Definition 8.2 will be used in all of our examples without further comment. The main property of this set of words is that all of the two-letter subwords are distinct.

Lemma 8.3 (No two-letter pieces)**.** *Let $W_1, W_2, \ldots, W_n$ be the n words of length n defined in Equation* (5)*. Every two-letter sequence $a_i a_j$ (with $1 \leq i, j \leq n$ and $i \neq j$) occurs in exactly one of these words and in exactly one place in that word.*

Proof. Since the sequence of differences between subscripts of consecutive letters in each word is the sequence $1, 2, \ldots, (n-1)$, we see that a two-letter subword $a_i a_j$ can only occur beginning on the $(j-i)^{th}$ letter of some word. However, the n words are uniquely determined by their $(j-i)^{th}$ letters, and so $a_i a_j$ can only occur in exactly one word and in exactly one place in that word. Conversely, it will occur at that location. □

Example 8.4 (Not locally quasiconvex)**.** Our first family of examples will be $C(p)$-$T(3)$ presentations which are not locally quasiconvex and have a local ratio of 1. For a fixed n, consider the presentation

$$G = \langle a_1, \ldots, a_n, t \mid \{(a_{1r})^t = W_r : 1 \leq r \leq n\} \rangle$$

and let X be the standard 2-complex of this presentation. Notice that G is an ascending HNN extension of a finitely generated free group. It follows from Lemma 8.3

that all the pieces in this presentation have length 1 and that the presentation therefore satisfies $C(p)$-$T(3)$ where $p = n + 3$.

Consider the weight function which assigns a weight of 0 to each side at the 1-cell labeled t, and assigns a weight of 1 to all of the other sides. The weight of each 2-cell is $n+1$, and the perimeter of each piece is $n+1$ or 0 depending on whether its label is a_i or t. Thus $\mathsf{Global}(X) = \mathsf{Local}(X) = 1$. This fails to satisfy the conditions of Theorem 7.9, since $\mathsf{Local}(X)$ would have needed to be at most $\frac{1}{2} - \frac{1}{4p-18}$.

We will show that G is not locally quasiconvex by exhibiting a specific subgroup which is finitely generated but not quasiconvex. Let H be the subgroup generated by the set $A = \{a_1, \ldots, a_n\}$. To see that H is not quasiconvex we consider the set of equalities $\{a_1^{t^n} = \phi^n(a_1)\}$ ordered by n where ϕ is the endomorphism of H induced by conjugation by t. Note that the length of $\phi^n(a_1)$ grows exponentially, whereas the length of $a_1^{t_n}$ grows linearly. Since A is a free group and $\phi^n(a_1)$ is reduced, $\phi^n(a_1)$ is a geodesic in H relative to the generating set A. Since this diverges exponentially from its geodesic length in G relative to the generating set $A \cup \{t\}$, H cannot be a quasiconvex subgroup of G.

More generally, let $\phi\colon F \to F$ be an injective endomorphism of a finitely generated free group, and let G be the mapping torus of ϕ. If G is word-hyperbolic, then it is not difficult to show that A is not a quasiconvex subgroup of G. For instance, it follows from [9] that F is not quasiconvex because it has infinite height, which means roughly that F has infinitely many distinct conjugates that intersect. As mentioned in the introduction, Feighn and Handel have recently shown in [7] that these groups are indeed coherent.

A slight modification of the presentations given above will produce incoherent examples. To show that these examples are incoherent, we will need the following theorem from [18].

Theorem 8.5. *Let $G = \langle a_1, \ldots, a_m, t \mid a_1^t = U_1, \ldots, a_m^t = U_m, V\rangle$ where U_i and V are words in $a_i^{\pm 1}$, and let X be the standard 2-complex of the presentation. If X is aspherical, then the subgroup $H = \langle a_1, \ldots, a_m\rangle$ is not finitely presented.*

Example 8.6 (Incoherent)**.** Our second family of examples are $C(p)$-$T(3)$ presentations which are incoherent and have a local ratio of 1. Let n be an even number and consider the presentation

$$G = \langle a_1, \ldots, a_n, t \mid \big\{(a_{1r})^t = W_r' : 1 \le r \le n\big\}, a_1 a_2^{-1} a_3 a_4^{-1} \cdots a_n^{-1}\rangle.$$

Notice that this presentation differs from the presentation in Example 8.4 by the addition of a single relator. As before, it is easy to check that all the pieces of this presentation have length 1, and that the presentation satisfies $C(p)$-$T(3)$ for $p = n$. If we assign a weight of 0 to all of the sides of the 1-cell labeled t and a weight of 1 to all of the other sides, then the weight of each 2-cell will be n and the perimeter of each piece will be either n or 0. Thus the local ratio of this presentation is 1. The words W_r have been replaced by W_r' since these words give a slightly lower ratio between the perimeter of a piece and the weight of a 2-cell. Finally, since this presentation satisfies the conditions of Theorem 8.5, the subgroup H generated by the a_i is finitely generated but not finitely presented, and thus G is incoherent.

In contrast to the examples given above, our best theorems along these lines are valid only for residually finite groups whose presentations have local ratios less than $\frac{1}{2}$. Since we have been unable to narrow this gap we pose the following question.

Problem 8.7. Is there a sequence of incoherent groups given by small-cancellation presentations where the (local or global) ratios of the presentations approach the values given in Theorem 7.8 and Theorem 7.9? For example, is there a sequence of examples X_p such that for each p, X_p is a weighted $C(p)$-$T(3)$ complex, $\pi_1 X_p$ is incoherent, and $\lim_{p\to\infty} \mathsf{Local}(X_p) = \frac{1}{2}$? The same question can be posed with respect to local quasiconvexity. Alternatively, can the values in Theorem 7.8 and Theorem 7.9 be raised in this regard?

9. $C(p)$-$T(4)$ COUNTEREXAMPLES

For $C(p)$-$T(4)$ complexes we are able to obtain slightly stronger results. The next family of examples is residually finite but not locally quasiconvex. The residual finiteness of their fundamental groups will be a consequence of the following theorem from [19]. Recall that the condition $C'(1/6)$ means that if P is a piece in ∂R, then the length of P is less than one-sixth of the length of ∂R.

Theorem 9.1. *If $\langle a_1, \dots | W_1, \dots \rangle$ is a finite $C'(1/6)$ presentation in which all of the W_i are positive words with the same length, then the group is residually finite.*

The finite generation will be a consequence of the following theorem.

Theorem 9.2. *Let X be a cell complex, let $\widetilde{X}$ be its universal cover, let $\phi : \pi_1 X \to \mathbb{Z}$ be an epimorphism and let $H = \mathsf{kernel}(\phi)$. If there is a ϕ-equivarient Morse function $f : \widetilde{X} \to \mathbb{R}$, such that the ascending and descending links of this map are connected but not simply connected, then H is a finitely generated but not finitely presented subgroup of $\pi_1 X$.*

For a proof of this theorem, see [6, Theorem 4.7], and for background and definitions, see [3, Theorem 4.1].

Example 9.3 (Not locally quasiconvex)**.** Our third family of examples consists of $C(p)$-$T(4)$ presentations whose groups are residually finite but not locally quasiconvex and which have a local ratio of $1 - \frac{1}{p}$. For a fixed n, consider the presentation

$$G = \big\langle a_1, \dots, a_n, x \mid \{W_r = x : 1 \le r \le n\} \big\rangle$$

and let X be the 2-complex of this presentation. It follows from Lemma 8.3 that all of the pieces in this presentation have length 1, and thus the presentation satisfies $C(p)$ for $p = n + 1$. The presentation also satisfies $T(4)$ since substituting x^{-1} for x would yield a presentation in which each relator is a positive word. Using the unit weighting, the weight of each 2-cell is $n + 1$ and the perimeter of each 1-cell is n. Thus $\mathsf{Global}(X) = \mathsf{Local}(X) = 1 - \frac{1}{p}$. For $p > 6$, the finitely presented group is residually finite by Theorem 9.1.

We will now show that G is not locally quasiconvex by exhibiting a specific subgroup which is finitely generated but not quasiconvex. Let $\phi\colon G \to \mathbb{Z}$ denote the surjective homomorphism induced by $a_i \mapsto 1, x \mapsto n$. By Theorem 9.2 or [4], $\mathsf{kernel}(\phi)$ is finitely generated. In order to apply Theorem 9.2, we need to show that the ascending and descending links of ϕ (relative to the presentation of G) are connected. Since the 1-cells in the ascending link correspond to the corners $x^{-1}a_{1r}$ for $1 \le r \le n$, this shows that the ascending link is connected. Similarly, the 1-cells in the descending link correspond to the corners $a_{nr}x^{-1}$ for $1 \le r \le n$. This shows that the descending link is connected. To see that $\mathsf{kernel}(\phi)$ is not quasiconvex,

note that it is an infinite normal subgroup of infinite index in a word-hyperbolic group. By [1] or [9] such subgroups cannot be quasiconvex.

Finally, we note that since the ascending and descending links are trees, Theorem 9.2 allows us to conclude that $\mathsf{kernel}(\phi)$ is a finitely generated free group F. Consequently $G \cong F \rtimes \mathbb{Z}$. This affirmatively answers a question of Gersten's, who asked in [8] whether an $F \rtimes \mathbb{Z}$ group could ever be the fundamental group of a compact negatively curved 2-complex.

Example 9.4 (Incoherent)**.** A slight modification of the presentations in Example 9.3 provides examples of $C(p)$-$T(4)$ complexes with local ratio 1 whose fundamental groups are residually finite but incoherent. Let X be the 2-complex of the presentation

$$\langle a_1, \ldots, a_n, x \mid \{W_r = x : 1 \leq r \leq n\}\rangle$$

from Example 9.3. Similarly, let Y and Z be additional copies of this 2-complex but with different labels on the 1-cells. In particular, we label the 1-cells of Y and Z using the letters $\{b_1, \ldots, b_n, y\}$ and $\{c_1, \ldots, c_n, z\}$ respectively. We now form a new complex C which is the union of X, Y, and Z obtained by identifying three pairs of 1-cells. The three identifications are $a_1 = b_1$, $b_2 = c_2$, and $c_3 = a_3$. To see that all of the pieces have length 1, note that if a two-letter word occurs in two distinct relators, then they must belong to two of the three subpresentations corresponding to X, Y, and Z. But any two of the subpresentations have only a single generator in common. Consequently, the two-letter subword must be of the form uu for some generator u. This is impossible because each letter appears at most once in each relator. Thus C is a $C(p)$ complex with $p = n+1$. The presentation corresponding to C will be $T(4)$ as before since the 1-cells can be oriented so that each of the relators is a positive word. By Theorem 9.1, the group $\pi_1 C$ is residually finite for $p > 6$.

If we place a weight of 1 at sides which are incident at any of the 1-cells $a_1 = b_1$, $b_2 = c_2$, or $c_3 = a_3$, and we place a weight of 2 at all other sides, then it is easy to check that the weight of each 2-cell is $2n$ and that the perimeter of each 1-cell is $2n$. Thus $\mathsf{Global}(X) = \mathsf{Local}(X) = 1$. We will now show that G is incoherent by exhibiting a specific subgroup which is finitely generated but not finitely presented. Consider the homomorphism $\phi\colon \pi_1 C \to \mathbb{Z}$ defined so that its restriction to the subgroups corresponding to X, Y, and Z are the same homomorphisms as in Example 9.3. The ascending and descending links of ϕ (relative to this presentation) are connected but not simply-connected, and therefore by Theorem 9.2, $\mathsf{kernel}(\phi)$ is finitely generated but not finitely presented.

In contrast to the examples given above, our best theorems along these lines are valid only for residually finite presentations with a local ratio less than $\frac{2}{3}$. This gap is smaller than the previous one, but we have not been able to close it either. Thus Problem 8.7 can also be asked for $C(p)$-$T(4)$ complexes.

10. $C(4)$-$T(q)$ COUNTEREXAMPLES

With our final family of examples we show that for $C(4)$-$T(q)$ complexes, Theorem 7.6 is asymptotically sharp as q increases. In particular, we will establish that for every q, there is a compact $C(4)$-$T(q)$ complex X' in which all minimal fans of type 2 are spread out, with a local ratio of 1, and an incoherent fundamental group. This is in sharp contrast with Theorem 7.6, which shows that a compact

$C(4)$-$T(q)$ complex in which all minimal fans of type 2 are spread out and with a local ratio less than $1 - \frac{1}{q-2}$ must have a coherent fundamental group. Thus, as q increases, the gap between the theorem and these counterexamples disappears completely. The construction of these complexes begins with a presentation which violates our assumption that attaching maps are immersions.

Definition 10.1 (X and Γ)**.** Let X be the 2-complex of the presentation $\langle a \mid a^2a^{-2}\rangle$ and let R be its unique 2-cell. Inside X we will define a graph Γ, but note that the vertices and edges of Γ will not be 0-cells or 1-cells in X. The graph Γ has two vertices: one at the center of R and one at the center of the unique 1-cell of X. Also, it has four edges connecting these two vertices, corresponding to the four sides of R. When the 0-cell of X is removed, the complement deformation-retracts onto Γ. Figure 5 illustrates what the graph Γ looks like when pulled back to R via its characteristic map.

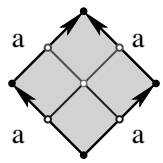

FIGURE 5. The graph Γ in the 2-cell R before it is attached to the 1-skeleton of X

The subpath aa^{-1} is the *ascending portion* of ∂R and the subpath $a^{-1}a$ is the *descending portion*. These will define *ascending and descending subgraphs* of Γ. Specifically, let Γ_A [respectively Γ_D] denote the subgraph of Γ consisting of the union of the two vertices of Γ and the two edges of Γ which correspond to the sides of R along the ascending [descending] portion of ∂R.

Each covering map $\Gamma' \to \Gamma$ uniquely determines a branched cover $X' \to X$ such that $X' - (X')^{(0)}$ deformation-retracts onto Γ' and the branching only occurs at the 0-cell of X. Let v denote the 0-cell of X and observe that 'pushing out from v' yields an immersion $\mathsf{Link}(v) \to \Gamma'$ which sends each edge of $\mathsf{Link}(v)$ to a length 2 path in Γ. Similarly, given a 0-cell u in X', there is an immersion $\mathsf{Link}(u) \to \Gamma$ which sends each edge of $\mathsf{Link}(u)$ to a length 2 path in Γ'.

The next two lemmas relate various properties of Γ' to properties of the corresponding branched cover X'. We first need a few definitions from graph theory.

Definition 10.2 (Graphs)**.** Let Γ be a finite graph. It has *girth at least g* if it does not contain a cycle of length less than g. It is *d-regular* if every vertex has valence exactly d. A *Hamiltonian cycle* in Γ is a cycle which passes through each vertex exactly once. A pair of Hamiltonian cycles in Γ which do not contain an edge in common is called *edge-disjoint*.

Lemma 10.3. *Let $\Gamma' \to \Gamma$ be a covering map and let X' be the corresponding branched cover of X. If Γ' has girth at least $2q$, $q \geq 4$, then X' is a $C(4)$-$T(q)$ complex whose attaching maps are immersions and with* $\mathsf{Global}(X') = \mathsf{Local}(X') = 1$ *using the unit perimeter. In addition, all minimal fans of type 2 are spread out in X'.*

Proof. If X' had a 2-cell whose attaching map was not an immersion, then Γ' would contain a cycle of length 4. Similarly, if X' contained a piece of length 2, then Γ' would contain a cycle of length 4. Since Γ' has no cycles of length less than $2q$, $q \geq 4$, all pieces in X' have length 1, its attaching maps are immersions, and X' satisfies the $C(4)$ condition. Moreover, since all pieces in X' are length 1, the perimeter of a piece in X' is 4, the weight of a 2-cell in X' is 4, and thus $\mathsf{Global}(X') = \mathsf{Local}(X') = 1$. Let u be a 0-cell in X'. Since Γ' has no cycles of length $\leq 2q$, the graph $\mathsf{Link}(u)$ has no cycles of length $\leq q$, and consequently X' satisfies the $T(q)$ condition.

Finally, every fan of type 2 is either a spur, a 0-shell, a 1-shell, or a pointed fan with $i \geq q-3$. The first three instances are trivially spread out in X' and the fourth contains a pointed fan with $i = q-3$. Let $F \to X'$ be a pointed fan in X with $i = q-3$. If F is not spread out, then two of the sides along Q are sent to identical sides in X' and, in particular, two of the 2-cells in F are sent to identical 2-cells in X'. Next, consider the preimage of Γ' in F. The two 2-cells in F which are sent to identical 2-cells in X' will contain preimages of the same vertex v, and they will be connected in this preimage by a path of length at most $2(q-4)$ edges. Since F has minimal area, this path is sent to a closed immersed path in Γ'. This shows that Γ' contains a closed immersed cycle of length less than $2q$, contradicting our assumptions about Γ'. We can thus conclude that all minimal fans of type 2 are spread out in X'. □

Lemma 10.4. *Let $\Gamma' \to \Gamma$ be a covering map and let X' be the corresponding branched cover of X. If the preimages in Γ' of Γ_A and Γ_D form a pair of edge-disjoint Hamiltonian cycles in Γ', then $\pi_1 X'$ is incoherent.*

Proof. From the orientation of the 1-cell of X, there is an induced orientation on the 1-cells of X', and each 2-cell of X' is attached by a path of the form $wx^{-1}y^{-1}z$. Consequently there is a homomorphism $\psi\colon \pi_1 X' \to \mathbb{Z}$ induced by sending each positively oriented 1-cell to the positive generator of $\mathbb{Z}$. The ascending and descending links relative to this Morse map are the circles which immerse to A and D. Since the ascending and descending links are connected but not simply-connected, by Theorem 9.2, $\mathsf{kernel}(\psi)$ is finitely generated but not finitely presented, and this proves that $\pi_1 X'$ is incoherent. □

Theorem 10.5. *For every q, there exists a compact $C(4)$-$T(q)$ complex X' with local ratio 1 and an incoherent fundamental group. In addition, every minimal fan of type 2 in X' is spread out in X'.*

Proof. By Lemma 10.3 and Lemma 10.4 it is sufficient to show that for each q, a 4-regular bipartite graph with girth at least $2q$ and a pair of edge-disjoint Hamiltonian cycles exists. To see this, note that given such a graph Γ', there exists a covering map $\Gamma' \to \Gamma$ such that the preimage of Γ_A is one of the Hamiltonian cycles and the preimage of Γ_D is the other.

We are extremely grateful to Nick Wormald for explaining to us that a graph fitting this description does indeed exist (as he and his coauthors have shown in [10]). The techniques involved are standard within random graph theory and we will merely sketch the line of argument below. Start with n distinguished blue vertices, n distiguished red vertices, and let Ω denote the configuration space of all 4-regular bipartite graphs with two distinguished edge-disjoint Hamiltonian cycles. Next make Ω a probability space by endowing it with the uniform probability

distribution. From the main result in [10], this space is contiguous to a random 4-regular bipartite graph with uniform distribution (as n goes to infinity). The probability that the girth of one of these is at least g is asymptotically a (nonzero) constant (from the short cycle distribution result given in [17]). Hence the same holds for Ω, and consequently, such a configuration exists. $\square$

Finally, we note that a similar argument using bipartite p-regular graphs would yield examples of incoherent $C(p)$-$T(q)$ complexes with similar properties.

11. Fans of large type

The results up to this point have been derived using fans of type 2 or type 3. In this section we investigate the behavior of fans of type k for arbitrary k. For simplicity we consider only global ratios. Arguments using coefficients, such as those used in Section 7, might be able to convert these results to ones which use local ratios instead, but we have not pursued that possibility. Our main goal in this section is to find the greatest upper bound on $\mathsf{Global}(X)$ which ensures that all fans of type k will be perimeter reducing for *some* value of k using Lemma 6.1, thus enabling us to apply Theorem 4.5 to conclude that its fundamental group is locally quasiconvex. For each value of k there is a bound which works. In this section we calculate these values and their limit. Since the results in this section use fans of type k for arbitrarily large k, there are no compact 2-complexes in which all of these fans are spread out. As a consequence, we only prove results for complexes with residually finite fundamental groups. We begin with a rough description of fans of type k and we refer the reader to [15] for a thorough development.

Definition 11.1 (Fans of type k)**.** For each choice of p, q, and k satisfying the Euclidean restrictions there is a collection of fans denoted by $\mathcal{F}^k_{pq}$, which is defined recursively starting with small values of k. Since p and q are usually understood in context, we usually call these *fans of type* k. For $k = 1, 2$, or 3, the fans of type k have already been described in Example 2.6. The general procedure for producing the next list of fans from the previous lists involves what we call *determined fans*. Let $F \to D$ be a fan in a $C(p)$-$T(q)$ disc diagram. The fan F is a *fan of type* $\mathcal{F}^k_{pq}$ *in* D provided that either it is a spur, or it is a 0-shell, or it is determined by a fan F' of type $\mathcal{F}^{k-1}_{qp}$ in the dual E of D. Rather than define determined fans precisely, we illustrate the process in Figure 6. In this figure the dual diagrams have been superimposed on the original diagrams. Illustrated from left to right are a spur determining a 1-shell, a 2-shell determining a pointed fan, and a pointed fan determining a broad fan.

There is a close relationship between F and F'. In particular, the 2-cells of F correspond with the 0-cells in the interior of the outer path of F' and under this correspondence, 0-cells of valence j become j-shells. Iterating this relationship, we can say quite a bit about the types of i-shells which arise in fans of type k. The following is Lemma 7.12 of [15], and we have included it so that the reader has a quantitative feel for the types of 2-cells which arise in fans of type k. In particular, note that any fan of type k is either a spur, 0-shell, or 1-shell, or else it is formed entirely as the union of 2-shells, 3-shells and 4-shells.

Lemma 11.2 (Fan valences)**.** *Let D be a $C(p)$-$T(q)$ diagram and let F be a fan of type k in D where F is not a spur,* 0*-shell, or* 1*-shell. Let U be the* 0*-cells in the interior of the outer path of F.*

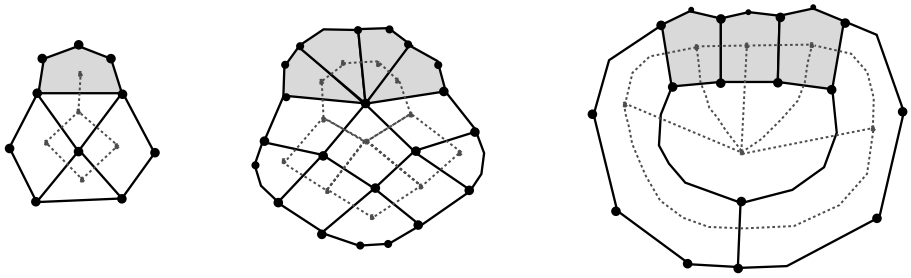

FIGURE 6. Determined fans

(1) *If $p \geq 4$, $q \geq 4$, and $k \geq 0$, then F is a 1-separated broad fan. That is, every 2-cell in F is a 2-shell or 3-shell, the first and last 2-cells are 2-shells, and there do not exist consecutive 3-cells. In addition, every 0-cell in U has valence ≤ 3, the first and last 0-cells in U have valence 2, and there do not exist consecutive 0-cells in U with valence 3.*

(2) *If $p = 3$, $q \geq 6$ and k is odd, then F is a 1-separated broad fan. That is, every 2-cell in F is a 2-shell or 3-shell, the first two and last two 2-cells are 2-shells, and there do not exist consecutive 3-shells. In addition, every 0-cell in U has valence ≤ 4, the first and last 0-cells have valence ≤ 3, and there do not exist consecutive 0-cells in U with valence 4.*

(3) *If $p \geq 6$, $q = 3$, and k is even, then every 2-cell in F is a 2-shell, 3-shell, or 4-shell, the first and last 2-cell is a 2-shell or 3-shell, and there do not exist consecutive 4-shells. In addition, every 0-cell in U has valence ≤ 3, the first two and last two 0-cells in U have valence 2, and there do not exist consecutive 0-cells in U with valence 3.*

For perimeter calculations it will be useful to distinguish between two different classes of fans of type k.

Definition 11.3 (Perpetual and transient fans of type k)**.** The fans of type k that are not spurs naturally split into two categories. The fans that are originally determined by a 0-shell or 1-shell are *perpetual* and those that are originally determined by a 2-shell or 3-shell are *transient*. To make this precise we need to consider sequences of fans F_j in diagrams D_j for $j = 1, \ldots, \ell$ where for each j, D_j is the dual of D_{j+1} and F_j determines F_{j+1}. A fan of type k either arises from such a sequence where $\ell \leq k$ and F_1 is a 0-shell or 1-shell, or $\ell = k$ and F_1 is a 2-shell or 3-shell. If a fan F_ℓ descends from a 0-shell or 1-shell, then F_ℓ is a fan of type k for arbitrarily large values of k and is thus called *perpetual*. On the other hand, if F_ℓ is the descendant of a 2-shell or 3-shell, then F_ℓ is a fan of type ℓ and not a fan of any other type, and we thus call F_ℓ a *transient fan of type k*.

Notation 11.4 (F_j, D_j, and the number of i-shells)**.** For the remainder of the section we consider sequences of fans F_j in diagrams D_j, $j \geq 1$, and suppose that for each j, the diagram D_j is the dual of D_{j+1} and the fan F_j determines the fan F_{j+1}. To fix notation, let b_j, c_j, and d_j denote the numbers of 2-shells, 3-shells, and 4-shells in F_j for each j. Notice that if D_j is a $C(p)$-$T(q)$ diagram, then for $i < j$, the diagram D_i is either $C(p)$-$T(q)$ or $C(q)$-$T(p)$ depending on the parity of i and j. Since we are interested in fans for fixed p and q our attention will be

focused on every other fan in such a sequence. In other words, we focus on either the odd values of j, or on the even values.

The information in Lemma 11.2 can be translated into equations and inequalities governing the numbers of 2-shells, 3-shells and 4-shells in a fan of type k, which we present using Notation 11.4.

Lemma 11.5 (Recursive relations)**.** *Let F_j be a fan in a $C(p)$-$T(q)$ diagram that is determined by the fan F_{j-1} in its $C(q)$-$T(p)$ dual. If $p \geq 4, q \geq 4$, then*

$$b_j \geq (q-3)b_{j-1} + (q-4)c_{j-1}, \qquad c_j = b_{j-1} + c_{j-1} - 1, \quad \textit{and} \quad d_j = 0.$$

If $p = 3$ and $q \geq 6$, then

$$b_j \geq (q-3)b_{j-1} + (q-4)c_{j-1} + (q-5)d_{j-1},$$

$$c_j = b_{j-1} + c_{j-1} + d_{j-1} - 1, \quad \textit{and} \quad d_j = 0.$$

If $p \geq 6$ and $q = 3$, then

$$b_j \geq 0, \quad c_j + 2d_j = b_{j-1} + c_{j-1} - 1, \quad \textit{and} \quad 0 \leq d_j \leq c_{j-1}.$$

Proof. All of the inequalities follow easily from Lemma 11.2. In the first two cases there are no 4-shells in F_j since the outer path of E has no valence 4 0-cells in its interior. The number of 3-shells in F_j is one less than the total number of 2-cells in F_{j-1}. The number of 2-shells in F_j equals the number of valence 2 0-cells in the interior of the outer path of F_{j-1} and each i-shell in F_{j-1} contributes at least $q-(i+1)$ such 0-cells.

In the third case, the number of 2-shells is clearly nonnegative, and each valence 4 0-cell in the outer path of F_{j-1} is incident with a unique 3-shell of F_{j-1}. This explains the first and last inequality. In the middle equation, the righthand side is one less than the number of 2-cells in F_{j-1} and the lefthand side counts the high valence 2-cells of F_j. To see the correspondence, identify each 2-cell in F_{j-1} with the leftmost vertex of the outer path that is contained in its boundary cycle and then, if possible, identify this vertex with the corresponding 2-cell in F_j. The leftmost 2-cell of F_{j-1} is the only one for which this identification is not possible. Moreover, each i-shell of F_j corresponds to a vertex of valence i in F_{j-1} and hence it is identified with exactly $(i-1)$ 2-cells of F_{j-1} as claimed. □

Remark 11.6 (Solutions and fans). We note that if b_{j-1}, c_{j-1} and d_{j-1} have values which can be realized by a fan F_{j-1} in a $C(q)$-$T(p)$-diagram, then every possible triple (b_j, c_j, d_j) which satisfies the appropriate system of equations and inequalities can be realized by a fan F_j in a $C(p)$-$T(q)$-diagram. In fact, every solution can be realized simply by varying the number of sides possessed by the 2-cells in F_{j-1}. This means that if we are looking for a fan F_k of type k where the ratio $\frac{b_k+c_k+d_k}{b_k+2c_k+3d_k+1}$ is as small as possible, we can forget about the fans themselves and focus on the iterative solutions to these systems.

We begin by analyzing the case where p and q are both at least 4 since d_j is always 0 in this case, and we start with a change of variables to simplify the system further. Continuing with the notation of Lemma 11.5, let F_j be a fan in a $C(p)$-$T(q)$ diagram which is part of a sequence of fans and diagrams and define $B_j = b_j - 1$ and $C_j = c_j + 1$. When we rewrite our recursion in these new variables

the constant terms disappear. Thus we find that $B_j \geq (q-3)B_{j-1} + (q-4)C_{j-1}$, that $C_j = B_{j-1} + C_{j-1}$ and that the fraction we are trying to minimize is

$$\frac{B_j + C_j}{B_j + 2C_j} = \frac{\frac{B_j}{C_j} + 1}{\frac{B_j}{C_j} + 2} = \frac{1 + \frac{C_j}{B_j}}{1 + 2\frac{C_j}{B_j}} \tag{6}$$

so long as $B_j, C_j > 0$. It is therefore sufficient to minimize the ratio B_j/C_j, or equivalently, to maximize the ratio C_j/B_j. Define matrices

$$M_p = \begin{bmatrix} (p-3) & (p-4) \\ 1 & 1 \end{bmatrix}, \qquad M_q = \begin{bmatrix} (q-3) & (q-4) \\ 1 & 1 \end{bmatrix}.$$

We can now view this as a geometric problem in the Euclidean plane. Plot the points (B_i, C_i) for $i = 1, \ldots, j$, and think of them as column vectors. We wish to maximize the slope of the line from the origin to the final point (B_j, C_j). For each $i \leq j$, the point (B_i, C_i) is related to the previous point as follows. First apply a linear transformation to the point (B_{i-1}, C_{i-1}) (i.e. left multiply the column vector $\begin{bmatrix} B_{i-1} \\ C_{i-1} \end{bmatrix}$ by the matrix M_q or M_p) and then increase the x-coordinate if we wish. The result is a point (B_i, C_i). Recall that p and q switch roles in the dual diagram, which is why both matrices arise. Moreover, which matrix is used alternates depending on the parity of i. Since M_p and M_q are both matrices with nonnegative entries and positive determinant, the first quadrant is mapped into itself in an orientation-preserving way. Thus increasing the x-coordinate at any point in the process will result in a lower value for the final slope. In particular, the maximum slope achieved by a fan F_j is found by applying $M_q M_p$ to the point (B_{j-2}, C_{j-2}) arising from the fan F_{j-2} that achieves the maximum slope for $j-2$. Because of the dynamincs of the linear transformation $M_q M_p$ on the plane (in particular, because it has two distinct real eigenvalues and maps the first quadrant properly into itself), the maximal slope of a fan F_j when j is large will closely approximate the slope of the eigenvector corresponding to the largest positive eigenvalue of $M_q M_p$. Because it simplifies later equations, we will calculate the reciprocal of this slope which we call z_{pq}. It is now routine to calculate z_{pq} by solving

$$M_q M_p \begin{bmatrix} z_{pq} \\ 1 \end{bmatrix} = \begin{bmatrix} ((q-3)(p-2)-1)z_{pq} + (q-3)(p-3) - 1 \\ (p-2)z_{pq} + (p-3) \end{bmatrix} = \lambda \begin{bmatrix} z_{pq} \\ 1 \end{bmatrix}.$$

Eliminating λ and simplifying we find that z_{pq} satisfies the equation

$$(p-2)z_{pq}^2 - (p-2)(q-4)z_{pq} - \big((p-3)(q-3) - 1\big) = 0.$$

One explicit form for the positive solution is

$$z_{pq} = \frac{1}{2}\left[(q-4) + (q-2)\sqrt{1 - \frac{4}{(p-2)(q-2)}}\,\right]. \tag{7}$$

In other words, z_{pq} is the average of $(q-4)$ and a number less than $(q-2)$. This proves the following lemma. Keep in mind that ζ and z_{pq} represent the *reciprocals* of slopes so that slopes larger but arbitrarily close to the asymptotic slope result in reciprocals which are less than but arbitrarily close to z_{pq}.

Lemma 11.7. *Given $p, q \geq 4$ and a real number $\zeta < z_{pq}$ as defined in Equation (7), there exists a number k such that for every transient fan F of type k in a $C(p)$-$T(q)$*

diagram, the inequality $\zeta C < B$ holds, where $(B+1)$ is the number of 2-shells and $(C-1)$ is the number of 3-shells in F.

Notice that the lemma is decidedly false for $\zeta = z_{pq}$ since a 2-shell leads to coordinates $(B_1, C_1) = (0, 1)$, which is clearly above the line through the origin with slope $1/z_{pq}$. As a consequence, the fans derived from it which have the largest possible slope also lie above this line (since the matrix $M_q M_p$ is an orientation preserving linear transformation which fixes this line). We next prove a similar result for fans which are ultimately derived from 0-shells and 1-shells.

Lemma 11.8. *Fix $p, q \geq 4$ and define z_{pq} as in Equation (7). For every perpetual fan F_j in a $C(p)$-$T(q)$ diagram with $j \geq 2$, the inequality $z_{pq}C < B$ holds, where $(B+1)$ is the number of 2-shells and $(C-1)$ is the number of 3-shells in F_j.*

Proof. We continue with the notation established above. Because $M_p M_q$ is an orientation preserving linear transformation which fixes the line through the origin with slope $1/z_{pq}$, it is sufficient to show that some earlier point (B_i, C_i) lies below this line. This is because once one of these points is below the line, all of the later points must also lie below the line. There are two cases depending on whether j is even or odd.

If j is even, then consider the fan F_2. Since F_1 is either a 0-shell or a 1-shell, F_2 is a pointed fan with $i \geq q-2$. Thus $b_2 \geq q-2$ and $c_2 = 0$, and the largest slope occurs when $b_2 = q-2$. This corresponds to the point $(B_2, C_2) = (q-3, 1)$. From the description of z_{pq} as an average of $(q-4)$ and a number less than $(q-2)$, it is clear that $B_2/C_2 = q-3 > z_{pq}$, which completes the proof when j is even.

When j is odd, we consider the fan F_3 and the inequalities are slightly more delicate. Since F_1 is a 0-shell or a 1-shell, F_2 is a pointed fan in a $C(q)$-$T(p)$ diagram with $i \geq p-2$. We cannot directly use this information since F_2 is a fan in a $C(q)$-$T(p)$ diagram. As we argued above, the fan F_3 will have the largest possible slope when F_2 has the largest possible slope. This F_2 has coordinates $(B_2, C_2) = (p-3, 1)$ and multiplying by M_q yields the best coordinates for the fan F_3. Thus $(B_3, C_3) = \big((q-3)(p-3) + (q-4), (p-3)+1\big)$. It only remains to show that this point lies below the fixed line. Since $1 - 4x < 1 - 4x + x^2$ when $x > 0$, we know that $\sqrt{1-4x} < 1 - 2x$. Thus

$$z_{pq} < \frac{1}{2}\left[(q-4) + (q-2)\left(1 - \frac{2}{(p-2)(q-2)}\right)\right] = (q-3) - \frac{1}{p-2} = \frac{B_3}{C_3},$$

and this completes the proof when j is odd. □

The following is our main result for the case $p, q \geq 4$. Note that the case $p = q = 4$ is excluded since the hyperbolic restrictions are necessary in order to appeal to Theorem 4.5.

Theorem 11.9. *Let X be a compact weighted $C(p)$-$T(q)$-complex with a residually finite fundamental group and $p \geq 4, q > 4$ or $p > 4, q \geq 4$. Then $\pi_1 X$ is locally quasiconvex whenever $\mathsf{Global}(X) < \frac{z_{pq}+1}{z_{pq}+2}$ where z_{pq} is the value defined in Equation (7).*

Proof. First note that the function $f(x) = \frac{x+1}{x+2}$ is monotonically increasing for $x > 0$. Thus, if $\mathsf{Global}(X) < \frac{z_{pq}+1}{z_{pq}+2}$, then it is also less than $\frac{\zeta+1}{\zeta+2}$ for some $\zeta < z_{pq}$. It then follows from Lemma 11.7 that we can choose k large enough so that all

transient fans F of type k satisfy $\zeta C < B$, where $b = B + 1$ is the number of 2-shells in F and $c = C - 1$ is the number of 3-shells. In particular,

$$\frac{b+c}{b+2c+1} = \frac{B+C}{B+2C} > \frac{\zeta+1}{\zeta+2} > \mathsf{Global}(X)$$

so that by Lemma 6.1 these fans are perimeter-reducing. Next, 0-shells and 1-shells are perimeter reducing since $\mathsf{Global}(X) < 1$. The only other fans of type k are perpetual fans, and these satisfy $z_{pq}C < B$ by Lemma 11.8, and the same sequence of inequalities (with ζ replaced by z_{pq}) shows that these are also perimeter-reducing. Therefore every fan of type k is perimeter-reducing and the result follows from Theorem 4.5. □

We now shift our attention to fans in the $C(p)$-$T(3)$ and $C(3)$-$T(q)$ diagrams. Since both types of diagrams are involved in the inductive definition of fans of type k, it makes sense to eliminate q and to discuss $C(3)$-$T(p)$ diagrams instead. As mentioned above, the recursion here is slightly more complicated. We begin by establishing some notation along with a change of variables in order to eliminate the constants from the inequalities. We will need the following matrices:

$$N_p = \begin{bmatrix} (p-4) & (p-5) \\ 1 & 1 \end{bmatrix}, \qquad N' = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Following the notation of Lemma 11.5, let F_j be a fan in a diagram D_j which satisfies $C(p)$-$T(3)$ when j is odd and $C(3)$-$T(p)$ when j is even. Moreover, suppose that we have sequence of such fans and diagrams in which each fan determines the next fan and the dual of each diagram is the previous one. As usual, let b_j, c_j, and d_j denote the numbers of 2-shells, 3-shells, and 4-shells in F_j, respectively. By Lemma 11.5, we already know that d_j is 0 when j is even. When j is even we define $B_j = b_j - 1$ and $C_j = c_j + 1$ (and $D_j = d_j = 0$) as before, and when j is odd we set $B_j = b_j$, $C_j = c_j - 1$ and $D_j = d_j + 1$. (We apologize for using D_j to denote both the disc diagram containing F_j and a variant of the number of 4-shells in F_j.)

When j is even (and F_j is a fan in a $C(3)$-$T(p)$ diagram), the ratio we wish to minimize is $\frac{b_j+c_j}{b_j+2c_j+1} = \frac{B_j+C_j}{B_j+2C_j}$. As in Equation (6), minimizing this is equivalent to minimizing B_j/C_j or maximizing C_j/B_j. In particular, it is clear that the rewritten relation $B_j \geq (p-3)B_{j-1} + (p-4)C_{j-1} + (p-5)D_{j-1}$ should be an equality for this to be achieved. Thus, when trying to find the fan F_j with j odd that minimizes this ratio we can assume that

$$\frac{B_j}{C_j} = \frac{(p-3)B_{j-1} + (p-4)C_{j-1} + (p-5)D_{j-1}}{B_{j-1} + C_{j-1} + D_{j-1}}.$$

Notice that we can view B_j/C_j as a weighted average of $(p-3)$, $(p-4)$ and $(p-5)$ with B_{j-1}, C_{j-1} and D_{j-1} over their sum providing the nonnegative weights. The values of B_{j-1}, C_{j-1} and D_{j-1} are in turn dependent on B_{j-2} and C_{j-2} with some choices involved, namely $B_{j-1} \geq 0$ and $0 \leq D_{j-1} \leq C_{j-2}$. If we are trying to minimize B_j/C_j, then we should clearly choose $B_{j-1} = 0$ since this minimizes the weight given to the largest number. Similarly, D_{j-1} should be as large as possible since this maximizes the weight given to the smallest number. Thus, when minimizing B_j/C_j, we have $B_{j-1} = 0$, $C_{j-1} = B_{j-2} - C_{j-2}$ and $D_{j-1} = C_{j-2}$. In particular, we have

$$\begin{bmatrix} B_j \\ C_j \end{bmatrix} = N_p \begin{bmatrix} C_{j-1} \\ D_{j-1} \end{bmatrix} = N_p N' \begin{bmatrix} B_{j-2} \\ C_{j-2} \end{bmatrix}.$$

The analysis now proceeds as before. The matrix N_pN' is an orientation preserving linear transformation with two distinct real eigenvalues. Under iteration, the slope of the line from the origin through the point (B_j, C_j) will closely approximate the slope of the eigenvector corresponding to the largest eigenvalue of N_pN'. Because it makes later equations simpler, we again calculate the reciprocal of this slope which we call z_{3p}. We calculate z_{3p} by solving

$$N_pN' \begin{bmatrix} z_{3p} \\ 1 \end{bmatrix} = \begin{bmatrix} (p-4)z_{3p} - 1 \\ z_{3p} \end{bmatrix} = \lambda \begin{bmatrix} z_{3p} \\ 1 \end{bmatrix}.$$

Eliminating λ we find that z_{3p} satisfies the equation $z_{3p}^2 - (p-4)z_{3p} + 1 = 0$. Of the two possible solutions, the one corresponding to the largest eigenvalue is

$$z_{3p} = \frac{1}{2}\left[(p-4) + \sqrt{(p-4)^2 - 4}\,\right]. \tag{8}$$

This proves the following lemma.

Lemma 11.10. *Given $p \geq 6$ and a real number $\zeta < z_{3p}$ as defined in Equation* (8)*, there exists an even number k such that for every transient fan F of type k in a $C(3)$-$T(p)$ diagram, the inequality $\zeta C < B$ holds, where $(B+1)$ is the number of 2-shells and $(C-1)$ is the number of 3-shells in F.*

Notice that the lemma is false for $\zeta = z_{3p}$ since a pointed fan with $i = p-4$ leads to coordinates $(B_2, C_2) = (p-5, 1)$, which is above the line through the origin with slope $1/z_{3p}$. To see this, note that $(p-6)^2 < (p-6)(p-2) = (p-4)^2 - 4$, so $(p-6) < \sqrt{(p-4)^2-4}$, $2p - 10 < p - 4 + \sqrt{(p-4)^2-4}$, and $p - 5 < z_{3p}$. Consequencely, the fans derived from this which have the largest possible slope also lie above this line (since the matrix N_pN' is an orientation preserving linear transformation which fixes the line). Conversely, we now show that the points corresponding to perpetual fans lie below this line.

Lemma 11.11. *Fix $p \geq 6$ and define z_{3p} as in Equation* (8)*. For every fan F_j in a $C(3)$-$T(p)$ diagram, $j \geq 2$ and even, which is ultimately derived from a 0-shell or a 1-shell, the inequality $z_{3p}C < B$ holds, where $b = B+1$ is the number of 2-shells and $c = C-1$ is the number of 3-shells in F_j.*

Proof. Because N_pN' is an orientation preserving linear transformation which fixes the line through the origin with slope $1/z_{3p}$, it is sufficient to show that some earlier point (B_i, C_i) lies below this line. Consider the fan F_2. Since F_1 is either a 0-shell or a 1-shell, F_2 is a pointed fan with $i \geq q-2$. Thus $b_2 \geq q-2$ and $c_2 = 0$, and the largest slope occurs when $b_2 = q-2$. This corresponds to the point $(B_2, C_2) = (q-3, 1)$. On the other hand, by Equation (8), z_{3p} is the average of $(p-4)$ and a number less than $(p-4)$, and thus less than $(p-3)$. In other words, $B_2/C_2 = q-3 > z_{3p}$, which completes the proof. □

The proof of our main theorem for $C(3)$-$T(p)$ complexes is identical to the proof of Theorem 11.9, except for the obvious substitutions, and will be omitted. Notice that p must be strictly greater than 6 in order to satisfy the hyperbolic restrictions.

Theorem 11.12. *Let X be a compact weighted $C(3)$-$T(p)$-complex with a residually finite fundamental group and $p > 6$. Then $\pi_1 X$ is locally quasiconvex whenever* $\mathsf{Global}(X) < \frac{z_{3p}+1}{z_{3p}+2}$ *where z_{3p} is the value defined in Equation* (8)*.*

Finally, we consider the situation of a fan F_j in a $C(p)$-$T(3)$ diagram with j odd. Because we are trying to minimize the ratio $\frac{b_j+c_j+d_j}{b_j+2c_j+3d_j+1}$, we can assume that $b_j = 0$. Thus, the ratio we are focusing on is

$$\frac{c_j + d_j}{2c_j + 3d_j + 1} = \frac{C_j + D_j}{2C_j + 3D_j} = \frac{1 + \frac{D_j}{C_j}}{2 + 3\frac{D_j}{C_j}} = \frac{\frac{C_j}{D_j} + 1}{2\frac{C_j}{D_j} + 3}$$

so long as $C_j, D_j > 0$. It is therefore sufficient to minimize the ratio C_j/D_j or, equivalently, maximize the ratio D_j/C_j. In particular, this means that D_j should be as large as possible, i.e. $D_j = C_{j-1}$. Based on the recurrence relations, we have

$$\frac{C_j}{D_j} = \frac{B_{j-1} - C_{j-1}}{C_{j-1}} = \frac{B_{j-1}}{C_{j-1}} - 1.$$

Thus C_j/D_j is minimized exactly when B_{j-1}/C_{j-1} is minimized. As discussed above this occurs when $B_{j-2} = 0$ and the matrix N_p is used to convert C_{j-2} and D_{j-2} into B_{j-1} and C_{j-1}. Putting this altogether we find that

$$\begin{bmatrix} C_j \\ D_j \end{bmatrix} = N'N_p \begin{bmatrix} C_{j-2} \\ D_{j-2} \end{bmatrix}.$$

If we view (C_j, D_j) as a point in the plane, then the analysis is as before. The matrix $N'N_p$ is an orientation preserving linear transformation with distinct real eigenvalues. Under iteration, the slope of the line from the origin through the point (C_j, D_j) will closely approximate the slope of the eigenvector corresponding to the largest eigenvalue of $N'N_p$. We calculate the reciprocal of this slope, which we call z_{p3}, by solving

$$N'N_p \begin{bmatrix} z_{p3} \\ 1 \end{bmatrix} = \begin{bmatrix} (p-5)z_{p3} + (p-6) \\ z_{p3} + 1 \end{bmatrix} = \lambda \begin{bmatrix} z_{p3} \\ 1 \end{bmatrix}.$$

Eliminating λ we find that z_{p3} satisfies the equation $z_{p3}^2 - (p-6)z_{p3} - (p-6) = 0$. The positive solution is

$$z_{p3} = \frac{1}{2}\left[(p-6) + \sqrt{(p-6)^2 + 4(p-6)}\,\right]. \tag{9}$$

This proves the following lemma:

Lemma 11.13. *Given $p \geq 6$ and a real number $\zeta < z_{p3}$ as defined in Equation* (9)*, there exists an odd number k such that for every transient fan F of type k in a $C(p)$-$T(3)$ diagram, the inequality $\zeta D < C$ holds, where $c = C + 1$ is the number of 3-shells and $d = D - 1$ is the number of 4-shells in F.*

As expected, this lemma is false for $\zeta = z_{p3}$ since a 3-shell leads to coordinates $(C_1, D_1) = (0, 1)$, which is clearly above the line through the origin with slope $1/z_{p3}$. Consequently, the fans derived from this 3-shell with the largest possible slope also lie above this line (since the matrix $N'N_p$ is an orientation preserving linear transformation which fixes the line). Conversely, we now show that the points corresponding to perpetual fans lie below this line.

Lemma 11.14. *Fix $p \geq 6$ and define z_{p3} as in Equation* (9)*. For every perpetual fan F_j in a $C(p)$-$T(3)$ diagram with $j \geq 1$ and odd, the inequality $z_{p3}D < C$ holds, where $c = C + 1$ is the number of 3-shells and $d = D - 1$ is the number of 4-shells in F.*

Proof. Because $N'N_p$ is an orientation preserving linear transformation which fixes the line through the origin with slope $1/z_{p3}$, it is sufficient to show that some earlier point (C_i, D_i) lies below this line. Consider the fan F_3. Since F_1 is either a 0-shell or a 1-shell, the fan F_2 is a pointed fan with $i \geq p-2$. As we argued above, the fan F_3 will have the largest possible slope when F_2 has the largest possible slope. This F_2 has coordinates $(p-3, 1)$ and multiplying by N' yields the best coordinates for the F_3. Thus $(C_3, D_3) = (p-4, 1)$. It only remains to show that this point lies below the fixed line, but this is clear since $(p-2) > \sqrt{(p-6)(p-2)}$ so that $2p-8 > (p-6) + \sqrt{(p-6)(p-2)}$ and $p-4 > z_{p3}$. □

The following is our main theorem for $C(p)$-$T(3)$ complexes.

Theorem 11.15. *Let X be a compact weighted $C(p)$-$T(3)$-complex with a residually finite fundamental group and $p > 6$. Then $\pi_1 X$ is locally quasiconvex whenever* $\mathsf{Global}(X) < \frac{z_{p3}+1}{2z_{pq}+3}$ *where z_{p3} is the value defined in Equation* (9).

Proof. First note that the function $f(x) = \frac{x+1}{2x+3}$ is monotonically increasing for $x > 0$. Thus, if $\mathsf{Global}(X) < \frac{z_{p3}+1}{2z_{p3}+3}$, then it is also less than $\frac{\zeta+1}{2\zeta+3}$ for some $\zeta < z_{p3}$. It then follows from Lemma 11.13 that we can choose k large enough so that all transient fans F of type k satisfy $\zeta D < C$, where $c = C+1$ is the number of 3-shells in F and $d = D-1$ is the number of 4-shells. In particular,

$$\frac{b+c+d}{b+2c+3d+1} \geq \frac{C+D}{2C+3D} > \frac{\zeta+1}{2\zeta+3} > \mathsf{Global}(X)$$

so that by Lemma 6.1 these fans are perimeter-reducing. Next, 0-shells and 1-shells are perimeter reducing since $\mathsf{Global}(X) < 1$. The only other fans of type k are perpetual and thus satisfy $z_{pq}D < C$ by Lemma 11.14, and the same sequence of inequalities (with ζ replaced by z_{p3}) shows that these are also perimeter-reducing. Therefore every fan of type k is perimeter reducing, and the result follows from Theorem 4.5. □

Finally, we conclude the section by summarizing the bounds from Theorems 11.9, 11.12 and 11.15 in a table for small values of p and q, and we then compare these results with the earlier results using only fans of types 2 or 3. Note that the decimals in Table 1 have been truncated rather than rounded, so if X is a $C(p)$-$T(q)$ complex with $\pi_1 X$ residually finite, and $\mathsf{Global}(X) < r$ where r is the entry in the p-th row and q-th column, then $\pi_1 X$ is locally quasiconvex.

TABLE 1. Bounds on $\mathsf{Global}(X)$ which imply local quasiconvexity for small values of p and q

	3	4	5	6	7	8	9	∞
3	-	-	-	.666	.783	.825	.852	1
4	-	.500	.702	.773	.816	.844	.865	1
5	-	.612	.723	.784	.822	.849	.869	1
6	.333	.630	.731	.788	.825	.851	.870	1
7	.419	.639	.735	.791	.827	.852	.871	1
8	.440	.644	.738	.792	.828	.853	.872	1
9	.452	.648	.740	.793	.829	.854	.872	1
∞	1/2	2/3	3/4	4/5	5/6	6/7	7/8	1

For $p = q = 5$, Theorem 11.9 provides local quasiconvexity when $\mathsf{Global}(X) < .723$, whereas Theorem 7.9 gives local quasiconvexity only when $\mathsf{Global}(X) < 5/11 \sim .454$. For $p = 3, q = 7$, Theorem 11.12 provides local quasiconvexity when $\mathsf{Global}(X) < .783$, whereas Theorem 7.8 gives local quasiconvexity only when $\mathsf{Global}(X) < .75$ For $p = 7, q = 3$, Theorem 11.15 provides local quasiconvexity when $\mathsf{Global}(X) < .419$ whereas Theorem 7.9 gives local quasiconvexity only when $\mathsf{Global}(X) < .40$. In each case, the theorems in this section are improvements, but the improvement is not always large. The main benefits arise when both p and q are large.

12. Coxeter groups

In this final section we give simple criteria for the coherence and local quasiconvexity of Coxeter groups which depend only on the exponents and the number of generators. Although these results will use only the easier versions of the previous results, they illustrate that a similar type of behavior is present in the realm of Coxeter groups. Recall that a *Coxeter group* is a group with a presentation of the form

$$\langle a_1, \ldots, a_r \mid a_i^2, (a_i a_j)^{m_{ij}} \ (i \neq j) \ \rangle$$

where the m_{ij} are integers greater than 1. In the proof we will need the following basic facts about Coxeter presentations. See [5] and [11] for details.

Lemma 12.1. *Let $\langle a_1, \ldots, a_r \mid a_i^2, (a_i a_j)^{m_{ij}} \ (i \neq j)\rangle$ be a Coxeter group. In this group the element a_i has order 2 for all i, the element $a_i a_j$ has order m_{ij} for all $i \neq j$, and the group itself is virtually torsion-free and residually finite.*

Our main result on Coxeter groups is the following.

Theorem 12.2. *The Coxeter group $\langle a_1, \ldots, a_r \mid a_i^2, (a_i a_j)^{m_{ij}} \ (i \neq j)\rangle$ is coherent provided $m_{ij} \geq r$ for all $i \neq j$. Similarly, if $m_{ij} > r$ for all $i \neq j$, then the group is locally quasiconvex.*

Proof. Let X be the standard 2-complex of the presentation. By Lemma 12.1 there is a finite group G and a map $\pi_1 X \to G$ such that for each i, the image of a_i has order 2, and for each i, j, the image of $a_i a_j$ has order m_{ij}. Let $\widehat{X}$ denote the cover of X corresponding to the kernel of this quotient. We now form a 2-complex $\overline{X}$ which is a quotient $\widehat{X}$ as follows. For each i, j and for each lift to $\widehat{X}$ of the path $(a_i a_j)^{m_{ij}}$ we identify the m_{ij} 2-cells attached along this path. Then, for each i and for each lift of the path a_i^2 to $\widehat{X}$, we identify the two 2-cells attached along this path, and then retract the resulting bigon to one of its 1-cells. Finally, notice that neither of these operations changes the fundamental group, so that $\pi_1 \widehat{X} \cong \pi_1 \overline{X}$.

Let m be the minimum of the m_{ij}. When $2 \geq m \geq r$, the theorem is trivial, so assume that $m \geq 3$. Since all pieces in $\overline{X}$ are of length 1, $\overline{X}$ satisfies $C(2m)$-$T(3)$. It will be sufficient to use the unit perimeter. Since each generator a_i of $\pi_1 X$ occurs in at most $r - 1$ relators (other than $a_i^2 = 1$), every 1-cell e in $\overline{X}$ satisfies $\mathbf{P}(e) \leq r - 1 \leq m - 1$. Since, in addition, the weight of each 2-cell is $2m_{ij} \geq 2m$, the global ratio of X is at most $\frac{m-1}{2m}$. For $m = 3$, the global ratio is bounded above by $\frac{1}{3}$ and $\pi_1 X$ is coherent by Theorem 5.3, so assume $m > 3$. In this case we will use Theorem 7.9, which is applicable since $p \geq 8$, $q = 3$, and $k = 3$ satisfy

the hyperbolic restrictions, and since $\pi_1\overline{X}$ is residually finite by Lemma 12.1. In particular, it would be sufficient to show that

$$\mathsf{Global}(X) \leq \frac{m-1}{2m} < \frac{(2m)-5}{2(2m)-9}.$$

The first inequality was established above, and the second inequality is true if and only if $m > 3$, as is easy to verify. Finally, local quasiconvexity is trivial for $m \leq 3$, and it follows by the exact same argument (using Theorem 7.9) for $m > 3$. □

The theorem proves that the group $\langle a, b, c \mid a^2, b^2, c^2, (ab)^x, (ac)^y, (bc)^z\rangle$ will be coherent whenever $x, y, z \geq 2$ and locally quasiconvex whenever $x, y, z > 2$. A 4-generator Coxeter group will be coherent provided that all exponents in the defining relations are at least 3. On the other hand, not every Coxeter group is coherent. The group F_2 is an index 2 subgroup of the Coxeter group $G = \langle a, b, c \mid a^2, b^2, c^2\rangle$, so $F_2 \times F_2$ is a subgroup of $G \times G$ (which is also a Coxeter group). Since $F_2 \times F_2$ is incoherent, the group $G \times G$ is incoherent as well.

Misha Kapovich [12] has proposed the following example where all exponents are ≥ 3: Let Q be the orbihedron whose underlying space is a 3-dimensional hyperbolic cube whose dihedral angles are $\frac{\pi}{3}$, and all of whose faces are reflectors. Then $\pi_1 Q$ is a Coxeter group all of whose exponents are 3 or ∞. The defining graph of the presentation for $\pi_1 Q$ corresponds to an octahedron, where each 0-cell corresponds to a generator of order 2 and the product of two generators has exponent 3 provided they are connected by a 1-cell.

This Q has a finite cover which fibers as a surface S bundled over the circle, where $\pi_1 S$ is finitely generated. The reason that the orbifold Q virtually fibers is because it has a finite cover that covers the figure-8 knot complement, which is well known to fiber. To see this, observe that Q can be subdivided into five ideal tetrahedrons with dihedral angles $\frac{\pi}{3}$. Thus, $\widetilde{Q}$ is tessellated in the same way as the universal cover of the figure-8 knot complement with its usual subdivision by ideal tetrahedrons. Consequently, $\pi_1 Q$ and the fundamental group of the figure-8 knot complement are finite index subgroups of the automorphism group of this tessellation and so their intersection is a finite index subgroup in both. We are grateful to Nathan Dunfield for this explanation.

Let C denote the suborbihedron consisting of a square which cuts Q in half. Since Q virtually fibers, $\pi_1 S \cap \pi_1 Q$ is infinitely generated. Now if we form the double $D = Q \cup_{C=C'} Q'$, then $\pi_1 D$ is still a Coxeter group all of whose exponents are 3 or ∞. However, since its second homology is infinitely generated, the subgroup $\langle \pi_1 S, \pi_1 S'\rangle$ is not finitely presented and so the group $\pi_1 D$ is incoherent.

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