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COMPLETELY ISOMETRIC REPRESENTATIONS OF $M_{cb}A(G)$ AND $UCB(\hat{G})^*$

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ABSTRACT. Let G be a locally compact group. It is shown that there exists a natural completely isometric representation of the completely bounded Fourier multiplier algebra $M_{cb}A(G)$, which is dual to the representation of the measure algebra $M(G)$, on $\mathcal{B}(L_2(G))$. The image algebras of $M(G)$ and $M_{cb}A(G)$ in $\mathcal{CB}^\sigma(\mathcal{B}(L_2(G)))$ are intrinsically characterized, and some commutant theorems are proved. It is also shown that for any amenable group G, there is a natural completely isometric representation of $UCB(\hat{G})^*$ on $\mathcal{B}(L_2(G))$, which can be regarded as a duality result of Neufang's completely isometric representation theorem for $LUC(G)^*$.

1. INTRODUCTION

In this paper we assume that G is a locally compact group with a fixed left Haar measure μ_G. We will simply write $d\mu_G(t) = dt$ if there is no confusion. Ghahramani showed in [15, Theorem 2] that if G contains at least two elements, the convolution algebra $L_1(G)$ (and thus the measure algebra $M(G)$) cannot be isometrically isomorphic to a subalgebra of operators on any Hilbert space. Therefore, the representation of the measure algebra $M(G)$ has to be considered on some other spaces different from Hilbert spaces.

The first such representation result was studied by Wendel [46], in which he showed that $M(G)$ is isometrically isomorphic to the right centralizer algebra $RC(L_1(G))$ of $L_1(G)$. More precisely, Wendel showed that every measure $\mu \in M(G)$ uniquely corresponds to a bounded right centralizer

$$m_\mu : f \in L_1(G) \mapsto f * \mu \in L_1(G)$$

on $L_1(G)$. If we let $\Phi_\mu = m_\mu^*$ denote the adjoint of m_μ, then Φ_μ is a bounded weak* continuous operator on $L_\infty(G)$ commuting with left translations (i.e., $\Phi_\mu(l_g f) = l_g\Phi_\mu(f)$). On the other hand, every such operator on $L_\infty(G)$ is implemented by a measure in this way. Therefore, if we denote by $\mathcal{B}_l^\sigma(L_\infty(G))$ the space of all bounded weak* continuous maps on $L_\infty(G)$ commuting with left translations, then

$$\Phi : \mu \in M(G) \mapsto \Phi_\mu \in \mathcal{B}_l^\sigma(L_\infty(G)) \tag{1.1}$$

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is an isometric isomorphism from $M(G)$ onto $\mathcal{B}_l^\sigma(L_\infty(G))$ (see [1, §1.6]). Let $LUC(G)^*$ denote the dual space of all left uniformly continuous bounded functions on G. Then $LUC(G)^*$ is a Banach algebra containing $M(G)$ as a Banach subalgebra. It was shown by Curtis and Figà-Talamanca (cf. Theorem 3.3 in [5]) that there is a similar isometric isomorphism from $LUC(G)^*$ onto the space of all bounded operators on $L_\infty(G)$ which commute with the left convolution action of $L_1(G)$ on $L_\infty(G)$. Note that the proof given in [5] assumes G to be unimodular. The general case follows from a more general result due to Lau (see Theorem 1, together with Lemma 1 and Remark 3 in [28]).

It is also known that $M(G)$ and $LUC(G)^*$ can be nicely represented on the space $\mathcal{B}(L_2(G))$ of all bounded linear operators on the Hilbert space $L_2(G)$. Størmer showed in [45] that for any abelian group G, there exists an isometric homomorphism Θ_l from $M(G)$ into $\mathcal{B}^\sigma_{\mathcal{R}(G)}(\mathcal{B}(L_2(G)))$, the space of all normal bounded $\mathcal{R}(G)$-bimodule morphisms on $\mathcal{B}(L_2(G))$, which is given by

$$\Theta_l(\mu)(a) = \int_G \lambda(s)a\lambda(s)^* d\mu(s) \tag{1.2}$$

for $\mu \in M(G)$ and $a \in B(L_2(G))$. This result was extended to general (not necessarily abelian) groups by Ghahramani [15] and was further studied by Neufang in his Ph.D. thesis [32]. Neufang showed that each $\Theta_l(\mu)$ is actually completely bounded and Θ_l is an isometric homomorphism from $M(G)$ into $\mathcal{CB}^\sigma_{\mathcal{R}(G)}(\mathcal{B}(L_2(G)))$, the space of all normal (i.e. weak* continuous) completely bounded $\mathcal{R}(G)$-bimodule morphisms on $\mathcal{B}(L_2(G))$. Moreover, Neufang successfully characterized the range space of the representation (1.2) in $\mathcal{CB}^\sigma_{\mathcal{R}(G)}(\mathcal{B}(L_2(G)))$ by showing that $\Theta_l(M(G))$ is equal to the space $\mathcal{CB}^{\sigma,L_\infty(G)}_{\mathcal{R}(G)}(\mathcal{B}(L_2(G)))$ of all normal completely bounded $\mathcal{R}(G)$-bimodule morphisms on $\mathcal{B}(L_2(G))$, which map $L_\infty(G)$ into $L_\infty(G)$ (see [32] and [34]). Neufang also introduced and studied the representation of the Banach algebra $LUC(G)^*$ on $\mathcal{B}(L_2(G))$ in [32] and [33].

The aim of this paper is to investigate the corresponding representations of the completely bounded Fourier multiplier algebra $M_{cb}A(G)$ and the Banach algebra $UCB(\hat{G})^*$ introduced by Granirer [19] (see §6 for details), since $M_{cb}A(G)$ and $UCB(\hat{G})^*$, when G is amenable, can be regarded as the natural dual objects of $M(G)$ and $LUC(G)^*$, respectively. Our main results show that there exist natural completely isometric representations of $M_{cb}A(G)$ and $UCB(\hat{G})^*$ when G is amenable on $\mathcal{B}(L_2(G))$. The advantage of this investigation is that it allows us to compare and study the connection of these representations with the corresponding representations of $M(G)$ and $LUC(G)^*$ on the same space $\mathcal{B}(L_2(G))$.

Since operator space techniques will play an important role, we first recall some necessary definitions and notations on operator spaces in §2. Readers are referred to the recent books [11], [35], and [37] for more details. In §3, we recall the representation theorem of $M(G)$ by considering the weak*-weak* continuous completely isometric homomorphism $\Theta_r : M(G) \to \mathcal{CB}^\sigma_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$ induced by the right regular representation

$$\Theta_r(\mu)(a) = \int_G \rho(s)a\rho(s)^* d\mu(s). \tag{1.3}$$

With this setup, we may significantly simplify our calculations, and we will be able to obtain some intriguing commutant theorems in §5. We provide a proof, which is simpler than Neufang's original argument, for the equality

$$\Theta_r(M(G)) = \mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G))) \tag{1.4}$$

in Theorem 3.2. Moreover, we show in Proposition 3.3 and Proposition 3.4 that Θ_r preserves the natural involutions and matrix orders on these two spaces. We also characterize the range space $\Theta_r(L_1(G))$ in Theorem 3.6.

We study the representation of $M_{cb}A(G)$ in §4. Using the techniques developed in Spronk's Ph.D. thesis [42] and published in [43], we show that $M_{cb}A(G)$ can be completely isometrically identified with the space $V^\infty_{\rm inv}(G,m)$ of all left invariant measurable Schur multipliers. It follows that we obtain a weak*-weak* continuous completely isometric isomorphism

$$\hat{\Theta} : M_{cb}A(G) \cong \mathcal{CB}^{\sigma, \mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G))), \tag{1.5}$$

which preserves the natural involutions and the matrix orders on these two spaces (see Theorem 4.3 and Theorem 4.5). In particular, if G is an abelian group, we can write $\mathcal{L}(G) = L_\infty(\hat{G})$ and $L_\infty(G) = \mathcal{L}(\hat{G})$. In this case, (1.5) can be expressed in the following duality form:

$$\Theta_r(M(\hat{G})) = \mathcal{CB}^{\sigma, L_\infty(\hat{G})}_{\mathcal{L}(\hat{G})}(\mathcal{B}(L_2(\hat{G}))).$$

In §5, we show some commutant results for $\Theta_r(M(G))$ and $\hat{\Theta}(M_{cb}A(G))$ in $\mathcal{CB}^\sigma(\mathcal{B}(L_2(G)))$ and some double commutant results for $\Theta_r(M(G))$ and $\hat{\Theta}(M_{cb}A(G))$ in $\mathcal{CB}(\mathcal{B}(L_2(G)))$, respectively. Finally, we show in §6 that for any amenable group G, there is a natural completely isometric homomorphism of $UCB(\hat{G})^*$ into $\mathcal{CB}^{\mathcal{L}(G)}_{L_\infty(G)}\mathcal{B}(L_2(G))$, which can be regarded as a duality result of Neufang's completely isometric representation theorem for $LUC(G)^*$.

2. Operator spaces and completely bounded maps

In this paper, we let X and Y be operator spaces and let $\mathcal{CB}(X,Y)$ denote the space of all completely bounded maps from X into Y. Then there exists a canonical operator space matrix norm on $\mathcal{CB}(X,Y)$ given by the identification

$$M_n(\mathcal{CB}(X,Y)) = \mathcal{CB}(X, M_n(Y)). \tag{2.1}$$

With this operator space structure, $\mathcal{CB}(\mathcal{B}(H)) = \mathcal{CB}(\mathcal{B}(H), \mathcal{B}(H))$ is a *completely contractive Banach algebra* since the composition multiplication $\Phi \circ \Psi$ on $\mathcal{CB}(\mathcal{B}(H))$ satisfies

$$\|[\Phi_{ij} \circ \Psi_{kl}]\|_{cb} \le \|[\Phi_{ij}]\|_{cb} \|[\Psi_{kl}]\|_{cb}$$

for all $[\Phi_{ij}] \in M_m(\mathcal{CB}(\mathcal{B}(H)))$ and $[\Psi_{kl}] \in M_n(\mathcal{CB}(\mathcal{B}(H)))$. There is a canonical involution on $\mathcal{CB}(\mathcal{B}(H))$ given by

$$\Phi^*(a) = \Phi(a^*)^*.$$

This involution is an isometrically conjugate automorphism on $\mathcal{CB}(\mathcal{B}(H))$ since

$$(\Phi \circ \Psi)^* = \Phi^* \circ \Psi^*.$$

Moreover, for each $n \in \mathbb{N}$ there exists a natural order on the matrix space

$$M_n(\mathcal{CB}(\mathcal{B}(H))) = \mathcal{CB}(\mathcal{B}(H), M_n(\mathcal{B}(H)))$$

given by the positive cone $\mathcal{CB}(\mathcal{CB}(H), M_n(\mathcal{B}(H)))^+$ of all completely positive maps from $\mathcal{B}(H)$ into $M_n(\mathcal{B}(H))$. This determines a matrix order on $\mathcal{CB}(\mathcal{B}(H))$.

If $\mathcal{M}$ is a von Neumann on a Hilbert space H, we let

$$\mathcal{CB}_{\mathcal{M}}(B(H)) = \mathcal{CB}_{\mathcal{M}}(\mathcal{B}(H), \mathcal{B}(H))$$

denote the space of all completely bounded $\mathcal{M}$-bimodule morphisms on $\mathcal{B}(H)$ and let $\mathcal{CB}^{\sigma}_{\mathcal{M}}(B(H))$ denote the space of all normal completely bounded $\mathcal{M}$-bimodule morphisms in $\mathcal{CB}_{\mathcal{M}}(B(H))$. Then $\mathcal{CB}^{\sigma}_{\mathcal{M}}(\mathcal{B}(H)) \subseteq \mathcal{CB}_{\mathcal{M}}(\mathcal{B}(H))$ are completely contractive Banach subalgebras of $\mathcal{CB}(\mathcal{B}(H))$ with a natural involution and a matrix order inherited from $\mathcal{CB}(\mathcal{B}(H))$. In general, $\mathcal{CB}^{\sigma}_{\mathcal{M}}(\mathcal{B}(H)) \neq \mathcal{CB}_{\mathcal{M}}(\mathcal{B}(H))$ (see Hofmeier and Wittstock [25]). But the two spaces are equal in some special cases (trivially when H is finite dimensional or when $\mathcal{M} = \mathcal{B}(H)$). Moreover, if G is a discrete group, then $\ell_\infty(G) = \ell_\infty(G)'$ is a finite atomic von Neumann algebra standardly represented on $\ell_2(G)$. We can conclude from [25, Lemma 3.5] that

$$\mathcal{CB}^{\sigma}_{\ell_\infty(G)}(\mathcal{B}(\ell_2(G))) = \mathcal{CB}_{\ell_\infty(G)}(\mathcal{B}(\ell_2(G))). \tag{2.2}$$

Inspection of the proof shows that we do not need to assume $\ell_2(G)$ to be separable (which is an assumption made throughout in [25] for the Hilbert spaces occurring). Similarly, we have

$$\mathcal{CB}^{\sigma}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G))) = \mathcal{CB}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G))) \tag{2.3}$$

for any compact group G since in this case, $\mathcal{L}(G) = \prod_\pi M_{n(\pi)} \otimes I_{n(\pi)}$ and $\mathcal{L}(G)' = \prod_\pi I_{n(\pi)} \otimes M_{n(\pi)}$ are finite atomic von Neumann algebras standardly represented on $L_2(G) = \sum^\oplus S^2_{n(\pi)}$, where we let $S^2_{n(\pi)}$ denote the Hilbert space of all $n(\pi) \times n(\pi)$ Hilbert-Schmidt matrices and let $n(\pi)$ denote the dimension of irreducible representations $\pi : G \to M_{n(\pi)}$ of G. Then we may obtain (2.3) by considering the central projections $z_{n(\pi)} = I_{n(\pi)} \otimes I_{n(\pi)} \in \mathcal{L}(G) \cap \mathcal{L}(G)'$ from $L_2(G)$ onto $S^2_{n(\pi)}$. Again, we do not have to assume the separability of $L_2(G)$. The result is true for arbitrary compact groups. Actually, the corresponding result holds for general discrete Kac algebras.

It is important to note that the mapping spaces $\mathcal{CB}^{\sigma}_{\mathcal{M}}(\mathcal{B}(H))$ and $\mathcal{CB}_{\mathcal{M}}(\mathcal{B}(H))$ can be completely identified with the extended (or weak*) Haagerup tensor product $\mathcal{M}' \otimes^{eh} \mathcal{M}'$ and the normal Haagerup tensor product $\mathcal{M}' \otimes^{\sigma h} \mathcal{M}'$ of $\mathcal{M}'$, the commutant of $\mathcal{M}$ in $\mathcal{B}(H)$, respectively. We assume that readers are familiar with the *Haagerup tensor product* $X \otimes^h Y$ (for instance, see details in [11], [35], and [37]). The definition for the *extended Haagerup tensor product* $X \otimes^{eh} Y$ can be found in [8] and [12]. For the convenience of the reader, let us recall that the extended Haagerup tensor norm of an element $[u_{ij}] \in M_n(X \otimes^{eh} Y)$ is defined by

$$\|[u_{ij}]\|_{eh,n} = \inf\{\|[x_{ik}]\|_{M_{n,I}(X)} \|[y_{kj}]\|_{M_{I,n}(Y)}\},$$

where the infimum is taken over all possible representations $[u_{ij}] = [x_{ik}] \odot [y_{kj}]$ with $[x_{ik}] \in M_{n,I}(X)$ and $[y_{kj}] \in M_{I,n}(Y)$. The index I in the above definition could be an infinite set (or a countable set if X and Y are operator spaces on separable Hilbert spaces). In this case, the notion $[u_{ij}] = [x_{ik}] \odot [y_{kj}]$ means that

$$\langle [u_{ij}], f \otimes g \rangle = \left[\sum_{k \in I} f(x_{ik}) g(y_{kj}) \right]$$

for all $f \in X^*$ and $g \in Y^*$. For dual operator spaces X^* and Y^*, $X^* \otimes^{eh} Y^*$ can be completely isometrically identified with the *weak* Haagerup tensor product* $X^* \otimes^{w^*h} Y^*$ introduced by Blecher and Smith [2] via the following complete isometries:

$$X^* \otimes^{eh} Y^* = (X \otimes^h Y)^* = X^* \otimes^{w^*h} Y^*.$$

The *normal Haagerup tensor product*

$$X^* \otimes^{\sigma h} Y^* = (X \otimes^{eh} Y)^*$$

for dual operator spaces was first introduced by Effros and Kishimoto [7]. It was shown by Effros and Ruan [12, §5] that the identity map on $X^* \otimes Y^*$ extends to a completely isometric inclusion

$$X^* \otimes^{eh} Y^* \hookrightarrow X^* \otimes^{\sigma h} Y^* \tag{2.4}$$

and the image space $X^* \otimes^{eh} Y^*$ is completely contractively complemented in $X^* \otimes^{\sigma h} Y^*$ since the adjoint map $(\iota_{X\otimes Y})^*$ of $\iota_{X\otimes Y} : X \otimes^h Y \hookrightarrow X \otimes^{eh} Y$ induces a completely contractive projection from $X^* \otimes^{\sigma h} Y^*$ onto $X^* \otimes^{eh} Y^*$.

Given $u = \sum_{k\in I} x_k \otimes y_k \in \mathcal{M}' \otimes^{eh} \mathcal{M}'$, we can define a normal completely bounded $\mathcal{M}$-bimodule morphism

$$T(u)(a) = \sum_{k\in I} x_k\, a\, y_k \qquad (\text{weak}^* \text{ limit}) \tag{2.5}$$

on $\mathcal{B}(H)$. It was shown by Haagerup [20] (also see [9] and [40]) that

$$T : u \in \mathcal{M}' \otimes^{eh} \mathcal{M}' \mapsto T(u) \in \mathcal{CB}^\sigma_{\mathcal{M}}(\mathcal{B}(H))$$

determines a weak*-weak* continuous completely isometric isomorphism from $\mathcal{M}' \otimes^{eh} \mathcal{M}'$ onto $\mathcal{CB}^\sigma_{\mathcal{M}}(\mathcal{B}(H))$ with respect to the completely contractive Banach algebra structure on $\mathcal{M}' \otimes^{eh} \mathcal{M}'$ given by

$$(x \otimes y) \circ (\tilde{x} \otimes \tilde{y}) \mapsto x\tilde{x} \otimes \tilde{y}y.$$

Moreover, there is an isometric involution

$$(x \otimes y)^* = y^* \otimes x^*$$

and a matrix order on $\mathcal{M}' \otimes^{eh} \mathcal{M}'$ given by the positive cones

$$\begin{aligned} M_n(\mathcal{M}' \otimes^{eh} \mathcal{M}')^+ = \{[u_{ij}] \in M_n(\mathcal{M} \otimes^{eh} \mathcal{M}') :& \\ [u_{ij}] = x^* \odot x \text{ for some } & x = [x_{kj}] \in M_{I,n}(\mathcal{M}')\}. \end{aligned}$$

It is easy to see that T preserves the involution and the matrix order on these spaces.

We can similarly define a completely contractive Banach algebra, an isometric involution, and a matrix order on the normal Haagerup tensor product $\mathcal{M}' \otimes^{\sigma h} \mathcal{M}'$. It was shown by Effros and Kishimoto [7] that there is a natural extension of T to a weak*-weak* continuous completely isometric isomorphism $\tilde{T}$ from $\mathcal{M}' \otimes^{\sigma h} \mathcal{M}'$ onto $\mathcal{CB}_{\mathcal{M}}(\mathcal{B}(H))$. Moreover, $\tilde{T}$ preserves the involution and the matrix order on $\mathcal{M}' \otimes^{\sigma h} \mathcal{M}'$ and $\mathcal{CB}_{\mathcal{M}}(\mathcal{B}(H))$. Therefore, we can completely identify these spaces

$$\mathcal{M}' \otimes^{eh} \mathcal{M}' \cong \mathcal{CB}^\sigma_{\mathcal{M}}(\mathcal{B}(H)) \text{ and } \mathcal{M}' \otimes^{\sigma h} \mathcal{M}' \cong \mathcal{CB}_{\mathcal{M}}(\mathcal{B}(H)) \tag{2.6}$$

via T and $\tilde{T}$, respectively. The complement of $\mathcal{M}' \otimes^{eh} \mathcal{M}'$ in $\mathcal{M}' \otimes^{\sigma h} \mathcal{M}'$ exactly corresponds to the space $\mathcal{CB}^s_{\mathcal{M}}(\mathcal{B}(H))$ of all *singular* completely bounded $\mathcal{M}$-bimodule morphisms on $\mathcal{B}(H)$.

Finally we note that there is a commutant theorem for $\mathcal{CB}^\sigma_{\mathcal{M}}(\mathcal{B}(H))$ (respectively, for $\mathcal{CB}_{\mathcal{M}}(\mathcal{B}(H))$) in $\mathcal{CB}(\mathcal{B}(H))$. If $\mathcal{V}$ is a subspace of $\mathcal{CB}(\mathcal{B}(H))$, we let

$$\mathcal{V}^c = \{\Psi \in \mathcal{CB}(\mathcal{B}(H)) : \Psi \circ \Phi = \Phi \circ \Psi \text{ for all } \Phi \in \mathcal{V}\}$$

denote the *commutant* of $\mathcal{V}$ in $\mathcal{CB}(\mathcal{B}(H))$. Then we have

$$\mathcal{CB}^{\sigma}_{\mathcal{M}}(\mathcal{B}(H))^c = \mathcal{CB}_{\mathcal{M}'}(\mathcal{B}(H)), \tag{2.7}$$

and if, in addition, $\mathcal{M}$ is standardly represented on H, then

$$\mathcal{CB}_{\mathcal{M}}(\mathcal{B}(H))^c = \mathcal{CB}^{\sigma}_{\mathcal{M}'}(\mathcal{B}(H)). \tag{2.8}$$

Combining (2.7) and (2.8), we obtain the following double commutant theorem:

$$\mathcal{CB}^{\sigma}_{\mathcal{M}}(\mathcal{B}(H))^{cc} = \mathcal{CB}^{\sigma}_{\mathcal{M}}(\mathcal{B}(H)) \tag{2.9}$$

when $\mathcal{M}$ is standardly represented on H. (2.7) is due to Effros and Exel [6, §3]; (2.8) was proved by Hofmeier and Wittstock [25, Proposition 3.1 and Remark 4.3] in case H is separable, and was extended to the non-separable situation by Magajna [31, §2]. We remark that, trivially, we also have

$$\mathcal{CB}_{\mathcal{M}}(\mathcal{B}(H))^{cc} = \mathcal{CB}_{\mathcal{M}}(\mathcal{B}(H)).$$

3. Representation of $M(G)$

The measure algebra $M(G)$ of all bounded complex-valued (Radon) measures on G is a Banach algebra with the multiplication defined by

$$\mu * \nu(f) = \int_G \int_G f(st)d\mu(s)d\nu(t) \tag{3.1}$$

for every bounded continuous function $f \in C_b(G)$ and $\mu, \nu \in M(G)$. We may identify $L_1(G)$ with a norm closed ideal in $M(G)$, which consists of all absolutely continuous measures with respect to the Haar measure. It follows from the definition (3.1) that there is an $M(G)$-bimodule action on $L_1(G)$. Taking the dual, we obtain an $M(G)$-bimodule structure on $L_\infty(G)$, which is defined by

$$\langle \mu \cdot f,\, h\rangle = \langle f,\, h * \mu\rangle \quad\text{and}\quad \langle f \cdot \mu,\, h\rangle = \langle f,\, \mu * h\rangle \tag{3.2}$$

for all $h \in L_1(G)$. More precisely, we have

$$\begin{aligned} \mu \cdot f(s) &= \int_G f(st)d\mu(t) = \int_G l_s f(t)d\mu(t) \quad\text{and} \\ f \cdot \mu(t) &= \int_G f(st)d\mu(s) = \int_G r_t f(s)d\mu(s), \end{aligned} \tag{3.3}$$

where we let $l_s f$ and $r_t f$ denote the *left translation* and *right translation*

$$l_s f(t) = f(st) \quad\text{and}\quad r_t f(s) = f(st).$$

Let $\lambda : G \to \mathcal{B}(L_2(G))$ denote the *left regular representation*, and let $\rho : G \to \mathcal{B}(L_2(G))$ denote the *right regular representation* defined by

$$\lambda(s)\xi(t) = \xi(s^{-1}t) \quad\text{and}\quad \rho(s)\xi(t) = \Delta(s)^{1/2}\xi(ts)$$

for $\xi \in L_2(G)$, $s, t \in G$, and the Haar modular function $\Delta : G \to (0, +\infty)$. We let

$$\mathcal{L}(G) = \overline{\operatorname{span}\{\lambda(s) : s \in G\}}^{s.o.t} \quad\text{and}\quad \mathcal{R}(G) = \overline{\operatorname{span}\{\rho(s) : s \in G\}}^{s.o.t}$$

denote the *left group von Neumann algebra* and the *right group von Neumann algebra* generated by λ and ρ, respectively. Then $\mathcal{L}(G)$ and $\mathcal{R}(G)$ are standardly represented on $L_2(G)$ and satisfy the commutant relations

$$\mathcal{L}(G)' = \mathcal{R}(G) \quad\text{and}\quad \mathcal{R}(G)' = \mathcal{L}(G). \tag{3.4}$$

With the conjugate linear isometry

$$J\xi(s) = \Delta(s^{-1})^{\frac{1}{2}}\overline{\xi(s^{-1})}$$

on $L_2(G) = L_2(\mathcal{L}(G))$, we obtain a natural *-anti-isomorphism

$$\lambda(t) \in \mathcal{L}(G) \mapsto J\lambda(t)^* J = \rho(t^{-1}) \in \mathcal{R}(G) \tag{3.5}$$

such that $J\mathcal{L}(G)J = \mathcal{R}(G)$.

We obtain the following result known as Heisenberg's theorem (see [13, Corollary 4.1.5] for a proof for general Kac algebras). We provide a simple proof here for the convenience of the reader.

Lemma 3.1. *Let G be a locally compact group. Then we have*

$$L_\infty(G) \cap \mathcal{R}(G) = \mathbb{C}1 = L_\infty(G) \cap \mathcal{L}(G). \tag{3.6}$$

Taking the commutants, we obtain

$$L_\infty(G) \vee \mathcal{L}(G) = \mathcal{B}(L_2(G)) = L_\infty(G) \vee \mathcal{R}(G), \tag{3.7}$$

where $L_\infty(G) \vee \mathcal{L}(G)$ denotes the von Neumann algebra generated by $L_\infty(G)$ and $\mathcal{L}(G)$, and $L_\infty(G) \vee \mathcal{R}(G)$ denotes the von Neumann algebra generated by $L_\infty(G)$ and $\mathcal{R}(G)$.

Proof. If $f \in L_\infty(G) \cap \mathcal{R}(G)$, then we get

$$l_s f = \lambda(s)^* f \lambda(s) = f$$

for all $s \in G$. This implies that f is a constant function on G and thus shows that $L_\infty(G) \cap \mathcal{R}(G) = \mathbb{C}1$. Taking the commutant, we obtain

$$\mathcal{B}(L_2(G)) = (L_\infty(G) \cap \mathcal{R}(G))' = L_\infty(G) \vee \mathcal{L}(G).$$

A similar argument shows that $L_\infty(G) \cap \mathcal{L}(G) = \mathbb{C}1$ and

$$\mathcal{B}(L_2(G)) = (L_\infty(G) \cap \mathcal{L}(G))' = L_\infty(G) \vee \mathcal{R}(G).$$

□

A function $f \in L_\infty(G)$ is said to be *left uniformly continuous* if the left translation map $s \in G \to l_s f \in L_\infty(G)$ is continuous. We note that in some books (such as [23]) these functions are called *right uniformly continuous* since this definition is equivalent to saying that we have $|f(s) - f(t)| < \varepsilon$ for st^{-1} in some neighborhood of e. In this paper, let us stay with the first notion and let $LUC(G)$ denote the space of all bounded left uniformly continuous functions on G. Then it is easy to see that

$$C_0(G) \subseteq LUC(G) \subseteq C_b(G).$$

Since the right regular representation $\rho : G \to \mathcal{B}(L_2(G))$ is strong operator continuous, for any $a \in \mathcal{B}(L_2(G))$ and $\xi, \eta \in L_2(G)$,

$$f_{a,\xi,\eta}(t) = \langle \rho(t) a \rho(t)^* \xi \mid \eta \rangle$$

defines a bounded left uniformly continuous function on G with $\|f_{a,\xi,\eta}\|_\infty \leq \|a\| \, \|\xi\| \, \|\eta\|$. Actually, for fixed $\xi, \eta \in L_2(G)$ the family of functions $\{f_{a,\xi,\eta}\}$ is *equi-left uniformly continuous in $a \in Ball(\mathcal{B}(L_2(G)))$*, i.e.

$$\|l_s f_{a,\xi,\eta} - l_t f_{a,\xi,\eta}\|_\infty \to 0 \tag{3.8}$$

uniformly in a when $s \to t$ in G. For any $\mu \in M(G)$,

$$(\xi, \eta) \in L_2(G) \times L_2(G) \mapsto \int_G \langle \rho(t) a \rho(t)^* \xi \mid \eta \rangle d\mu(t) \in \mathbb{C}$$

is a bounded sesquilinear form on $L_2(G) \times L_2(G)$ and thus determines a bounded linear operator, which is denoted by $\Theta_r(\mu)(a)$, on $L_2(G)$ such that

$$\langle \Theta_r(\mu)(a)\xi \mid \eta \rangle = \int_G \langle \rho(t) a \rho(t)^* \xi \mid \eta \rangle d\mu(t).$$

We simply write

$$\Theta_r(\mu)(a) = \int_G \rho(t) a \rho(t)^* d\mu(t). \tag{3.9}$$

In particular, if we let δ_t denote the Dirac measure at t we can write

$$\Theta_r(\delta_t)(a) = \rho(t) a \rho(t)^*. \tag{3.10}$$

Then Θ_r is a well-defined weak*-weak* continuous isometric homomorphism from $M(G)$ into $\mathcal{B}^\sigma(\mathcal{B}(L_2(G)))$. Moreover, for each $\mu \in M(G)$, $\Theta_r(\mu)$ is a completely bounded $\mathcal{L}(G)$-bimodule morphism on $\mathcal{B}(L_2(G))$ with $\|\Theta_r(\mu)\|_{cb} = \|\Theta_r(\mu)\| = \|\mu\|$ (see [33]). If $f \in L_\infty(G)$, we have

$$\Theta_r(\mu)(f) = \int_G \rho(t) f \rho(t)^* d\mu(t) = \mu \cdot f \in L_\infty(G).$$

This shows that $\Theta_r(\mu)$ maps $L_\infty(G)$ into $L_\infty(G)$. In particular, if $f \in LUC(G)$ (respectively, $f \in C_b(G)$), then $\Theta_r(\mu)(f) = \mu \cdot f$ is contained in $LUC(G)$ (respectively, in $C_b(G)$). In this case, we can consider the point evaluation

$$\langle \delta_e, \Theta_r(\mu)(f) \rangle = \mu \cdot f(e) = \int_G f(t) d\mu(t) = \mu(f). \tag{3.11}$$

Let us use $\mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$ to denote the space of all normal completely bounded $\mathcal{L}(G)$-bimodule morphisms on $\mathcal{B}(L_2(G))$ which map $L_\infty(G)$ into $L_\infty(G)$, and let us assume that $M(G) = C_0(G)^*$ is equipped with the MAX operator space matrix norm. We are now ready to state the following result of Størmer [45], Ghahramani [15], and Neufang (see [32] and [34]) for Θ_r in the completely isometric form.

Theorem 3.2. *The map Θ_r is a weak*-weak* continuous completely isometric isomorphism from $M(G)$ onto $\mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$.*

Proof. We only need to show that Θ_r is onto and is completely isometric. Suppose that we are given a map $\Phi \in \mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$. Then we need to construct a measure $\mu \in M(G)$ such that $\Theta_r(\mu) = \Phi$. We carry out this construction in the following three steps.

Step 1. We claim that $\Phi(1) = \alpha 1$ for some $\alpha \in \mathbb{C}$. Since Φ is $L_\infty(G)$-invariant, we have $\Phi(1) \in L_\infty(G)$. Moreover since Φ is an $\mathcal{L}(G)$-bimodule morphism, we have

$$\lambda(s)\Phi(1) = \Phi(\lambda(s)) = \Phi(1)\lambda(s)$$

for all $s \in G$. It follows from Lemma 3.1 that $\Phi(1) \in L_\infty(G) \cap \mathcal{R}(G) = \mathbb{C}1$.

Step 2. Construct a measure $\mu \in M(G)$. Since $\Phi \in \mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$, we have

$$l_s \Phi(f) = \lambda(s)^* \Phi(f) \lambda(s) = \Phi(\lambda(s)^* f \lambda(s)) = \Phi(l_s f).$$

This implies that for every $f \in LUC(G)$, $\Phi(f)$ is a bounded left uniformly continuous function on G. Then, the point evaluation $\Phi(f)(e) = \langle \delta_e, \Phi(f) \rangle$ defines a bounded linear functional

$$f \in C_0(G) \mapsto \Phi(f)(e) \in \mathbb{C}$$

on $C_0(G)$. By the Riesz representation theorem, we obtain a unique bounded complex-valued measure $\mu \in M(G)$ which satisfies $\mu(f) = \Phi(f)(e)$.

Step 3. We claim that $\Theta_r(\mu) = \Phi$. For any $f \in C_0(G)$, we have

$$\begin{aligned}\Theta_r(\mu)(f)(s) &= \int_G f(st)d\mu(t) = \mu(l_s f) = \Phi(l_s f)(e) \\ &= \lambda(s)^* \Phi(f) \lambda(s)(e) = l_s \Phi(f)(e) = \Phi(f)(s).\end{aligned}$$

This shows that $\Theta_r(\mu)(f) = \Phi(f)$ for any $f \in C_0(G)$. Since $C_0(G)$ is weak* dense in $L_\infty(G)$, the normality of $\Theta_r(\mu)$ implies that $\Theta_r(\mu) = \Phi$ on $L_\infty(G)$. We also have $\Theta_r(\mu) = \Phi$ on $\mathcal{L}(G)$ since for any $s \in G$,

$$\Theta_r(\mu)(\lambda(s)) = \lambda(s)\Theta_r(\mu)(1) = \lambda(s)\Phi(1)(e) = \lambda(s)\Phi(1) = \Phi(\lambda(s)).$$

Since $\mathrm{span}\{f\lambda_s : f \in L_\infty(G), s \in G\}$ is actually a *-subalgebra of $\mathcal{B}(L_2(G))$, its σ-weak closure is equal to the von Neumann algebra generated by $L_\infty(G)$ and $\mathcal{L}(G)$ and thus equals $\mathcal{B}(L_2(G))$ by Lemma 3.1; therefore, we can conclude that $\Theta_r(\mu) = \Phi$ on $\mathcal{B}(L_2(G))$.

Since $M(G) = C_0(G)^*$ is equipped with the MAX operator space matrix norm, Θ_r is clearly a complete contraction. For any $[\mu_{ij}] \in M_n(M(G)) = M_n(C_0(G)^*) = \mathcal{CB}(C_0(G), M_n)$, we can express the matrix norm of $[\mu_{ij}]$ as follows:

$$\|[\mu_{ij}]\|_{M_n(M(G))} = \sup\{\|[\mu_{ij}(f)]\|_{M_n} : f \in C_0(G), \|f\|_\infty \le 1\}. \tag{3.12}$$

Then we can conclude from (3.11) and (3.12) that

$$\begin{aligned}\|[\Theta_r(\mu_{ij})]\|_{cb} &\ge \sup\{\|[\Theta_r(\mu_{ij})(f)]\|_{M_n(C_b(G))} : f \in C_0(G), \|f\|_\infty \le 1\} \\ &\ge \sup\{\|[\langle \delta_e, \Theta_r(\mu_{ij})(f)\rangle]\|_{M_n} : f \in C_0(G), \|f\|_\infty \le 1\} \\ &= \sup\{\|[\mu_{ij}(f)]\|_{M_n} : f \in C_0(G), \|f\|_\infty \le 1\} = \|[\mu_{ij}]\|.\end{aligned}$$

This shows that Θ_r is a completely isometric isomorphism from $M(G)$ onto $\mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$. $\square$

Considering the complex conjugation on $M(G)$, we obtain the following result.

Proposition 3.3. *For any $\mu \in M(G)$, we have*

$$\Theta_r(\bar{\mu}) = \Theta_r(\mu)^*.$$

Therefore, $\mu \in M(G)$ is a real-valued measure if and only if $\Theta_r(\mu)$ is a self-adjoint map in $\mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$.

Proof. Let $\mu \in M(G)$. For any $a \in \mathcal{B}(L_2(G))$, we have

$$\begin{aligned}\Theta_r(\mu)^*(a) = \Theta_r(\mu)(a^*)^* &= (\int_G \rho(t) a^* \rho(t)^* d\mu(t))^* \\ &= \int_G \rho(t) a \rho(t)^* d\bar{\mu}(t) = \Theta_r(\bar{\mu})(a).\end{aligned}$$

This shows that $\bar{\mu} = \mu$ if and only if $\Theta_r(\mu)^* = \Theta_r(\mu)$. $\square$

It is worthy to note that Θ_r does not preserve the anti-automorphic involution on $M(G)$ given by

$$d\mu^*(t) = d\overline{\mu}(t^{-1}) = \Delta(t)^{-1} d\overline{\mu}(t).$$

With this involution, $\Theta_r(\mu^*) \in \mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$ can be expressed as

$$\Theta_r(\mu^*)(a) = \int_G \rho(t)^* a \rho(t) \Delta(t)^{-1} d\overline{\mu}(t).$$

For each $n \in \mathbb{N}$, there is a natural order on $M_n(M(G))$ given by the positive cone

$$M_n(M(G))^+ = \mathcal{CB}(C_0(G), M_n)^+$$

of all completely positive maps from $C_0(G)$ into M_n. The following proposition shows that Θ_r also preserves the matrix orders on $M(G)$ and $\mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$.

Proposition 3.4. *Given $[\mu_{ij}] \in M_n(M(G))$, the following are equivalent:*

(1) *$[\mu_{ij}]$ is positive in $M_n(M(G))^+$;*

(2) *$[\Theta_r(\mu_{ij})]$ is a completely positive map in*

$$\mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)), M_n(\mathcal{B}(L_2(G))));$$

(3) *$[\Theta_r(\mu_{ij})]$ is a positive map in $\mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)), M_n(\mathcal{B}(L_2(G))))$.*

Therefore, Θ_r preserves the matrix orders on $M(G)$ and $\mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$.

Proof. It is easy to verify that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Let us assume $[\Phi_{ij}] = [\Theta_r(\mu_{ij})]$ is a positive map in

$$M_n(\mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G))))^+ = \mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)), M_n(\mathcal{B}(L_2(G))))^+.$$

It follows from (3.11) that for any positive function $f \in C_0(G)^+$,

$$[\mu_{ij}(f)] = [\Theta_r(\mu_{ij})(f)(e)] = [\Phi_{ij}(f)(e)]$$

is a positive matrix in M_n. Then $[\mu_{ij}]$ is a positive and thus completely positive map (by Stinespring [44]) from $C_0(G)$ into M_n. $\square$

In the following, let us write $RUC(G)$ for the space of right uniformly continuous functions $f \in L_\infty(G)$, i.e., the translation map $s \in G \to r_s f \in L_\infty(G)$ is continuous. Note that these functions are called left uniformly continuous in [23]; cf. the discussion after Lemma 3.1 above.

Lemma 3.5. *For $\mu \in M(G)$, the following conditions are equivalent:*

(1) *$\mu \in L_1(G)$;*

(2) *$\Theta_r(\mu)$ maps $L_\infty(G)$ into $RUC(G)$;*

(3) *$\Theta_r(\mu)$ maps $L_\infty(G)$ into $C_b(G)$.*

Proof. (1) $\Rightarrow$ (2). This is well known. Indeed, if $d\mu = g dt$ with $g \in L_1(G)$, then for any $f \in L_\infty(G)$, one easily checks that $\Theta_r(\mu)(f) = (g * \check{f})\check{}$, where $\check{f}(s) = f(s^{-1})$. Hence, by [23, Theorem (20.16)], we have $\Theta_r(\mu)(f) \in RUC(G)$.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (1). If G is a compact group, this is proved in [24, Theorem (35.13)]. However, inspection of the proof shows that compactness is only needed in order to apply [23, Theorem (8.7)], which holds for all compactly generated, locally compact groups G. This observation can be used to modify the proof given for [24, Theorem

(35.13)] and obtain our assertion for all locally compact groups G. We shall now outline this procedure as follows.

Assume towards a contradiction that $\mu \notin L_1(G)$. Write $\mu = \mu_a + \mu_s$, where μ_a is absolutely continuous and μ_s is singular with respect to the Haar measure. By assumption, we have $\mu_s \neq 0$. Without loss of generality, we may assume $\|\mu_s\| = 1$. Since $\mu_s \in M(G)$, there is a σ-compact set $H_0 \subseteq G$ with $\mu_s(H_0^c) = 0$. Let $H_1 \subseteq G$ be an open σ-compact subgroup of G. Define $\mathcal{H}$ to be the subgroup of G generated by H_0 and H_1. Obviously, $\mathcal{H}$ is σ-compact and open, since $\mathcal{H}^\circ \neq \varnothing$. Hence, $\mathcal{H}$ is an open σ-compact subgroup of G, in particular locally compact in the relative topology.

Denote by $\lambda_{\mathcal{H}}$ the restriction to $\mathcal{H}$ of our fixed left Haar measure, which defines a left Haar measure on $\mathcal{H}$. Define $\mu' = \mu_s|_{\mathcal{H}}$. Of course, we still have $\|\mu'\| = 1$, in particular $\mu' \neq 0$. Furthermore, it is easy to see that μ' is singular with respect to $\lambda_{\mathcal{H}}$. We can now, in order to finish the argument, follow the proof of [24, Theorem (35.13)] with the group G replaced by our $\mathcal{H}$ and the measure ν replaced by our μ' (using, as mentioned above, [23, Theorem (8.7)] for the compactly generated group $\mathcal{H}$). □

As a consequence of Lemma 3.5, we can obtain the following theorem, which characterizes the image space of $\Theta_r(L_1(G))$ in $\mathcal{CB}^{\sigma}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$.

Theorem 3.6. *We have*

$$\Theta_r(L_1(G)) = \mathcal{CB}^{\sigma,(L_\infty(G),RUC(G))}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G))) = \mathcal{CB}^{\sigma,(L_\infty(G),C_b(G))}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G))),$$

where $\mathcal{CB}^{\sigma,(L_\infty(G),RUC(G))}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$, *resp.* $\mathcal{CB}^{\sigma,(L_\infty(G),C_b(G))}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$, *denotes the space of all normal completely bounded* $\mathcal{L}(G)$*-bimodule morphisms which map* $L_\infty(G)$ *into* $RUC(G)$, *resp.* $C_b(G)$.

Remark 3.7. It is known from abstract harmonic analysis that there is a (completely) isometric injection

$$M(G) \hookrightarrow LUC(G)^*.$$

Neufang proved in [33] that the map Θ_r can be extended to a (completely) isometric homomorphism $\tilde{\Theta}_r$ from $LUC(G)^*$ into $\mathcal{CB}^{L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$ so that

$$\Theta_r(M(G)) = \tilde{\Theta}_r(LUC(G)^*) \cap \mathcal{CB}^{\sigma}(\mathcal{B}(L_2(G))).$$

However, it is still left open whether the map $\tilde{\Theta}_r$ is onto $\mathcal{CB}^{L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$ for general locally compact groups. The difficulty is of course the missing normality for maps in $\mathcal{CB}^{L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$. The techniques used in the proof of Theorem 3.2 fail in this case. In the special case when G is a compact group, this is true since every continuous function on G is bounded and uniformly continuous, and thus $M(G) = LUC(G)^*$. It follows from (2.3) that

(3.13)
$$\tilde{\Theta}_r(LUC(G)^*) = \Theta_r(M(G)) = \mathcal{CB}^{\sigma,L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G))) = \mathcal{CB}^{L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G))).$$

4. Representation of $M_{cb}A(G)$

We let $A(G)$ denote the *Fourier algebra* of G, which consists of all coefficient functions

$$\psi(s) = \langle \lambda(s)\xi \mid \eta \rangle \tag{4.1}$$

of the left regular representation λ. It was shown by Eymard [14] that $A(G)$ with the pointwise multiplication and norm

$$\|\psi\|_{A(G)} = \inf\{\|\xi\| \, \|\eta\| : \ \psi(s) = \langle \lambda(s)\xi \mid \eta \rangle\}$$

is a commutative Banach algebra. Moreover, $A(G)$ can be isometrically identified with the predual $\mathcal{L}(G)_*$ of $\mathcal{L}(G)$ and thus has a canonical operator space matrix norm with which $A(G)$ is a completely contractive Banach algebra (see [39]). The multiplication on $A(G)$ induces a commutative completely contractive $A(G)$-bimodule structure $\langle \omega \cdot x, \omega' \rangle = \langle x, \omega' \cdot \omega \rangle$ on $\mathcal{L}(G)$.

A function $\varphi : G \to \mathbb{C}$ is called a *multiplier* of $A(G)$ if the induced (pointwise) multiplication map

$$m_\varphi(\psi) = \varphi\psi$$

maps $A(G)$ into $A(G)$. It is known that m_φ is automatically bounded on $A(G)$. A multiplier φ is completely bounded if $\|m_\varphi\|_{cb} < \infty$. We let $M_{cb}A(G)$ denote the space of all completely bounded multipliers of $A(G)$. Then $M_{cb}A(G)$ is a completely contractive Banach subalgebra of $\mathcal{CB}(A(G))$. The complex conjugation on $M_{cb}A(G)$ corresponds to the involution on $\mathcal{CB}(A(G))$. Taking the Banach space dual, we obtain the completely isometric, but anti-isomorphic identification $\mathcal{CB}(A(G)) = \mathcal{CB}^\sigma(\mathcal{L}(G))$ such that $\varphi \in M_{cb}A(G)$ if and only if the adjoint map

$$M_\varphi = (m_\varphi)^* : \lambda(s) \in \mathcal{L}(G) \mapsto \varphi(s)\lambda(s) \in \mathcal{L}(G)$$

is a weak* continuous completely bounded map on $\mathcal{L}(G)$. This provides us with a completely isometrically isomorphic identification of $M_{cb}A(G)$ with the subalgebra $\mathcal{CB}^\sigma_{A(G)}(\mathcal{L}(G)))$ of all weak* continuous completely bounded $A(G)$-bimodule maps on $\mathcal{L}(G)$, and thus induces a natural matrix order on $M_{cb}A(G)$ given by the positive cones

$$M_n(M_{cb}A(G))^+ = \{[\varphi_{ij}] \in M_n(M_{cb}A(G)) \quad \text{such that } [M_{\varphi_{ij}}] : \mathcal{L}(G) \to M_n(\mathcal{L}(G)) \text{ is completely positive}\}.$$

Using an (unpublished) result of Gilbert [18], Bożejko and Fendler [4] showed that $M_{cb}A(G)$ is isometrically isomorphic to $B_2(G)$, the space of all *Herz-Schur multipliers* on G. This shows that a function $\varphi : G \to \mathbb{C}$ is contained in $M_{cb}A(G)$ with $\|\varphi\|_{M_{cb}A(G)} \le 1$ if and only if there exist a Hilbert space H and two bounded continuous maps $\xi, \eta : G \to H$ such that

$$\varphi(s^{-1}t) = \langle \eta(t) \mid \xi(s) \rangle = \xi(s)^*\eta(t) \tag{4.2}$$

and

$$\sup\{\|\xi(s)\|_H\} \sup\{\|\eta(t)\|_H\} \le 1.$$

If we replace $\xi(s)$ and $\eta(t)$ by $\check{\xi}(s) = \xi(s^{-1})$ and $\check{\eta}(t) = \eta(t^{-1})$, then (4.2) can be expressed as

$$\varphi(st^{-1}) = \langle \check{\eta}(t) \mid \check{\xi}(s) \rangle = \check{\xi}(s)^*\check{\eta}(t). \tag{4.3}$$

In this case, we have the same norm control

$$\sup\{\|\check{\xi}(s)\|_H\} \sup\{\|\check{\eta}(t)\|_H\} \le 1.$$

Haagerup gave a complete argument (including a proof of Gilbert's result) in [21, Appendix], and Jolissaint [26] provided a very short elegant proof of this result by using the representation theorem for completely bounded maps. The matricial form of (4.3) for discrete groups can be found in Pisier [37, Theorem 8.3]. Spronk showed in [43, Theorem 5.3] that such a matricial form still holds for general locally compact groups. We state their result in the following lemma, which can be proved by simply applying Jolissaint's representation argument to the normal completely bounded map $[M_{\varphi_{ij}}] : \mathcal{L}(G) \to M_n(\mathcal{L}(G)) \subseteq M_n(B(L_2(G)))$.

Lemma 4.1. *Let $\varphi_{ij} : G \to \mathbb{C}$ be functions on G with $1 \leq i, j \leq n$. The following are equivalent:*

(1) $[\varphi_{ij}] \in M_n(M_{cb}A(G))$ *with* $\|[\varphi_{ij}]\|_{M_n(M_{cb}A(G))} \leq 1$*;*
(2) *there exist a Hilbert space H and bounded continuous maps $\xi_i, \eta_j : G \to H$ such that*

$$[\varphi_{ij}(st^{-1})] = [\langle \eta_j(t) \mid \xi_i(s) \rangle] = [\xi_i(s)^* \eta_j(t)]$$

and

$$\sup\{\|[\xi_1(s) \cdots \xi_n(s)]\|_{\mathcal{B}(\mathbb{C}^n, H)}\} \sup\{\|[\eta_1(t) \cdots \eta_n(t)]\|_{\mathcal{B}(\mathbb{C}^n, H)}\} \leq 1.$$

Moreover, $[\varphi_{ij}] \in M_n(M_{cb}A(G))^+$ if and only if we can choose $\eta_j = \xi_j$ for all $j = 1, \cdots, n$.

In all of these cases, the functions φ_{ij} are automatically continuous.

A map $\xi : G \to H$ is called *weakly continuous* if $s \in G \mapsto \langle \xi(s) \mid \eta \rangle = \eta^* \xi(s)$ is continuous for all $\eta \in H$. A *continuous Schur multiplier* is a function $u : G \times G \to \mathbb{C}$ such that there exist a Hilbert space and bounded weakly continuous maps $\xi, \eta : G \to H$ with

$$u(s, t) = \langle \eta(t) \mid \xi(s) \rangle = \xi^*(s) \eta(t).$$

Given a function $\varphi : G \to \mathbb{C}$, we let $u_\varphi : G \times G \to \mathbb{C}$ denote the function defined by

$$u_\varphi(s, t) = \varphi(st^{-1}).$$

Lemma 4.1 shows that for every $\varphi \in M_{cb}A(G)$, u_φ is a continuous Schur multiplier. Let $V^b(G)$ denote the space of all continuous Schur multipliers on $G \times G$. Then $V^b(G)$ has a canonical operator space matrix norm given by

$$\begin{aligned}\|[u_{ij}]\|_{M_n(V^b(G))} = \inf\{ &\sup\{\|[\xi_1(s) \cdots \xi_n(s)]\|_{\mathcal{B}(\mathbb{C}^n, H)}\} \\ &\sup\{\|[\eta_1(t) \cdots \eta_n(t)]\|_{\mathcal{B}(\mathbb{C}^n, H)}\}\},\end{aligned}$$

where the infimum is taken over all representations $[u_{ij}(s, t)] = [\xi_i^*(s) \eta_j(t)]$. It was shown by Spronk [43, Proposition 3.6] that we may completely isometrically identify $V^b(G)$ with $C_b(G) \otimes^{eh} C_b(G)$. To see this, let us assume that $[u_{ij}]$ is a matrix element in $M_n(V^b(G))$ with norm $\|[u_{ij}]\|_{M_n(V^b(G))} < 1$. Then there exist bounded weakly continuous maps $\xi_i, \eta_j : G \to H$ such that

$$[u_{ij}(s, t)] = [\xi_i^*(s) \eta_j(t)]$$

and

$$\sup\{\|[\xi_1(s) \cdots \xi_n(s)]\|_{\mathcal{B}(\mathbb{C}^n, H)}\} \sup\{\|[\eta_1(t) \cdots \eta_n(t)]\|_{\mathcal{B}(\mathbb{C}^n, H)}\} < 1.$$

We may fix an orthonormal basis $\{h_k\}_{k \in I}$ for the Hilbert space H and thus identify H with $\ell_2(I)$. In this case the identity operator 1_H on H can be expressed as $1_H = \sum_{k \in I} h_k \otimes h_k^*$. Let us define

$$v_{ik}(s) = \langle h_k \mid \xi_i(s) \rangle = \xi_i^*(s) h_k \quad \text{and} \quad w_{kj}(t) = \langle \eta_j(t) \mid h_k \rangle = h_k^* \eta_j(t).$$

These are continuous functions in $C_b(G)$ such that $[v_{ik}] \in M_{n,I}(C_b(G))$ and $[w_{kj}] \in M_{I,n}(C_b(G))$ with

$$\begin{aligned}\|[v_{ik}]\|_{M_{n,I}(C_b(G))} &= \sup\{\|[\xi_1(s)\cdots\xi_n(s)]^*\|_{\mathcal{B}(\ell_2(I),\mathbb{C}^n)}\}\\ &= \sup\{\|[\xi_1(s)\cdots\xi_n(s)]\|_{\mathcal{B}(\mathbb{C}^n,H)}\}\end{aligned}$$

and

$$\begin{aligned}\|[w_{kj}]\|_{M_{I,n}(C_b(G))} &= \sup\{\|[\eta_1(t)\cdots\eta_n(t)]\|_{\mathcal{B}(\mathbb{C}^n,\ell_2(I))}\}\\ &= \sup\{\|[\eta_1(t)\cdots\eta_n(t)]\|_{\mathcal{B}(\mathbb{C}^n,H)}\}.\end{aligned}$$

It follows that we can express

$$[u_{ij}] = [v_{ik}] \odot [w_{kj}] \in M_n(C_b(G) \otimes^{eh} C_b(G)) \tag{4.4}$$

with norm $\|[u_{ij}]\|_{M_n(C_b(G)\otimes^{eh}C_b(G))} < 1$. We note that the expression of $[u_{ij}]$ in (4.4) is well defined and is independent of the choice of an orthonormal basis $\{h_k\}_{k\in I}$. The converse is obvious since we may simply take $H = M_{I,1} = \ell_2(I)$.

Motivated by this, Spronk also considered measurable Schur multipliers in [42] by letting

$$V^\infty(G,m) = L_\infty(G) \otimes^{eh} L_\infty(G),$$

i.e., a function $u : G\times G \to \mathbb{C}$ is a *measurable Schur multiplier* if and only if u can be identified with an element in $L_\infty(G)\otimes^{eh} L_\infty(G)$. The advantage of this approach is that we may apply the completely isometric isomorphism

$$T : u \in L_\infty(G) \otimes^{eh} L_\infty(G) \mapsto T(u) \in \mathcal{CB}^\sigma_{L_\infty(G)}(\mathcal{B}(L_2(G)))$$

defined in (2.5).

A continuous (respectively, measurable) Schur multiplier u is *right invariant* if for every $g\in G$,

$$r_g u(s,t) = u(sg,tg) = u(s,t)$$

for all $(s,t)\in G\times G$ (respectively, for almost every $(s,t)\in G\times G$). It is easy to see that for any $\varphi \in M_{cb}A(G)$, the corresponding Schur multiplier u_φ is right invariant. If we let $V^b_{\mathrm{inv}}(G)$ and $V^\infty_{\mathrm{inv}}(G,m)$ denote the spaces of all right invariant continuous Schur multipliers and right invariant measurable Schur multipliers on $G\times G$, respectively, we obtain the following result, in which the equivalence of (1), (2) and (2′) has been observed by Spronk [43, Theorem 5.3]. We provide a very different, but much simpler proof here.

Theorem 4.2. *Let $u_{ij} \in V^\infty(G,m) = L_\infty(G)\otimes^{eh} L_\infty(G)$ with $1\le i,j\le n$. Then the following are equivalent:*

(1) *there exists $[\varphi_{ij}] \in M_n(M_{cb}A(G))$ such that $\|[\varphi_{ij}]\|_{M_n(M_{cb}A(G))} \le 1$ and $[u_{ij}] = [u_{\varphi_{ij}}]$;*
(2) *$[u_{ij}] \in M_n(V^b_{\mathrm{inv}}(G))$ with $\|[u_{ij}]\|_{M_n(V^b_{\mathrm{inv}}(G))} \le 1$;*
(2′) *$[u_{ij}] \in M_n(V^\infty_{\mathrm{inv}}(G,m))$ with $\|[u_{ij}]\|_{M_n(V^\infty_{\mathrm{inv}}(G,m))} \le 1$;*
(3) *$[T(u_{ij})] \in M_n(\mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G))))$ with $\|[T(u_{ij})]\|_{cb} \le 1$.*

Proof. (1) $\Rightarrow$ (2) is an immediate consequence of Lemma 4.1, and (2) $\Rightarrow$ (2′) is obvious since we have the completely isometric injections

$$C_b(G)\otimes^{eh} C_b(G) \hookrightarrow L_\infty(G)\otimes^{eh} L_\infty(G) \quad\text{and thus}\quad V^b_{\mathrm{inv}}(G) \hookrightarrow V^\infty_{\mathrm{inv}}(G,m).$$

(2′) $\Rightarrow$ (3) Let us assume that $[u_{ij}] = \left[\sum_{k\in I} v_{ik}\otimes w_{kj}\right]$ is a contractive element in $M_n(V^\infty_{\mathrm{inv}}(G,m))$. Then $T_n([u_{ij}]) = [T(u_{ij})]$ is a normal completely contractive

$L_\infty(G)$-bimodule morphism contained in $M_n(\mathcal{CB}^\sigma_{L_\infty(G)}(\mathcal{B}(L_2(G))))$. For any $s \in G$ and $[\zeta_j] \in L_2(G)^n$, we have

$$[T(u_{ij})(\lambda(s))]\,[\zeta_j]\,(t) = \left[\sum_{k\in I}(v_{ik}\lambda(s)w_{kj}\zeta_j)(t)\right] = \left[\sum_{k\in I} v_{ik}(t)w_{kj}(s^{-1}t)\zeta_j(s^{-1}t)\right]$$
$$= \left[u_{ij}(t, s^{-1}t)(\lambda(s)\zeta_j)(t)\right] = [u_{ij}(s,e)\lambda(s)\zeta_j]\,(t).$$

This shows that $T_n([u_{ij}])(\lambda(s)) = [u_{ij}(s,e)\lambda(s)] \in M_n(\mathcal{L}(G))$. By the normality of $T_n([u_{ij}])$, we can conclude that $T_n([u_{ij}])$ maps $\mathcal{L}(G)$ into $M_n(\mathcal{L}(G))$. Therefore, $T_n([u_{ij}]) \in M_n(\mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G))))$.

(3) $\Rightarrow$ (1) Let us assume (3). Since T is a complete isometry from $L_\infty(G) \otimes^{eh} L_\infty(G)$ onto $\mathcal{CB}^\sigma_{L_\infty(G)}(\mathcal{B}(L_2(G)))$, we may assume that $[u_{ij}] = \left[\sum_{k\in I} v_{ik} \otimes w_{kj}\right]$ is a contractive element in $M_n(L_\infty(G) \otimes^{eh} L_\infty(G))$. As we calculated above,

$$[T(u_{ij})(\lambda(s))] = \left[\sum_{k\in I} v_{ik}\lambda(s)w_{kj}\right] = \left[\sum_{k\in I} v_{ik}\,(l_{s^{-1}}w_{kj})\lambda(s)\right]$$

for any $s \in G$. Since $\sum_{k\in I} v_{ik}\,(l_{s^{-1}}w_{kj}) \in L_\infty(G)$ and $\sum_{k\in I} v_{ik}\,(l_{s^{-1}}w_{kj}) = T(u_{ij})(\lambda(s))\lambda(s)^* \in \mathcal{L}(G)$, we can conclude from Lemma 3.1 that $\sum_{k\in I} v_{ik}\,(l_{s^{-1}}w_{kj})$ are scalar multiples of identity contained in $L_\infty(G) \cap \mathcal{L}(G) = \mathbb{C}1$. We let $\varphi_{ij} : G \to \mathbb{C}$ be functions determined by

$$\sum_{k\in I} v_{ik}\,(l_{s^{-1}}w_{kj}) = \varphi_{ij}(s)1. \tag{4.5}$$

Then these functions φ_{ij} satisfy

$$[T(u_{ij})(\lambda(s))] = [\varphi_{ij}(s)\lambda(s)].$$

Since $T(u_{ij})$, restricted to $\mathcal{L}(G)$, is a normal completely bounded map on $\mathcal{L}(G)$, its preadjoint determines a completely bounded map on $A(G)$. Therefore, φ_{ij} defines a completely bounded multiplier on $A(G)$ such that $T(u_{ij})_{|\mathcal{L}(G)} = M_{\varphi_{ij}}$. It follows from (4.5) that we have

$$u_{ij}(s,t) = \sum_{k\in I} v_{ik}(s)\,w_{kj}(t) = \sum_{k\in I} v_{ik}(s)\,(l_{ts^{-1}}w_{kj})(s) = \varphi_{ij}(st^{-1}).$$

This completes the proof. ☐

Theorem 4.3. *We have the completely isometric isomorphisms*

$$M_{cb}A(G) \cong V^b_{\mathrm{inv}}(G) \cong V^\infty_{\mathrm{inv}}(G,m) \cong \mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G))), \tag{4.6}$$

which preserve the natural involutions and matrix orders on these completely contractive Banach algebras.

In particular, we may completely identify $M_{cb}A(G)$ with $\mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G)))$ via the map $\hat{\Theta}$ given by

$$\hat{\Theta}(\varphi) = T(u_\varphi).$$

Proof. Let us first note that for any

$$[\Phi_{ij}] \in M_n(\mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G)))) \subseteq M_n(\mathcal{CB}^{\sigma}_{L_\infty(G)}(\mathcal{B}(L_2(G)))),$$

there exists $[u_{ij}] = \left[\sum_{k\in I} v_{ik} \otimes w_{kj}\right] \in M_n(L_\infty(G) \otimes^{eh} L_\infty(G))$ such that $[\Phi_{ij}] = [T(u_{ij})]$. Then we may apply Lemma 4.1 and Theorem 4.2 to obtain the result. ☐

It is known from work of Haagerup and Kraus [22] that $M_{cb}A(G)$ is a dual space with a predual $Q(G)$, which is the completion of $L_1(G)$ under the norm induced from $M_{cb}A(G)^*$, i.e., we may define the $Q(G)$-norm

$$\|f\|_{Q(G)} = \sup\left\{\left|\int_G f(t)\varphi(t)dt\right| : \ \varphi \in M_{cb}A(G), \ \|\varphi\|_{M_{cb}A(G)} \leq 1\right\}$$

for any $f \in L_1(G)$. We may obtain a canonical operator space matrix norm on $Q(G)$ such that $M_{cb}A(G)$ is completely isometric to the operator dual of $Q(G)$ (see Kraus and Ruan [27]). Since the *co-involution* $\kappa : f \in L_\infty(G) \to \check{f} \in L_\infty(G)$ is a normal unital *-isomorphism on $L_\infty(G)$, its preadjoint

$$\kappa_*(f)(t) = f(t^{-1})\Delta(t^{-1})$$

defines an isometric anti-isomorphism on $L_1(G)$. Let m denote the convolution multiplication on the convolution algebra $L_1(G)$. The following is similar to a result of Spronk [43, Theorem 6.5].

Proposition 4.4. *The map*

$$m \circ (\mathrm{id} \otimes \kappa_*) : f \otimes g \in L_1(G) \otimes L_1(G) \mapsto f * \kappa_*(g) \in L_1(G)$$

extends to a complete quotient map, which is denoted by $m_{id\otimes\kappa_*}$, from $L_1(G) \otimes^h L_1(G)$ onto $Q(G)$.

Proof. Let us first recall (from the discussion above) that

$$\Gamma_{\hat{\Theta}} = T^{-1} \circ \hat{\Theta} : \varphi \in M_{cb}A(G) \mapsto u_\varphi \in L_\infty(G) \otimes^{eh} L_\infty(G)$$

defines a completely isometric injection from $M_{cb}A(G)$ into $L_\infty(G) \otimes^{eh} L_\infty(G)$. For any $\varphi \in M_{cb}A(G)$ and $f, h \in L_1(G)$, we have

$$\begin{aligned}
\langle \varphi, \ m \circ (\mathrm{id} \otimes \kappa_*)(f \otimes h)\rangle &= \int_G \varphi(t) \int_G f(s)\kappa_*(h)(s^{-1}t)dsdt \\
&= \int_G \int_G \varphi(t)f(s)h(t^{-1}s)\Delta(t^{-1}s)dsdt \\
&= \int_G \int_G \varphi(sg^{-1})f(s)h(g)dsdg \qquad (\text{with } \ g = t^{-1}s) \\
&= \langle u_\varphi, f \otimes h\rangle = \langle \Gamma_{\hat{\Theta}}(\varphi), f \otimes h\rangle.
\end{aligned}$$

This shows that $m \circ (\mathrm{id} \otimes \kappa_*)$ is equal to $\Gamma^*_{\hat{\Theta}}|_{L_1(G)\otimes L_1(G)}$, the restriction of $\Gamma^*_{\hat{\Theta}}$ to $L_1(G) \otimes L_1(G)$, and thus extends to a complete quotient map from $L_1(G) \otimes^h L_1(G)$ onto $Q(G)$. □

Theorem 4.5. *$\hat{\Theta}$ is a weak*-weak* continuous completely isometric isomorphism from $M_{cb}A(G)$ onto $\mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G)))$.*

Proof. From Proposition 4.4, we see that $\Gamma_{\hat{\Theta}}$ is just the adjoint of $m_{id\otimes\kappa_*}$ and thus is weak*-weak* continuous from $M_{cb}A(G)$ into $L_\infty(G) \otimes^{eh} L_\infty(G)$. Since T is a weak*-weak* continuous completely isometric isomorphism from $L_\infty(G) \otimes^{eh} L_\infty(G)$ onto $\mathcal{CB}^{\sigma}_{L_\infty(G)}(\mathcal{B}(L_2(G)))$, we can conclude that $\hat{\Theta} = T \circ \Gamma_{\hat{\Theta}}$ is a weak*-weak* continuous completely isometric isomorphism from $M_{cb}A(G)$ onto $\mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G)))$. □

We have seen from Theorem 3.2 and Theorem 4.5 that $M(G)$ and $M_{cb}A(G))$ can be completely isometrically identified as the completely contractive Banach algebras $\Theta_r(M(G)) = \mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$ and $\hat{\Theta}(M_{cb}A(G)) = \mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G)))$ on $\mathcal{B}(L_2(G))$, respectively. Moreover, we may easily obtain the following result.

Corollary 4.6. *We have* $\Theta_r(M(G)) \cap \hat{\Theta}(M_{cb}A(G)) = \mathbb{C}\, id_{\mathcal{B}(L_2(G))}$.

Proof. Let $\Phi \in \Theta_r(M(G)) \cap \hat{\Theta}(M_{cb}A(G))$. We have

$$\Phi(f) = f\Phi(1) = \Phi(1)f \quad \text{and} \quad \Phi(\lambda(s)) = \lambda(s)\Phi(1) = \Phi(1)\lambda(s)$$

for all $f \in L_\infty(G)$ and $s \in G$. Since $\mathrm{span}\{f\lambda_s \colon f \in L_\infty(G), s \in G\}$ is σ-weakly dense in $\mathcal{B}(L_2(G))$ (cf. step 3 in the proof of Theorem 3.2), the normality of Φ implies that

$$\Phi(x) = x\Phi(1) = \Phi(1)x$$

for all $x \in \mathcal{B}(L_2(G))$. Therefore, Φ is a scalar multiple of $id_{\mathcal{B}(L_2(G))}$. □

Let $C^*(G)$ denote the *full group* C^**-algebra* of G and $B(G) = C^*(G)^*$ the *Fourier-Stieltjes algebra* of G. The (non-degenerate) universal representation $\pi_u : L_1(G) \to C^*(G)$ induces a complete contraction from $Q(G)$ into $C^*(G)$. The adjoint map $(\pi_u)^*$ of π_u is exactly the canonical inclusion of $B(G)$ into $M_{cb}A(G)$. It is known (see Bożejko [3] and Losert [30]) that a locally compact group G is amenable if and only if $M_{cb}A(G) = B(G)$ (or equivalently, $Q(G) = C^*(G) = C^*_\lambda(G)$). Then the following corollary is an immediate consequence of Proposition 4.4, which extends the result of Pisier [36] for discrete groups.

Corollary 4.7. *A locally compact group G is amenable if and only if $\pi_u \circ m_{id\otimes\kappa_*}$ defines a complete quotient map from $L_1(G) \otimes^h L_1(G)$ onto $C^*(G)$ (or equivalently, $\lambda \circ m_{id\otimes\kappa_*} = m_{\lambda\otimes\hat{\kappa}\circ\lambda}$ defines a complete quotient map from $L_1(G) \otimes^h L_1(G)$ onto $C^*_\lambda(G)$, where $\hat{\kappa}$ is the* co-involution *$\hat{\kappa}(\lambda(s)) = \lambda(s^{-1})$ on $C^*_\lambda(G)$).*

We note that the right regular representation ρ is unitarily equivalent to the left regular representation λ. More precisely, there exists a self-adjoint unitary operator V on $L_2(G)$ defined by

$$V\xi(s) = \xi(s^{-1})\Delta(s^{-1})^{\frac{1}{2}}$$

such that $\rho(s) = V^*\lambda(s)V$. Using this unitary equivalence, we may completely isometrically identify $\mathcal{R}(G)$ with $\mathcal{L}(G)$ and thus completely isometrically identify the operator predual $\mathcal{R}(G)_*$ of $\mathcal{R}(G)$ with the Fourier algebra $A(G)$. Recall that $T : \mathcal{R}(G) \otimes^{eh} \mathcal{R}(G) \to \mathcal{CB}^\sigma_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$, defined in (2.5), is a completely isometric isomorphism. Then

$$\Gamma_{\Theta_r}(\mu) = T^{-1} \circ \Theta_r(\mu) = \int_G \rho(t) \otimes \rho(t)^* d\mu(t)$$

defines a weak*-weak* continuous completely isometric homomorphism Γ_{Θ_r} from $M(G)$ into $\mathcal{R}(G) \otimes^{eh} \mathcal{R}(G)$. We may obtain the following duality result to Proposition 4.4.

Proposition 4.8. *The preadjoint of Γ_{Θ_r} determines a complete quotient map*

$$m_{\iota\otimes\kappa\circ\iota} : \varphi \otimes \psi \in A(G) \otimes^h A(G) \mapsto \varphi\check{\psi} \in C_0(G)$$

from $A(G) \otimes^h A(G)$ onto $C_0(G)$.

Proof. Since $C_0(G)$ is a commutative C^*-algebra, the canonical inclusion $\iota : A(G) \to C_0(G)$ and the co-involution κ, restricted to $C_0(G)$, are complete contractions. Then

$$m \circ (\iota \otimes \kappa \circ \iota) : \varphi \otimes \psi \in A(G) \otimes^h A(G) \mapsto \varphi \check{\psi} \in C_0(G)$$

extends to a complete contraction $m_{\iota\otimes\kappa\circ\iota}$ from $A(G) \otimes^h A(G)$ into $C_0(G)$. It is routine to verify that $m_{\iota\otimes\kappa\circ\iota} = (\Gamma_{\Theta_r})_*$. Therefore, this map is a complete quotient map from $A(G) \otimes^h A(G)$ onto $C_0(G)$. □

Remark 4.9. In Theorem 3.6, we characterized the image space $\Theta_r(L_1(G))$ in $\mathcal{CB}^{\sigma}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$. It is an intriguing question to ask whether we can obtain an analogous characterization for $\hat{\Theta}(A(G))$ in $\mathcal{CB}^{\sigma}_{L_\infty(G)}(\mathcal{B}(L_2(G)))$. To consider this question, we need to assume G to be an amenable group. This allows us to completely isometrically identify $A(G)$ with a norm closed ideal in $B(G) = M_{cb}A(G)$. Since $C_b(G) = MC_0(G)$ is the multiplier C^*-algebra of $C_0(G)$, it is natural to conjecture that we have

$$\hat{\Theta}(A(G)) = \mathcal{CB}^{\sigma,(\mathcal{L}(G),MC^*_\lambda(G))}_{L_\infty(G)}(\mathcal{B}(L_2(G))),$$

where $MC^*_\lambda(G)$ is the multiplier C^*-algebra of the *reduced group C^*-algebra* $C^*_\lambda(G)$.

5. Commutant theorems

It is known from Theorem 3.2 that we have $\Theta_r(M(G)) = \mathcal{CB}^{\sigma,L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$. The following theorem shows that this space can be identified with the commutant of $\hat{\Theta}(M_{cb}A(G))$ in $\mathcal{CB}^{\sigma}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$.

Theorem 5.1. *We have*

$$\Theta_r(M(G)) = \hat{\Theta}(M_{cb}A(G))^c \cap \mathcal{CB}^{\sigma}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G))).$$

Proof. Let us first show that for every $\mu \in M(G)$, $\Theta_r(\mu)$ is contained in

$$\hat{\Theta}(M_{cb}A(G))^c = T(V^\infty_{\mathrm{inv}}(G,m))^c.$$

Given any $u = \sum_{k\in I} v_k \otimes w_k \in V^\infty_{\mathrm{inv}}(G,m)$, we have

$$\begin{aligned}
\Theta_r(\mu)(T(u)(a)) &= \int_G \rho(t) \left(\sum_{k\in I} v_k a w_k \right) \rho(t)^* d\mu(t) \\
&= \int_G \left(\sum_{k\in I} \rho(t) v_k a w_k \rho(t)^* \right) d\mu(t) \\
&= \int_G \left(\sum_{k\in I} (r_t v_k)(\rho(t) a \rho(t)^*)(r_t w_k) \right) d\mu(t) \\
&= \sum_{k\in I} (r_t v_k) \left(\int_G (\rho(t) a \rho(t)^*) d\mu(t) \right) (r_t w_k) \\
&= T(r_t u)(\Theta_r(\mu)(a)) = T(u)(\Theta_r(\mu)(a))
\end{aligned}$$

for all $a \in \mathcal{B}(L_2(G))$. This shows that

$$\Theta_r(M(G)) \subseteq \hat{\Theta}(M_{cb}A(G))^c \cap \mathcal{CB}^{\sigma}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G))).$$

Now assume that Φ is a map contained in $\hat{\Theta}(M_{cb}A(G))^c \cap \mathcal{CB}^{\sigma}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$. We claim that $\Phi(f) \in L_\infty(G)$ for all $f \in L_\infty(G)$. To see this, let us first fix an orthonormal basis $\{h_k\}_{k\in I}$ of $L_2(G)$. For any $\xi, \eta \in L_2(G)$, $\psi(s) = \xi^*\lambda(s)\eta$ is an element in $A(G) \subseteq M_{cb}A(G)$, and we can write

$$u_\psi = \sum_{k\in I} v_k \otimes w_k \in L_\infty(G) \otimes^{eh} L_\infty(G)$$

with $v_k(s) = \xi^*\lambda(s)h_k$ and $w_k(t) = h_k^*\lambda(t^{-1})\eta$. For any $f \in L_\infty(G)$ and $s, t \in G$, we have

$$\begin{aligned}\rho(s)T(u_\psi)(f)\rho(t) &= \rho(s)\left(\sum_{k\in I} v_k f w_k\right)\rho(t)\\ &= \sum_{k\in I}(r_s v_k)(r_s f)(r_s w_k)\rho(s)\rho(t) = \sum_{k\in I}\psi(e)\rho(s)f\rho(t).\end{aligned}$$

This implies that

$$xT(u_\psi)(f)y = \psi(e)xfy \tag{5.1}$$

for all $x, y \in \mathcal{R}(G)$. Since $\Phi \in \mathcal{CB}^\sigma_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$, there exists $z = \sum_{j\in J} x_j \otimes y_j \in \mathcal{R}(G) \otimes^{eh} \mathcal{R}(G)$ such that

$$\Phi(a) = \sum_{j\in J} x_j a y_j$$

for all $a \in \mathcal{B}(L_2(G))$. It follows from (5.1) that

$$\Phi(T(u_\psi)(f)) = \sum_{j\in J} x_j T(u_\psi)(f) y_j = \psi(e)\Phi(f). \tag{5.2}$$

In particular, if we let $\psi_{ik}(s) = h_i^*\lambda(s)h_k$, then

$$\begin{aligned}\psi_{ik}(e)\Phi(f) &= \Phi(T(u_{\psi_{ik}})(f)) = T(u_{\psi_{ik}})(\Phi(f)) = \sum_{j\in I}\psi_{ij}\Phi(f)\psi_{jk}\\ &= [h_i^*\lambda h_j](1_I \otimes \Phi(f))[h_j^*\lambda^* h_k].\end{aligned} \tag{5.3}$$

We may regard

$$\lambda = [\psi_{ik}] : s \in G \to [\psi_{ik}(s)] = [h_i^*\lambda(s)h_k] \in M_I(\mathbb{C})$$

as an element in $M_I(C_b(G))$. Then $\check{\lambda}$ defined by

$$\check{\lambda}(s) = [\check{\psi}(s^{-1})] = [h_i^*\lambda(s^{-1})h_k]$$

is also an element in $M_I(C_b(G))$ such that

$$\lambda\check{\lambda} = 1_I \otimes 1 = \check{\lambda}\lambda \in M_I(C_b(G)).$$

Since $\psi_{ik}(e) = \delta_{ik}$, we can conclude from (5.3) that

$$1_I \otimes \Phi(f) = [\psi_{ik}(e)\Phi(f)] = [h_i^*\lambda h_j](1_I \otimes \Phi(f))[h_j^*\lambda^* h_k] = \lambda(1_I \otimes \Phi(f))\check{\lambda}.$$

This shows that

$$\lambda(1_I \otimes \Phi(f)) = (1_I \otimes \Phi(f))\lambda.$$

Using the orthonormal basis $\{h_k\}_{k\in I}$, we may express elements ξ and $\eta \in L_2(G)$ by scalar vectors $[\alpha_k]$ and $[\beta_k]$ in $\ell_2(I)$, i.e. we can write $\xi = \sum_{k\in I}\alpha_k h_k$ and $\eta = \sum_{k\in I}\beta_k h_k$. Then we can write

$$\psi(s) = \xi^*\lambda(s)\eta = \sum_{i,j\in I}\bar{\alpha}_i h_i^*\lambda(s)h_j\beta_j = \sum_{i,j\in I}\bar{\alpha}_i\psi_{ij}(s)\beta_j.$$

From this we can conclude that

$$\psi\Phi(f) = [\bar{\alpha}_i]\lambda(1_I \otimes \Phi(f))[\beta_j] = [\bar{\alpha}_i](1_I \otimes \Phi(f))\lambda[\beta_j] = \Phi(f)\psi.$$

This shows that $\Phi(f)$ commutes with all functions in $A(G)$ and thus commutes with all functions in $L_\infty(G) = \overline{A(G)}^{s.o.t} \subseteq \mathcal{B}(l_2(G))$. Therefore, $\Phi(f)$ is contained in $L_\infty(G)' = L_\infty(G)$. $\square$

Correspondingly, we may obtain the following theorem which shows that $\hat{\Theta}(M_{cb}A(G))$ can be identified with the commutant of $\Theta_r(M(G))$ in $\mathcal{CB}^\sigma_{L_\infty(G)}(\mathcal{B}(L_2(G)))$.

Theorem 5.2. *We have*

$$\hat{\Theta}(M_{cb}A(G)) = \Theta_r(M(G))^c \cap \mathcal{CB}^\sigma_{L_\infty(G)}(\mathcal{B}(L_2(G))).$$

Proof. Let us consider $T(V^\infty_{\mathrm{inv}}(G,m))$. Since T is a completely isometric isomorphism from $L_\infty(G) \otimes^{eh} L_\infty(G)$ onto $\mathcal{CB}^\sigma_{L_\infty(G)}(\mathcal{B}(L_2(G)))$, it suffices to show that an element $u = \sum_{k\in I} v_k \otimes w_k \in L_\infty(G) \otimes^{eh} L_\infty(G)$ is contained in $V^\infty_{\mathrm{inv}}(G,m)$ if and only if $T(u)\circ\Theta_r(\mu) = \Theta_r(\mu)\circ T(u)$ for all $\mu \in M(G)$. Let us first consider the Dirac measures δ_t. Given any $a \in \mathcal{B}(L_2(G))$, we have

$$\begin{aligned} T(u)(\Theta_r(\delta_t)(a)) &= \sum_{k\in I} v_k\rho(t)a\rho(t)^* w_k = \sum_{k\in I}\rho(t)(r_{t^{-1}}v_k)a(r_{t^{-1}}w_k)\rho(t)^* \\ &= \Theta_r(\delta_t)(T(r_{t^{-1}}u)(a)). \end{aligned}$$

Hence $u \in V^\infty_{\mathrm{inv}}(G,m)$ if and only if $T(u)\circ\Theta_r(\delta_t) = \Theta_r(\delta_t)\circ T(u)$ for all $t \in G$. Moreover, since

$$\Theta_r(\mu)(a) = \int_G \rho(t)a\rho(t)^* d\mu(t) = \int_G \Theta_r(\delta_t)(a)d\mu(t)$$

and $T(u)$ is a normal map, we can conclude that $u \in V^\infty_{\mathrm{inv}}(G,m)$ if and only if $T(u)\circ\Theta_r(\mu) = \Theta_r(\mu)\circ T(u)$ for all $\mu \in M(G)$. $\square$

Remark 5.3. Let $\tilde{\Theta}_r : LUC(G)^* \to \mathcal{CB}^{L_\infty(G)}_{\mathcal{L}(G)}(L_2(G)))$ be the completely isometric homomorphism discussed in Remark 3.7. It was shown by Neufang [32, Theorem 3.5.3] that, for non-compact, second countable groups G, the commutant of $\tilde{\Theta}_r(LUC(G)^*)$ in $\mathcal{CB}(\mathcal{B}(L_2(G)))$ is contained in $\mathcal{CB}^\sigma(\mathcal{B}(L_2(G)))$. Then we can easily conclude from Theorem 5.2 that in this case,

$$\hat{\Theta}(M_{cb}A(G)) = \tilde{\Theta}_r(LUC(G)^*)^c \cap \mathcal{CB}_{L_\infty(G)}(\mathcal{B}(L_2(G))).$$

Theorem 5.4. *We have*

$$\hat{\Theta}(M_{cb}A(G)) = \hat{\Theta}(M_{cb}A(G))^{cc}$$

or equivalently,

$$\mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G))) = \mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G)))^{cc}.$$

Proof. Combining Theorem 4.3 and Theorem 5.2, we obtain

$$\mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G))) = \left(\Theta_r(M(G))^c \cap \mathcal{CB}^\sigma_{L_\infty(G)}(\mathcal{B}(L_2(G)))\right).$$

Taking the double commutant, we get

$$\mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G)))^{cc} = \Big(\Theta_r(M(G))^c \cap \mathcal{CB}^{\sigma}_{L_\infty(G)}(\mathcal{B}(L_2(G)))\Big)^{cc}$$
$$\subseteq \Theta_r(M(G))^{ccc} = \Theta_r(M(G))^c.$$

Moreover, we have

$$\mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G)))^{cc} \subseteq \mathcal{CB}^{\sigma}_{L_\infty(G)}(\mathcal{B}(L_2(G)))^{cc} = \mathcal{CB}^{\sigma}_{L_\infty(G)}(\mathcal{B}(L_2(G)))$$

by (2.9). The above argument shows that

$$\mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G)))^{cc} \subseteq \Theta_r(M(G))^c \cap \mathcal{CB}^{\sigma}_{L_\infty(G)}(\mathcal{B}(L_2(G))) = \mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G))).$$

This completes the proof. □

Similarly, we can apply Theorem 3.2 and Theorem 5.1 to obtain the following double commutant theorem, which was first proved in Neufang [32] by using a different argument (in a more general setting). We leave the details to the reader.

Theorem 5.5. *We have*

$$\Theta_r(M(G)) = \Theta_r(M(G))^{cc}$$

or equivalently,

$$\mathcal{CB}^{\sigma,L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G))) = \mathcal{CB}^{\sigma,L_\infty(G)}_{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))^{cc}.$$

Remark 5.6. As we discussed in (1.1), there is an isometric isomorphism Φ from the measure $M(G)$ into the space $\mathcal{B}(L_\infty(G))$ of all bounded maps on $L_\infty(G)$. Restricting Φ to $L_1(G)$, Ghahramani and Lau proved in [16, Theorem 5.1] that the bicommutant of $\Phi(L_1(G))$ in $\mathcal{B}(L_\infty(G))$ is equal to $\Phi(M(G))$. Motivated by this result, it is natural to conjecture that

$$\Theta_r(L_1(G))^{cc} = \Theta_r(M(G)).$$

Since $\Theta_r(L_1(G)) \subseteq \Theta_r(M(G))$, it follows from Theorem 5.5 that

$$\Theta_r(L_1(G))^{cc} \subseteq \Theta_r(M(G))^{cc} = \Theta_r(M(G)).$$

However, we cannot prove the equality at this moment. Considering the duality, it is also natural to conjecture that for amenable groups G, we have

$$\hat{\Theta}(A(G))^{cc} = \hat{\Theta}(M_{cb}A(G)).$$

6. Representation of $UCB(\hat{G})^*$

Let us first recall from §4 that there is a commutative completely contractive $A(G)$-bimodule structure on $\mathcal{L}(G)$ given by $\langle \omega \cdot x, \omega' \rangle = \langle x, \omega' \cdot \omega \rangle$. We let $A(G) \cdot \mathcal{L}(G) = \{\omega \cdot x : \ \omega \in A(G), x \in \mathcal{L}(G)\}$, and we define $UCB(\hat{G})$ to be the norm closure of $A(G) \cdot \mathcal{L}(G)$ in $\mathcal{L}(G)$. Then $UCB(\hat{G})$ is a linear subspace of $\mathcal{L}(G)$ (see [19, footnote (2) on page 373]), and elements in $UCB(\hat{G})$ are called *bounded uniformly continuous linear functionals* on $\hat{G}$. Moreover, it was shown by Lau [29] that $UCB(\hat{G})$ is a C^*-subalgebra of $\mathcal{L}(G)$, which contains the *reduced group C^*-algebra* $C^*_\lambda(G)$, and we have $UCB(\hat{G}) = C^*_\lambda(G)$ when G is a discrete group. In particular, if G is an amenable group, $A(G) \cdot \mathcal{L}(G)$ is automatically norm closed in $\mathcal{L}(G)$ and we have the equality $UCB(\hat{G}) = A(G) \cdot \mathcal{L}(G)$ (see [19]).

The dual space $UCB(\hat{G})^*$ is a Banach algebra with the multiplication given by

$$\langle m \diamond n,\, x\rangle = \langle m,\, n \diamond x\rangle \tag{6.1}$$

for $m, n \in UCB(\hat{G})^*$ and $x \in UCB(\hat{G})$, where $n \diamond x$ defined by

$$\langle n \diamond x,\, \omega\rangle = \langle n,\, \omega \cdot x\rangle \qquad (\forall \omega \in A(G)) \tag{6.2}$$

is an operator contained in $UCB(\hat{G})$. Since we need to use the *right action* $\omega \cdot x = (id \otimes \omega)\hat{\Gamma}(x)$ on $UCB(\hat{G})$ (especially, later on in Lemma 6.1 and Theorem 6.2), we use this diamond product $\diamond$ to distinguish it from the first Arens' product $\circ$ defined by the *left action* $x \cdot \omega = (\omega \otimes id)\hat{\Gamma}(x)$. Given any $n \in UCB(\hat{G})^*$, there exists a norm preserving extension $\tilde{n} \in \mathcal{L}(G)^*$. We may find a net of elements $n_\alpha \in A(G)$ such that $\|n_\alpha\| \le \|n\|$ and $n_\alpha(x) \to \tilde{n}(x)$ for all $x \in \mathcal{L}(G)$. It follows that $n_\alpha(x) \to n(x)$ for all $x \in UCB(\hat{G})$. If m is another element in $UCB(\hat{G})^*$, we let $m_\beta \in A(G)$ such that $m_\beta(x) \to m(x)$ for all $x \in UCB(\hat{G})$. Then we can write (6.1) as

$$\langle m \diamond n,\, x\rangle = \langle m, n \diamond x\rangle = \lim_\beta \langle m_\beta,\, n \diamond x\rangle = \lim_\beta \langle n,\, m_\beta \cdot x\rangle = \lim_\beta \lim_\alpha \langle n_\alpha \cdot m_\beta,\, x\rangle. \tag{6.3}$$

On the other hand, the first Arens' product $m \circ n$ can be expressed as

$$\langle m \circ n,\, x\rangle = \langle m, n \circ x\rangle = \lim_\beta \langle m_\beta,\, n \circ x\rangle = \lim_\beta \langle n,\, x \circ m_\beta\rangle = \lim_\beta \lim_\alpha \langle m_\beta \cdot n_\alpha,\, x\rangle.$$

Since $A(G)$ is a commutative Banach algebra, we atually have $m \diamond n = m \circ n$ for $m, n \in UCB(\hat{G})^*$. However, the two multiplications are different on A^{**} for general non-commutative completely contractive Banach algebras A.

It is easy to see from (6.3) that $UCB(\hat{G})^*$ with the canonical dual operator space structure is a completely contractive Banach algebra. In general, $UCB(\hat{G})^*$ is not necessarily commutative since Lau has proved in [29, Theorem 5.5] that for any amenable group G, $UCB(\hat{G})^*$ is commutative if and only if G is discrete. If G is an amenable group, then $M_{cb}A(G) = B(G)$ is completely isometrically isomorphic to a Banach subalgebra of $UCB(\hat{G})^*$ and $UCB(\hat{G})^*$ acts naturally as the duality of $LUC(G)^*$. Our goal of this section is to study the representation theorem for $UCB(\hat{G})^*$.

To do this, we need to recall some useful notions from Kac algebras (see [13] for details). It is known from the theory of Kac algebras that for any locally compact group G, there is an important *fundamental unitary operator* W on $L_2(G \times G)$ defined by $W\zeta(s,t) = \zeta(s, st)$ for all $\zeta \in L_2(G \times G)$. The operator W is contained in $L_\infty(G)\bar{\otimes}\mathcal{L}(G)$ and satisfies the *pentagonal relation*

$$W_{23}W_{13}W_{12} = W_{12}W_{23}, \tag{6.4}$$

where we let $W_{12} = W \otimes 1$, $W_{23} = 1 \otimes W$ and $W_{13} = (\sigma \otimes 1)W_{23}(\sigma \otimes 1)$, and we let σ be the *flip map* $\sigma\zeta(s,t) = \zeta(t,s)$ on $L_2(G \times G)$. Given any $f \in L_1(G)$, we can write

$$\lambda(f) = \langle f \otimes id, W^*\rangle \quad \text{and} \quad \hat{\kappa}(\lambda(f)) = \langle f \otimes id, W\rangle$$

by using the *right slice map* $\langle f \otimes id, x \otimes y\rangle = f(x)y$ from $L_\infty(G)\bar{\otimes}\mathcal{L}(G)$ into $\mathcal{L}(G)$ induced by f.

We denote by $\hat{W} = \sigma W^* \sigma$ the *dual fundamental unitary operator* of W. Then $\hat{W}$ also satisfies the pentagonal relation (6.4), and we may define a normal unital completely isometric *-homomorphism

$$\hat{\Gamma}(x) = \hat{W}(1 \otimes x)\hat{W}^* \tag{6.5}$$

from $\mathcal{B}(L_2(G))$ into $\mathcal{B}(L_2(G))\bar{\otimes}\mathcal{B}(L_2(G))$. The pentagonal relation implies that $\hat{\Gamma}$ is *co-associative*, i.e., it satisfies

$$(\hat{\Gamma} \otimes id) \circ \hat{\Gamma} = (id \otimes \hat{\Gamma}) \circ \hat{\Gamma}. \tag{6.6}$$

The preadjoint of $\hat{\Gamma}$ defines an associative completely contractive multiplication

$$m_{\hat{\Gamma}} = \hat{\Gamma}_* : \mathcal{T}(L_2(G)) \times \mathcal{T}(L_2(G)) \to \mathcal{T}(L_2(G)) \tag{6.7}$$

on $\mathcal{T}(L_2(G)) = \mathcal{B}(L_2(G))_*$. This also determines a completely contractive $\mathcal{T}(L_2(G))$-bimodule action

$$\omega \cdot x = \langle id \otimes \omega, \hat{\Gamma}(x)\rangle \quad \text{and} \quad x \cdot \omega = \langle \omega \otimes id, \hat{\Gamma}(x)\rangle$$

on $\mathcal{B}(L_2(G))$. If we restrict $\hat{\Gamma}$ to $\mathcal{L}(G)$, we obtain the *co-multiplication*

$$\hat{\Gamma}(\lambda(s)) = \lambda(s) \otimes \lambda(s)$$

on $\mathcal{L}(G)$ and $m_{\hat{\Gamma}}$ induces the completely contractive multiplication on $A(G)$.

Lemma 6.1. *Let G be an amenable group. For any $\omega \in \mathcal{T}(L_2(G))$ and $x \in \mathcal{B}(L_2(G))$, $\omega \cdot x$ is an operator contained in $UCB(\hat{G})$.*

(1) *The map*

$$\mathcal{S} : \omega \otimes x \in \mathcal{T}(L_2(G)) \otimes \mathcal{B}(L_2(G)) \to \omega \cdot x = \langle id \otimes \omega, \hat{\Gamma}(x)\rangle \in UCB(\hat{G})$$

extends to a complete quotient from the operator projective tensor product $\mathcal{T}(L_2(G))\hat{\otimes}\mathcal{B}(L_2(G))$ onto $UCB(\hat{G})$.

(2) *The corresponding map*

$$\mathcal{S}_{\mathcal{L}(G)} : \omega \otimes x \in A(G) \otimes \mathcal{L}(G) \to \omega \cdot x = \langle id \otimes \omega, \hat{\Gamma}(x)\rangle \in UCB(\hat{G})$$

extends to a complete quotient from the operator projective tensor product $A(G)\hat{\otimes}\mathcal{L}(G)$ onto $UCB(\hat{G})$.

Proof. Let us first prove (1). Assume that $\omega = \omega_{\xi,\eta}$ such that ξ and η are vectors in $C_c(G) \subseteq L_2(G)$ with supports contained in a compact, symmetric set C. Since G is an amenable group, there exists a net of non-empty compact sets K_α such that

$$\int_G \frac{|\chi_{K_\alpha}(st) - \chi_{K_\alpha}(t)|\, dt}{\mu_G(K_\alpha)} = \frac{\mu_G(s^{-1}K_\alpha \Delta K_\alpha)}{\mu_G(K_\alpha)} \to 0 \tag{6.8}$$

uniformly for $s \in C$. Then $\xi_\alpha = \frac{1}{\mu_G(K_\alpha)^{\frac{1}{2}}}\chi_{K_\alpha}$ is a net of unit vectors in $L_2(G)$ such that

$$\int_G \xi_\alpha(st)\xi_\alpha(t)dt \to 1$$

uniformly for $s \in C$ by (6.8), and thus

$$\begin{aligned}
\|\hat{W}^*(\xi_\alpha \otimes \xi) - (\xi_\alpha \otimes \xi)\|^2 &= \|W(\xi \otimes \xi_\alpha) - (\xi \otimes \xi_\alpha)\|^2 \\
&= 2\|\xi\|^2 - 2Re\left(\langle W(\xi \otimes \xi_\alpha) \mid (\xi \otimes \xi_\alpha)\rangle\right) \\
&= 2\|\xi\|^2 - 2Re\left(\int_G |\xi(s)|^2 \left(\int_G \xi_\alpha(st)\xi_\alpha(t)dt\right) ds\right) \\
&= 2\|\xi\|^2 - 2Re\left(\int_C |\xi(s)|^2 \left(\int_G \xi_\alpha(st)\xi_\alpha(t)dt\right) ds\right) \to 0.
\end{aligned}$$

This shows that

$$\|\hat{W}^*(\xi_\alpha \otimes \xi) - (\xi_\alpha \otimes \xi)\| \to 0. \tag{6.9}$$

Similarly, we can prove that

$$\|\hat{W}^*(\xi_\alpha \otimes \eta) - (\xi_\alpha \otimes \eta)\|^2 \to 0. \tag{6.10}$$

Let x be an arbitrary operator in $\mathcal{B}(L_2(G))$. Since $\hat{W} \in \mathcal{L}(G)\bar{\otimes}L_\infty(G)$, it is easy to see that

$$a = \omega_{\xi,\eta} \cdot x = \langle \hat{W}(1 \otimes x)\hat{W}^*(id \otimes \xi) \mid (id \otimes \eta)\rangle$$

is an operator contained in $\mathcal{L}(G)$. Using the pentagonal rule, we can write

$$\begin{aligned}\omega_{\xi_\alpha,\xi_\alpha} \cdot a &= \langle \hat{W}(1 \otimes a)\hat{W}^*(id \otimes \xi_\alpha) \mid (id \otimes \xi_\alpha)\rangle \\ &= \langle \hat{W}_{12}\hat{W}_{23}W_{12}^*(1 \otimes 1 \otimes x)W_{12}\hat{W}_{23}^*\hat{W}_{12}^*(id \otimes \xi_\alpha \otimes \xi) \mid (id \otimes \xi_\alpha \otimes \eta)\rangle \\ &= \langle \hat{W}_{23}\hat{W}_{13}(1 \otimes 1 \otimes x)\hat{W}_{13}^*\hat{W}_{23}^*(id \otimes \xi_\alpha \otimes \xi) \mid (id \otimes \xi_\alpha \otimes \eta)\rangle,\end{aligned}$$

which clearly converges to

$$a = \langle \hat{W}(1\otimes x)\hat{W}^*(id\otimes\xi) \mid (id\otimes\eta)\rangle = \langle \hat{W}_{13}(1\otimes 1\otimes x)\hat{W}_{13}^*(id\otimes\xi_\alpha\otimes\xi) \mid (id\otimes\xi_\alpha\otimes\eta)\rangle$$

in norm by (6.9) and (6.10). This shows that $a = \omega_{\xi,\eta} \cdot x \in UCB(\hat{G})$. Since $C_c(G)$ is norm dense in $L_2(G)$, we get $\omega_{\xi,\eta} \cdot x \in UCB(\hat{G})$ for all $\xi, \eta \in L_2(G)$ and $x \in \mathcal{B}(L_2(G))$. Moreover, since every ω in $\mathcal{T}(L_2(G))$ can be expressed as a norm convergent series $\omega = \sum_{n=1}^\infty \omega_{\xi_n,\eta_n}$ in $\mathcal{T}(L_2(G))$, we can conclude that $\omega \cdot x \in UCB(\hat{G})$.

It is easy to see that the map

$$\mathcal{S} : \omega \otimes x \in \mathcal{T}(L_2(G)) \otimes \mathcal{B}(L_2(G)) \to \omega \cdot x = \langle id \otimes \omega, \hat{\Gamma}(x)\rangle \in UCB(\hat{G})$$

extends to a complete contraction from $\mathcal{T}(L_2(G))\hat{\otimes}\mathcal{B}(L_2(G))$ into $UCB(\hat{G})$. Moreover, for each $n \in \mathbb{N}$, $M_n(UCB(\hat{G}))$ is a commutative completely contractive $A(G)$-bimodule with the module structure given by $\omega \cdot [x_{ij}] = [\omega \cdot x_{ij}]$. Using Herz's idea (see [19, footnote (2) on page 373]), it is easy to show that $A(G) \cdot M_n(UCB(\hat{G})) = M_n(UCB(\hat{G}))$. For any $[x_{ij}] \in M_n(UCB(\hat{G}))$ with $\|[x_{ij}]\| < 1$, we may apply Cohen's factorization theorem (since $A(G)$ has a contractive approximate identity) to find $\omega \in A(G)$ with $\|\omega\| < 1$ and $[\tilde{x}_{ij}] \in M_n(UCB(\hat{G}))$ with $\|[\tilde{x}_{ij}]\| < 1$ such that

$$[x_{ij}] = \omega \cdot [\tilde{x}_{ij}] = [\mathcal{S}(\omega \otimes \tilde{x}_{ij})].$$

This shows that $\mathcal{S}$ is a complete quotient map from $\mathcal{T}(L_2(G))\hat{\otimes}\mathcal{B}(L_2(G))$ onto $UCB(\hat{G})$.

To see (2), it is obvious (from the definition) that

$$\mathcal{S}_{\mathcal{L}(G)} : \omega \otimes x \in A(G) \otimes \mathcal{L}(G) \to \omega \cdot x = \langle id \otimes \omega, \hat{\Gamma}(x)\rangle \in UCB(\hat{G})$$

extends to a complete contraction from the operator projective tensor product $A(G)\hat{\otimes}\mathcal{L}(G)$ into $UCB(\hat{G})$. The complete quotient property can be proved by using the same argument as the one given at the end of the above proof. □

In the rest of this section, let us assume that G is an amenable group. By Lemma 6.1, we may always apply $n \in UCB(\hat{G})^*$ to $\omega \cdot x$ for any $\omega \in \mathcal{T}(L_2(G))$ and $x \in \mathcal{B}(L_2(G))$. Then, for any $n \in UCB(\hat{G})^*$ and $x \in \mathcal{B}(L_2(G))$, we may obtain a bounded linear operator, which is denoted by $\tilde{\hat{\Theta}}(n)(x)$, in $\mathcal{B}(L_2(G))$ such that

$$\begin{aligned}\langle \tilde{\hat{\Theta}}(n)(x)\xi \mid \eta\rangle &= \langle n, \omega_{\xi,\eta} \cdot x\rangle = \lim_\alpha \langle n_\alpha, \omega_{\xi,\eta} \cdot x\rangle \\ &= \lim_\alpha \langle n_\alpha \otimes \omega_{\xi,\eta}, \hat{\Gamma}(x)\rangle = \lim_\alpha \langle n_\alpha \cdot \omega_{\xi,\eta}, x\rangle,\end{aligned} \tag{6.11}$$

where the module action $n_\alpha \cdot \omega_{\xi,\eta} \in \mathcal{T}(L_2(G))$ of $A(G)$ on $\mathcal{T}(L_2(G))$ is induced by the co-multiplication $\hat{\Gamma}$. If we let $\{h_k\}_{k\in I}$ be an orthonormal basis of $L_2(G)$, then we may identify $L_2(G)$ with $\ell_2(I)$ and may write

$$\begin{aligned}\tilde{\hat{\Theta}}(n)(x) &= \left[h_k^* \tilde{\hat{\Theta}}(n)(x) h_l\right] = \left[\langle \tilde{\hat{\Theta}}(n)(x) h_l \mid h_k\rangle\right] \\ &= \left[\langle n \otimes \omega_{h_l,h_k}, \hat{\Gamma}(x)\rangle\right] = \langle n \otimes id, \hat{\Gamma}(x)\rangle.\end{aligned} \tag{6.12}$$

It is easy to see that for each $n \in UCB(\hat{G})^*$, $\tilde{\hat{\Theta}}(n)$ is a completely bounded linear map on $\mathcal{B}(L_2(G))$ with $\|\tilde{\hat{\Theta}}(n)\|_{cb} \le \|n\|$, and it follows from the definition that $\tilde{\hat{\Theta}}$ is equal to the adjoint of $\mathcal{S}$. Then we can conclude from Lemma 6.1 that $\tilde{\hat{\Theta}}$ is a weak*-weak* continuous complete isometry from $UCB(\hat{G})^*$ into $\mathcal{CB}(\mathcal{B}(L_2(G)))$. Let $\mathcal{CB}_{L_\infty(G)}^{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$ denote the space of all completely bounded $L_\infty(G)$-bimodule morphisms which map $\mathcal{L}(G)$ into $\mathcal{L}(G)$. We now obtain the following completely isometric representation theorem for $UCB(\hat{G})^*$, which can be regarded as the dual result to Neufang's representation theorem [33] indicated in Remark 3.7.

Theorem 6.2. *Let G be an amenable group. Then $\tilde{\hat{\Theta}}$ is a weak*-weak* completely isometric homomorphism from $UCB(\hat{G})^*$ into $\mathcal{CB}_{L_\infty(G)}^{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$ such that $\tilde{\hat{\Theta}}|_{M_{cb}A(G)} = \hat{\Theta}$.*

Moreover, $\tilde{\hat{\Theta}}$ preserves the natural involutions and matrix orders on $UCB(\hat{G})^*$ and $\mathcal{CB}_{L_\infty(G)}^{\mathcal{L}(G)}(\mathcal{B}(L_2(G)))$.

Proof. Let us first show that $\tilde{\hat{\Theta}}$ is an algebraic homomorphism from $UCB(\hat{G})^*$ into $\mathcal{CB}(\mathcal{B}(L_2(G)))$. Suppose that we are given $m, n \in UCB(\hat{G})^*$. Then for any $x \in \mathcal{B}(L_2(G))$ and $\xi, \eta \in L_2(G)$,

$$\begin{aligned}\langle \tilde{\hat{\Theta}}(m \diamond n)(x)\xi \mid \eta\rangle &= \langle m \diamond n,\, \omega_{\xi,\eta} \cdot x\rangle = \langle m \diamond n,\, \langle id \otimes \omega_{\xi,\eta}, \hat{\Gamma}(x)\rangle\rangle \\ &= \lim_\beta \lim_\alpha \langle n_\alpha \otimes m_\beta \otimes \omega_{\xi,\eta},\, (\hat{\Gamma} \otimes id)\hat{\Gamma}(x)\rangle \\ &= \lim_\beta \lim_\alpha \langle n_\alpha \cdot m_\beta \cdot \omega_{\xi,\eta},\, x\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle \tilde{\hat{\Theta}}(m)(\tilde{\hat{\Theta}}(n)(x))\xi \mid \eta\rangle &= \langle m,\, \omega_{\xi,\eta} \cdot \tilde{\hat{\Theta}}(n)(x)\rangle = \lim_\beta \langle m_\beta \otimes \omega_{\xi,\eta},\, \hat{\Gamma}(\tilde{\hat{\Theta}}(n)(x))\rangle \\ &= \lim_\beta \langle m_\beta \cdot \omega_{\xi,\eta},\, \tilde{\hat{\Theta}}(n)(x)\rangle \\ &= \lim_\beta \langle n,\, (m_\beta \cdot \omega_{\xi,\eta}) \cdot x\rangle = \lim_\beta \lim_\alpha \langle n_\alpha \otimes (m_\beta \cdot \omega_{\xi,\eta}),\, \hat{\Gamma}(x)\rangle \\ &= \lim_\beta \lim_\alpha \langle n_\alpha \cdot m_\beta \cdot \omega_{\xi,\eta}, x\rangle.\end{aligned}$$

This shows that

$$\tilde{\hat{\Theta}}(m \diamond n) = \tilde{\hat{\Theta}}(m) \circ \tilde{\hat{\Theta}}(n).$$

Therefore, $\tilde{\hat{\Theta}}$ is a completely contractive homomorphism from $UCB(\hat{G})$ into $\mathcal{CB}(\mathcal{B}(L_2(G)))$.

Since $\hat{W}$ is contained in $\mathcal{L}(G)\bar{\otimes}L_\infty(G)$, for any $f, g \in L_\infty(G)$ and $x \in \mathcal{B}(L_2(G))$, we have

$$\hat{W}(1 \otimes fxg)\hat{W}^* = (1 \otimes f)W(1 \otimes x)\hat{W}^*(1 \otimes g)$$

and thus obtain

$$\tilde{\hat{\Theta}}(n)(fxg) = f\tilde{\hat{\Theta}}(n)(x)g$$

from (6.12). This shows that $\tilde{\hat{\Theta}}(n)$ is an $L_\infty(G)$-bimodule morphism in $\mathcal{CB}(\mathcal{B}(L_2(G)))$. For any $x \in \mathcal{L}(G)$ we have $\hat{\Gamma}(x) \in \mathcal{L}(G)\bar{\otimes}\mathcal{L}(G) = \mathcal{CB}(A(G), \mathcal{L}(G))$. It follows that

$$\tilde{\hat{\Theta}}(n)(x) : \omega \in A(G) \to \lim_\alpha \langle n_\alpha \otimes \omega, \hat{\Gamma}(x)\rangle \in \mathbb{C}$$

defines an element in $\mathcal{L}(G)$. Therefore, $\tilde{\hat{\Theta}}(n)$ maps $\mathcal{L}(G)$ into $\mathcal{L}(G)$.

Suppose that we are given $\varphi \in M_{cb}A(G)$ and $\xi, \eta \in L_2(G)$. From the proof of Theorem 4.2, we get

$$\langle \hat{\Theta}(\varphi)(\lambda(s))\xi \mid \eta\rangle = \varphi(s)\langle \lambda(s)\xi \mid \eta\rangle = \langle \varphi \cdot \omega_{\xi,\eta},\, \lambda(s)\rangle$$

for all $s \in G$ and

$$\langle \hat{\Theta}(\varphi)(f)\xi \mid \eta\rangle = \langle \hat{\Theta}(\varphi)(1)f\xi \mid \eta\rangle = \varphi(1)\langle f\xi \mid \eta\rangle = \langle \varphi \cdot \omega_{\xi,\eta},\, f\rangle$$

for any $f \in L_\infty(G)$. By the normality of $\hat{\Theta}$ this implies that

$$\langle \hat{\Theta}(\varphi)(x)\xi \mid \eta\rangle = \langle \varphi \cdot \omega_{\xi,\eta},\, x\rangle = \langle \varphi,\, \omega_{\xi,\eta} \cdot \hat{\Gamma}(x)\rangle = \langle \tilde{\hat{\Theta}}(\varphi)(x)\xi \mid \eta\rangle$$

for all $x \in \mathcal{B}(L_2(G))$. This shows $\tilde{\hat{\Theta}}|_{M_{cb}A(G)} = \hat{\Theta}$. It follows from (6.12) that we can write

$$\hat{\Theta}(\varphi)(x) = \langle \varphi \otimes id, \hat{\Gamma}(x)\rangle = \langle \varphi \otimes id, \hat{W}(1 \otimes x)\hat{W}^*\rangle$$

for $\varphi \in M_{cb}A(G)$ and $x \in \mathcal{B}(L_2(G))$.

Since $UCB(\hat{G})$ is a C^*-algebra, there exist a natural involution and a matrix order on its dual space $UCB(\hat{G})^*$. We leave it to the reader to verify that $\tilde{\hat{\Theta}}$ preserves the involutions and matrix orders on $UCB(\hat{G})^*$ and $\mathcal{CB}^{\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G)))$. □

We note that as a consequence of Theorem 4.5 and Theorem 6.2, it is easy to show that if G is an amenable group, then

$$\hat{\Theta}(M_{cb}A(G)) = \tilde{\hat{\Theta}}(UCB(\hat{G})^*) \cap \mathcal{CB}^\sigma(\mathcal{B}(L_2(G))),$$

i.e. the image space of $M_{cb}A(G)$ is exactly the *normal part* of $\tilde{\hat{\Theta}}(UCB(\hat{G})^*)$. This is dual to Neufang's result [33, Proposition 3.4 (i)], which shows that $\Theta_r(M(G)) = \tilde{\Theta}_r(LUC(G)^*) \cap \mathcal{CB}^\sigma(\mathcal{B}(L_2(G)))$ is the *normal part* of $\tilde{\Theta}_r(LUC(G)^*)$.

Let G be an amenable group. For any $n \in UCB(\hat{G})^*$, we may define a bounded linear map $n_{\mathcal{L}(G)}$ on $\mathcal{L}(G)$ given by

$$\langle n_{\mathcal{L}(G)}(x),\, \omega\rangle = \langle n,\, \omega \cdot x\rangle = \langle n,\, \langle id \otimes \omega, \hat{\Gamma}(x)\rangle\rangle \tag{6.13}$$

for all $x \in \mathcal{L}(G)$ and $\omega \in A(G)$. The map $n_{\mathcal{L}(G)}$ is equal to the map n_R discussed in [29, §6]. Since $A(G)$ is commutative it is also equal to Lau's map n_L. Lau showed that this determines an isometric isomorphism from $UCB(\hat{G})^*$ onto the space $\mathcal{B}_{A(G)}(\mathcal{L}(G))$ of all bounded $A(G)$-bimodule morphisms on $\mathcal{L}(G)$. We note that for each $n \in UCB(\hat{G})^*$, we actually have $n_{\mathcal{L}(G)} = \tilde{\hat{\Theta}}(n)|_{\mathcal{L}(G)}$, the restriction of $\tilde{\hat{\Theta}}(n)$ to $\mathcal{L}(G)$. Therefore, each $n_{\mathcal{L}(G)}$ is a completely bounded $A(G)$-bimodule morphism contained in $\mathcal{CB}_{A(G)}(\mathcal{L}(G))$, and it can be shown by applying Lemma 6.1 that the map

$$n \in UCB(\hat{G})^* \to n_{\mathcal{L}(G)} = \tilde{\hat{\Theta}}(n)|_{\mathcal{L}(G)} \in \mathcal{CB}_{A(G)}(\mathcal{L}(G))$$

is a completely isometric isomorphism from $UCB(\hat{G})^*$ onto $\mathcal{CB}_{A(G)}(\mathcal{L}(G))$. Therefore, we may obtain the following completely bounded characterization of $UCB(\hat{G})^*$.

Proposition 6.3. *Let G be an amenable group. Then we have the completely isometric isomorphism*

$$UCB(\hat{G})^* \cong \mathcal{CB}_{A(G)}(\mathcal{L}(G)).$$

Remark 6.4. Similar to the situation described in Remark 3.7, it is still an open question whether the map $\tilde{\hat{\Theta}}$ studied in Theorem 6.2 is onto $\mathcal{CB}^{\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G)))$ for general locally compact amenable groups. This is true when G is an amenable discrete group, since in this case we have $UCB(\hat{G}) = C^*_\lambda(G) = Q(G)$ and thus $UCB(\hat{G})^* = C^*_\lambda(G)^* = M_{cb}A(G)$. It follows from (2.2) that

(6.14)
$$\tilde{\hat{\Theta}}(UCB(\hat{G})^*) = \hat{\Theta}(M_{cb}A(G)) = \mathcal{CB}^{\sigma,\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G))) = \mathcal{CB}^{\mathcal{L}(G)}_{L_\infty(G)}(\mathcal{B}(L_2(G))).$$

The following result extends the map $m \circ (\lambda \otimes \hat{\kappa} \circ \lambda)$ in Corollary 4.7 to the extended Haagerup tensor product $L_1(G) \otimes^{eh} L_1(G)$.

Proposition 6.5. *Let G be an amenable group. The map $m \circ (\lambda \otimes \hat{\kappa} \circ \lambda)$ extends to a complete quotient map, which is denoted by $\tilde{m}_{\lambda\otimes\hat{\kappa}\circ\lambda}$, from $L_1(G) \otimes^{eh} L_1(G)$ onto $UCB(\hat{G})$.*

Proof. We first note from §2 that T has a natural weak*-weak* continuous completely isometric extension $\tilde{T}$ from $L_\infty(G) \otimes^{\sigma h} L_\infty(G)$ onto $\mathcal{CB}_{L_\infty(G)}(\mathcal{B}(L_2(G)))$. Then $\Gamma_{\tilde{\hat{\Theta}}} = \tilde{T}^{-1} \circ \tilde{\hat{\Theta}}$ is a weak*-weak* continuous completely isometric homomorphism from $UCB(\hat{G})^*$ into $L_\infty(G) \otimes^{\sigma h} L_\infty(G)$. The preadjoint of $\Gamma_{\tilde{\hat{\Theta}}}$ defines a complete quotient from $L_1(G) \otimes^{eh} L_1(G)$ onto $UCB(\hat{G})$. Since $\tilde{\hat{\Theta}}|_{M_{cb}A(G)} = \hat{\Theta}$, it is easy to see from Proposition 4.4 and Corollary 4.7 that this map is the natural extension of $m \circ (\lambda \otimes \hat{\kappa} \circ \lambda)$. □

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