CONSTRUCTING TILTING MODULES

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Abstract. We investigate the structure of (infinite dimensional) tilting modules over hereditary artin algebras. For connected algebras of infinite representation type with Grothendieck group of rank \( n \), we prove that for each \( 0 \leq i < n - 1 \), there is an infinite dimensional tilting module \( T_i \) with exactly \( i \) pairwise non-isomorphic indecomposable finite dimensional direct summands. We also show that any stone is a direct summand in a tilting module. In the final section, we give explicit constructions of infinite dimensional tilting modules over iterated one-point extensions.

The study of finite dimensional tilting modules of projective dimension at most one over finite dimensional algebras was initiated by Brenner and Butler [11] and continued by Happel and Ringel [17]. Since then, many variations of this concept have been introduced and used successfully, for example: Tilting modules of higher projective dimension, tilting modules over rings, tilting complexes in derived categories, tilting objects in hereditary categories or in cluster categories. In this paper, we use the term tilting module as follows: Let \( R \) bear in and \( T \) be a right \( R \)-module. Then \( T \) is a tilting module provided that (T1) \( \text{p.dim} T \leq 1 \), (T2) \( \text{Ext}^1_R(T, T^{(I)}) = 0 \) for any set \( I \), and (T3) there is a short exact sequence \( 0 \to R \to T_0 \to T_1 \to 0 \) where \( T_0 \) and \( T_1 \) are direct summands in a direct sum of (possibly infinitely many) copies of \( T \). Equivalently, \( T \) is tilting if and only if \( \text{Gen}(T) = \{T\}^\perp \), [13].

Here, \( \text{Gen}(T) \) denotes the class of all homomorphic images of direct sums of copies of \( T \), and, for a class of modules \( \mathcal{C} \),

\[
\mathcal{C}^\perp = \text{Ker} \text{Ext}^1_R(\mathcal{C}, -) = \{M \in \text{Mod-}R \mid \text{Ext}^1_R(C, M) = 0 \text{ for all } C \in \mathcal{C}\}.
\]

If \( T \) is a tilting module, then \( \{T\}^\perp \) is a torsion class in \( \text{Mod-}R \), the tilting class generated by \( T \). If \( T' \) is another tilting module, then \( T \) is said to be equivalent to \( T' \) if \( \{T\}^\perp = \{T'\}^\perp \).

Though our definition of a tilting module allows infinitely generated modules, there is an implicit finiteness property connected with tilting, recently proved by Bazzoni and Herbera in [9, Theorem 2.4]. Namely, any tilting module \( T \) is of finite type, that is, there exists a set \( S \) consisting of finitely presented modules of projective dimension at most 1 such that \( S^\perp = \{T\}^\perp \).

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It has been shown in [25, Theorem 2.1] that for an artin algebra $A$ there is a bijection between the set of equivalence classes of tilting $A$-modules of finite type and the set of torsion classes $\{ T \mid D(A) \in T \}$ in $\text{mod-} A$ where $D(A)$ denotes the injective cogenerator in $\text{mod-} A$. The bijection is defined by $T \mapsto \{ T \}^\perp \cap \text{mod-} A$.

Combining these two results, we get for an artin algebra $A$ a bijection between equivalence classes of tilting modules in $\text{Mod-} A$ and torsion classes in $\text{mod-} A$ containing all injective modules. We therefore call such torsion classes $T$ in $\text{mod-} A$ tilting torsion classes. (This notion generalizes the classical case where only finite dimensional tilting modules were considered, and hence the extra condition of $T$ being generated by a single module had to be assumed; cf. [3], [19].)

Our first goal here is to investigate direct summands of tilting modules, in particular, the finite dimensional ones, and to use these investigations to construct tilting modules with various additional properties.

We will show that indecomposable finite dimensional direct summands of a tilting module $T \in \text{Mod-} A$ coincide with the indecomposable Ext-projective modules in $T = \{ T \}^\perp \cap \text{mod-} A$. Then we will study the open problem of the number of indecomposable Ext-projective modules in tilting torsion classes in $\text{mod-} A$ for hereditary artin algebras, and prove

**Theorem A.** Let $H$ be a connected hereditary artin algebra of infinite representation type with $n$ simple modules.

1. If $T$ is a tilting $H$-module with at least $n - 1$ pairwise non-isomorphic finite dimensional indecomposable direct summands, then $T$ is equivalent to a finite dimensional tilting $H$-module $\hat{T}$ (so that $T$ has exactly $n$ such direct summands).

2. For any natural number $i$ with $0 \leq i \leq n - 2$, there exists an infinite dimensional tilting $H$-module $T_i$ with exactly $i$ pairwise non-isomorphic finite dimensional indecomposable direct summands.

The proof of Theorem A will be presented in Section 2.

As a direct consequence, we obtain

**Corollary.** Let $H$ be a connected hereditary artin algebra with $n$ simple modules and $T$ a tilting torsion class in $\text{mod-} H$.

1. If $H$ is representation finite, then $T$ contains $n$ pairwise non-isomorphic indecomposable Ext-projective modules.

2. If $H$ is representation infinite and $0 \leq i \leq n$ with $i \neq n - 1$, then there exists a tilting torsion class in $\text{mod-} H$ with $i$ pairwise non-isomorphic indecomposable Ext-projective modules.

It should be mentioned that the Ext-projective torsion modules are central in the construction of torsion pairs in [4].

In Section 3, we study the problem of which modules in $\text{Mod-} A$ occur as direct summands of tilting $A$-modules. Of course, any such module has projective dimension $\leq 1$, and no self-extensions.

Until now, only partial cases were considered. In [13], a module $X$ is called partial tilting if $\text{Gen}(X) = \{ X \}^\perp$ and $\{ X \}^\perp$ is a torsion class in $\text{Mod-} A$, that is, if $X$ can be completed to a tilting module $T$ by the Bongartz construction.
Restricting to right hereditary right artinian rings, we show

**Theorem B.** Let \( H \) be a right hereditary right artinian ring and \( X \) be an \( H \)-module with \( \text{Ext}^1_H(X, X^{(I)}) = 0 \) for every set \( I \). If \( X \) is finitely generated over its endomorphism ring, then \( X \) is a direct summand in a tilting \( H \)-module.

The assumption that \( X \) is finitely generated over its endomorphism ring, frequently called finendo, is quite natural: by [13, Proposition 2.5.], each tilting module is finendo.

Let \( H \) be a connected hereditary algebra. The endofinite modules, that is, the modules of finite length over their endomorphism rings, clearly are finendo. Since any endofinite module \( X \) is \( \Sigma \)-pure-injective (see [14, 26]) the pure embedding \( X^{(I)} \rightarrow X^I \) splits, hence \( \text{Ext}^1_H(X, X) = 0 \) implies in this case the stronger condition of \( \text{Ext}^1_H(X, X^{(I)}) = 0 \). Since endofinite modules are direct sums of indecomposable endofinite modules [14, 4.5] with local endomorphism rings and nilpotent radical [14, 4.2, 4.4], we may restrict to indecomposable endofinite modules \( X \) with \( \text{Ext}^1_H(X, X) = 0 \). Following Schofield, such an \( H \)-module \( X \) is called a stone. Stones are studied in [15] in the case where the ground-field \( k \) is algebraically closed: There always exist infinite dimensional stones when \( H \) is connected and representation-infinite. They are in 1-1 correspondence with imaginary indivisible Schur roots, whereas finite dimensional stones correspond to real Schur roots. The correspondence is defined as follows: Let \( X \) be a stone and \( E = \text{End}_H(X) \). Then \( E \) is a division algebra; see [15]. If \( (P_1, \ldots, P_n) \) is a complete family of pairwise non-isomorphic indecomposable projective \( H \)-modules, then the assignment \( X \mapsto \text{dim}_E \text{Hom}_H(P_i, X) \) defines this bijection. Since \( \text{dim}_E \text{Hom}_H(P_i, X) \) is finite and each finite dimensional \( H \)-module \( M \) has a finite dimensional projective cover, \( \text{dim}_E \text{Hom}_H(M, X) \) is also finite, for \( M \) finite dimensional.

If \( H \) is tame, there is a unique imaginary indivisible Schur root, and the generic module, \( G \), is the unique infinite dimensional stone (originally studied in [31]). If \( H \) is wild, there always exist infinitely many such roots; consequently, there exist infinitely many pairwise non-isomorphic infinite dimensional stones. We immediately get from Theorem B:

**Corollary.** Let \( H \) be a hereditary algebra, and \( X \) be a stone. Then \( X \) is a direct summand in a tilting \( H \)-module.

The proof of Theorem B is done in several steps. One of these steps (Proposition 3.3) deals with the problem of when a tilting \( R/I \)-module \( \overline{T} \), where \( I \) is an idempotent ideal in \( R \), can be completed to a tilting \( R \)-module \( T \) with \( \{T\}^\perp = \{\overline{T}\}^\perp \). This situation is studied again in the special case when \( B \) is a finite dimensional \( k \)-algebra, \( \overline{T} \in \text{Mod-}B \) a tilting \( B \)-module and \( A \) an iterated one-point extension of \( B \). We obtain rather explicit results concerning this case in Section 4.

For background on representation theory of finite dimensional modules, we refer to [5, 6, 32], of infinite dimensional modules to [15, 26, 31], and on wild hereditary algebras to [23].

1. Preliminaries

For a commutative artinian ring \( k \), a \( k \)-algebra \( A \) is called an artin algebra if it is finitely generated as a \( k \)-module. Additionally, we will assume that \( A \) is a faithful \( k \)-module, and that \( A \) is connected. This means that 0 and 1 are the only central idempotents in \( A \), in particular, \( k \) is a local ring.
By Mod-$A$, we denote the category of all (right $A$-) modules, and by mod-$A$ the subcategory of all finitely presented modules. ind-$A$ will denote a (fixed) representative set of the class of all finitely generated indecomposable modules. If $I$ is the injective hull of the simple $k$-module, then the functor $D = \text{Hom}_k(-, I)$ defines a duality between mod-$A$ and $A$-mod. Also, $\tau_A = D\text{Tr}$ and $\tau_A^\perp = \text{Tr}D$ denote the Auslander-Reiten translations in mod-$A$.

The Auslander-Reiten quiver, $\Gamma(A)$, is a directed graph whose set of vertices is ind-$A$, and whose arrows are induced by the Auslander-Reiten sequences $0 \to \tau X \to E \to X \to 0$ for $X \in \text{ind}$-$A$ non-projective, and by the embeddings rad$X \subseteq X$ for $X \in \text{ind}$-$A$ projective. For more details, see [6].

Moreover, if $A$ is hereditary, then $k$ is a field, hence $A$ is a finite dimensional $k$-algebra and $\tau_A^\perp = \text{Ext}^1_A(D(A), -)$ and $\tau_A = D\text{Ext}^1_A(-, A) \cong \text{Tor}^1_A(D(A), -)$ are endo-functors on mod-$A$.

Assume $A$ is hereditary and representation-infinite. Then $\Gamma(A)$ is partitioned into three types of modules: a module $X \in \text{ind}$-$A$ is preprojective (preinjective) if $\tau^m X = 0$ for some $m \geq 0$; $X$ is regular if $\tau^m \tau^{-m} X \cong X$ for all integers $m$. A module $M \in \text{mod}$-$A$ is preprojective (preinjective, and regular) if either $M = 0$, or each indecomposable direct summand of $M$ is isomorphic to a preprojective (preinjective, and regular) module in ind-$A$. The set of all $M \in \text{mod}$-$A$ that are preprojective (preinjective, and regular) will be denoted by $\mathcal{P}$ ($\mathcal{I}$, and $\mathcal{R}$). To avoid ambiguity, occasionally for example $\mathcal{P}(A)$ instead of just $\mathcal{P}$ will be written.

The Auslander-Reiten quiver $\Gamma(A)$ consists of infinitely many (connected) components: one preprojective component whose vertices are the elements of $\mathcal{P} \cap \text{ind}$-$A$, one preinjective component with the elements of $\mathcal{I} \cap \text{ind}$-$A$ as vertices, and an infinite set of regular components (with vertices in $\mathcal{R} \cap \text{ind}$-$A$).

If $A$ is tame hereditary, all regular components are tubes, all of them homogeneous, up to finitely many. If $A$ is wild hereditary, all regular components are of type $ZA_\infty$. In both cases, the modules at the border of the regular components are called quasi-simple. If $Y$ is an arbitrary indecomposable module contained in a regular component $C$, there exists a unique quasi-simple module $X$ in $C$ and a chain of irreducible monomorphisms

$$(*) \quad X = X(1) \to X(2) \to \cdots \to X(r) = Y,$$

which we will consider as inclusions. The number $r$ is called the quasi-length of $Y$, and $X(i)/X(i-1) \cong \tau^{-i+1}X$ holds for $1 < i \leq r$; see [30].

Dually, there exists a unique quasi-simple module $Z$ in $C$ and a chain of irreducible epimorphisms of the same length $r$

$$(**) \quad Y = (r)Z \to (r-1)Z \to \cdots \to (1)Z = Z.$$

Given an abelian category $A$, we call a pair $(T, F)$ of classes of objects in $A$ a torsion pair if $\text{Hom}(T, F) = 0$, and both classes are maximal with respect to this property. This means that for any object $M \in A$, there is a short exact sequence, called the canonical short exact sequence

$$0 \to t(M) \to M \to f(M) \to 0,$$

with $t(M) \in T$ and $f(M) \in F$.

An object $P \in T$ is called Ext-projective (in $T$), provided $\text{Ext}^1_A(P, T) = 0$, that is, $T \subseteq \{P\}^\perp$. 

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If $H$ is a finite dimensional hereditary algebra with $n$ simple modules and $X$ is a finite dimensional $H$-module with $\text{Ext}_H^1(X,X) = 0$, we consider the full subcategory $\{X\}^{\perp \geq 0}$ of $\text{Mod}-H$, defined by the modules $\{M \mid \text{Hom}_H(X,M) = 0 = \text{Ext}_H^1(X,M)\}$. It is easy to check that $\{X\}^{\perp \geq 0}$ is an exact and extension closed abelian subcategory of $\text{Mod}-H$. It is well known that for $X$ indecomposable, $\{X\}^{\perp \geq 0}$ is equivalent to a module category $\text{Mod}-C$, where $C$ is a finite dimensional hereditary algebra with $n - 1$ simple modules. Indeed, if $X = eH$ is indecomposable projective, then $\{X\}^{\perp \geq 0} \cong \text{Mod-}(H/HeH)$. If $X$ is not projective, and $0 \to H \to Y \to X^r \to 0$ is the Bongartz universal short exact sequence [10], then $Y$ is a finite dimensional projective generator in $\{X\}^{\perp \geq 0}$, hence $X^{\perp \geq 0}$ is equivalent to $\text{Mod-End}_H(Y)$.

More generally, if $X'$ is a finite dimensional partial tilting module with $r$ pairwise non-isomorphic indecomposable direct summands, then $\{X'\}^{\perp \geq 0}$ is equivalent to $\text{Mod-C}'$, where $C'$ is a finite dimensional hereditary with $n - r$ pairwise non-isomorphic simple modules. Indeed, let $0 \to H \to E \to X'^r \to 0$ be the Bongartz universal short exact sequence. Then the factor-module $Y$ of $E$ by the $X$-trace of $E$ is a small projective generator in $\{X'\}^{\perp \geq 0}$, hence $\{X'\}^{\perp \geq 0} \cong \text{Mod-C}'$, where $C' = \text{End}_H(Y)$; see [16, Proposition 3.8]. Especially, if $X'$ has $n - 1$ pairwise non-isomorphic indecomposable direct summands, then $C'$ is a hereditary artin algebra with one simple module $S$, which means $C'$ is a full matrix ring over the division algebra $K = \text{End}(S)$. We refer to [16] for more details on perpendicular categories.

2. Finite dimensional direct summands of tilting modules

The first aim of this section is to study finitely generated indecomposable direct summands of a tilting module $T$ in $\text{Mod-A}$, for $A$ an artin algebra. For this purpose, we consider the torsion class $T = \{T\}^{\perp} \cap \text{mod-A}$ in $\text{Mod-A}$, and its corresponding torsion free class $\mathcal{F}$.

**Lemma 2.1.** Let $A$ be an artin algebra and $T$ a tilting $A$-module. A finitely generated $A$-module $X$ is in $\text{add}T$ if and only if $X$ is Ext-projective in $T$.

**Proof.** If $X \in \text{add}T$, then $X$ is Ext-projective in $\{T\}^{\perp}$, hence also in $T$.

Conversely, let $X$ be Ext-projective in $T$. Then $\tau_A X$ is in $\mathcal{F}$ by [7, 19]. Since the injective cogenerator $DA$ is a torsion module, $\text{Hom}_A(DA, \tau_A X) = 0$ follows. Hence $\text{p.dim}X \leq 1$ by [32, 2.4]. Combining [9, Theorem 2.4] and [25, Theorem 2.1], we get $\{T\}^{\perp} = (\tau_A^{-1} \mathcal{F})^{\perp}$. Since $X \in \tau_A^{-1} \mathcal{F}$, we have $X \in \{T\}^{\perp} \subset \{X\}^{\perp}$. Consequently $X \in \text{add}T$ by [13, Lemma 2.2].

The following lemma is well known:

**Lemma 2.2.** Let $R$ be a ring and $X$ an $R$-module with a local endomorphism ring. If $X$ is a direct summand of $Y = \bigoplus_{i=1}^r Y_i$, then $X$ is a direct summand of some direct summand $Y_i$.

**Proposition 2.3.** Let $A$ be an artin algebra and $T$ a tilting $A$-module. If $\{X_i \mid 1 \leq i \leq r\}$ is a set of pairwise non-isomorphic indecomposable Ext-projective modules in $T$, then $\bigoplus_i X_i$ is a direct summand of $T$.

**Proof.** By Lemma 2.1 $X_i$ is in $\text{add}T$ with a local endomorphism ring. Therefore, by Lemma 2.2, $X_i$ is a direct summand of $T$, for each $i$.

The proof now is done by induction. $X_1$ is a direct summand of $T$. Assume $T = (\bigoplus_{j=1}^{t-1} X_j) \oplus Y$ for $1 < t \leq r$. Since $X_t$ is a direct summand of $T$, but not a
direct summand of the indecomposable modules \( X_j \), for \( 1 \leq j < t \), again by Lemma 2.2 \( X_i \) is a direct summand of \( Y \). Consequently \( \bigoplus_{j=1}^t X_j \) is a direct summand of \( T \).

**Proof of Theorem A(1).** Let \( X_1, \ldots, X_{n-1} \) be the pairwise non-isomorphic indecomposable finite dimensional direct summands of \( T \) and \( X = \bigoplus_i X_i \). Clearly, \( \{ T \} \perp \subset \{ X \} \perp \).

It follows from [4] and [13] that \( T \mapsto T \cap \{ X \}^{\perp \geq 0} \) defines a bijection between the torsion classes \( T \) in \( \text{Mod-}H \) containing \( X \) as an Ext-projective module, and the torsion classes in \( \{ X \}^{\perp \geq 0} \). Since \( X \) has \( n - 1 \) pairwise non-isomorphic indecomposable direct summands, the category \( X^{\perp \geq 0} \) is equivalent to \( \text{Mod-}K \), where \( K \) is a \( k \)-division algebra; see Section 1. \( \text{Mod-}K \) only contains the trivial torsion classes \( \{ 0 \} \) and \( \text{Mod-}K \). Hence \( \text{Gen}(X) \) and \( \{ X \}^{\perp} \) are the only torsion classes in \( \text{Mod-}H \), containing \( X \) as an Ext-projective module. The torsion class \( \{ X \}^{\perp} \) is the tilting torsion class of a finite dimensional tilting module \( T' \). If \( X \) is not sincere, then \( \text{Gen}(X) \) is not a tilting torsion class since there are indecomposable injective \( H \)-modules not contained in \( \text{Gen}(X) \). Therefore \( T \in \text{Add}(T') \), and hence \( T \) is equivalent to \( T' \) in this case. If \( X \) is sincere, then \( \text{Gen}(X) \) also is the tilting torsion class of a finite dimensional tilting module \( T'' \); see [18]. Hence \( T \in \text{Add}(T') \) or \( T \in \text{Add}(T'') \) in the second case.

The proof of part (2) is by induction on \( n \), but requires some preparation:

**Proposition 2.4.** Let \( H \) be a representation-infinite connected hereditary artin algebra with \( n \) simple modules. Then the following holds true:

1. There exists a quasi-simple regular \( H \)-module \( X \) with \( \text{Ext}^1_H(X, X) = 0 \), if and only if \( n > 2 \).
2. Let \( n > 2 \), \( X \) be quasi-simple regular without self-extensions, and \( \{ X \}^{\perp \geq 0} \cong \text{Mod-}C \) where \( C \) is hereditary with \( n - 1 \) simple modules. Then \( C \) is connected to the same representation type (tame or wild) as \( H \).

If \( 0 \rightarrow \tau_H X \rightarrow Z \rightarrow X \rightarrow 0 \) is the Auslander-Reiten sequence ending in \( X \), then \( Z \in \{ X \}^{\perp \geq 0} \) and \( Z \) is quasi-simple regular in \( \text{mod-}C \).

If \( H \) is tame and \( X \) belongs to a tube of rank \( p > 1 \), then \( Z \) belongs to a tube of rank \( p - 1 \) in \( \text{mod-}C \).

**Proof.** The first part is proved in [33], and the second in [34].

Let \( H \) be a hereditary artin algebra and \( X \) a finite dimensional \( H \)-module without self-extensions. Then \( \{ X \}^{\perp \geq 0} \cong \text{Mod-}C \), where \( C \) is a hereditary artin algebra. To simplify the notation, we will identify these categories.

For \( Y \in \{ X \}^{\perp \geq 0} \cap \text{mod-}H = \text{mod-}C \) with \( \text{Ext}^1_C(Y, Y) = 0 = \text{Ext}^1_H(Y, X) \), we consider the full subcategory \( \{ Y_C \}^{\perp \geq 0} \) of \( \text{Mod-}C \). This category then coincides with the full subcategory \( \{ X \oplus Y \}^{\perp \geq 0} \) in \( \text{Mod-}H \).

**Lemma 2.5.** Let \( H \) be a wild connected hereditary artin algebra with \( n > 2 \) simple modules, and \( i \) be a natural number with \( 1 \leq i \leq n - 2 \). Then there exists a regular partial tilting module \( X = \bigoplus_{j=1}^i X_j \), where the modules \( X_j \) are indecomposable in \( \text{mod-}H \), and for each \( 1 \leq j < i \), the following holds:

1. There exists an epimorphism \( \phi_j : X_{j+1} \rightarrow X_j \);
2. \( X_{j+1} \) is a quasi-simple regular module in \( \{ \bigoplus_{a \leq j} X_a \}^{\perp \geq 0} \).
Proof. The proof is by induction on $i$. For $i = 1$, we just choose a quasi-simple regular $H$-module $X_1$ without self-extensions; see Proposition 2.4(1).

Let $i > 1$ and choose a quasi-simple regular $H$-module $X_1$ with $\text{Ext}_H^1(X_1, X_1) = 0$. We identify $\{X_1\}^{1\geq 0}(\cong \text{Mod-}C)$ with $\text{Mod-}C$. Then $C$ is a connected wild hereditary artin algebra with $n - 1$ simple modules; see Proposition 2.4(2). By induction, we find a regular partial tilting module $Y = Y_1 \oplus \cdots \oplus Y_{i-1}$ in $\text{Mod-}C$ satisfying (1) and (2).

Let $0 \to \tau_H X_1 \to Z_1 \to X_1 \to 0$ be the Auslander-Reiten sequence, ending in $X_1$. We know that $Z_1$ is a regular $C$-module. By [20, 28] there exists a natural number $m$ with the following properties:

(a) $0 = \text{Ext}_C^1(\tau_C^{-m}Y, Z_1) = \text{Ext}_H^1(\tau_C^{-m}Y, Z_1)$.

(b) There is an epimorphism $\psi$: $\tau_C^{-m}Y_1 \to Z_1$.

Since $\tau_C^{-m}Y \in \{X_1\}^{1\geq 0}$, we have $\text{Ext}_H^1(X_1, \tau_C^{-m}Y) = 0$. From the choice of $m$ we get $0 = \text{Ext}_H^1(\tau_C^{-m}Y, Z_1) = \text{Ext}_H^1(\tau_C^{-m}Y, X_1)$. Hence $X = X_1 \oplus \tau_C^{-m}Y$ is a partial tilting module. Defining $X_{j+1} = \tau_C^{-m}Y_j$, for $1 \leq j \leq i-1$, a straightforward computation yields the properties (1) and (2) for $X$. \hfill \Box

Lemma 2.6. Let $H$ be a hereditary artin algebra and $X$ be a finitely generated indecomposable $H$-module without self-extensions. Let $\{X\}^{1\geq 0} = \text{Mod-}C$, and $T'$ be a tilting $C$-module which generates $X$. Then $T = T' \oplus X$ is a tilting $H$-module.

Proof. Since $H$ is hereditary, $\text{p.dim}T \leq 1$. Since $X$ is finite dimensional, $\text{Ext}_H^1(X, T) = 0$ implies $\text{Ext}_H^1(X, T^{(I)}) = 0$ for all sets $I$.

From $0 = \text{Ext}_C^1(T', T^{(I)}) \cong \text{Ext}_H^1(T', T^{(I)})$ we deduce $\text{Ext}_H^1(T', X^{(I)}) = 0$ for all sets $I$, since $X$ is generated by $T'$; hence (T2) holds. We have to show that condition (T3) holds for $T$.

Let

$$0 \to H \xrightarrow{u} P \xrightarrow{v} X' \to 0$$

be Bongartz’s universal short exact sequence [10]. Since $X$ is indecomposable, $P$ is a projective generator in $\{X\}^{1\geq 0} = \text{Mod-}C$. Since $T'$ is a tilting $C$-module, there exists a short exact sequence

$$0 \to P \xrightarrow{f} T_1 \xrightarrow{g} T_2 \to 0,$$

with $T_1, T_2 \in \text{Add}(T')$.

Forming the pushout of $f$ and $v$, we get the following commutative diagram, with exact rows and columns:

$$
\begin{array}{cccc}
0 & 0 & & 0 \\
\downarrow & \downarrow & & \downarrow \\
0 & \to & H & \xrightarrow{u} & P & \xrightarrow{v} & X' & \to & 0 \\
\| & \uparrow{\downarrow{f}} & \| & \downarrow & & \downarrow & \| & \downarrow \\
0 & \to & H & \xrightarrow{h} & T_1 & \xrightarrow{l} & Q & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & \| & \downarrow & \| & \downarrow {\downarrow{h}} \\
T_2 & \xrightarrow{=} & T_2 & \| & \downarrow & \| & \downarrow & \| & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

Since $\text{Ext}_H^1(T_2, X) = 0$, the last column splits, that is, $Q = T_2 \oplus X'$. So $T$ is a tilting $H$-module. \hfill \Box
Proof of Theorem A(2). For $i = 0$, let $P$ be the set of preprojective modules in mod-$H$. Then $P^\perp$ is a tilting torsion class, generated by a countably generated tilting module $T_\infty$, [28, 25]. Since $T_\infty = P^\perp \cap \text{ind}-H$ consists of the indecomposable regular and preinjective modules, there is no non-zero Ext-projective module in $T_\infty$. Consequently $T_0 = T_\infty$ has no indecomposable finite dimensional direct summand; see Lemma 2.1.

For $i > 0$ first let $H$ be wild and consider the partial tilting module $X = X_1 \oplus \cdots \oplus X_i$ from Lemma 2.5. Let $\{ \bigoplus_{1 \leq j \leq i-1} X_j \}^{\perp_{\geq 0}} = \text{Mod-}C'$. Then $X_i$ is a quasi-simple regular $C'$-module, with Auslander-Reiten sequence $0 \to \tau_{C'} X_i \to Z_i \to X_i \to 0$ in mod-$C'$. Moreover we know that $Z_i$ is quasi-simple regular in $\{ X \}^{\perp_{\geq 0}} = \text{Mod-}C''$. Choose in Mod-$C''$ the infinite dimensional tilting module (without indecomposable finite dimensional direct summands) $T''_\infty$, with $\{ T''_\infty \}^\perp = P(\text{C''})^\perp$ as in the case of $i = 0$. Since $Z_i$ is a regular $C''$-module, it is generated by $T''_\infty$. Hence there exists an epimorphism $\phi: T''_\infty \to Z_i \to X_i$, for some $l > 0$. Consequently we have a chain of epimorphisms

$$T''_\infty \to X_i \to \cdots \to X_1.$$ 

Let $T_i = T''_\infty \oplus X_i \oplus \cdots \oplus X_1$. It follows from Lemma 2.6 by induction that $T_i$ is a tilting $H$-module.

If $H$ is tame hereditary, then there exist $1 \leq r \leq 3$ inhomogeneous tubes $T_1, \ldots, T_r$ of rank $p_1, \ldots, p_r$ where $\sum (p_i - 1) = n - 2$. Let $Y_s \in T_s$ be quasi-simple and choose natural numbers $0 \leq i_s < p_s$ with $\sum_{s=1}^r i_s = i$. For each $s$ we consider the chain of irreducible epimorphism in $T_s$ (see (**) )

$$(i_s + 1)Y_s \to (i_s)Y_s \to \cdots \to (1)Y_s = Y_s.$$ 

Let $Y(s) = \bigoplus_{t=1}^{i_s} (t) Y_s$ and $X = \bigoplus Y(s)$. Then $X$ is a partial tilting $H$-module with $i$ indecomposable direct summands. Let $\{ X \}^{\perp_{\geq 0}} \cong \text{Mod-}C''$. Then $C''$ is a connected tame hereditary algebra with $n-i$ simple modules. Moreover, the modules $(i_s+1)Y_s$ are quasi-simple regular $C''$-modules, contained in tubes of rank $p_i - i_s$. In analogy to the first case, we chose the tilting $C''$-module $T''_\infty$ with $\{ T''_\infty \}^\perp = P(C'')^\perp$. Since $T''_\infty$ generates all the modules $(i_s+1)Y_s$, it generates $X$, hence again we get for $T_i = T''_\infty \oplus X$ that $\text{Ext}^1_H(T_i, T_i^{(1)}) = 0$, for all sets $I$. As in the wild case, Lemma 2.6 and induction give that $T_i$ is a tilting module in mod-$H$.

Finally, we have to show in both cases that an indecomposable finite dimensional direct summand $Y$ of $T_i = T''_\infty \oplus X$ is isomorphic to some direct summand of $X$. Since $T''_\infty$ has no finite dimensional indecomposable direct summand, $Y$ is a direct summand of $X$ by Lemma 2.2. Therefore $Y$ is isomorphic to some indecomposable direct summand of $X$ by the Krull-Schmidt theorem. \qed 

3. Direct summands of tilting modules

In this section, Theorem B will be proved. Most steps of the proof work in more generality: Recall, that for a class $C$ of $R$-modules, a morphism $f: M \to C_0$ with $C_0 \in C$ is called a $C$-preenvelope or also $C$-approximation, if for any $C \in C$ the induced morphism $\text{Hom}_R(f, C): \text{Hom}_R(C_0, C) \to \text{Hom}_R(M, C)$ is surjective. The preenvelope $f$ is called an envelope or minimal left approximation, if for any endomorphism $g$ of $C_0$ the condition $gf = f$ implies that $g$ is an automorphism. In this case the image of $f$ cannot be contained in a proper direct summand of $C_0$. 


Lemma 3.2. \(Let\) \(f\) \(\in\) \(\text{ann}\) \((M)\) \(\implies\) \(f\) \(\text{is}\) \(\text{faithful}\), \(\text{hence} \Hom_R(H, M)\) \(\cong\) \(\text{Ext}_R^1(H, M)\). \(\text{Since} \text{p.dim} R \leq 1\), \(\text{and} \text{condition}\) \((T1)\) \(\text{holds}\). \(The\) \(\text{exact\ sequence}\) \((\varepsilon)\)
\[0 \to R \overset{f}{\to} P \to Q \to 0.\]

\(Let\) \(T = M \oplus Q\). \(Clearly, \text{Gen}(T) = \text{Gen}(M)\). \(Since\) \(\text{p.dim} M \leq 1\), \(also\) \(\text{p.dim Q} \leq 1\), \(and\) \(condition\) \((T1)\) \(\text{holds}\). \(The\) \(\text{exact\ sequence}\) \((\varepsilon)\) \(\text{yields\ condition}\) \((T3)\).

\(It\) \(\text{remains\ to\ prove\ condition\} (T2). \(From\ \text{Ext}_R^1(M, M^{(I)}) = 0\) \(\text{and}\) \((\varepsilon)\) \(\text{one\ gets}\)
\[
\text{Ext}_R^1(M, T^{(I)}) = 0 \quad \text{for\ any}\ \text{set}\ I.
\]
\(Since\ \text{p.dim} Q \leq 1\), \(it\ \text{remains\ to\ prove\ that}\)
\[
\text{Ext}_R^1(Q, M^{(I)}) = 0 \quad \text{for\ any\ set}\ I.
\]

\(However, f\) \(\text{is\ an\ Add}(M)\)-\(preenvelope, \text{so\ the\ map}\)
\[
\Hom_R(f, M^{(I)}) \text{is\ surjective, \and\ we\ have\ the\ exact\ sequence}
\]
\[
\Hom_R(P, M^{(I)}) \overset{\Hom_R(f, M^{(I)})}{\longrightarrow} \Hom_R(R, M^{(I)}) \to \text{Ext}_R^1(R, M^{(I)}) \to \text{Ext}_R^1(P, M^{(I)}).
\]
\(Since\ \text{Ext}_R^1(P, M^{(I)}) = 0, \text{we\ conclude\ that}\)
\[
\text{Ext}_R^1(Q, M^{(I)}) = 0. \quad \square
\]

\(In\ \text{general, \it it\ is\ not\ easy\ to\ check\ whether\ a\ module\ (without\ self-extensions)\ is\ faithful. \text{We\ will\ see\ that\ this\ becomes\ easier\ when\ the\ underlying\ ring\ is\ a\ right\ hereditary\ artinian\ ring,\ generalizing\ a\ well\ known\ result\ on\ finite\ dimensional\ modules\ over\ hereditary\ artin\ algebras; \text{see\ for\ example\ [23, 8.3].}}\)

\(First, \text{recall\ that\ a\ module\ M\ over\ a\ ring\ R\ is\ sincere\ if}\ \Hom_R(P, M) \neq 0\ \text{for\ all\ non-zero\ projective\ modules}\ P. \text{Clearly,\ any\ faithful\ module\ is\ sincere. \If\ R\ is\ a\ right\ perfect\ ring, \text{then\ a\ module}\ M\ \text{is\ sincere\ if\ and\ only\ if\ all\ simple\ modules\ appear\ as\ subfactors\ of}\ M.}\)

Lemma 3.2. \(Let\ H\ be\ a\ right\ hereditary\ right\ artinian\ ring\ and\ M\ be\ a\ module\ \text{with} \text{Ext}_H^1(M, M) = 0. \text{Then}\ M\ \text{is\ faithful\ if\ and\ only\ if\ M\ is\ sincere.}\)

\(Proof. \text{It\ suffices\ to\ show\ that\ M\ is\ faithful\ provided\ M\ is\ sincere. \Let\ L = \text{ann}_H(M). \text{Since}\ H\ \text{is\ right\ artinian, \the\ module}\ H/L\ \text{is\ finitely\ cogenerated. \So\ there\ is\ a\ finite\ subset}\ F \subseteq M\ \text{such\ that}\ L = \bigcap_{m \in F} \text{ann}_H(m), \text{and}\ L\ \text{is\ the}\)
\(\text{kernel\ of\ the\ map}\ f : H \to M^F\ \text{defined\ by}\ f(h) = (mh)_{m \in F}. \text{Denote\ by}\ I\ \text{the}\ \text{image}\ \text{of}\ f, \text{and\ consider\ the\ short\ exact\ sequence}\)
\[0 \to L \hookrightarrow H \overset{p}{\to} I \to 0.\]
\(By\ construction, \text{the\ map}\ \Hom_H(p, M) : \Hom_H(I, M) \to \Hom_H(H, M)\ \text{is}\ \text{surjective, \hence}\ \Hom_H(L, M) \cong \text{Ext}_H^1(I, M). \text{Since}\ I\ \text{is\ a\ submodule\ of}\ M^F\ \text{and}\ \text{Ext}_H^1(M^F, M) = 0, \text{we\ infer\ that}\ \Hom_H(L, M) \cong \text{Ext}_H^1(I, M) = 0. \text{Since}\ H\ \text{is}\ right\ hereditary, \text{the\ module}\ L\ \text{is\ projective. \But\ M\ \text{sincere, \so}\ L = 0, \text{and}\ M\ \text{is}\ faithful.} \quad \square\)
Proposition 3.3. Let $R$ be a ring, $I^2 = I$ an idempotent two-sided ideal of $R$, finitely generated as right ideal, and $\bar{T}$ a tilting $R$-module where $\bar{R} = R/I$. Assume that $\text{p.dim}_R \bar{T} \leq 1$.

Then $\bar{T}$ is a direct summand in a tilting module $T$ with $T^\perp = \bar{T}^\perp$.

Proof. First, since $I$ is idempotent, we have $\text{Ext}_R^1(A, B) = \text{Ext}_R^1(A, B)$ for all $\bar{R}$-modules $A$ and $B$. In particular, $\text{Ext}_R^1(\bar{T}, T(X)) = 0$ for any set $X$.

Since $\text{p.dim}_R \bar{T} \leq 1$, it remains only to prove that the class $\bar{T}^\perp$ is closed under arbitrary direct sums (then $\bar{T}$ is partial tilting, and $T$ is obtained by the Bongartz construction; cf. [13, Lemma 1.8]).

Assume that $\text{Ext}_R^1(\bar{T}, M_j) = 0$ for a family of $R$-modules $M_j$ $(j \in J)$. Since $\bar{T}^\perp \cap \text{Mod-}\bar{R}$ is a class of $\bar{R}$-modules of finite type [9, Theorem 2.4], there is a set, $S$, of finitely presented $\bar{R}$-modules of projective dimension $\leq 1$ such that $\bar{R} \in S$, and $\bar{T}^\perp \cap \text{Mod-}\bar{R} = S^\perp \cap \text{Mod-}\bar{R}$.

By [35, Theorem 2.2], $\bar{T}$ is a direct summand in an $\bar{R}$-module, $T'$, such that there are an ordinal $\lambda$ and an increasing chain, $(T_\alpha | \alpha < \lambda)$, consisting of $\bar{R}$-submodules of $T'$ and satisfying $T_0 = 0$, $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$ for each limit ordinal $\alpha < \lambda$, $T' = \bigcup_{\alpha<\lambda} T_\alpha$, and $T_{\alpha+1}/T_\alpha$ is isomorphic to an element of $S$ for each $\alpha + 1 < \lambda$. Similarly, by [35, Theorem 3.3], each $S \in S$ is a direct summand in an $\bar{R}$-module $N_S$ such that $N_S$ is an extension of a free $\bar{R}$-module by a direct sum of copies of $T$.

By assumption on $\bar{T}$, there is a short exact sequence $0 \to \bar{R} \to A \to B \to 0$ where $A, B \in \text{Add}(\bar{T})$. So $\text{Ext}_R^1(\bar{R}, M_j) = 0$, hence $\text{Ext}_R^1(N_S, M_j) = 0$, and consequently $\text{Ext}_R^1(S, M_j) = 0$, for all $S \in S$ and $j \in J$. Since $I$ is finitely generated as a right ideal, a finitely presented $\bar{R}$-module remains finitely presented, when considered as an $R$-module. Since $S \in S$ is finitely presented as an $R$-module, we have $\text{Ext}_R^1(S, \bigoplus_{j \in J} M_j) = 0$, and hence $\text{Ext}_R^1(T', \bigoplus_{j \in J} M_j) = 0$. Then also $\text{Ext}_R^1(\bar{T}, \bigoplus_{j \in J} M_j) = 0$.

The condition $I^2 = I$ cannot be dropped in Proposition 3.3, since otherwise we can easily have $\text{Ext}_R^1(\bar{T}, \bar{T}) \neq 0 = \text{Ext}_R^1(\bar{T}, \bar{T})$ even if $\bar{T}$ is a free $\bar{R}$-module (just take $R = \mathbb{Z}$ and $I = \mathbb{Z}n$ for an integer $n > 1$).

Proof of Theorem B. Let $X$ be as in Theorem B.

(a) Assume $X$ is sincere. Then $X$ is faithful by Lemma 3.2. By Proposition 3.1, there exists a tilting module $T = X \oplus Y$ with $\{T\}^\perp = \text{Gen}(X)$.

(b) If $X$ is not sincere, let $H'$ be the support ring of $X$, that is, let $H' = H/(HeH)$ where $e$ is the sum of all primitive idempotents $\epsilon_i$ with $X\epsilon_i = 0$. Then $HeH$ is an idempotent ideal in $H$, finitely generated as a right $H$-module. Since $\text{Ext}_{H'}^1(A, B) = \text{Ext}_H^1(A, B)$ for all $H'$-modules $A$ and $B$, the ring $H'$ is a right hereditary and right artinian factor ring of $H$, and $X$ is a sincere $H'$-module with $\text{Ext}_{H'}^1(X, X^{(1)}) = 0$. So part (a) applies and we get a tilting $H'$-module $T' = X \oplus Y'$. The tilting $H'$-module $T'$ can be completed to a tilting $H$-module $T = X \oplus Y' \oplus Y''$ by Proposition 3.3.

Remark 3.4. If $X$ is a stone, then $E = \text{End}_H(X)$ is a skew field (see [15]), and for each finite dimensional $H$-module $M$ the dimension $\dim_E \text{Hom}_H(M, X)$ is finite. If $\{g_1, \ldots, g_r\}$ is an $E$-basis of $\text{Hom}_H(M, X)$, the map $g = (g_1, \ldots, g_r): M \to X^t$ clearly is an $\text{Add}(X)$-envelope of $M$. If moreover $X$ is sincere, hence faithful, and $\{f_1, \ldots, f_r\}$ is a basis of $\text{Hom}_H(H, X)$, considered as an $E$-vector space, then the
morphism \( f = (f_1, \ldots, f_r): H \to X^r \) is the Add(X)-envelope of \( H \), and there is a tilting module \( T = X \oplus Y \) induced by the short exact sequence

\[
0 \to H \xrightarrow{f} X^r \to Y \to 0.
\]

It is easy to check that \( f: H \to X^r \) is an Add(T)-envelope, simultaneously.

If \( P \) is a direct summand of \( H \), say \( H = P \oplus P' \), by the Snake lemma we receive the following exact and commutative diagram:

\[
\begin{array}{cccccc}
0 & \to & P & \xrightarrow{g'} & X^r & \to & Z' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H & \xrightarrow{f} & X^r & \to & Y & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
P' & & 0 & & 0 & & 0 & & 0
\end{array}
\]

As in the case of finite dimensional tilting modules one checks that \( Z' \in \text{Add}(T) \). Moreover, \( g': P \to X^r \) is an Add(T)- and an Add(X)-envelope, simultaneously. Since \( P \) has an Add(X)-envelope, the first row induces a short exact sequence

\[
0 \to P \xrightarrow{g} X^t \to Z \to 0
\]

where \( g: P \to X^t \), for some \( t \leq r \), is an Add(X)-envelope and \( Z \in \text{Add}(T) \).

In the case when \( H \) is a tame connected hereditary algebra, \( G \) is the generic module and \( \dim_E G = r \), in the short exact sequence

\[
0 \to H \to G^r \to Y \to 0,
\]

the tilting module \( T = G \oplus Y \) is equivalent to \( G \oplus (\bigoplus_{\lambda \in k \cup \infty} R_\lambda) \) where \( \{R_\lambda\} \) is the set of all Prüfer modules. For more details, we refer to [1].

4. Iterated One-Point Extensions

In some cases, the torsion class \( \{T\}^+ \) in Proposition 3.3 can be described rather precisely. Let \( B \) be a finite dimensional \( k \)-algebra and \( 0 \neq R \in \text{mod-}B \). The one-point extension \( A = B[R] \) of \( B \) by \( R \) is the generalized lower triangular matrix ring

\[
A = B[R] = \begin{pmatrix} B & 0 \\ R & k \end{pmatrix}.
\]

Then \( e_\omega = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) is a primitive idempotent in \( A \) and the indecomposable projective \( A \)-module \( P_\omega = e_\omega A \) has radical \( \begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix} \). The injective \( A \)-module \( S_\omega = D(Ae_\omega) \) is simple, and \( (1_A - e_\omega)A(1_A - e_\omega) \cong B \cong A/(Ae_\omega A) \).

Following [32], we will identify right \( A \)-modules \( M \) with the triples \( (M_0, V, \phi) \) where \( M_0 \in \text{Mod-}B \), \( V \in \text{Mod-}k \) and \( \phi: V \to \text{Hom}_B(R, M_0) \) is a \( k \)-linear map. Also, we will identify \( \text{Mod-}B \) with the full subcategory \( \{(M_0, 0, 0) \mid M_0 \in \text{Mod-}B\} \)
of Mod-$A$. Consequently $B$ can be considered as a right ideal in $A$, and for a $B$-module $M_0$ we have $\text{p.dim}_B M_0 = \text{p.dim}_A M_0$.

For the global dimension of $A = B[R]$ one has $\text{g.d. } A = \max\{\text{g.d. } B, 1 + \text{p.dim} R\}$. Consequently $A$ is hereditary, if and only if $B$ is hereditary and $R$ is projective.

For $M = (M_0, V, \phi) \in \text{Mod}-A$ we have a short exact sequence

$$\eta : 0 \to M_0 \to M \to S\omega((\dim V)) \to 0$$

in $\text{Mod}-A$. It is also shown in [32] that for an indecomposable module $X \in \text{mod}-B$ and for the Auslander-Reiten translations $\tau_A$, respectively $\tau_B$, one has

$$\tau_A X = (\tau_B X, \text{Hom}_B(R, \tau_B X), 1).$$

A finite dimensional $K$-algebra $A$ is called an \textit{iterated one-point extension} (or, more precisely, an \textit{m-fold one-point extension}) of $B$ if there is a sequence

$$B = A_0, A_1, \ldots, A_m = A$$

such that $A_i = A_{i-1}[R_{i-1}]$ is a one-point extension of $A_{i-1}$ by some module $R_{i-1} \in \text{mod-}A_{i-1}$. In this case, for an $A$-module $M$ we again denote by $M_0 = M B$ the biggest submodule of $M$, having support in (the right ideal) $B \cong A/(A(1_A - 1_B)A)$.

Let $A$ be an iterated one-point extension of $B$ and $\tilde{T}$ be a tilting $B$-module. Then $\text{p.dim}_B T = \text{p.dim}_A \tilde{T} \leq 1$, hence Proposition 3.3 applies. Thus there exists an $A$-module $C$ such that $T = \tilde{T} \oplus C$ is a tilting $A$-module with $\{T\}^\perp = \{\tilde{T}\}^\perp$. The $B$-submodule $T_0$ of $T$ then is of the form $T_0 = \tilde{T} \oplus C_0$. The tilting $B$-module $\tilde{T}$ induces a torsion pair $(\tilde{T}, \mathcal{F})$ in mod-$B$. The torsion pair in mod-$A$, induced by $\tilde{T}$, is denoted by $(T, \mathcal{F})$.

**Proposition 4.1.** Let $A$ be an iterated one-point extension of some finite dimensional $k$-algebra $B$. Let $\tilde{T}$ be a tilting $B$-module and $T = \tilde{T} \oplus C$ a tilting $A$-module with $\{T\}^\perp = \{\tilde{T}\}^\perp$. Then the following hold:

1. $\{\tilde{T}\}^\perp = \{M \in \text{Mod-}A \mid M_0 \in \text{Gen}(\tilde{T}) \subset \text{Mod-}B\}$.
2. $T = \{M \in \text{mod-}A \mid M_0 \in \tilde{T}\}$.

For $A$ hereditary, one has moreover:

3. $\text{Add}(\tilde{T}) = \text{Add}(T_0)$.
4. $\mathcal{F} = \mathcal{F} \cap \text{mod-}B$.
5. $T$ contains indecomposable finite dimensional direct summands if and only if $\tilde{T}$ does.

**Proof.** By induction it is enough to consider the case where $A = B[R]$ is a one-point extension of $B$. Recall that for an $A$-module $M = (M_0, V, \phi)$, there is the short exact sequence

$$\eta : 0 \to M_0 \to M \to S\omega((I)) \to 0,$$

where $I = \dim V$ and $S\omega$ is injective.

1. We apply the functor $\text{Hom}_A(\tilde{T}, -)$ to $\eta$ and get $0 = \text{Hom}_A(\tilde{T}, S\omega((I))) \to \text{Ext}^1_A(\tilde{T}, M_0) \to \text{Ext}^1_A(\tilde{T}, M) \to \text{Ext}^1_A(\tilde{T}, S\omega((J))) = 0$. Hence $M \in \{\tilde{T}\}^\perp$ if and only if $M_0 \in \{T\}^\perp$ if and only if $M_0 \in \text{Gen}(\tilde{T})$.

2. Follows from (1), since $\text{Gen}(\tilde{T}) \cap \text{mod-}B = \tilde{T}$.

For the remaining parts of the proof, assume that $A$ is hereditary.

3. Consider the short exact sequence

$$\rho : 0 \to T_0 \to T \to S\omega((J)) \to 0.$$
For any set \( L \), \( \text{Ext}^1_A(T,T^{(L)}) = 0 \), so \( \text{Ext}^1_A(T_0,T^{(L)}) = 0 \). Since \( \text{Hom}_A(T_0,S_\omega) = 0 \), we get \( 0 \to \text{Ext}^1_A(T_0,T^{(L)}) \to \text{Ext}^1_A(T_0,T^{(L)}) = 0 \), hence \( \text{Ext}^1_B(T_0,T^{(L)}) = \text{Ext}^1_A(T_0,T^{(L)}) = 0 \). But \( T \) is a tilting \( B \)-module, and \( T_0 = \overline{T} \oplus C_0 \). Since \( B \) is hereditary, \( \text{p.dim}_{R} \) \( T \) is at most one and \( \text{Ext}^1_B(C_0,T^{(j)}) = 0 \), the module \( C_0 \) is Ext-projective in \( \text{Gen}(T) \), hence \( \pi \) is a split epimorphism. Therefore \( \text{Add}(T_0) = \text{Add}(\overline{T}) \) holds.

(4) Since \( \mathcal{F} \cap \text{mod-}B = \{ M \in \text{mod-}B \mid \text{Hom}_A(T,M) = 0 \} \), we get \( \mathcal{F} \supseteq \mathcal{F} \cap \text{mod-}B \). Let \( M \in \mathcal{F} \). Since \( T_0 \in \text{Add}(\overline{T}) \), we get \( \text{Hom}_A(T_0,M) = 0 \). An application of \( \text{Hom}_A(-,M) \) to the short exact sequence \( 0 \to L \to T^{(j)} \to \pi, C_0 \), with \( L \in \text{Gen}(T) \); see [13, Lemma 1.2.]. Since \( C_0 \) has projective dimension at most one and \( \text{Ext}^1_B(C_0,T^{(j)}) = 0 \), the module \( C_0 \) is Ext-projective in \( \text{Gen}(T) \), hence \( \pi \) is a split epimorphism. Therefore \( \text{Add}(T_0) = \text{Add}(\overline{T}) \) holds.

(5) Since \( \overline{T} \) is a direct summand of \( T \), it is enough to show that if \( T \) has an indecomposable finite dimensional direct summand \( X \), then so does \( T \). Let \( T = X \oplus T' \). Since \( \text{Ext}^1_A(S_\omega,T) \neq 0 \), we get \( X \not\cong S_\omega \). Therefore the submodule \( X_0 \) of \( X \) is non-zero. We have \( T_0 = X_0 \oplus T'' \in \text{Add}(\overline{T}) \), hence \( 0 \neq X_0 \in \text{add}(\overline{T}) \). By Proposition 2.3 each indecomposable direct summand of \( X_0 \) is therefore a direct summand in \( \overline{T} \).

If \( A \) is not hereditary, part (3) and (5) of the proposition no longer hold:

**Example 4.2.** Let \( B \) be connected wild hereditary, and \( \overline{T} \) the tilting \( B \)-module without non-zero finite dimensional direct summands, such that \( \overline{T} \) consists of the direct sums of regular and preinjective \( B \)-modules. Let \( R \neq 0 \) be a regular \( B \)-module. The one-point extension \( A = B[R] \) has global dimension two. The new projective \( B[R] \)-module \( P_\omega \) is in \( T \), since so are \( R \) and \( S_\omega \). Therefore it is Ext-projective in \( T \) and consequently a direct summand of \( T \). It is easy to see that \( T \) is equivalent to \( P_\omega \oplus \overline{T} \), hence (3) and (5) fail. Nevertheless we get (4) in this example since \( \mathcal{F} = \mathcal{F} \cap \text{mod-}B = \mathcal{F} \).

If \( A = B[R] \) is a one-point extension and \( \overline{T} \) is a finite dimensional tilting \( B \)-module, then there exists (a unique) indecomposable \( A \)-module \( X_0 \), such that \( T = \overline{T} \oplus X_0 \) is a tilting \( A \)-module; see [18]. If \( A \) additionally is hereditary, which means \( B \) is hereditary and \( R \) is a projective \( B \)-module, one can prove more.

**Proposition 4.3.** Let \( A = B[R] \) be a hereditary \( k \)-algebra, and \( \overline{T} \) be a tilting \( B \)-module. If \( R \) admits an \( \text{Add}(\overline{T}) \)-envelope, then there exists an indecomposable \( A \)-module \( X_0 \), such that \( T = \overline{T} \oplus X_0 \) is a tilting \( A \)-module with \( \{ T \}^\perp = \{ \overline{T} \}^\perp \).

**Proof.** Since \( R \) is a projective \( B \)-module, there exists a short exact sequence

\[
0 \to R \xrightarrow{f} \overline{T}_1 \xrightarrow{g} \overline{T}_2 \to 0,
\]

with \( \overline{T}_1, \overline{T}_2 \in \text{Add}(\overline{T}) \). The morphism \( f \) clearly is an \( \text{Add}(\overline{T}) \)-preenvelope. Since \( R \) admits an \( \text{Add}(\overline{T}) \)-envelope, we may assume that \( f \) is already an \( \text{Add}(\overline{T}) \)-envelope; see [13]. Let \( P_\omega \) be the indecomposable projective \( A \)-module with radical \( R \), let \( i: R \to P_\omega \) be the inclusion and let \( S_\omega \) be the new simple \( A \)-module, which is injective. Forming the pushout of \( f \) and \( i \), we get the following exact and commutative
Proposition 4.4. Let $H$ be a connected finite dimensional hereditary algebra, $(T,\mathcal{F})$ a tilting torsion pair in mod-$H$, and $\mathcal{E} = \{T_1, \ldots, T_r\}$ a complete set of indecomposable Ext-projective modules in $T$. Then the following hold:

(1) Let $X$ be an indecomposable non-projective torsion module. Then $\tau_H X \in T$, if and only if $\text{Hom}_H(X, T_i) = 0$ for all $i$. If $\tau_H X$ is not torsion, then the add($\mathcal{E}$)-envelope $X \to \bigoplus_{i=1}^r T_i^{s_i}$ is surjective.

(2) Let $Y$ be an indecomposable non-injective torsion-free module. Then $\tau_H^{-1} Y \in \mathcal{F}$, if and only if $\text{Hom}_H(\tau_H T_i, Y) = 0$ for all $i$. If $\tau_H^{-1} Y$ is not torsion-free, then the add($\tau_H^{-1} \mathcal{E}$)-cover $\bigoplus_{i=1}^r \tau_H T_i^{t_i} \to Y$ is injective.

(3) If there exists a preprojective module which is not torsion-free, then there exists an indecomposable preprojective Ext-projective torsion module.

\[
\begin{array}{cccccc}
0 & 0 & \downarrow & \downarrow & \\
0 & \to & R & \xrightarrow{f} & \bar{T}_1 & \xrightarrow{g} & \bar{T}_2 & \to & 0 \\
1 & \downarrow & i & \downarrow i' & \| & \\
0 & \to & P_\omega & \xrightarrow{f'} & X_0 & \xrightarrow{g'} & \bar{T}_2 & \to & 0 \\
& & \downarrow & \downarrow & \\
& & S_\omega & \equiv & S_\omega & \equiv & \\
& & 0 & \to & 0 & \\
\end{array}
\]

and we consider $i'$ to be the inclusion. Since $\text{Hom}_A(\bar{T}_1, S_\omega) = 0$, we get $k \cong \text{Hom}_A(S_\omega, S_\omega) \cong \text{Hom}_A(X_0, S_\omega)$. Let $X_0 = U \oplus V$ with $\text{Hom}_A(U, S_\omega) \neq 0$. Then $0 = \text{Hom}_A(V, S_\omega) \cong \text{Hom}_A(P_\omega, V)$, since $S_\omega$ is injective. Therefore $f'(P_\omega) \subset U$, hence we get $\bar{T}_2 \cong (U/f(\omega(P_\omega)) \oplus V$. From $\text{Hom}_A(V, S_\omega) = 0$ we deduce $\bar{T}_1 = i'(\bar{T}_1) = \oplus U' \oplus V$, with $U' \subset U$ and additionally $f(R) \subset U'$. But $f: R \to \bar{T}_1$ is an Add($\bar{T}$)-envelope, consequently $V = 0$ follows. This implies that $X_0$ is indecomposable.

Let $I$ be any set. Since Mod-$B$ is closed under extensions in Mod-$A$, we get $0 = \text{Ext}_B^1(T, T(I)) = \text{Ext}_A^1(\bar{T}, T(I))$. From the second column of the diagram we infer $\text{Ext}_A^1(X_0, \bar{T}(I)) = 0$, since $S_\omega$ is injective. The second row shows $\text{Ext}_A^1(\bar{T}, X_0(I)) = 0$ and $\text{Ext}_A^1(X_0, X_0(I)) = 0$. Hence $\text{Ext}_A^1(T, T(I)) = 0$ holds for $T = T \oplus X_0$. Since $\bar{T}$ is a tilting $B$-module, there exists a short exact sequence $0 \to B \to \bar{T}' \to \bar{T}'' \to 0$, with $\bar{T}', \bar{T}'' \in \text{Add}(\bar{T})$. Since $A = B \oplus P_\omega$, the short exact sequence

$$0 \to A \to \bar{T}' \oplus X_0 \to \bar{T}'' \oplus \bar{T}_2 \to 0$$

shows (T3).
(4) Let \( H \) be wild hereditary and \( R \) an indecomposable regular \( H \)-module. If \( \tau^i_H R \) is torsion (respectively, torsion-free) for all integers \( i \), then all regular and preinjective modules are torsion (all regular and preprojective modules are torsion-free, respectively).

(5) Let \( H \) be wild hereditary and assume that all preprojective modules are torsion-free and all preinjective modules are torsion. Then each elementary module \( E \) is either torsion or torsion-free.

Example 4.5. Let \( K \) be an algebraically closed field and \( H = KQ \) be the path algebra of the quiver

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
3 & \rightarrow & 1
\end{array}
\]

Then \( H \) is a connected wild hereditary algebra with three simple modules. Also, \( H \) is a one-point extension of the Kronecker algebra \( A = H/(He_3H) \) by the projective module \( P_2 \), so Proposition 4.1 applies in this setting.

By [22], for regular \( H \)-modules \( X \) and \( Y \) with \( \text{Hom}_H(X,Y) \neq 0 \), for all \( i \geq 2 \) we get \( \text{Hom}_H(X,\tau^i_H Y) \neq 0 \), since \( H \) has three simple modules. The elementary \( H \)-modules form the \( \tau \)-orbit of the indecomposable module \( X \) with dimension vector \( \dim X = (5,4,0) \), and the \( \tau \)-orbits of the \( P(1,K) \)-family of indecomposable modules \( Y_\rho, \rho \in P(1,K) \) with \( \dim Y_\rho = (1,1,0) \), by [29].

(1) Let \( X \) be the elementary module with \( \dim X = (5,4,0) \). In \( \{X\}^{\perp \geq 0} = \text{Mod}-C \) we consider the tilting module \( T_\infty \), defined by \( \{T_\infty\}^{\perp} = P(C)^{\perp} \); see [28, 25]. This tilting \( C \)-module generates all regular and preinjective \( C \)-modules. Let \( 0 \rightarrow \tau^i_H X \rightarrow (2)X \rightarrow X \rightarrow 0 \) be the Auslander-Reiten sequence, ending in \( X \). Then \( (2)X \) is a regular \( C \)-module, hence it is generated by \( T_\infty \). Consequently \( X \) is also generated by \( T_\infty \), hence \( T = T_\infty \oplus X \) is a tilting \( H \)-module, by Lemma 2.6. Let \( T = T^\perp \cap \text{ind}-H \). By construction \( X \) is the only indecomposable Ext-projective module in \( T \), since \( T_\infty \) has no non-zero finite dimensional direct summands; see Lemma 2.2.

Consequently all indecomposable preinjective \( H \)-modules are contained in \( T \), whereas all preprojective \( H \)-modules are torsion-free. Since \( \text{Hom}_H(X,\tau^i_H X) \neq 0 \) for all \( i > 1 \), the elementary modules \( \tau^i_H X \in T \), for \( i > 1 \). Dually, all the modules \( \tau^{-i}_H X \), for \( i < 0 \), are torsion-free, since \( \text{Hom}_H(\tau^{-i}_H X, \tau^i_H X) \neq 0 \). Similarly, from \( \text{Hom}_H(X,\tau^i_H Y_\rho) \neq 0 \) for \( i \geq 0 \), we conclude \( \tau^i_H Y_\rho \in T \), for all \( i \geq 0 \) and all \( \rho \in P(1,K) \). The modules \( \tau^i_H Y_\rho \), for \( i < 0 \), are torsion-free, since they map non-trivially to \( \tau_H X \). Since each indecomposable regular module has a filtration with elementary subquotients, we conclude that each regular component \( C \) contains a quasi-simple module \( U \), such that the cone \( (\rightarrow U) \) of predecessors of \( U \) in the component \( C \) consists of torsion modules. Here, \( V \) is a predecessor of \( U \) in \( C \) if there exists a path of arrows in the AR-quiver of \( H \) of length \( \geq 0 \) from \( V \) to \( U \). Similarly, there exists a quasi-simple module \( V = \tau^{-r}_H U \) for some \( r > 0 \), such that all modules in the cone \( (V \rightarrow) \) of successors of \( V \) in \( C \) are torsion-free. Since \( (2)X \) is generated by \( T_\infty \), we get \( (2)X \in T \). Since the modules \( \tau^j_H (j)X \in T \), for all \( j \geq 1 \) and we have short exact sequences \( \tau^j_H ((i-2)X \rightarrow (i)X \rightarrow (i-1)X \rightarrow 0 \), for all \( i \geq 0 \), we conclude from \( X, (2)X \in T \) by induction that \( (i)X \in T \) for all \( i \).

All regular \( C \)-modules are generated by \( T_\infty \), hence they are torsion modules. If \( R \) is an indecomposable regular \( C \)-module, then \( \text{Hom}_C(\tau^{-m}_C R, (2)X) \neq 0 \), for \( m \gg 0 \), by [8]. Since \( 0 \neq \text{Hom}_C(\tau^{-m}_C R, (2)X) \cong \text{Hom}_H(\tau^{-m}_C R, X) \), we conclude
from Proposition 4.4 that $\tau_H \tau_C^{-m} R \notin T$. Hence there exist infinitely many indecomposable regular torsion modules $\tau_C^{-m} R$, not contained in the torsion cones $(\to U)$, described above; we call them the isolated torsion modules. By definition they are those indecomposable torsion modules $M$, such that $\tau_H^r M \notin T$, for some $r > 0$.

We first show that each regular component $C$ contains at most one isolated quasi-simple torsion module. Indeed, suppose $U$ and $V = \tau_H^{-t} U$, with $t > 0$, are quasi-simple isolated torsion modules in $C$. We may assume that $\tau_H U \notin T$, hence by Proposition 4.4 $\text{Hom}_H(U, X) \neq 0$. Consequently $\text{Hom}_H(V, \tau_H^{-1} X) \neq 0$, but $\tau_H^{-1} X$ is torsion-free, for $i > 0$, a contradiction.

Next we show that all isolated torsion modules, with the exception of the modules $(i)X$, are quasi-simple. Indeed, assume $M \notin (r)X$ is an indecomposable isolated torsion module of quasi-length $r > 1$. Then there exists a quasi-simple regular module $S$ and a chain of irreducible epimorphisms

$$M = (r)S \to (r-1)S \to \cdots \to (2)S \to S.$$ 

Therefore all the modules $(i)S$, for $1 \leq i \leq r$, are indecomposable torsion modules. Consider the Auslander-Reiten sequence $0 \to \tau_H S \xrightarrow{f} (2)S \xrightarrow{g} S \to 0$. Since $M$ is an isolated torsion module, so is $S$. Therefore $S$ is the only quasi-simple isolated torsion module in $C$, which implies $\tau_H S$ is not torsion. Since $S \neq X$, the module $\tau_H S$ is not torsion-free. Let $0 \to t(\tau_H S) \to \tau_H S \xrightarrow{f} (\tau_H S) \to 0$ be the canonical short exact sequence, where $t(\tau_H S)$ is the torsion submodule of $\tau_H S$ and $f(\tau_H S)$ is torsion-free. Since the epimorphism $\tau_H S \xrightarrow{f} (\tau_H S)$ is not a split monomorphism, it factorises through the source map $f$: $\tau_H S \to (2)S$. But $(2)S$ is a torsion module, a contradiction.

Dually, we call an indecomposable regular torsion-free module $F$, isolated torsion-free, if $\tau_H^{-1} F$ is not torsion-free, for some $r > 0$. We now show that $\tau_H X$ is the only isolated torsion-free module. Indeed, suppose $M \notin \tau_H X$ is isolated torsion-free and $\tau_H^{-1} M$ is not torsion-free By Proposition 4.4 there exists a non-zero morphism $h: \tau_H X \to M$, which clearly is not a split mono. Hence it factorises through the source map $f$: $\tau_H X \to (2)X$, but $(2)X$ is a torsion module, a contradiction.

(2) Let $S_3$ be the simple injective $H$-module, corresponding to the vertex 3 of the quiver. Let $A = S_3^{1 \to 0} \subset \text{Mod-}H$. Then $A$ is equivalent to $\text{Mod-}A$, where $A$ is the Kronecker-algebra, the minimal projective generator of $A$ is the direct sum $P_1 \oplus P_3$ of projective modules, and an $H$-module $M$ is in $A$ if and only if $\text{Hom}_M(M, S_2) = 0$.

The equivalence $F$: $\text{Mod-}A \to A$ is given (in term of representations) by

$$U \xrightarrow{f} V \quad \quad U \xleftarrow{g} V \quad \quad U \xrightarrow{1_U} U \xleftarrow{g} V \xrightarrow{f}$$

Let $T' = G \oplus \bigoplus \lambda R_\lambda$ be the tilting $A$-module, where $G$ is the generic $A$-module and \{ $R_\lambda \mid \lambda \in \mathcal{P}(1, K)$ \} the set of all Prüfer $A$-modules; see [1, Example 1.4]. Then $T' \cap \text{mod-}A$ consists of the preinjective $A$-modules. Let $T_1 \in A$ be the image of $T'$ under the equivalence $F$. The indecomposable injective module $I_2$ is preinjective in $A$. Indeed, it is the simple injective object in $A$, hence it is generated by $T_1$. Consequently $S_3$ is also generated by $T_1$. By Lemma 2.6 $T = T_1 \oplus S_3$ is an $H$-tilting module with exactly one finite dimensional indecomposable direct
summand $S_3$. Additionally, we know from the construction of $\hat{T}$ that $\{\hat{T}\}^\perp = \mathrm{Gen}(F(G))$. It is easy to check that $T = \hat{T}^\perp \cap \mathrm{ind}-H$ consists of the module $S_3$ and the countably many indecomposable $H$-modules $Z_i$ without self-extensions, where $\dim Z_i = (i, i + 1, i + 1)$, for $i \geq 0$. Indeed, for $Z \in T$ with $Z \not\in S_3$, we get $\text{Hom}_H(S_3, Z) = 0$ which means that $Z \in T \cap A$. Note that the modules $Z_i$ are regular for $i \geq 2$, that $Z_2$ is elementary, and that $Z_0$ and $Z_1$ are injective.

Since $S_2 = \tau_H S_3$, the simple module $S_2$ is torsion-free [19]. If $I \not\in S_2$ is an indecomposable preinjective module, then $\text{Hom}_H(Z_5, I) \neq 0$, hence $S_2$ is the only indecomposable preinjective torsion-free module. Since $S_3$ is the unique indecomposable Ext-projective module in $T$, all preprojective $H$-modules are torsion-free. The elementary modules $X$ with $\dim X = (5, 4, 0)$ and the elementary modules $Y_\lambda$ with $\dim Y_\lambda = (1, 1, 0)$ have filtrations with composition factors $S_1$ and $S_2$, therefore they are torsion-free. If $R$ is an indecomposable regular torsion-free module, then $\tau_H R$ is torsion-free by Proposition 4.4, for all $r \geq 0$, since $\text{Hom}_H(S_2, R') = 0$ for all regular modules $R'$, which means that there are no isolated regular torsion-free modules. Consequently all elementary modules $\tau_H r Y_\lambda$ and $\tau_H r X$ are torsion-free.

Again we conclude that each regular component $C$ contains a quasi-simple module $U$, such that the cone $(U \rightarrow)$ consists of the indecomposable torsion-free modules in $C$. All torsion modules are isolated.

This example has another interpretation: We now consider the stone $X$, corresponding to the imaginary Schur root $(1, 1, 1)$. Then $X$ is the image of the generic $A$-module $G$ under the functor $F$: Mod-$A \rightarrow A$. Therefore $X$ is the representation

\[
\begin{align*}
K(t) \xrightarrow{1} K(t) \xrightarrow{1} K(t)
\end{align*}
\]

and $E = \text{End}_H(X) = K(t)$, the rational function field. Since $X$ is sincere, hence faithful, we get by Proposition 3.1 a tilting module $T = X \oplus Y$ with $\{T\}^\perp = \text{Gen}(X)$. Since also $\{\hat{T}\}^\perp = \text{Gen}(X)$, the tilting modules $T$ and $\hat{T}$ are equivalent.

(3) Let $A = H/(He_3H)$ be the Kronecker algebra. We identify Mod-$A$, respectively mod-$A$, with the full subcategories of Mod-$H$, respectively mod-$H$, consisting of $H$-modules, having support in the full subquiver $Q$, defined by the vertices 1 and 2. In Mod-$A$ we consider the tilting $A$-module $T' = G \oplus \bigoplus R_\lambda$, where $G$ is the generic module and $\{R_\lambda\}$ is the set of all Prüfer $A$-modules. Then $T'^\perp \cap \text{mod}-A$ consists of the preinjective $A$-modules. Let $T$ be a tilting $H$-module with $\{T\}^\perp = \{T'^\perp\}$.

Then we infer from Proposition 4.1:

(1) $M \in \{T\}^\perp$ if and only if $M_0 = MA \in \text{Gen}(T')$. If additionally $M$ is finite dimensional, then $M \in \{T\}^\perp$ if and only if $M_0$ is a preinjective $A$-module.

(2) $T$ has no finite dimensional indecomposable direct summands.

Let $(T, \mathcal{F})$ be the torsion pair in mod-$H$, induced by $T$. From Lemma 2.1 and Proposition 4.4 we therefore deduce:

(a) $\tau_H T \subset T$ and $\tau_H \mathcal{F} \subset \mathcal{F}$.

(b) All preprojective modules are torsion free, and all preinjective modules are torsion.

The elementary $H$-module $X'$ with $\dim X' = (1, 2, 0)$ is in $T$, hence $\tau_H^r X' \in T$, for all $r \geq 0$. The elementary $H$-modules $Y_\lambda$ with $\dim Y_\lambda = (1, 1, 0)$ are not torsion, but the elementary modules $\tau_H^2 Y_\lambda$ have the biggest $A$-submodule $X'$, hence they
are torsion modules. Consequently all the modules $\tau^r_H Y_\lambda$, for $r \geq 2$ are torsion modules.

Again we conclude that each regular component $C$ contains quasi-simple modules $U$ and $V = \tau^r_H U$, for some $r > 0$, such that the cone $(\rightarrow U)$ consists of torsion modules in $C$ and the cone $(V \rightarrow)$ of the torsion free modules in $C$.

We know from Remark 3.4 that the projective $A$-module $P_2$ admits an Add($T'$)-envelope of the form

$$0 \rightarrow P_2 \rightarrow G \rightarrow T'_1 \rightarrow 0$$

with $T'_1 \in \text{Add}(T')$, since $\dim_E \text{Hom}_A(P_2, G) = 1$, for $E = \text{End}_A(G) \cong k(t)$. Consequently, the tilting module $T$ with $\{T\} = \{T'\}$ can be chosen by Proposition 4.3 as $T = T' \oplus X_0$, with $X_0$ indecomposable. Indeed, $X_0$ is the middle term of the non-split short exact sequence $0 \rightarrow G \rightarrow X_0 \rightarrow S_2 \rightarrow 0$. Additionally, a direct calculation shows $\text{End}_H(X_0) \cong k$.

References

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