A CONNECTEDNESS RESULT IN POSITIVE CHARACTERISTIC

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Dedicated to Professor Paul Roberts on the occasion of his sixtieth birthday

Abstract. Let \((R, m)\) be a complete local ring of dimension at least two, which contains a separably closed coefficient field of positive characteristic. Using a vanishing theorem of Peskine-Szpiro, Lyubeznik proved that the local cohomology module \(H^1_m(R)\) is Frobenius-torsion if and only if the punctured spectrum of \(R\) is connected in the Zariski topology. We give a simple proof of this theorem and, more generally, a formula for the number of connected components in terms of the Frobenius action on \(H^1_m(R)\).

1. Introduction

All rings considered in this note are commutative and Noetherian. We give a simple proof of the following result due to Lyubeznik:

Theorem 1.1 ([Ly2, Corollary 4.6]). Let \((R, m)\) be a complete local ring of dimension at least two, with a separably closed coefficient field of positive characteristic. Then the \(e\)-th iteration of the Frobenius map

\[ F: H^1_m(R) \rightarrow H^1_m(R) \]

is zero for \(e \gg 0\) if and only if \(\text{Spec } R \setminus \{m\}\) is connected in the Zariski topology.

We also obtain, by similar methods, the following theorem:

Theorem 1.2. Let \((R, m)\) be a complete local ring of positive dimension, with an algebraically closed coefficient field of positive characteristic. Then the number of connected components of \(\text{Spec } R \setminus \{m\}\) is

\[ 1 + \dim_K \bigcap_{e \in \mathbb{N}} F^e(H^1_m(R)) \].

In Section 5 we describe how this provides an algorithm to determine the number of geometrically connected components of projective algebraic sets defined over a finite field: computer algebra algorithms for primary decomposition can be used to determine the number of connected components over finite extensions of the fields \(\mathbb{F}_p\) or \(\mathbb{Q}\), but not over the algebraic closures of these fields. In the case of characteristic zero, de Rham cohomology allows for the computation of the number...
of geometrically connected components via $D$-module methods, [Wal], and we show that the Frobenius provides analogous methods in the case of positive characteristic.

Theorem 1.1 is obtained in [Ly2] as a corollary of the following two theorems of Lyubeznik and Peskine-Szpiro:

**Theorem 1.3** ([Ly2, Theorem 1.1]). Let $(A, \mathfrak{M})$ be a regular local ring containing a field of positive characteristic, and let $\mathfrak{A}$ be an ideal of $A$. Then $H^i_{\mathfrak{M}}(A) = 0$ if and only if there exists an integer $e \geq 1$ such that the $e$-th Frobenius iteration

$$F^e : H^i_{\mathfrak{M}}(A/\mathfrak{A}) \to H^i_{\mathfrak{M}}(A/\mathfrak{A})$$

is the zero map.

**Theorem 1.4** ([PS, Chapter III, Theorem 5.5]). Let $(A, \mathfrak{M})$ be a complete regular local ring with a separably closed coefficient field of positive characteristic, and let $\mathfrak{A}$ be an ideal of $A$. Then $H^i_{\mathfrak{M}}(A) = 0$ for $i \geq \dim A - 1$ if and only if $\dim(A/\mathfrak{A}) \geq 2$ and $\Spec(A/\mathfrak{A}) \setminus \{\mathfrak{M}\}$ is connected.

Our proof of Theorem 1.1 is “simple” in the sense that it does not rely on vanishing theorems such as those of [PS]—indeed, the only ingredient, aside from elementary considerations, is the local duality theorem. Results analogous to Theorem 1.4 were proved by Hartshorne in the projective case [HaR, Theorem 7.5], and by Ogus in equicharacteristic zero using de Rham cohomology [Og, Corollary 2.11]. Combining these results, one has:

**Theorem 1.5.** Let $(A, \mathfrak{M})$ be a regular local ring containing a field, and let $\mathfrak{A}$ be an ideal of $A$. Then $H^i_{\mathfrak{M}}(A) = 0$ for $i \geq \dim A - 1$ if and only if

1. $\dim(A/\mathfrak{A}) \geq 2$, and
2. $\Spec(A/\mathfrak{A}) \setminus \{\mathfrak{M}\}$ is formally geometrically connected (see Definition 2.1).

Huneke and Lyubeznik [HL, Theorem 2.9] gave a characteristic free proof of this using a generalization of a result of Faltings, [Fa, Satz 1]. Some other applications of local cohomology theory which yield strong results on the connectedness properties of algebraic varieties may be found in the papers [BR] and [HH], where the authors obtain generalizations of Faltings’ connectedness theorem.

For the convenience of the reader, we include an Appendix with some facts about Frobenius actions; see Section 6.

2. Preliminary remarks

*Notation.* When $R$ is the homomorphic image of a ring $A$, we use upper-case letters $\mathfrak{P}, \mathfrak{Q}, \mathfrak{M}, \mathfrak{A}, \mathfrak{B}$ for ideals of $A$, and corresponding lower-case letters $p, q, m, a, b$ for their images in $R$.

**Definition 2.1.** Let $(R, \mathfrak{m})$ be a local ring. A field $K \subseteq R$ is a coefficient field for $R$ if the composition $K \hookrightarrow R \twoheadrightarrow R/\mathfrak{m}$ is an isomorphism. Every complete local ring containing a field has a coefficient field.

We recall some notions from [Ra, Chapitre VIII]. Let $(R, \mathfrak{m}, K)$ be a local ring and let $\overline{f}(T) \in K[T]$ denote the image of a polynomial $f(T) \in R[T]$. Then $R$ is Henselian if for every monic polynomial $f(T) \in R[T]$, every factorization of $\overline{f}(T)$ as a product of relatively prime monic polynomials in $K[T]$ lifts to a factorization of $f(T)$ as a product of monic polynomials in $R[T]$. Hensel’s Lemma is precisely the statement that every complete local ring is Henselian. The Henselization of a local
ring $R$ is a local ring $R^h$, with the property that every local homomorphism from $R$ to a Henselian local ring factors uniquely through $R^h$. The ring $R^h$ is obtained by taking the direct limit of all local étale extensions $S$ of $R$ for which $(R, m) \to (S, n)$ induces an isomorphism of residue fields $R/m \cong S/n$.

A local ring $(R, m, K)$ is said to be strictly Henselian if it is Henselian and its residue field $K$ is separably closed. It is easily seen that $R$ is strictly Henselian if and only if every monic polynomial $f(T) \in R[T]$ for which $\overline{f(T)} \in K[T]$ is separable splits into linear factors in $R[T]$. Every local ring has a strict Henselization $R^h$, such that every local homomorphism from $R$ to a strictly Henselian ring factors through $R^h$. The strict Henselization of a field $K$ is its separable closure $K^{\text{sep}}$.

In general, the strict Henselization of a local ring $(R, m, K)$ is obtained by fixing an embedding $\iota : K \to K^{\text{sep}}$, and taking the direct limit of local étale extensions $(S, n, L)$ of $(R, m, K)$ with $L \hookrightarrow K^{\text{sep}}$, for which the induced map $K \to L \to K^{\text{sep}}$ agrees with $\iota : K \to K^{\text{sep}}$.

The punctured spectrum of a local ring $(R, m)$ is the set $\text{Spec } R \setminus \{m\}$, with the topology induced by the Zariski topology on $\text{Spec } R$. We say that the punctured spectrum of $R$ is formally geometrically connected if the punctured spectrum of $R^h$ is disconnected if and only if the minimal primes of $R$, and there is an edge between minimal primes $p$ and $p'$ if and only if $\text{rad}(p + p) \neq m$.

It follows that the punctured spectrum of $R$ is connected if and only if the graph $\Gamma$ is connected. If the graph $\Gamma$ is connected, take a spanning tree, i.e., a connected acyclic subgraph, containing all the vertices of $\Gamma$. This spanning tree must contain a vertex $p_i$ with only one edge, so $\Gamma \setminus \{p_i\}$ is connected as well.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be incomparable prime ideals of a local domain $A$. Then their images $p_1, \ldots, p_n$ are precisely the minimal primes of the ring $R = A/\langle \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n \rangle$.

From the above discussion, we conclude that if the punctured spectrum of $R$ is...
connected, then there exists $i$ such that the punctured spectrum of the ring
\[ A/(\mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_i \cap \cdots \cap \mathfrak{P}_n) \]
is connected as well.

Theorems 1.1 and 1.2 assert that connectedness issues for $\text{Spec } R \setminus \{m\}$ are determined by the Frobenius action on $H^1_m(R)$. We next record an observation about the length of $H^1_m(R)$.

**Proposition 2.4.** Let $(R, m)$ be a local ring which is a homomorphic image of a Gorenstein domain. Then $H^1_m(R)$ has finite length if and only if $\text{ann}_R p = 0$ for every prime ideal $p$ of $R$ with $\text{dim } R/p = 1$.

**Proof.** If $\text{dim } R = 0$, then $H^1_m(R) = 0$, and $R$ has no primes with $\text{dim } R/p = 1$. If $\text{dim } R = 1$, then $H^1_m(R)$ has infinite length and $\text{dim } R/p = 1$ for some minimal prime $p$ of $R$. For the rest of the proof we hence assume that $\text{dim } R \geq 2$.

Let $R = A/\mathfrak{M}$ where $A$ is a Gorenstein domain. Localizing $A$ at the inverse image of $m$, we may assume that $(A, \mathfrak{M})$ is a local ring. Using local duality over $A$, the module $H^1_m(R) = H^1_m(A/\mathfrak{M})$ has finite length if and only if $\text{Ext}^1_A(A/\mathfrak{M}, A)$ has finite length as an $A$-module. Since $\text{Ext}^1_A(A/\mathfrak{M}, A)$ is finitely generated, this is equivalent to the vanishing of
\[ \text{Ext}^1_A(A/\mathfrak{M}, A) \mathfrak{P} = \text{Ext}^1_A(A/\mathfrak{M}, A_\mathfrak{P}) \]
for all $\mathfrak{P} \in \text{Spec } A \setminus \{\mathfrak{M}\}$. Using local duality over the Gorenstein local ring $(A_\mathfrak{P}, \mathfrak{P}A_\mathfrak{P})$, this is equivalent to the vanishing of
\[ H^1_\mathfrak{P}(A_\mathfrak{P}) = H^1_\mathfrak{P}(A_\mathfrak{P}, A_\mathfrak{P}) \]
for all $\mathfrak{P} \in \text{Spec } A \setminus \{\mathfrak{M}\}$. This local cohomology module vanishes for $\mathfrak{P} \notin V(\Omega)$. Since $\text{dim } A_\mathfrak{P} - \text{dim } A + 1 \leq 0$ for $\mathfrak{P} \in \text{Spec } A \setminus \{\mathfrak{M}\}$, we need only consider primes $\mathfrak{P} \in V(\Omega)$ with $\text{dim } A_\mathfrak{P} = \text{dim } A - 1$. Since $A$ is a catenary local domain, $\text{dim } A_\mathfrak{P}$ equals $\text{dim } A - 1$ precisely when $\text{dim } A/\mathfrak{P} = 1$, equivalently $\text{dim } R/p = 1$. Hence $H^1_m(R)$ has finite length if and only if $H^1_{pR_\mathfrak{P}}(R_\mathfrak{P}) = H^1_{pR_\mathfrak{P}}(R_\mathfrak{P})$ vanishes for all $\mathfrak{P} \in \text{Spec } R$ with $\text{dim } R/p = 1$, i.e., if and only if $\text{ann}_R p = 0$ for all $p$ with $\text{dim } R/p = 1$. \(\square\)

### 3. Main results

**Theorem 3.1.** Let $(R, m)$ be a strictly Henselian local domain containing a field of positive characteristic. If $R$ is a homomorphic image of a Gorenstein domain and $\text{dim } R \geq 2$, then $H^1_m(R)$ is $F$-torsion.

**Proof.** Suppose there exists $\eta \in H^1_m(R)$ which is not $F$-torsion. Since $R$ is a domain, Proposition 2.4 implies that $H^1_m(R)$ has finite length. Hence for all integers $e \gg 0$, the element $F^e(\eta)$ belongs to the $R$-module spanned by $\eta, F(\eta), F^2(\eta), \ldots, F^{e-1}(\eta)$. Amongst all equations of the form
\[ F^{e+k}(\eta) + r_1 F^{e+k-1}(\eta) + \cdots + r_e F^k(\eta) = 0 \tag{3.1.1} \]
with $r_i \in R$ for all $i$, choose one where the number of nonzero coefficients $r_i$ that occur is minimal. We claim that $r_e$ must be a unit. Note that $H^1_m(R)$ is killed by $m^{q'}$ for some $q' = p^e$. If $r_e \in m$, then applying $F^{e'}$ to equation (3.1.1), we get
\[ F^{e'+e+k}(\eta) + r_1^{q'} F^{e'+e+k-1}(\eta) + \cdots + r_e^{q'} F^{e'+k}(\eta) = 0. \]
But \( r_e'F^{e'+k} \langle \eta \rangle \in m'^e H^1_m(R) = 0 \), so this is an equation with fewer nonzero coefficients, contradicting the minimality assumption. This shows that \( r_e \in R \) is a unit. Since \( \eta \) is not \( F \)-torsion, neither is \( F^k(\eta) \), so after replacing \( \eta \) if necessary, we have an equation of the form
\[
F^e(\eta) + r_1 F^{e-1}(\eta) + \cdots + r_e \eta = 0
\]
where \( r_e \) is a unit and \( \eta \in H^1_m(R) \) is not \( F \)-torsion. Let \( \eta = [(y_1/x_1, \ldots, y_d/x_d)] \) where \( H^1_m(R) \) is regarded as the cohomology of a Čech complex on a system of parameters \( x_1, \ldots, x_d \) for \( R \). Then (3.1.2) implies that there exists \( r_{e+1} \in R \) such that each \( y_i/x_i \in R_{x_i} \) is a root of the polynomial
\[
f(T) = T^p^e + r_1 T^{p^{e-1}} + \cdots + r_e T + r_{e+1} \in R[T].
\]
Now \( f'(T) = r_e \) is a unit, so \( f(T) \in R/m[T] \) is a separable polynomial. Since \( R \) is strictly Henselian, the polynomial \( f(T) \) splits in \( R[T] \), and hence any root of \( f(T) \) in the fraction field of \( R \) is an element of \( R \). In particular, \( y_1/x_1 = \cdots = y_d/x_d \) is an element of \( R \), and so \( \eta = 0 \). \( \square \)

We next prove the connectedness criterion, Theorem 1.1. By Proposition 6.1, the module \( H^1_m(R) \) is \( F \)-torsion if and only if there exists \( e \) such that \( F^e(H^1_m(R)) = 0 \). In view of this, the following theorem is equivalent to Theorem 1.1.

**Theorem 3.2.** Let \((R, m)\) be a local ring with \( \dim R > 0 \), which contains a field of positive characteristic. Then \( H^1_m(R) \) is \( F \)-torsion if and only if \( \dim R \geq 2 \) and the punctured spectrum of \( R \) is formally geometrically connected.

**Proof.** Quite generally, for a local ring \((R, m)\) we have \( H^1_m(\hat{R}) = H^1_m(R) \). Moreover, \( S = R_{\hat{m}} \) is a faithfully flat extension of \( R \), and \( H^1_m(R) \otimes_R S \cong H^1_m(S) \) is \( F \)-torsion if and only if \( H^1_m(R) \) is \( F \)-torsion. Hence we may assume that \( R \) is a complete local ring with a separably closed coefficient field.

Suppose that \( H^1_m(R) \) is \( F \)-torsion. The local cohomology module \( H^1_{mR} \) is not \( F \)-torsion by Proposition 6.2, so \( \dim R \geq 2 \). Let \( a \) and \( b \) be ideals of \( R \) such that \( a + b \) is \( m \)-primary and \( a \cap b = 0 \). Let
\[
x_1 = y_1 + z_1, \quad \ldots, \quad x_d = y_d + z_d
\]
be a system of parameters for \( R \) where \( y_i \in a \) and \( z_i \in b \). Since \( ab \subseteq a \cap b = 0 \), we have \( y_i z_j = 0 \) for all \( i, j \), and hence
\[
y_i (y_j + z_j) = y_j (y_i + z_i).
\]
These relations give an element of \( H^1_m(R) \) regarded as the cohomology of a Čech complex on \( x_1, \ldots, x_d \), namely
\[
\eta = \left[ \left( \frac{y_1}{x_1}, \ldots, \frac{y_d}{x_d} \right) \right] \in H^1_m(R).
\]
The hypotheses imply that \( F^e(\eta) = 0 \) for some \( e \), so there exists \( q = p^e \) and \( r \in R \) such that \( (y_i/x_i)^q = r = \frac{r}{R_{x_i}} \) for all \( 1 \leq i \leq d \). Hence there exists \( t \in \mathbb{N} \) such that
\[
x_i^t y_i^q = r x_i^{q+t}, \text{ i.e., } \quad (y_i + z_i)^t y_i^q = r(y_i + z_i)^{q+t}.
\]
But \( y_i z_i = 0 \), so these equations simplify to give \((1 - r)y_i^{q+t} = r z_i^{q+t} \). Since \( R \) is a local ring, either \( r \) or \( 1 - r \) must be a unit. If \( r \) is a unit, then \( z_i^{q+t} \in a \) for all \( i \),
and so \(a\) is \(m\)-primary. Similarly if \(1 - r\) is a unit, then \(b\) is \(m\)-primary. This proves that the punctured spectrum of \(R\) is connected.

For the converse, assume that \(\dim R \geq 2\) and that the punctured spectrum of \(R\) is connected. Let \(n\) denote the nilradical of \(R\). Note that \(\Spec R\) is homeomorphic to \(\Spec R/n\). Moreover, \(n\) supports a Frobenius action and is \(F\)-torsion. The long exact sequence of local cohomology relating \(H^n_m(R)\) and \(H^n_m(R/n)\) implies that if \(H^n_m(R/n)\) is \(F\)-torsion, then so is \(H^n_m(R)\), and hence there is no loss of generality in assuming that \(R\) is reduced. Let \(R = A/(\mathcal{P}_1 \cap \cdots \cap \mathcal{P}_n)\) where \(\mathcal{P}_1, \ldots, \mathcal{P}_n\) are incomparable prime ideals of a power series ring \(A = \mathbb{K}[[x_1, \ldots, x_m]]\) over a separably closed field \(\mathbb{K}\). We use induction on \(n\) to prove that \(H^1_m(R)\) is \(F\)-torsion; the case \(n = 1\) follows from Theorem 3.1, so we assume \(n > 1\) below.

If \(\dim R/p_i = 1\) for some \(i\), then \(\Spec R \setminus \{p_i\}\) is the disjoint union of \(V(p_i) \setminus \{m\}\) and \(V(p_1 \cap \cdots \cap p_i \cap \cdots \cap p_n) \setminus \{m\}\), contradicting the connectedness assumption. Hence \(\dim R/p_i \geq 2\) for all \(i\). By Remark 2.3, after relabeling the minimal primes if necessary, we may assume that the punctured spectrum of \(A/\mathcal{Q}\) is connected where \(\mathcal{Q} = \mathcal{P}_2 \cap \cdots \cap \mathcal{P}_n\). The short exact sequence
\[
0 \rightarrow A/(\mathcal{P}_1 \cap \mathcal{Q}) \rightarrow A/\mathcal{P}_1 + A/\mathcal{Q} \rightarrow A/(\mathcal{P}_1 + \mathcal{Q}) \rightarrow 0
\]
induces a long exact sequence of local cohomology modules containing the piece
\[
H^0_{2\mathbb{K}}(A/(\mathcal{P}_1 + \mathcal{Q})) \rightarrow H^1_{2\mathbb{K}}(A/(\mathcal{P}_1 \cap \mathcal{Q})) \rightarrow H^1_{2\mathbb{K}}(A/\mathcal{P}_1) \oplus H^1_{2\mathbb{K}}(A/\mathcal{Q})
\]
Since \(\rad(\mathcal{P}_1 + \mathcal{P}_2) \neq \mathbb{M}\) for some \(i > 1\), it follows that \(\dim A/(\mathcal{P}_1 + \mathcal{Q}) \geq 1\). Proposition 6.2 now implies that \(H^0_{2\mathbb{K}}(A/(\mathcal{P}_1 + \mathcal{Q}))\) is \(F\)-torsion. By the inductive hypothesis, \(H^1_{2\mathbb{K}}(A/\mathcal{P}_1)\) and \(H^1_{2\mathbb{K}}(A/\mathcal{Q})\) are \(F\)-torsion as well. The exact sequence (3.2.1) implies that \(H^1_{2\mathbb{K}}(A/(\mathcal{P}_1 \cap \mathcal{Q}))\) is \(F\)-torsion.

The following lemma will be used in the proof of Theorem 1.2.

**Lemma 3.3.** Let \((R, m)\) be a complete local domain with an algebraically closed coefficient field of positive characteristic. Then \(H^1_m(R)_{st}\), the \(F\)-stable part of the module \(H^1_m(R)\), is zero.

**Proof.** If \(\dim R = 0\), then \(H^1_m(R) = 0\), and if \(\dim R \geq 2\), then the assertion follows from Theorem 3.1. The remaining case is \(\dim R = 1\). Theorem 6.3 implies that \(H^1_m(R)_{st}\) has a vector space basis \(\eta_1, \ldots, \eta_i\) such that \(F(\eta_i) = \eta_i\).

Let \(\eta \in H^1_m(R)_{st}\) be an element with \(F(\eta) = \eta\). Considering \(H^1_m(R)\) as the cohomology of a suitable Čech complex, let \(\eta\) be the class of \(y/x\) in \(R_x/R = H^1_m(R)\), where \(y \in R\) and \(x \in m\). Since \(F(\eta) = \eta\), there exists \(r \in R\) such that
\[
\left(\frac{y}{x}\right)^r - \frac{y}{x} - r = 0,
\]
and so \(y/x \in R_x\) is a root of the polynomial \(f(T) = T^r - T - r \in R[T]\). The polynomial \(f(T) \in K[T]\) is separable and \(R\) is strictly Henselian, so \(f(T)\) splits in \(R[T]\). Since \(y/x\) is a root of \(f(T)\) in the fraction field of \(R\), it must then be an element of \(R\), and hence \(\eta = 0\).

**Proof of Theorem 1.2.** We may assume \(R\) to be reduced by Proposition 6.5. First consider the case where the punctured spectrum of \(R\) is connected. If \(\dim R \geq 2\), then \(H^1_m(R)\) is \(F\)-torsion by Theorem 3.2, so \(H^1_m(R)_{st} = 0\). If \(\dim R = 1\), then \(R\) is a domain, and Lemma 3.3 implies that \(H^1_m(R)_{st} = 0\).

We continue by induction on the number of connected components of the punctured spectrum of \(R\). If the punctured spectrum of \(R\) is disconnected, then \(R = \ldots \)
$A/(\mathfrak{A} \cap \mathfrak{B})$ where $(A, \mathfrak{M})$ is a power series ring over the field $K$, and $\mathfrak{A}$ and $\mathfrak{B}$ are radical ideals of $A$ which are not $\mathfrak{M}$-primary, but $\mathfrak{A} + \mathfrak{B}$ is $\mathfrak{M}$-primary. There is a short exact sequence

$$0 \to A/(\mathfrak{A} \cap \mathfrak{B}) \to A/\mathfrak{A} \oplus A/\mathfrak{B} \to A/(\mathfrak{A} + \mathfrak{B}) \to 0.$$ 

Since $H^0_{\mathfrak{M}}(A/\mathfrak{A}) = H^0_{\mathfrak{M}}(A/\mathfrak{B}) = H^1_{\mathfrak{M}}(A/(\mathfrak{A} + \mathfrak{B})) = 0$, the resulting exact sequence of local cohomology gives us

$$0 \to H^0_{\mathfrak{M}}(A/(\mathfrak{A} \cap \mathfrak{B})) \to H^0_{\mathfrak{M}}(A/(\mathfrak{A} + \mathfrak{B})) \to H^0_{\mathfrak{M}}(A/\mathfrak{A}) \oplus H^1_{\mathfrak{M}}(A/\mathfrak{B}) \to 0.$$

By Theorem 6.4, we have a $K$-vector space isomorphism

$$H^1_\text{st}(R) \cong H^0_{\mathfrak{M}}(A/(\mathfrak{A} \cap \mathfrak{B})_{\text{st}}) \cong H^0_{\mathfrak{M}}(A/(\mathfrak{A} + \mathfrak{B})_{\text{st}}) \oplus H^1_{\mathfrak{M}}(A/\mathfrak{A})_{\text{st}} \oplus H^1_{\mathfrak{M}}(A/\mathfrak{B})_{\text{st}}.$$

Since $H^0_{\mathfrak{M}}(A/(\mathfrak{A} + \mathfrak{B}))_{\text{st}} = K$ by Proposition 6.2, the inductive hypothesis completes the proof.

We next record the graded versions of the results proved in this section:

**Theorem 3.4.** Let $R$ be an $\mathbb{N}$-graded ring of positive dimension, which is finitely generated over a field $R_0 = K$ of characteristic $p > 0$.

1. If $R$ is a domain with $\dim R \geq 2$, and $K$ is separably closed, then $H^1_m(R)$ is $F$-torsion.
2. The module $H^1_m(R)$ is $F$-torsion if and only if $\dim R \geq 2$ and $\text{Proj} R$ is geometrically connected.
3. Let $K$ be a perfect field, and let $\overline{K}$ denote its algebraic closure. Then the number of connected components of $\text{Proj}(R \otimes_K \overline{K})$ is

$$1 + \dim_K H^1_m(R)_{\text{st}} = 1 + \dim_K ([H^1_m(R)]_0)_{\text{st}}.$$

**Proof.** (1) Note that $H^1_m(R)$ is a $\mathbb{Z}$-graded $R$-module, and that

$$F: [H^1_m(R)]_n \to [H^1_m(R)]_{n+p} \quad \text{for all } n \in \mathbb{Z}.$$

The module $H^1_m(R)$ has finite length, so all elements of $H^1_m(R)$ of positive or negative degree are $F$-torsion; it remains to show that elements $\eta \in [H^1_m(R)]_0$ are $F$-torsion as well. Let $\eta$ be an element of $[H^1_m(R)]_0$ which is not $F$-torsion. As in the proof of Theorem 3.1, after a change of notation we may assume that

$$F^e(\eta) + r_1 F^{e-1}(\eta) + \cdots + r_e \eta = 0$$

where all $r_i$ are in $[R]_0 = K$, and $r_e$ is nonzero. Let $\eta = [(y_1/x_1, \ldots, y_d/x_d)]$ where $H^1_m(R)$ is regarded as the cohomology of a homogeneous Čech complex. Then there exists $r_{e+1} \in K$ such that $y_i/x_i \in R_{x_i}$ is a root of the polynomial

$$f(T) = T^p^e + r_1 T^p^{e-1} + \cdots + r_e T + r_{e+1} \in K[T].$$

But $f(T)$ is a separable polynomial, so it splits in $K[T]$. The element $y_i/x_i = y_j/x_j$ is a root of $f(T)$ in the fraction field of $R$, so it must be one of the roots of $f(T)$ in $K$. It follows that $\eta = 0$, which completes the proof of (1).

The proof of (2) is now similar to that of Theorem 3.2 and is left to the reader. For (3), note that $F^e(H^1_m(R))$ is a $K$-vector space since $K$ is perfect, and that

$$\dim_K H^1_m(R)_{\text{st}} = \dim_{\overline{K}} H^1_m(R \otimes_K \overline{K})_{\text{st}}.$$

Thus we may assume $K = \overline{K}$, and the proof is similar to that of Theorem 1.2. \hfill \square
Remark 3.5. Theorem 3.4(3) generalizes, in the case of positive characteristic, the well-known fact that the number of connected components of \( X = \text{Proj} \, R \)
\[
\dim_k H^0(X, \mathcal{O}_X) = 1 + \dim_k (H^1_m(R))_0,
\]
where \( R \) is an \( \mathbb{N} \)-graded reduced ring of positive dimension, which is finitely generated over an algebraically closed field \( R_0 = K \). The point is that in this case the Frobenius is bijective on \( (H^1_m(R))_0 \). To see this, let
\[
\eta = \left[ \left( \frac{y_1}{x_1}, \ldots, \frac{y_d}{x_d} \right) \right] \in (H^1_m(R))_0
\]
be an element with \( F(\eta) = 0 \), where \( H^1_m(R) \) is computed as the cohomology of a suitable Čech complex. Then there exists a homogeneous element \( r \in R \) with \((y_i/x_i)^p = r \) in \( R_{x_i} \) for all \( 1 \leq i \leq d \). Such an element \( r \) must have degree zero, and hence must be an element of \( K \). But then \( r^{1/p} \in K \), and, since \( R \) is reduced, \( y_i/x_i = r^{1/p} \) for all \( i \). It follows that
\[
\eta = [(r^{1/p}, \ldots, r^{1/p})] = 0.
\]
To complete the argument, note that \( (H^1_m(R))_0 \) is a finite dimensional \( K \)-vector space, and that if \( \eta_1, \ldots, \eta_n \in (H^1_m(R))_0 \) are linearly independent, then so are \( F(\eta_1), \ldots, F(\eta_n) \). It follows that \( F: (H^1_m(R))_0 \to (H^1_m(R))_0 \) is surjective.

4. \( F \)-purity

A ring homomorphism \( \varphi: R \to S \) is pure if \( \varphi \otimes 1: R \otimes_R M \to S \otimes_R M \) is injective for every \( R \)-module \( M \). If \( R \) is a ring containing a field of characteristic \( p > 0 \), then \( R \) is \( F \)-pure if the Frobenius homomorphism \( F: R \to R \) is pure. The notion was introduced by Hochster and Roberts in the course of their study of rings of invariants in \([HR1, HR2]\).

Examples of \( F \)-pure rings include regular rings of positive characteristic and their pure subrings. If \( \mathfrak{a} \) is generated by square-free monomials in the variables \( x_1, \ldots, x_n \) and \( K \) is a field of positive characteristic, then \( K[x_1, \ldots, x_n]/\mathfrak{a} \) is \( F \)-pure.

Goto and Watanabe \([GW]\) classified one-dimensional \( F \)-pure rings: let \((R, \mathfrak{m}) \) be a local ring of positive characteristic such that \( R/\mathfrak{m} = K \) is algebraically closed, \( F: R \to R \) is finite, and \( \dim R = 1 \). Then \( R \) is \( F \)-pure if and only if
\[
\hat{R} \cong K[[x_1, \ldots, x_n]]/(x_ix_j \mid i < j).
\]

Two-dimensional \( F \)-pure rings have attracted a lot of attention: Watanabe \([Wat1]\) proved that \( F \)-pure normal Gorenstein local rings of dimension two are either rational double points, simple elliptic singularities, or cusp singularities. He also classified two-dimensional normal \( \mathbb{N} \)-graded rings \( R \) over an algebraically closed field \( R_0 \), in terms of \( \mathbb{Q} \)-divisors on the curve \( \text{Proj} \, R \), \([Wat2]\). In \([MS]\) Mehta and Srinivas obtained a classification of two-dimensional \( F \)-pure normal singularities in terms of the resolution of the singularity. Hara completed the classification of two-dimensional normal \( F \)-pure singularities in terms of the dual graph of the minimal resolution of the singularity, \([HaN]\).

The results of Section 3 imply that over separably closed fields, \( F \)-pure domains of dimension two are Cohen-Macaulay. The point is that if \( R \) is an \( F \)-pure ring, then the Frobenius action \( F: H^1_m(R) \to H^1_m(R) \) is an injective map.
Corollary 4.1. Let $R$ be a local ring with $\dim R \geq 2$, which contains a field of positive characteristic. If $R$ is $F$-pure and the punctured spectrum of $R$ is formally geometrically connected, then depth $R \geq 2$.

In particular, if $R$ is a complete local $F$-pure domain of dimension two, with a separably closed coefficient field, then $R$ is Cohen-Macaulay.

Proof. An $F$-pure ring is reduced, so $H_m^0(R) = 0$. By Theorem 3.1, $H^1_m(R)$ is $F$-torsion. Since $R$ is $F$-pure, it follows that $H^1_m(R) = 0$. \qed

In the graded case, we similarly have:

Corollary 4.2. Let $R$ be an $\mathbb{N}$-graded ring with $\dim R \geq 2$, which is finitely generated over a field $R_0$ of positive characteristic. If $R$ is $F$-pure and Proj $R$ is geometrically connected, then depth $R \geq 2$.

The ring $R$ below is a graded $F$-pure domain of dimension two, and depth one. The issue is that Proj $R$ is connected though not geometrically connected.

Example 4.3. Let $K$ be a field of characteristic $p > 2$, and $a \in K$ an element such that $\sqrt{a} \notin K$. Let $R = K[x, y, x\sqrt{a}, y\sqrt{a}]$. The domain $R$ has a presentation

$$R = K[x, y, u, v]/(u^2 - ax^2, v^2 - ay^2, uv - axy, vx - uy),$$

and if $K^{sep}$ denotes the separable closure of $K$, then

$$R \otimes_K K^{sep} \cong K^{sep}[x, y, u, v]/(u - x\sqrt{a}, v - y\sqrt{a})(u + x\sqrt{a}, v + y\sqrt{a}).$$

Using a change of variables, $R \otimes_K K^{sep} \cong K^{sep}[x', y', u', v']/(x', y')(u', v')$. Since $(x', y')(u', v')$ is a square-free monomial ideal, $R \otimes_K K^{sep}$ is $F$-pure and it follows that $R$ is $F$-pure. However, $R$ is not Cohen-Macaulay since $x, y$ is a homogeneous system of parameters with a non-trivial relation

$$(x\sqrt{a})y = (y\sqrt{a})x.$$

Using the Čech complex on $x, y$ to compute $H^1_m(R)$, we see that it is a 1-dimensional $K$-vector space generated by the element

$$\eta = \left[\left(\frac{x\sqrt{a}}{x}, \frac{y\sqrt{a}}{y}\right)\right] \in H^1_m(R)$$

corresponding to the relation above. Given $e \in \mathbb{N}$, let $p^e = 2k + 1$. Then

$$F^e(\eta) = a^k \eta,$$

which is a nonzero element of $H^1_m(R)$. Consequently $H^1_m(R)$ is not $F$-torsion, corresponding to the fact that Proj $R$ is not geometrically connected.

The corollaries obtained in this section imply that over a separably closed field, a graded or complete local $F$-pure domain of dimension two is Cohen-Macaulay. We record an example which shows that this is not true for rings of higher dimension.

Example 4.4. Let $K$ be a field of characteristic $p > 0$, and take

$$A = K[x_1, \ldots, x_d]/(x_1^d + \cdots + x_d^d)$$

where $d \geq 3$. Let $R$ be the Segre product of $A$ and the polynomial ring $B = K[s, t]$. Then $\dim R = d$, and the Künneth formula for local cohomology implies that

$$H^{d-1}_m(R) \cong (H^{d-1}_m(A))_0 \otimes_K [B]_0 \cong K,$$

so $R$ is not Cohen-Macaulay. If $p \equiv 1 \text{ mod } d$, then $A$ is $F$-pure by [HR2, Proposition 5.21]; hence $A \otimes_K B$ and its direct summand $R$ are $F$-pure as well.
5. Algorithmic aspects

Let $R$ be an $\mathbb{N}$-graded ring, which is finitely generated over a finite field $R_0 = K$. We wish to determine the number of geometrically connected components of the scheme $\text{Proj} R$, i.e., the number of connected components of $\text{Proj}(R \otimes_K K^{\text{sep}})$, or, equivalently, of $\text{Proj}(R \otimes_K K^{\text{sep}})$. While primary decomposition algorithms such as those of [EHV], [GTZ], or [SY], may be used to determine the connected components of $\text{Proj} R$, there is computationally no hope of "determining" the connected components over the algebraic closure, $K$. However, simply finding their number is much easier: by Theorem 3.4, this is $1 + \dim_K([H^1_m(R)]_0)_{\text{st}}$. Computing this number involves three steps.

1. Finding a good presentation of $[H^1_m(R)]_0$;
2. Determining the Frobenius action on $[H^1_m(R)]_0$ in terms of this presentation;
3. Computing the dimension of the $F$-stable part, $([H^1_m(R)]_0)_{\text{st}}$.

If $R = A/\mathfrak{a}$ for a polynomial ring $A$, we first replace $\mathfrak{a}$ by an ideal that has the same radical as $\mathfrak{a}$, but does not have the homogeneous maximal ideal $\mathfrak{m}$ as an associated prime. This can be done by saturating $\mathfrak{a}$ with respect to $\mathfrak{m}$; if desired, one may simply compute the radical of $\mathfrak{a}$, but this is often computationally expensive. Now, since $\mathfrak{m}$ is not associated to $\mathfrak{a}$, one can find a homogeneous system of parameters $x_1, \ldots, x_d$ for $R$ such that each $x_i$ is a nonzerodivisor on $R$.

The length $\ell$ of $[H^1_m(R)]_0$ may be computed by computing the length of its graded dual $[\text{Ext}^1_A(R, A(-n))]_0$, where $\dim A = n$. Of course, if this length is zero, then $X_K$ is connected. Consider the Koszul cohomology modules

$$H^1(x^t_1, \ldots, x^t_d; R) = \frac{\{(a_1, \ldots, a_d) \in R^d \mid a_i x^t_i = a_j x^t_j \text{ for all } i < j\}}{\{rx^t_1, \ldots, rx^t_d \mid r \in R\}}.$$  

These modules have an $\mathbb{N}$-grading, where for homogeneous elements $a_i \in R$, we define the degree of $[(a_1, \ldots, a_d)] \in H^1(x^t_1, \ldots, x^t_d; R)$ as

$$\deg[(a_1, \ldots, a_d)] = \deg a_i - \deg x^t_i,$$

which is independent of $i$. This ensures that for each $t$, the map

$$H^1(x^t_1, \ldots, x^t_d; R) \to H^1(x^{t+1}_1, \ldots, x^{t+1}_d; R)$$

preserves degrees. The module $H^1_m(R)$ is the direct limit of these Koszul cohomology modules, and the assumption that the $x_i$ are nonzerodivisors ensures that the maps in the direct limit system are injective. The modules $H^1(x^t_1, \ldots, x^t_d; R)$ may be computed for increasing values of $t$, until we arrive at an integer $N$ such that

$$\ell([H^1(x^N_1, \ldots, x^N_d; R)]_0) = \ell.$$

This gives us a presentation for $[H^1_m(R)]_0 = [H^1(x^N_1, \ldots, x^N_d; R)]_0$, in terms of which we now analyze the Frobenius map. Replacing the $x_i$ by their powers if needed, assume that $N = 1$. Let

$$\alpha = [(a_1, \ldots, a_d)] \in [H^1(x_1, \ldots, x_d; R)]_0,$$

in which case, $F(\alpha) = [(a_1^p, \ldots, a_d^p)] \in [H^1(x^p_1, \ldots, x^p_d; R)]_0$. Since the map

$$[H^1(x_1, \ldots, x_d; R)]_0 \to [H^1(x^p_1, \ldots, x^p_d; R)]_0$$
coming from the direct limit system is bijective, it follows that \( a_i^p \in x_i^{p-1} R \) for each \( 1 \leq i \leq d \). Setting \( b_i = a_i^p / x_i^{p-1} \), we arrive at
\[
F(\alpha) = [(b_1, \ldots, b_d)] \in [H^1(x_1, \ldots, x_d; R)]_0.
\]
Using this description of Frobenius action on the finite dimensional \( K \)-vector space \([H^1_m(R)]_0 = [H^1(x_1, \ldots, x_d; R)]_0\), it is now straightforward to compute the ranks of the vector spaces
\[
[H^1_m(R)]_0 \supseteq F([H^1_m(R)]_0) \supseteq F^2([H^1_m(R)]_0) \supseteq \ldots,
\]
and hence of the \( F \)-stable part, \( ([H^1_m(R)]_0)_{st} \).

6. Appendix: \( F \)-torsion modules and \( F \)-stable vector spaces

Let \( R \) be a commutative ring containing a field \( K \) of characteristic \( p > 0 \). A Frobenius action on an \( R \)-module \( M \) is an additive map \( F: M \rightarrow M \) such that \( F(rm) = r^p F(m) \) for all \( r \in R \) and \( m \in M \). In this case, \( F \) is a submodule of \( M \), and we have an ascending sequence of submodules of \( M \),
\[
\ker F \subseteq \ker F^2 \subseteq \ker F^3 \subseteq \ldots.
\]
The union of these is the \( F \)-nilpotent submodule of \( M \), denoted \( M_{nil} = \bigcup_{e \in \mathbb{N}} \ker F^e \). We say \( M \) is \( F \)-torsion if \( M_{nil} = M \).

**Proposition 6.1.** Let \((R, \mathfrak{m})\) be a local ring containing a field of positive characteristic, and let \( M \) be an Artinian \( R \)-module with a Frobenius action. Then there exists \( e \in \mathbb{N} \) such that \( F^e(M_{nil}) = 0 \).

Hence an Artinian module \( M \) is \( F \)-torsion if and only if \( F^e(M_{nil}) = 0 \) for some \( e \).

**Proof.** This is proved in [HS, Proposition 1.11] under the hypothesis that \( R \) is a complete local ring with a perfect coefficient field. The general case may be concluded from this, but a more elegant approach is via Lyubeznik’s theory of \( F \)-modules; see [Ly1, Proposition 4.4]. \( \square \)

If \( R \) is a ring containing a perfect field \( K \) of positive characteristic and \( M \) is an \( R \)-module with a Frobenius action, then \( F(M) \) is a \( K \)-vector space, and we have a descending sequence of \( K \)-vector spaces
\[
F(M) \supseteq F^2(M) \supseteq F^3(M) \supseteq \ldots.
\]
The \( F \)-stable part of \( M \) is the vector space \( M_{st} = \bigcap_{e \in \mathbb{N}} F^e(M) \).

**Proposition 6.2.** Let \((R, \mathfrak{m}, K)\) be a local ring of dimension \( d \) which contains a field of positive characteristic.

1. \( H^0_m(R) \) is \( F \)-torsion if and only if \( d > 0 \).
2. \( H^0_m(R) \) is not \( F \)-torsion.
3. If \( d = 0 \) and \( K \) is perfect, then \( H^0_m(R)_{st} = R_{st} = K \).

**Proof.** (1) If \( d = 0 \), then \( H^0_m(R) = R \), which is not \( F \)-torsion. If \( d > 0 \), then \( H^0_m(R) \) is contained in \( \mathfrak{m} \). Since every element of \( H^0_m(R) \) is killed by a power of \( \mathfrak{m} \), it follows that each element is nilpotent. (See also [Ly2, Corollary 4.6(a)].)

(2) View \( H^d_m(R) \) as the cohomology of a \( \check{C}ech \) complex on a system of parameters \( x_1, \ldots, x_d \) for \( R \), and let \( \eta = [1 + (x_1, \ldots, x_d)] \in H^d_m(R) \). For all \( e_0 \in \mathbb{N} \), the collection of elements \( F^e(\eta) \) with \( e > e_0 \) generates \( H^d_m(R) \) as an \( R \)-module. Hence \( F^{e_0}(\eta) \) cannot be zero by Grothendieck’s nonvanishing theorem.
Theorem 6.3. Let \((R, m)\) be a local ring with a perfect coefficient field \(K\) of positive characteristic. Let \(M\) be an Artinian \(R\)-module with a Frobenius action. Then \(M_{st}\) is a finite dimensional \(K\)-vector space, and \(F: M_{st} \to M_{st}\) is an automorphism of the Abelian group \(M_{st}\).

If \(K\) is algebraically closed, then there exists a \(K\)-basis \(e_1, \ldots, e_n\) for \(M_{st}\) such that \(F(e_i) = e_i\) for all \(1 \leq i \leq n\).

Proof. For the finiteness assertion, see [HS, Theorem 1.12] or [Ly1, Proposition 4.9]. It is easily seen that \(F: M_{st} \to M_{st}\) is an automorphism whenever \(M_{st}\) is finite dimensional. The existence of the special basis when \(K\) is algebraically closed follows from [Di, Proposition 5, page 233].

Theorem 6.4 ([HS, Theorem 1.13]). Let \((R, m)\) be a complete local ring with an algebraically closed coefficient field of positive characteristic. Let \(L, M, N\) be \(R\)-modules with Frobenius actions such that we have a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & L \\
& \downarrow F & \downarrow F & \downarrow F \\
0 & \longrightarrow & L \\
\end{array}
\]

\[
\begin{array}{ccc}
& & M \\
& \downarrow F & \downarrow F & \downarrow F \\
& & N \\
\end{array}
\]

with exact rows. If \(L\) is Noetherian and \(N\) is Artinian, then the \(F\)-stable parts form a short exact sequence

\[0 \to L_{st} \to M_{st} \to N_{st} \to 0.\]

Proposition 6.5. Let \((R, m, K)\) be a complete local ring with an algebraically closed coefficient field of positive characteristic. Let \(n\) denote the nilradical of \(R\). Then for all \(i \geq 0\), the natural map \(H^i_m(R) \to H^i_m(R/n)\), when restricted to \(F\)-stable subspaces, gives an isomorphism

\[H^i_m(R)_{st} \cong H^i_m(R/n)_{st}.\]

Proof. Let \(k\) be an integer such that \(n^{\geq k} = 0\). The short exact sequence

\[0 \to n \to R \to R/n \to 0\]

induces a long exact sequence of local cohomology modules

\[\cdots \to H^i_m(n) \xrightarrow{\alpha} H^i_m(R) \xrightarrow{\beta} H^i_m(R/n) \xrightarrow{\gamma} H^{i+1}_m(n) \to \cdots.\]

Consider an element \(\mu \in \ker(\beta) \cap H^i_m(R)_{st}\). Then \(\mu \in \image(\alpha)\), so \(F^k(\mu) = 0\). The Frobenius action on \(H^i_m(R)_{st}\) is an automorphism, so \(\mu = 0\), and hence the map \(H^i_m(R)_{st} \to H^i_m(R/n)_{st}\) is injective.

To complete the proof it suffices, by Theorem 6.3, to consider an element \(\eta \in H^i_m(R/n)_{st}\) with \(F(\eta) = \eta\), and prove that it lies in the image of \(H^i_m(R)_{st}\). Now \(\gamma(\eta) \in H^{i+1}_m(n)\) so \(F^k(\gamma(\eta)) = 0\), and therefore \(F^k(\eta) = \eta \in \ker(\gamma)\).

Let \(\eta = \beta(\mu)\) for some element \(\mu \in H^i_m(R)\). Then \(\beta(F(\mu) - \mu) = 0\), which implies that \(F(\mu) - \mu \in \image(\alpha)\). Consequently \(F^k(F(\mu) - \mu) = 0\), which shows that \(F^{k+1}(\mu) = F^k(\mu)\), and hence that \(F^k(\mu) \in H^i_m(R)_{st}\). Since \(\beta(F^k(\mu)) = F^k(\beta(\mu)) = F^k(\eta) = \eta\), we are done.
A CONNECTEDNESS RESULT IN POSITIVE CHARACTERISTIC

References


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