A SIMPLE FORMULA FOR AN ANALOGUE OF CONDITIONAL WIENER INTEGRALS AND ITS APPLICATIONS

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Abstract. Let \( C[0, T] \) denote the space of real-valued continuous functions on the interval \([0, T]\) and for a partition \( \tau : 0 = t_0 < t_1 < \cdots < t_n = T \) of \([0, T]\), let \( X_\tau : C[0, T] \to \mathbb{R}^{n+1} \) be given by \( X_\tau(x) = (x(t_0), x(t_1), \ldots, x(t_n)) \).

In this paper, with the conditioning function \( X_\tau \), we derive a simple formula for conditional expectations of functions defined on \( C[0, T] \) which is a probability space and a generalization of Wiener space. As applications of the formula, we evaluate the conditional expectation of functions of the form

\[
F_m(x) = \int_0^T (x(t))^m \, dt, \quad m \in \mathbb{N},
\]

for \( x \in C[0, T] \) and derive a translation theorem for the conditional expectation of integrable functions defined on the space \( C[0, T] \).

1. Introduction and preliminaries

Let \( C_0[0, T] \) be the space of continuous real-valued functions \( x \) on \([0, T]\) with \( x(0) = 0 \). It is well-known that the space \( C_0[0, T] \) equipped with Wiener measure is a probability space. On the space, Yeh introduced an inversion formula that a conditional expectation can be found by Fourier-transform ([9]). Also, in [10, 11], he obtained very useful results including the Kac-Feynman integral equation and the conditional Cameron-Martin translation theorem using the inversion formula. But Yeh’s inversion formula is very complicated in its applications when the conditioning function is vector-valued. In [6], Park and Skoug derived a simple formula for conditional Wiener integrals on \( C_0[0, T] \) with the conditioning function \( X : C_0[0, T] \to \mathbb{R}^n \) given by

\[
X(x) = (x(t_1), \ldots, x(t_n))
\]

where \( \tau : 0 = t_0 < t_1 < \cdots < t_n = T \) is a partition of the interval \([0, T]\). In their simple formula they expressed the conditional Wiener integral directly in terms of the ordinary Wiener integral. Using the formula, they generalized the Kac-Feynman formula and obtained a Cameron-Martin type translation theorem for conditional Wiener integrals.
On the other hand, let $C[0, T]$ denote the space of continuous real-valued functions on the interval $[0, T]$. Im and Ryu introduced a probability measure $w_\varphi$ on $(C[0, T], B(C[0, T]))$ where $B(C[0, T])$ denotes a Borel $\sigma$-algebra on $C[0, T]$ and $\varphi$ is a probability measure on $(\mathbb{R}, B(\mathbb{R}))$ ([4, 7]). This measure space is a generalization of Wiener space. In [4], they introduced a translation theorem of the $w_\varphi$-integral, which corresponds to the Cameron-Martin translation theorem on Wiener space ([2]). Also, Im and Ryu evaluated the conditional $w_\varphi$-integral of various functions on $C[0, T]$ with the conditioning function $X_\tau : C[0, T] \to \mathbb{R}^{n+1}$ given by

$$X_\tau(x) = (x(t_0), x(t_1), \ldots, x(t_n))$$

and derived a translation theorem of the conditional $w_\varphi$-integral when the conditioning function is $X_\tau(x) = x(t_n)$. But their processes were complicated in its proof.

In this paper, on the space $(C[0, T], B(C[0, T]), w_\varphi)$, we derive a simple formula for the conditional $w_\varphi$-integral of functions on $C[0, T]$ with the vector-valued conditioning function $X_\tau$ given by (1). This formula expresses the conditional $w_\varphi$-integral directly in terms of the non-conditional $w_\varphi$-integral. As applications of the formula, we evaluate the conditional $w_\varphi$-integral of functions of the form

$$F_m(x) = \int_0^T (x(t))^m dt$$

for any positive integer $m$ and using the translation theorem of $w_\varphi$-integral in [4], we derive a translation theorem for the conditional $w_\varphi$-integral of functions on $C[0, T]$ with the conditioning function $X_\tau$.

Throughout this paper, let $\mathbb{C}$ and $\mathbb{C}_+$ denote the set of complex numbers and that of complex numbers with positive real parts, respectively.

Now, we begin by introducing the probability space $(C[0, T], B(C[0, T]), w_\varphi)$.

For a positive real $T$, let $C = C[0, T]$ be the space of all real-valued continuous functions on the closed interval $[0, T]$ with the supremum norm. For $\tilde{t} = (t_0, t_1, \ldots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \leq T$, let $J_\tilde{t} : C[0, T] \to \mathbb{R}^{n+1}$ be the function given by

$$J_\tilde{t}(x) = (x(t_0), x(t_1), \ldots, x(t_n)).$$

For $B_j$ ($j = 0, 1, \ldots, n$) in $B(\mathbb{R})$, the subset $J_\tilde{t}^{-1}\left(\prod_{j=0}^n B_j\right)$ of $C[0, T]$ is called an interval and let $\mathcal{I}$ be the set of all such intervals. For a probability measure $\varphi$ on $(\mathbb{R}, B(\mathbb{R}))$, we let

$$m_\varphi\left(J_\tilde{t}^{-1}\left(\prod_{j=0}^n B_j\right)\right) = \int_{\mathcal{I}_0} \int_{\prod_{j=1}^n B_j} W_{n+1}(\tilde{t}; u_0, u_1, \ldots, u_n) d(u_1, \ldots, u_n) d\varphi(u_0),$$

where

$$(2) \quad W_{n+1}(\tilde{t}; u_0, u_1, \ldots, u_n) = \left[\prod_{j=1}^n \frac{1}{2\pi(t_j - t_{j-1})}\right]^\frac{1}{2} \exp\left\{ -\frac{1}{2} \sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}} \right\}.$$

$B(C[0, T])$, the Borel $\sigma$-algebra of $C[0, T]$, coincides with the smallest $\sigma$-algebra generated by $\mathcal{I}$ and there exists a unique probability measure $w_\varphi$ on $(C[0, T], B(C[0, T]))$. 

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such that \( w_{\varphi}(I) = m_{\varphi}(I) \) for all \( I \) in \( \mathcal{I} \) (\cite{4, 7, 8}). This measure \( w_{\varphi} \) is called an analogue of Wiener measure associated with the probability measure \( \varphi \).

By the change of variable theorem, we can easily prove the following theorem.

**Theorem 1.1** (\cite{3} Lemma 2.1). If \( f : \mathbb{R}^{n+1} \to \mathbb{C} \) is a Borel measurable function, then we have

\[
\int_{C} f(x(t_0), x(t_1), \cdots, x(t_n)) dw_{\varphi}(x)
\]

\[
\overset{=}{=} \int_{\mathbb{R}^{n+1}} f(u_0, u_1, \cdots, u_n) W_{n+1}(\vec{t}; u_0, u_1, \cdots, u_n) d\varphi(u_0)
\]

where \( \overset{=}{=} \) means that if either side exists, then both sides exist and they are equal.

Let \( C_0 = C_0[0, T] \) be a Wiener space with Wiener measure \( m_w \). Then Theorem 1.1 is reduced to the well-known Wiener integration theorem if \( \varphi \) is a Dirac measure concentrated at 0.

**Theorem 1.2** (Wiener integration theorem). If \( f : \mathbb{R}^{n} \to \mathbb{C} \) is a Borel measurable function, then we have

\[
\int_{C_0} f(x(t_1), \cdots, x(t_n)) dm_w(x)
\]

\[
\overset{=}{=} \int_{\mathbb{R}^{n}} f(u_1, \cdots, u_n) W_{n+1}(\vec{t}; 0, u_1, \cdots, u_n) d\varphi(u_0)
\]

where \( \overset{=}{=} \) means that if either side exists, then both sides exist and they are equal.

Let \( \{e_k : k = 1, 2, \cdots\} \) be a complete orthonormal subset of \( L_2[0, T] \) such that each \( e_k \) is of bounded variation. For \( f \) in \( L_2[0, T] \) and \( x \) in \( C[0, T] \), we let

\[
(f, x) = \lim_{n \to \infty} \int_{0}^{T} \left[ \sum_{k=1}^{n} e_k(t) \int_{0}^{T} f(s)e_k(s) ds \right] dx(t)
\]

if the limit exists. \( (f, x) \) is called the Paley-Wiener-Zygmund integral of \( f \) according to \( x \).

Let \( F : C[0, T] \to \mathbb{C} \) be integrable and let \( X \) be a random vector on \( C[0, T] \). Then, we have the conditional expectation \( E[F|X] \) of \( F \) given \( X \) from a well-known probability theory (\cite{5}). Further, there exists a \( P_X \)-integrable complex-valued function \( \psi \) on the value space of \( X \) such that \( E[F|X](x) = (\psi \circ X)(x) \) for \( w_{\varphi}\)-a.e. \( x \in C[0, T] \), where \( P_X \) is the probability distribution of \( X \). The function \( \psi \) is called the conditional \( w_{\varphi} \)-integral of \( F \) given \( X \) and it is also denoted by \( E[F|X] \).

### 2. A Simple Formula for the Conditional \( w_{\varphi} \)-Integral

In this section, we derive a simple formula for the conditional \( w_{\varphi} \)-integral of the functions on \( C[0, T] \) with the conditioning function \( X_T \) given by (1).

First, we define a stochastic process \( X_t(x) : C[0, T] \to \mathbb{R} \) by

\[
X_t(x) = x(t)
\]

for \( t \in [0, T] \). We have the following lemma which is useful to prove several results.
Lemma 2.1. Let $W_t$ be the standard Wiener process given by $W_t(x) = x(t)$ on $C_0[0,T]$. Then $X_t - X_s$ and $W_t - W_s$ have the same distribution if $0 \leq s \leq t \leq T$, that is, for any Borel subset $B$ of $\mathbb{R}$

$$w_\varphi(X_t - X_s \in B) = m_w(W_t - W_s \in B).$$

Moreover, if $0 \leq s < t \leq T$, then they are normally distributed with mean 0 and variance $t - s$. In particular, if $0 \leq s_1 < s_2 < s_3 < s_4 \leq T$, then $X_{s_4} - X_{s_3}$ and $X_{s_2} - X_{s_1}$ are stochastically independent on $C[0,T]$.

Proof. Suppose that $s = t$. Then for a Borel subset $B$ of $\mathbb{R}$

$$w_\varphi(X_t - X_s \in B) = m_w(W_t - W_s \in B),$$

where $\delta_0$ is Dirac measure concentrated at 0.

Now suppose that $0 < s < t$. By Theorem \[\square\] we have

$$w_\varphi(X_t - X_s \in B) = \int_C \chi_B(x(t) - x(s))dw_\varphi(x)$$

$$= \left[ \frac{1}{(2\pi)^2s(t-s)} \right]^{\frac{1}{2}} \int_\mathbb{R} \int_\mathbb{R} \chi_B(u_2 - u_1) \exp \left\{ -\frac{(u_1 - u_0)^2}{2s} \right\}$$

$$- \frac{(u_2 - u_1)^2}{2(t-s)} d(u_1, u_2) d\varphi(u_0)$$

where $\chi_B$ denotes the indicator function concentrated on $B$. Let $v_2 = u_2 - u_1$ and $v_1 = u_1 - u_0$. By the change of variable and Fubini’s theorems, we have

$$w_\varphi(X_t - X_s \in B) = \left[ \frac{1}{(2\pi)^2s(t-s)} \right]^{\frac{1}{2}} \int_\mathbb{R} \int_\mathbb{R} \chi_B(v_2) \exp \left\{ -\frac{v_1^2}{2s} - \frac{v_2^2}{2(t-s)} \right\} dv_1 dv_2$$

$$= \left[ \frac{1}{2\pi(t-s)} \right]^{\frac{1}{2}} \int_B \exp \left\{ -\frac{v_2^2}{2(t-s)} \right\} dv_2$$

$$= m_w(W_t - W_s \in B)$$

since $\varphi$ is a probability measure and $W_t - W_s$ is normally distributed with mean 0, variance $t - s$. If $0 = s < t$, the proof is similar to the case $0 < s < t$.

Since $W_{s_4} - W_{s_3}$ and $W_{s_2} - W_{s_1}$ are stochastically independent on $C_0[0,T]$, $X_{s_4} - X_{s_3}$ and $X_{s_2} - X_{s_1}$ are stochastically independent on $C[0,T]$. Now the proof is completed. \[\square\]

Remark 2.2. It is well-known that the Wiener process $W_t$ is the standard Brownian motion process. On the other hand, $X_t$ need not be a Brownian motion process since $X_0(x) = x(0)$ can take arbitrary values. Note that $W_t$ is normally distributed with mean 0 and variance $t$, but $X_t$ need not be if $\varphi$ is not Dirac measure $\delta_0$. See (2.6) of \[\square\] p. 803 and (6) of \[\square\] p. 4930.
For a given partition $\tau : 0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$ and for $x$ in $C[0, T]$, define the polygonal function $[x]$ on $[0, T]$ by

$$[x](t) = x(t_{j-1}) + \frac{t-t_{j-1}}{t_j-t_{j-1}}(x(t_j) - x(t_{j-1})), \quad t_{j-1} \leq t \leq t_j, \quad j = 1, \ldots, n.$$  

Similarly, for $\xi = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$, define the polygonal function $[\xi]$ on $[0, T]$ by

$$[\xi](t) = \xi_{j-1} + \frac{t-t_{j-1}}{t_j-t_{j-1}}(\xi_j - \xi_{j-1}), \quad t_{j-1} \leq t \leq t_j, \quad j = 1, \ldots, n.$$  

Then both $[x]$ and $[\xi]$ are continuous on $[0, T]$, their graphs are line segments on each subinterval $[t_{j-1}, t_j]$, and $[x](t_j) = x(t_j)$ and $[\xi](t_j) = \xi_j$ at each $t_j$.

To derive a simple formula for the conditional $w_\omega$-integral, we begin with letting for $t_{j-1} < t < t_j$

$$\alpha_j = \frac{t_j-t}{t_j-t_{j-1}}, \quad \beta_j = \frac{t-t_{j-1}}{t_j-t_{j-1}}, \quad \Gamma_j(t) = \frac{t_{j-1}}{(t_j-t)(t-t_{j-1})},$$

$$W_j(t, x) = x(t) - [x](t) \quad \text{for} \quad x \in C[0, T]$$

and

$$X_j(t, x) = x(t) - [x](t) \quad \text{for} \quad x \in C[0, T]$$

for each $j = 1, \ldots, n$.

The following lemma is useful to evaluate several conditional $w_\omega$-integrals.

**Lemma 2.3.** Let $t \in (t_{j-1}, t_j)$ for some $j \in \{1, \cdots, n\}$ and $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Then

$$\int_{C_0} f(W_j(t, x))dm_\omega(x) = \left[\Gamma_j(t)\right]^{1/2} \int f(u) \exp\left\{ -\frac{\Gamma_j(t)}{2} u^2 \right\} du$$

where $\Gamma_j(t)$ is given by (2). In particular, $W_j(t, \cdot)$ is normally distributed with mean $0$ and variance $\frac{1}{\Gamma_j(t)}$.

**Proof.** For convenience, let

$$\Gamma = \left[ \frac{1}{(2\pi)^2(t_j-t)(t-t_{j-1})} \right]^{1/2}.$$  

Note that

$$W_j(t, x) = x(t) - x(t_{j-1}) - \frac{t-t_{j-1}}{t_j-t_{j-1}}(x(t_j) - x(t) + x(t) - x(t_{j-1}))$$

$$= \alpha_j(x(t) - x(t_{j-1})) - \beta_j(x(t_j) - x(t))$$

where $\alpha_j$ and $\beta_j$ are given by (2). Now, for $j = 2, \cdots, n$, we have

$$\int_{C_0} f(W_j(t, x))dm_\omega(x) = \int_{\mathbb{R}^3} f(\alpha_j(u_2-u_1) - \beta_j(u_3-u_2)) \times W_4((0, t_{j-1}, t, t_j); 0, u_1, u_2, u_3)d(u_1, u_2, u_3)$$

by Theorem 1.2 where $W_4$ is given by (2). Let

$$v_2 = \alpha_j(u_2-u_1)$$

and

$$v_3 = -\beta_j(u_3-u_2).$$
Then we have by the change of variable theorem
\[
\int_{0}^{\Gamma_{j}(t)} f(W_{j}(t, x)) dm_{W_{j}}(x)\\
= \frac{\Gamma}{\alpha_{j} \beta_{j}} \left( \frac{1}{2\pi t_{j-1}} \right)^{\frac{1}{2}} \int_{\mathbb{R}^{2}} f(v_{2} + v_{3}) \exp \left\{ -\frac{1}{2\alpha_{j}^{2}(t - t_{j-1})} v_{2}^{2} - \frac{1}{2\beta_{j}^{2}(t - t_{j-1})} v_{3}^{2} \right\} d(u_{1}, v_{2}, v_{3})\\
= \frac{\Gamma}{\alpha_{j} \beta_{j}} \int_{\mathbb{R}^{2}} f(v_{2} + v_{3}) \exp \left\{ -\frac{1}{2\alpha_{j}^{2}(t - t_{j-1})} v_{2}^{2} - \frac{1}{2\beta_{j}^{2}(t - t_{j-1})} v_{3}^{2} \right\} d(v_{2}, v_{3}).
\]

Let \( u = v_{2} + v_{3} \). Note that \( \alpha_{j}^{2}(t - t_{j-1}) + \beta_{j}^{2}(t_{j} - t) = \frac{1}{1_{j}(t)} \) and hence
\[
1 - \beta_{j}^{2}\Gamma_{j}(t)(t_{j} - t) = \alpha_{j}^{2}\Gamma_{j}(t)(t - t_{j-1}).
\]

Again, by the change of variable theorem we have
\[
\int_{0}^{\Gamma_{j}(t)} f(W_{j}(t, x)) dm_{W_{j}}(x)\\
= \frac{\Gamma}{\alpha_{j} \beta_{j}} \int_{\mathbb{R}} f(u) \exp \left\{ -\frac{1}{2\alpha_{j}^{2}(t - t_{j-1})} (u - v_{3})^{2} - \frac{1}{2\beta_{j}^{2}(t_{j} - t)} v_{3}^{2} \right\} d(u, v_{3})\\
= \frac{\Gamma}{\alpha_{j} \beta_{j}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(u) \exp \left\{ -\frac{1}{2\alpha_{j}^{2}(t - t_{j-1})} u^{2} + \frac{\beta_{j}^{2}\Gamma_{j}(t)(t_{j} - t)}{2\beta_{j}^{2}(t - t_{j-1})} u^{2} - \frac{1}{2\beta_{j}^{2}(t_{j} - t)} v_{3}^{2} \right\} dv_{3}du\\
\times (t_{j} - t)^{2} u_{2}^{2} \right\} \right\} du_{3}du\\
= \int_{\mathbb{R}} f(u) \exp \left\{ -\frac{\alpha_{j}^{2}\Gamma_{j}(t)(t - t_{j-1})}{2\alpha_{j}^{2}(t - t_{j-1})} u^{2} \right\} du\\
= \frac{\Gamma_{j}(t)}{2\pi} \int_{\mathbb{R}} f(u) \exp \left\{ -\frac{\Gamma_{j}(t)}{2} u^{2} \right\} du
\]
which is the desired result. If \( j = 1 \), then the proof is similar to the case \( j \geq 2 \).

The following theorem gives an interesting observation for the process \( x(t) - [x](t) \) on \( C[0, T] \). In fact, \( x(t) - [x](t) \) is a Brownian bridge motion on each subinterval.

**Theorem 2.4.** For each \( j = 1, \cdots, n \), let \( W_{j} \) and \( X_{j} \) be given by (6) and (4), respectively. Then \( X_{j} \) is a Brownian bridge motion process on \( [t_{j-1}, t_{j}] \). Moreover for \( t \in (t_{j-1}, t_{j}) \), both \( W_{j}(t, \cdot) \) and \( X_{j}(t, \cdot) \) are normally distributed with mean 0 and variance \( \frac{1}{t_{j} - t_{j-1}} \), where \( \Gamma_{j}(t) \) is given by (3).

**Proof.** Note that \( W_{j} \) is a Brownian bridge motion process (Theorem 1). By Lemma 2.1 \( W_{j} \) and \( X_{j} \) have the same distribution so that \( X_{j} \) is a Brownian bridge motion process on the subinterval \( [t_{j-1}, t_{j}] \) since since \( X_{j}(t_{j-1}, x) = x(t_{j-1}) - [x](t_{j-1}) = 0 + x(t_{j}) - [x](t_{j}) = X_{j}(t_{j}, x) \) for \( x \in C[0, T] \).

The second result immediately follows from Lemmas 2.1 and 2.3.
By Theorem 2.4 and Corollary 1, we have the following corollary.

**Corollary 2.5.** The processes \( \{X_j(t, \cdot) : t_{j-1} \leq t \leq t_j\} \), where \( j = 1, \cdots, n \) are stochastically independent.

**Theorem 2.6.** Let \( Y_\tau : C[0, T] \to \mathbb{R}^{n+1} \) be given by
\[
Y_\tau(x) = (x(t_0), x(t_1) - x(t_0), \cdots, x(t_n) - x(t_0)).
\]
Then the process \( \{x(t) - [x](t) : 0 \leq t \leq T\} \) and \( Y_\tau \) are independent.

**Proof.** Define \( Y : [0, T] \times C[0, T] \to \mathbb{R} \) by
\[
Y(s, x) = X_s(x) - X_0(x) = x(s) - x(t_0)
\]
where \( X_s \) and \( X_0 \) are given by (1) with replacing \( t \) by \( s \) and \( x \), respectively.

Fix \( j \in \{1, \cdots, n\} \) and let \( X_j \) be given by (1). By Theorem 2.4 for \( t_{j-1} < t < t_j \), \( X_j(t, \cdot) \) is normally distributed with mean 0 and so is \( Y(s, \cdot) \) for \( s \in (0, t_{j-1}] \cup [t_j, T] \) by Lemma 2.1. Moreover, it is not difficult to show that the distributions of \( Y(0, \cdot) \), \( X_j(t_{j-1}, \cdot) \) and \( X_j(t, \cdot) \) are the Dirac measure \( \delta_0 \) concentrated at 0.

First, we show that \( X_j(t, \cdot) \) and \( Y(s, \cdot) \) are stochastically independent for \( t \in [t_{j-1}, t_j] \) and \( s \in [0, t_{j-1}] \cup [t_j, T] \), and so are \( X_j(t, \cdot) \) and \( X_0 \). To prove the assertions, let \( B_1 \) and \( B_2 \) be Borel subsets of \( \mathbb{R} \).

Suppose that \( t = t_{j-1} \) or \( t = t_j \). Then we have for \( s \in [0, t_{j-1}] \cup [t_j, T] \)
\[
w_\varphi(X_j(t, \cdot) \in B_1, Y(s, \cdot) \in B_2) = w_\varphi((X_j(t, \cdot) - 0) \in B_1, (Y(s, \cdot) - 0) \in B_2) = w_\varphi(X_j(t, \cdot) \in B_1)w_\varphi(Y(s, \cdot) \in B_2)
\]
which shows the independence of \( X_j(t, \cdot) \) and \( Y(s, \cdot) \). If we replace \( Y(s, \cdot) \) by \( X_0 \), then we have the independence of \( X_j(t, \cdot) \) and \( X_0 \).

Similarly, we can prove the independence of \( X_j(t, \cdot) \) and \( Y(0, \cdot) \) for \( t_{j-1} < t < t_j \). Now suppose that \( t_{j-1} < t < t_j \). If \( s \in (0, t_{j-1}] \), then we have
\[
\text{Cov}(X_j(t, \cdot), Y(s, \cdot)) = E[X_j(t, \cdot)Y(s, \cdot)] = E[\alpha_j(X_t - X_{t_{j-1}}) - \beta_j(X_{t_j} - X_t)](X_s - X_0) = 0
\]
by Lemma 2.1 where \( \alpha_j \) and \( \beta_j \) are given by (2). If \( s \in [t_j, T] \), then we also have
\[
\text{Cov}(X_j(t, \cdot), Y(s, \cdot)) = E[X_j(t, \cdot)Y(s, \cdot)] = E[\alpha_j(X_t - X_{t_{j-1}})^2 - \beta_j(X_{t_j} - X_t)^2] = \alpha_j(t - t_{j-1}) - \beta_j(t_j - t) = 0
\]
by Lemma 2.1 which shows the independence of \( X_j(t, \cdot) \) and \( Y(s, \cdot) \). Finally, let
\[
\Delta_t = \left[ \frac{1}{(2\pi)^3(t_j - t)(t - t_{j-1})t_{j-1}} \right]^{\frac{1}{2}},
\]
if \( j \in \{2, \cdots, n\} \). Then we have by Theorem 1.1
\[
w_\varphi(X_j(t, \cdot) \in B_1, X_0 \in B_2) = \int_C \chi_{B_1}(X_j(t, x))\chi_{B_2}(X_0(x))dw_\varphi(x)
\]
\[
= \Delta_t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \chi_{B_1}(\alpha_j(u_2 - u_1) - \beta_j(u_3 - u_2))\chi_{B_2}(u_0)\exp\left\{ -\frac{(u_1 - u_0)^2}{2t_{j-1}} - \frac{(u_2 - u_1)^2}{2(t - t_{j-1})} - \frac{(u_3 - u_2)^2}{2(t_j - t)} \right\}d(u_1, u_2, u_3)du_0.
\]
For $i = 1, 2, 3$, let $v_i = u_i - u_{i-1}$. By the change of variable theorem and Fubini’s theorem, we have

$$w_\phi(X_j(t, \cdot) \in B_1, X_0 \in B_2)$$

$$\Delta_i \int_{\mathbb{R}^3} \chi_B(\alpha_j v_2 - \beta_j v_3) \chi_B(u_0) \exp \left\{ - \frac{v_1^2}{2t_{j-1}} - \frac{v_2^2}{2(t-t_{j-1})} - \frac{v_3^2}{2(t_j-t)} \right\}$$

$$d(v_1, v_2, v_3) d\phi(u_0)$$

$$= \Delta_i \int_{\mathbb{R}^3} \chi_B(\alpha_j v_2 - \beta_j v_3) \exp \left\{ - \frac{v_1^2}{2t_{j-1}} - \frac{v_2^2}{2(t-t_{j-1})} - \frac{v_3^2}{2(t_j-t)} \right\}$$

$$d(v_1, v_2, v_3) d\phi(u_0) \int_{\mathbb{R}^3} \chi_B(u_0) d\phi(u_0)$$

$$= \Delta_i \int_{\mathbb{R}^3} \chi_B(\alpha_j (u_2 - u_1) - \beta_j (u_3 - u_2)) \exp \left\{ - \frac{(u_1 - u_0)^2}{2t_{j-1}} - \frac{(u_2 - u_1)^2}{2(t-t_{j-1})} \right\}$$

$$- \frac{(u_3 - u_2)^2}{2(t_j-t)} d(u_1, u_2, u_3) d\phi(u_0) \int_{\mathbb{R}^3} \chi_B(u_0) d\phi(u_0)$$

$$= w_\phi(X_j(t, \cdot) \in B_1) w_\phi(X_0 \in B_2)$$

which shows the independence of $X_j(t, \cdot)$ and $X_0$. If $j = 1$, then we can show that $X_j(t, \cdot)$ and $X_0$ are independent by a similar method.

Now, for each $j = 1, \cdots, n$, the process $\{X_j(t, \cdot) : t_{j-1} \leq t \leq t_j \}$ is independent of $Y_\tau$, which completes the proof. □

**Remark 2.7.** Since for $x \in C[0, T]$ and $t_{j-1} \leq t \leq t_j$

$$X_j(t, x) = \alpha_j(x(t) - x(t_{j-1})) - \beta_j(x(t_j) - x(t))$$

$$= \alpha_j(x(t) - x(t_0) - (x(t_{j-1}) - x(t_0)))$$

$$- \beta_j(x(t_j) - x(t_0) - (x(t) - x(t_0))),$$

we can also obtain Corollary 2.5 from Theorem 2.6.

**Theorem 2.8.** Let $X_\tau : C[0, T] \to \mathbb{R}^{n+1}$ be given by

$$(5) \quad X_\tau(x) = (x(t_0), x(t_1), \cdots, x(t_n)).$$

Then the process $\{x(t) - [x(t) : 0 \leq t \leq T] \}$ and $X_\tau$ are independent.

**Proof.** Take $j \in \{1, \cdots, n\}$. For $s \in [0, t_{j-1}] \cup [t_j, T]$, we have

$$X_s(x) = x(s) = (x(s) - x(t_0)) + x(t_0).$$

Note that $X_j(t, x) = x(t) - [x(t) : t_{j-1} \leq t \leq t_j]$ and $\{x(s) - x(t_0), x(t_0)\}$ are independent by Theorem 2.6. Define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(u_1, u_2) = u_1 + u_2$. Since $g$ is Borel measurable and $x(s) = g(x(s) - x(t_0), x(t_0))$, $X_j(t, \cdot)$ and $X_s$ are independent. Now the theorem follows. □

Applying the same method used in the proof of Theorem 2 in [6, p. 383] with Problem 4 of [11, p. 216], we have the following theorem from Theorem 2.8.

**Theorem 2.9.** Let $F : C[0, T] \to \mathbb{C}$ be integrable and $X_\tau$ be given by (5) of Theorem 2.8. Then for a Borel subset $B$ of $\mathbb{R}^{n+1}$ we have

$$(6) \quad \int_{X_\tau^{-1}(B)} F(x) dw_\phi(x) = \int_B E[F(x - [x] + [\xi])] dP_{X_\tau}(\xi)$$

where $P_{X_\tau}$ is the probability distribution of $X_\tau$ on $(\mathbb{R}^{n+1}, B(\mathbb{R}^{n+1}))$. 

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Remark 2.10. We emphasize that both \([x](t_0) = x(t_0)\) and \([\xi](t_0) = \xi_0\) need not be 0. Applying the same method used in the proof of Theorem 2 in [6, p. 383], we can prove (6) using Theorem 2.6 directly.

For a function \(F : C[0, T] \to \mathbb{C}\) and \(\lambda > 0\), let \(F^\lambda(x) = F(\lambda^{-\frac{1}{2}}x)\) and \(X^\lambda_t(x) = X_t(\lambda^{-\frac{1}{2}}x)\). Suppose that \(E[F^\lambda]\) exists for each \(\lambda > 0\). By the definition of the conditional \(w_\varphi\)-integral and (6), we have

\[
E[F^\lambda | X^\lambda_X](\xi) = E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\xi])] \quad \text{for } P_{X^\lambda}-\text{a.e. } \tilde{\xi} \in \mathbb{R}^{n+1},
\]

where \(P_{X^\lambda}\) is the probability distribution of \(X^\lambda_X\) on \((\mathbb{R}^{n+1}, \mathcal{B}((\mathbb{R}^{n+1}))\). If \(E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\xi])]\) has the analytic extension \(J^\lambda_x(F)(\tilde{\xi})\) on \(\mathbb{C}_+\) as a function of \(\lambda\), then it is denoted by

\[
E^{anw_\lambda}[F | X^\lambda_X](\tilde{\xi}) = J^\lambda_x(F)(\tilde{\xi})
\]

for \(\tilde{\xi} \in \mathbb{R}^{n+1}\). Moreover, if for a non-zero real \(q\), \(E^{anw_\lambda}[F | X^\lambda_X](\tilde{\xi})\) has a limit as \(\lambda\) approaches to \(-iq\) through \(\mathbb{C}_+\), then it is denoted by

\[
E^{anf_\lambda}[F | X^\lambda_X](\tilde{\xi}) = \lim_{\lambda \to -iq} E^{anw_\lambda}[F | X^\lambda_X](\tilde{\xi}).
\]

3. Evaluations of conditional \(w_\varphi\)-integrals

Throughout this section, let \(X_t\) be given by (5) and \(P_{X^\lambda}\) denote the probability distribution of \(X^\lambda_t\) on \((\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))\). Now, we evaluate conditional \(w_\varphi\)-integrals of functions on \(C[0, T]\) as applications of (6) in Theorem 2.9.

**Theorem 3.1.** Let \(F_m(x) = \int_0^T (x(t))^m dt\) (\(m \in \mathbb{N}\)) for \(x \in C[0, T]\) and suppose that \(\int_0^T |u|^m d\varphi(u) < \infty\). Then \(F_m\) is \(w_\varphi\)-integrable. Moreover, \(E[F_m | X^\lambda_X](\tilde{\xi})\) exists for \(P_{X^\lambda}-\text{a.e. } \tilde{\xi} \in \mathbb{R}^{n+1}\) and it is given by

\[
E[F_m | X^\lambda_X](\tilde{\xi}) = \sum_{j=1}^n \sum_{k=0}^M \frac{m!}{2^k k! (m-2k)!} \int_{\tau_j - \epsilon}^{\tau_j} (|\tilde{\xi}(t)|)^{m-2k} \frac{1}{\Gamma_j(t)} \, dt
\]

where \(\Gamma_j(t)\) is given by (2) and \(M = \frac{m}{2}\) if \(m\) is even and \(M = \frac{m-1}{2}\) if \(m\) is odd.

**Proof.** First, we show that \(F_m\) is \(w_\varphi\)-integrable. Indeed, we have

\[
\int_C |F_m(x)| dw_\varphi(x) \leq \int_0^T \int_C |x(t)|^m dw_\varphi(x) dt
\]

\[
= \int_{(0, T]} \left( \frac{1}{2\pi t} \right)^{\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} |u_1|^m \exp \left\{ -\frac{(u_1 - u_0)^2}{2t} \right\} du_1 d\varphi(u_0) dt
\]

\[
= \int_{(0, T]} \left( \frac{1}{2\pi t} \right)^{\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} |u + u_0|^m \exp \left\{ -\frac{u^2}{2t} \right\} dv d\varphi(u_0) dt
\]
by the change of variable theorem and Theorem 11, where \( v = u_1 - u_0 \). By the binomial expansion, we have

\[
\int_C |F_m(x)| dw_\varphi(x)
\]

\[
\leq \int_{(0,T)} \left[ \sum_{l=0}^{m} \left( \frac{m}{l} \right) \left( \frac{1}{2\pi t} \right)^{\frac{1}{2}} \int_{\mathbb{R}} |v|^l |u_0|^{m-l} \exp \left\{ -\frac{u^2}{2t} \right\} dv \right] d\varphi(u_0) dt
\]

\[
= 2 \int_{(0,T)} \left[ \sum_{l=0}^{m} \left( \frac{m}{l} \right) \left( \frac{1}{2\pi t} \right)^{\frac{1}{2}} \int_{0}^{\infty} v^l \exp \left\{ -\frac{v^2}{2t} \right\} dv \right] \left[ \int_{\mathbb{R}} |u_0|^{m-l} d\varphi(u_0) \right] dt.
\]

Now, for \( l = 0, 1, \cdots, m \), we have

\[
\int_{\mathbb{R}} |u_0|^{m-l} d\varphi(u_0) = \int_{|u_0|\leq 1} |u_0|^{m-l} d\varphi(u_0) + \int_{|u_0|> 1} |u_0|^{m-l} d\varphi(u_0)
\]

\[
\leq \int_{|u_0|\leq 1} d\varphi(u_0) + \int_{|u_0|> 1} |u_0|^m d\varphi(u_0)
\]

\[
\leq \int_{|u_0|\leq 1} d\varphi(u_0) + \int_{\mathbb{R}} |u_0|^m d\varphi(u_0) \equiv K < \infty
\]

by assumption. Let \( u = \frac{v^2}{2t} \). Then \( v = (2tu)^{\frac{1}{2}} \) and hence by the change of variable theorem it follows that

\[
\int_C |F_m(x)| dw_\varphi(x) \leq K \left( \frac{1}{\pi} \right)^{\frac{1}{2}} \int_{(0,T)} \left[ \sum_{l=0}^{m} \left( \frac{m}{l} \right) \left( 2t \right)^{\frac{l}{2}} \int_{0}^{\infty} u^{\frac{l}{2}-1} \exp \left\{ -u \right\} du \right] dt
\]

\[
= K \left( \frac{1}{\pi} \right)^{\frac{1}{2}} \sum_{l=0}^{m} \left( \frac{m}{l} \right) \left( 2T \right)^{\frac{l}{2}+1} \frac{1}{l+2} \Gamma \left( \frac{l+1}{2} \right) < \infty
\]

where \( \Gamma \) denotes the gamma function. Now, by Theorem 2.49 we have for \( P_{X_t} \)-a.e. \( \xi \in \mathbb{R}^{n+1} \)

\[
E[F_m(X_T)(\xi)]
\]

\[
= \int_{0}^{T} \int_{C} (x(t) - [x](t) + [\xi](t))^{m} dw_\varphi(x) dt
\]

\[
= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \int_{C} (X_j(t,x) + [\xi](t))^{m} dw_\varphi(x) dt
\]

\[
= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \int_{C} \left[ \sum_{l=0}^{m} \left( \frac{m}{l} \right) (X_j(t,x))^l ([\xi](t))^{m-l} \right] dw_\varphi(x) dt
\]

\[
= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left[ \sum_{l=0}^{m} \left( \frac{m}{l} \right) ([\xi](t))^{m-l} \left[ \frac{\Gamma_j(t)}{2\pi} \right] \right] \left( \frac{1}{2} \right) \int_{\mathbb{R}} u^l \exp \left\{ -\frac{\Gamma_j(t)}{2\pi} \frac{u^2}{2} \right\} du \right] dt
\]
by Lemma 2.3 and Theorem 2.4, where $\Gamma_j(t)$ and $X_j$ are given by (2) and (4), respectively. Let $M = \frac{n}{2}$ if $m$ is even and $M = \frac{n-1}{2}$ if $m$ is odd. Then we have
\[
E[F_m | X_T]\ell_j = \sum_{j=1}^{n} \left[ \sum_{k=0}^{M} \left( \frac{m}{2k} \right) \left( \frac{1}{2} \right)^{m-2k} \frac{\left( \frac{2}{\Gamma_j(t)} \right)^k}{\Gamma_j(t)} \int_{t_{j-1}}^{t_j} \right] \sum_{k=0}^{M} \left( \frac{m}{2k} \right) \left( \frac{1}{2} \right)^{m-2k} \frac{\left( \frac{2}{\Gamma_j(t)} \right)^k}{\Gamma_j(t)} \int_{t_{j-1}}^{t_j} \right)
\]
which is the desired result.

\[
\text{Remark 3.2.} \text{ The integrand in the result of Theorem 3.1 is a polynomial of degree } m \text{ with respect to } t \text{ so that the integral can always be evaluated.}
\]

In the following example, we evaluate $E[F_m | X_T], m = 1, 2, 3,$ as applications of Theorem 3.1

\[
\text{Example 3.3. Let } F_1(x) = \int_0^T x(t)dt \text{ for } x \in C[0, T] \text{ and suppose that } \int_\mathbb{R} |u|d\varphi(u) < \infty. \text{ Then for } P_X.-\text{a.e. } \xi = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}, \text{ we have}
\]
\[
E[F_1 | X_T]\ell_j = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left[ \xi_j - \xi_{j-1} \left( t - t_{j-1} \right) + \xi_{j-1} \right] dt
\]
which can also be obtained by an application of Corollary 4.5 in [4].

Let $F_2(x) = \int_0^T (x(t))^2 dt$ for $x \in C[0, T]$ and suppose that $\int_\mathbb{R} u^2d\varphi(u) < \infty$. Then for $P_X.-\text{a.e. } \xi = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1},$ we have
\[
E[F_2 | X_T]\ell_j = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left[ (t_j - t)(t - t_{j-1}) + \left( \frac{\xi_j - \xi_{j-1}}{t_j - t_{j-1}} \right)^2 + \left( \frac{\xi_j - \xi_{j-1}}{t_j - t_{j-1}} \right) \right] dt
\]
which is the result given by Corollary 4.10 of [4].
Let \( F_3(x) = \int_0^T (x(t))^3 \, dt \) for \( x \in C[0, T] \) and suppose that \( \int_R |u|^3 \, d\varphi(u) < \infty \).

Then for \( P_{\lambda, \varphi} \)-a.e. \( \xi = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1} \), we have

\[
E[F_3 | X_T](\xi) = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{3(t_j - t)(t - t_{j-1}) [\xi_j - \xi_{j-1}]^3}{t_j - t_{j-1}} (t - t_{j-1}) + \xi_{j-1}] \, dt
\]

\[
= \frac{1}{2} \sum_{j=1}^{n} \xi_{j-1}(t_j - t_{j-1})^2 + \frac{1}{4} \sum_{j=1}^{n} (t_j - t_{j-1})(\xi_j^3 + 2\xi_j^2 \xi_{j-1} + \xi_j \xi_{j-1}^2)
\]

\[
\quad + \frac{3}{2} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (t_j - t)(t - t_{j-1})^2 \, dt
\]

\[
= \frac{1}{4} \sum_{j=1}^{n} (t_j - t_{j-1})[(t_j - t_{j-1})(\xi_j + \xi_{j-1}) + \xi_j^3 + 2\xi_j^2 \xi_{j-1} + \xi_j \xi_{j-1}^2]
\]

from the integration by parts formula.

**Corollary 3.4.** Under the assumptions and notations given as in Theorem 3.1, \( E^{\text{ary}}[F_m | X_T](\xi) \) exists for \( \lambda \in \mathbb{C}_+ \) and for \( \xi \in \mathbb{R}^{n+1} \). Moreover, for a non-zero real \( q \), \( E^{\text{ary}}[F_m | X_T](\xi) \) exists for \( \xi \in \mathbb{R}^{n+1} \) and it is given by

\[
E^{\text{ary}}[F_m | X_T](\xi) = \sum_{j=1}^{M} \sum_{k=0}^{M} \left( \frac{i}{q} \right)^k \frac{m!}{2^k k!(m-2k)!} \int_{t_{j-1}}^{t_j} (|\xi|(t))^{m-2k} \left[ \frac{1}{\Gamma_j(t)} \right]^k \, dt.
\]

**Proof.** For \( \lambda > 0 \), we have

\[
E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\xi])] = \sum_{j=1}^{M} \sum_{k=0}^{M} \frac{1}{\lambda^{2k} k!(m-2k)!} \int_{t_{j-1}}^{t_j} (|\xi|(t))^{m-2k} \left[ \frac{1}{\Gamma_j(t)} \right]^k \, dt
\]

by an application of Theorem 3.1 where \( M = \frac{m}{2} \) if \( m \) is even and \( M = \frac{m-1}{2} \) if \( m \) is odd. Now, we have the corollary immediately. \( \Box \)

Now, we generalize the result in Example 4 of [4], which is also considered by Chang and Chang in [3].

**Theorem 3.5.** Let \( F(x) = \exp\{\int_0^T x(t) \, dt\} \) for \( x \in C[0, T] \) and suppose that \( F \) is \( w_\varphi \)-integrable. Then, for a.e. \( y \in C[0, T] \), we have

\[
\lim_{||y||_0 \to 0} E[F | X_T](X_T(y)) = F(y).
\]

**Proof.** For a.e. \( y \in C[0, T] \), we have

\[
E[F | X_T](X_T(y)) = \int_C F(x - [x] + [y]) \, d\varphi(x)
\]

\[
= \exp\left\{ \frac{1}{2} \sum_{j=1}^{n} (t_j - t_{j-1})(y(t_{j-1}) + y(t_j)) \right\}
\]

\[
\times \int_C \exp\left\{ \int_0^T (x(t) - [x](t)) \, dt \right\} \, d\varphi(x)
\]

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by Theorem 2.9 and the same process used in Example 3.3. Letting \( \|\tau\| \to 0 \), we have
\[
\lim_{\|\tau\| \to 0} E[F|X_\tau](X_\tau(y)) = \lim_{\|\tau\| \to 0} \exp\left\{ \frac{1}{2} \sum_{j=1}^{n} (t_j - t_j-1)(y(t_j-1) + y(t_j)) \right\} \\
\times \int_C \exp\left\{ \int_0^T (x(t) - [x](t)) dt \right\} dw_\varphi(x) = F(y)
\]
because \( \lim_{\|\tau\| \to 0} (x(t) - [x](t)) = 0 \) for \( x \in C[0, T] \).

\[ \square \]

4. A translation theorem for the conditional \( w_\varphi \)-integral

In this section, we derive a translation theorem for the conditional \( w_\varphi \)-integral, which is a generalization of [6, Theorem 4]. To derive the theorem, we need a translation theorem for the \( w_\varphi \)-integral on \( C[0, T] \).

**Theorem 4.1** ([4, Theorem 3.1]). Let \( h \) be in \( C[0, T] \) and of bounded variation. Let \( \alpha \in \mathbb{R} \) and let \( x_0(t) = \int_0^t h(s) ds + \alpha \) for \( 0 \leq t \leq T \) and let \( \varphi_\alpha \) be a measure on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) such that \( \varphi_\alpha(B) = \varphi(B + \alpha) \) for \( B \in \mathcal{B}(\mathbb{R}) \). Moreover, let \( F \) be a measurable function on \( C[0, T] \). Then, \( F(x + x_0) \) is also measurable on \( C[0, T] \) and we have
\[
\int_C F(x) dw_\varphi(x) = \exp\left\{ -\frac{1}{2} \|h\|^2 \right\} \int_C F(x + x_0) \exp\{-h, x\} dw_\varphi(x),
\]
where \( \hat{=} \) means that if one side exists, then both sides exist and they are equal.

Let \( E_{w_\varphi}[F|X_\tau] \) and \( E_{w_\varphi,\alpha}[F|X_\tau] \) denote the conditional \( w_\varphi \)-integral and the conditional \( w_{\varphi,\alpha} \)-integral of \( F \) given \( X_\tau \), respectively. In the following theorem we derive a translation theorem for the conditional \( w_\varphi \)-integral using Theorem 4.1.

**Theorem 4.2.** Let \( x_0 \) and \( \varphi_\alpha \) be given as in Theorem 4.1. Moreover, let \( F \) be defined and \( w_\varphi \)-integrable on \( C[0, T] \). Then we have for \( P_{X_\tau} \)-a.e. \( \xi = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1} \)
\[
E_{w_\varphi}[F|X_\tau](\xi) = \exp\left\{ \sum_{j=1}^{n} \frac{x_0(t_j) - x_0(t_{j-1})}{t_j - t_{j-1}} (\xi_j - \xi_{j-1}) \right\} \exp\left\{ -\frac{1}{2} \|x_0\|_2^2 \right\} \exp\{-h, x\} \int_C F(x_0 + \cdot)dw_\varphi(x_0) J(x) \]
where \( J(x) = \exp\{-\frac{1}{2} \|h\|_2^2 \} \exp\{-h, x\} \) for \( x \in C[0, T] \) and \( x_0 = (x_0(t_0), x_0(t_1), \cdots, x_0(t_n)) \).

**Proof.** By Theorems 2.9 and 4.1 we have for \( P_{X_\tau} \)-a.e. \( \xi = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1} \)
\[
E_{w_\varphi}[F|X_\tau](\xi) = \int_C F(x - [x] + \delta) dw_\varphi(x) = \int_C F(x_0 + x - [x] + \delta) J(x) dw_\varphi(x).
\]
But we have
\[
\exp\{-h, x\} = \exp\{-h, x - [x] + \delta \} = (h - [x] - \delta).
\]
and
\[
\exp\{ (h, [\xi - \bar{x}_0]) \}
= \exp\left\{ \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} h(t)d[\xi - \bar{x}_0]\{t\} \right\}
= \exp\left\{ \sum_{j=1}^{n} \frac{x_0(t_j) - x_0(t_{j-1})}{t_j - t_{j-1}}((\xi_j - \xi_{j-1}) - (x_0(t_j) - x_0(t_{j-1}))) \right\}.
\]

Hence we have
\[
E_{w_0}[F|X_\tau](\xi)
\]
\[
= \int_C F(x_0 + x - [x] + [\xi - \bar{x}_0])J(x - [x] + [\xi - \bar{x}_0])\exp\{ - (h, [x]) \}
\]
\[
\times \exp\left\{ \sum_{j=1}^{n} \frac{x_0(t_j) - x_0(t_{j-1})}{t_j - t_{j-1}}((\xi_j - \xi_{j-1}) - (x_0(t_j) - x_0(t_{j-1}))) \right\}dw_{\varphi_n}(x)
\]
\[
= E_{w_0}[F(x_0 + \cdot)J|X_\tau](\xi - \bar{x}_0)\exp\left\{ \sum_{j=1}^{n} \frac{x_0(t_j) - x_0(t_{j-1})}{t_j - t_{j-1}}((\xi_j - \xi_{j-1}) - (x_0(t_j) - x_0(t_{j-1}))) \right\}
\]
\[
\times \exp\{ - (h, [x]) \}\int_C dw_{\varphi_n}(x)
\]
since \( x - [x] \) and \([x]\) are independent by Theorem 2.1. Again, by independence (Lemma 2.1), we have
\[
\int_C \exp\{ - (h, [x]) \}dw_{\varphi_n}(x)
\]
\[
= \int_C \exp\left\{ - \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} h(t)d[x]\{t\} \right\}dw_{\varphi_n}(x)
\]
\[
= \int_C \exp\left\{ - \sum_{j=1}^{n} \frac{x_0(t_j) - x_0(t_{j-1})}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1})) \right\}dw_{\varphi_n}(x)
\]
\[
= \prod_{j=1}^{n} \left[ \frac{1}{2\pi(t_j - t_{j-1})} \right]^\frac{1}{2} \int_\mathbb{R} \exp\left\{ - \frac{x_0(t_j) - x_0(t_{j-1})}{t_j - t_{j-1}}v_j - \frac{1}{2(t_j - t_{j-1})}v_j^2 \right\}dv_j
\]
\[
= \exp\left\{ \sum_{j=1}^{n} \frac{(x_0(t_j) - x_0(t_{j-1}))^2}{2(t_j - t_{j-1})} \right\}
\]
so that we have
\[
E_{w_0}[F|X_\tau](\xi)
\]
\[
= \exp\left\{ \sum_{j=1}^{n} \frac{x_0(t_j) - x_0(t_{j-1})}{t_j - t_{j-1}}\left[ (\xi_j - \xi_{j-1}) - \frac{1}{2}(x_0(t_j) - x_0(t_{j-1})) \right] \right\}E_{w_0}[F(x_0 + \cdot)J|X_\tau](\xi - \bar{x}_0)
\]
which is the desired result. \( \square \)
Letting $\alpha = 0$ in Theorem 4.2 we have the following corollary from the theorem since $\varphi_\alpha = \varphi$.

**Corollary 4.3.** Under the assumptions and notations given as in Theorem 4.2 with one exception $\alpha = 0$, we have $x_0(t) = \int_0^t h(s)ds$ for $t \in [0, T]$ and

$$E_{w_\alpha}[F|X_T](\xi) = \exp \left\{ \sum_{j=1}^n x_0(t_j) \frac{x_0(t_{j-1})}{t_j - t_{j-1}} \left[ (\xi_j - \xi_{j-1}) - \frac{1}{2} (x_0(t_j) - x_0(t_{j-1})) \right] \right\} E_{w_\alpha}[F(x_0 + \cdot)J|X_T](\tilde{x}_0).$$

For notational convenience, let $E_{m_w}$ denote the (conditional) Wiener integral on $C_0[0, T]$.

Under the assumption that $V$ is a non-negative continuous function on $\mathbb{R}$ satisfying the condition

$$\int_{\mathbb{R}} V(\xi_1) \exp \left\{ -\frac{\xi_1^2}{2T} \right\} d\xi_1 < \infty$$

for every $T > 0$, we have for a.e. $\xi_1 \in \mathbb{R}$

$$E_{w_\alpha} \left[ \exp \left\{ -\int_0^T V(x(s))ds \right\} \left| x(t_0) = 0, x(T) = \xi_1 \right. \right] = E_{m_w} \left[ \exp \left\{ -\int_0^T V(x(s))ds \right\} \left| x(T) = \xi_1 \right. \right]$$

and for a.e. $(\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$

$$E_{w_\alpha} \left[ \exp \left\{ -\int_0^T V(x(s))ds \right\} \left| x(t_0) = 0, x(t_1) = \xi_1, \cdots, x(t_n) = \xi_n \right. \right] = E_{m_w} \left[ \exp \left\{ -\int_0^T V(x(s))ds \right\} \left| x(t_1) = \xi_1, \cdots, x(t_n) = \xi_n \right. \right]$$

by Theorems 2.3 and 2.9.

In [10], Yeh showed that the function $U$ defined on $\mathbb{R} \times (0, \infty)$ by

$$U(\xi_1, T) = \left( \frac{1}{2\pi T} \right)^{\frac{1}{2}} \exp \left\{ -\frac{\xi_1^2}{2T} \right\} E_{m_w} \left[ \exp \left\{ -\int_0^T V(x(s))ds \right\} \left| x(T) = \xi_1 \right. \right]$$

satisfies the Kac-Feynman integral equation

$$U(\xi_1, T) = \left( \frac{1}{2\pi T} \right)^{\frac{1}{2}} \exp \left\{ -\frac{\xi_1^2}{2T} \right\} \left[ \frac{1}{2\pi (T - s)} \right]^{\frac{1}{2}} \int_{\mathbb{R}} \exp \left\{ -\frac{\eta^2}{2(T - s)} \right\} V(\eta) U(\eta, s) d\eta ds$$

whose solution is given by

$$U(\xi_1, T) = \sum_{k=0}^{\infty} (-1)^k U_k(\xi_1, T), \quad (\xi_1, T) \in \mathbb{R} \times (0, \infty),$$
where the sequence \( \{ U_k \}_{k=0}^{\infty} \) is defined inductively by

\[
U_0(\xi_1, T) = \left( \frac{1}{2\pi T} \right)^{\frac{1}{2}} \exp \left\{ -\frac{\xi_1^2}{2T} \right\},
\]

\[
U_{k+1}(\xi_1, T) = \int_0^T \left[ \frac{1}{2\pi(T - s)} \right]^{\frac{1}{2}} \int_{\mathbb{R}} \exp \left\{ -\frac{(\xi_1 - \eta)^2}{2(T - s)} \right\} V(\eta)U_k(\eta, s)d\eta ds
\]

for \( k = 0, 1, \ldots \). Thus \( E_w \{ \exp \{ -\int_0^T V(x(s))ds \} \mid x(t_0) = 0, x(T) = \xi_1 \} \) can be found using (1) and (3).

Now, for \( \xi_1, \xi_2 \in \mathbb{R} \) and \( t \in (0, T] \), let

\[
(4) \quad U(\xi_1, \xi_2 - \xi_1, T - t) = \left[ \frac{1}{2\pi(T - t)} \right]^{\frac{1}{2}} \exp \left\{ -\frac{(\xi_2 - \xi_1)^2}{2(T - t)} \right\} \times E_{w_x} \left\{ \exp \left\{ -\int_t^T V(x(s))ds \right\} \mid x(t_0) = 0, x(t) = \xi_1, x(T) = \xi_2 \right\}.
\]

Using the same process in [6, pp. 388-389], (4) satisfies the integral equation

\[
U(\xi_1, \xi_2 - \xi_1, T - t) = \left[ \frac{1}{2\pi(T - t)} \right]^{\frac{1}{2}} \exp \left\{ -\frac{(\xi_2 - \xi_1)^2}{2(T - t)} \right\} - \int_0^{T-t} \left[ \frac{1}{2\pi(T - t - u)} \right]^{\frac{1}{2}} \times \int_{\mathbb{R}} \exp \left\{ -\frac{(\xi_2 - \gamma)^2}{2(T - t - u)} \right\} V(\gamma)U(\xi_1, \gamma - \xi_1, u)d\gamma du
\]

by (2) so that we have

\[
U(\xi_1, \xi_2 - \xi_1, T - t) = \sum_{k=0}^{\infty} (-1)^k U_k(\xi_2, \xi_2 - \xi_1, T - t),
\]

where

\[
U_0(\xi_1, \xi_2 - \xi_1, T - t) = \left[ \frac{1}{2\pi(T - t - u)} \right]^{\frac{1}{2}} \exp \left\{ -\frac{(\xi_2 - \xi_1)^2}{2(T - t)} \right\}
\]

and

\[
U_{k+1}(\xi_1, \xi_2 - \xi_1, T - t) = \int_0^{T-t} \left[ \frac{1}{2\pi(T - t - u)} \right]^{\frac{1}{2}} \int_{\mathbb{R}} \exp \left\{ -\frac{(\xi_2 - \gamma)^2}{2(T - t - u)} \right\} V(\gamma)U_k(\xi_1, \gamma - \xi_1, u)d\gamma du
\]

for \( k = 0, 1, 2, \ldots \). In particular, the solution \( U(\xi_1, T) \) of the Kac-Feynman integral equation is a special case, namely,

\[
U(\xi_1, T) = U(0, \xi_1, T).
\]
We are now ready to write out an expression for the multi-conditional expectation. Using (4.6) in [6], (1) and (2), we can write for a.e. $(\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$

$$E_{w_\varphi} \left[ \exp \left\{ - \int_0^T V(x(s)) ds \right\} \mid x(t_0) = 0, x(t_1) = \xi_1, \cdots, x(t_n) = \xi_n \right]$$

$$= \prod_{j=1}^n \left[ 2\pi (t_j - t_{j-1}) \right]^{\frac{1}{2}} \exp \left\{ \frac{(\xi_j - \xi_{j-1})^2}{2(t_j - t_{j-1})} \right\} U(\xi_{j-1}, \xi_j - \xi_{j-1}, t_j - t_{j-1})$$

where $\xi_0 = 0$ and $t_n = T$.

REFERENCES


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