TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 360, Number 10, October 2008, Pages 5229–5246
S 0002-9947(08)04603-5
Article electronically published on May 20, 2008

C_0-COARSE GEOMETRY OF COMPLEMENTS OF Z-SETS IN THE HILBERT CUBE

E. CUCHILLO-IBÁÑEZ, J. DYDAK, A. KOYAMA, AND M. A. MORÓN

Abstract. Motivated by the Chapman Complement Theorem, we construct an isomorphism between the topological category of compact Z-sets in the Hilbert cube Q and the C_0-coarse category of their complements. The C_0-coarse morphisms are, in this particular case, intrinsically related to uniformly continuous proper maps. Using that fact we are able to relate in a natural way some of the topological invariants of Z-sets to the geometry of their complements.

1. Introduction

In 1972 T. Chapman [3] published what many consider the deepest and most beautiful results in shape theory. The one that stands out is the following *Complement Theorem* in Shape Theory:

Two compact Z-sets in the Hilbert cube have the same shape if and only if their complements are homeomorphic.

More generally, he constructed an isomorphism of categories allowing one to describe Borsuk's Shape Theory in terms of proper maps of complements of Z-sets in the Hilbert cube.

Combining both results Chapman proved a rigidity theorem:

The complements of two compact Z-sets (or two contractible Q-manifolds admitting a boundary [5]*) in the Hilbert cube have the same weak proper homotopy type if and only if those complements are homeomorphic.*

Recently [1] suggested a possibility to describe topological invariants of compact Z-sets of the Hilbert cube in terms of metric-uniform invariants of their complements. In particular two questions were posed there. They were related to the canonical copy of a compact metric space inside its hyperspace with the Hausdorff metric. When the compact space is a non-degenerate Peano continuum those ques-

Received by the editors July 17, 2006.
2000 *Mathematics Subject Classification.* Primary 18B30, 54D35, 54E15; Secondary 54C55, 54E35, 54F45.
Key words and phrases. Covering dimension, asymptotic dimension, C_0-coarse structure, ANR-space, C_0-coarse morphism, uniformly continuous map, compact Z-set, Higson-Roe compactification and corona.
The first and fourth named authors were supported by the MEC, MTM2006-0825.

tions are about Z-copies in a Hilbert cube. When translated to our current context the questions become:

1. How can the covering dimension of a compact Z-set be described from the outside?
2. What is a manifold from the outside?

The above paragraphs provide the main motivation for this paper.

Following the spirit of the isomorphism of categories constructed by Chapman in [3] we prove that the topological category for compact Z-sets of the Hilbert cube is isomorphic to the C_0-coarse geometry category of their complements. The definition of this C_0-coarse geometry category requires a fixed metric on the Hilbert cube, but soon it is pointed out that the definition is, in fact, independent of the chosen (equivalent) metric. The C_0-coarse structure of a metric space was introduced by Wright [16] (see also [17]) for different purposes from those pursued herein. We recommend [14] for the basic definitions of coarse spaces. Nevertheless we will recall some of these definitions, if needed.

The objects in this category are the complements of Z-sets with the C_0-coarse structure induced by a metric in the Hilbert cube. The morphisms are the equivalence classes of *close coarse maps* between them. See [16], [17] and [14] for the basic definitions of C_0-coarse maps and C_0-close maps. We also obtain our rigidity result. In fact, after the construction of the isomorphism of categories, it is very easy to see that the complements of two compact Z-sets are C_0-coarse equivalent if and only if those complements are uniformly homeomorphic (with respect to the metrics induced by the one fixed on the Hilbert cube).

In view of the equivalence of categories described above it is natural to consider the problem of describing topological invariants of Z-sets in terms of C_0-coarse invariants of their complements. That is how the asymptotic dimension of such a coarse structure on the complement appears as the adequate tool to describe, from the outside, the topological dimension of a Z-set. To do it efficiently we use some results of [7]. In fact we prove that the Hilbert cube is the Higson-Roe compactification of the complement of a Z-set for this coarse structure. Moreover, its Higson-Roe corona is exactly that Z-set. That fact allows us to show that the dimension can be interpreted in terms of the asymptotic dimension of such a coarse space. See [14] and [7] for the definitions of asymptotic dimension, Higson-Roe compactification, and corona for arbitrary coarse categories. So, in some sense, we reinforce the relation between the topological and coarse definitions of dimension and, in particular, we solve the first question of [1]. We also reinterpret the concept of *external dimension* introduced by Mrozik [11] as a characterization of the C_0-asymptotic dimension of the complement of a Z-set analogous to that of the topological dimension using extensions of maps to spheres.

The concept of asymptotic dimension was defined by Gromov [8] to study asymptotic invariants of discrete groups, mainly the fundamental group of manifolds. For finitely generated groups Gromov used the *bounded* coarse geometry associated to the word metric defined by means of a finite set of generators. The main fact in [8] was that such coarse geometry does not depend on the chosen generating set. Motivated by this, some authors (see for example Dranishnikov [6], Roe [12, 13] and references there) began to develop the asymptotic topology or large-scale geometry. Our paper is related to [6] as we also connect topological invariants of compact spaces to coarse concepts (in our case to C_0-coarse concepts). In fact,

due to the isomorphism of categories we construct, this paper could help in the program of finding suitable translations of topological concepts to coarse geometry as pursued in [6], because it could be possible to relate all the topological invariants between compacta to the C_0-coarse structure of their complements, by means of the categorical isomorphism, and then to adapt them to the general definition of coarse geometry. We do not follow this line herein. On the contrary, due to the fact that, in our case, the coarse morphisms can be represented by proper uniformly continuous functions, we stay as close as possible to Topology in the procedure of translating topological invariants of compact Z-sets to C_0-coarse invariants of their complements.

In [14], Roe gives a definition of asymptotic dimension for general coarse spaces. Later on Grave [7] proposed other definitions (translating some characterizations from the topological framework). All of them are equivalent in the realm of proper coarse spaces.

We complete the paper by reinterpreting the ANR-property (a necessary property for being a manifold; see the second question above) and the group of autohomeomorphisms in terms of the C_0-coarse structure. We also take this opportunity to describe the homotopical category of compact metric spaces in terms of proper and uniformly continuous maps between their complements.

The authors are very grateful to the referee whose detailed suggestions improved the exposition of the paper significantly.

2. Equivalence of categories and applications

In this section we establish that the topological category of compacta (the most geometric topological spaces, namely compact and metrizable) can be described in terms of coarse geometry. More precisely, we need to use the so-called *C_0-coarse geometry* introduced in [15] (see also [16]). Later on we translate some topological invariants to the coarse geometry language.

As mentioned above we follow [14] for basic definitions and notation in Coarse Geometry.

Recall (see [15] or [14, page 22]) that for each metric space (X, d) the C_0-coarse structure is the collection $\mathcal{E}_0$ of all $E \subset X \times X$ for which the distance function d, restricted to E, tends to zero at infinity; that is to say, E is *controlled* if for any $\varepsilon > 0$ there is a compact subset $K \subset X$ such that $d(x, x') < \varepsilon$ whenever $(x, x') \in E \setminus K \times K$.

Consider the Hilbert cube (Q, d) with a fixed metric d reflecting its topology. Suppose that $X \subset Q$ is a compact Z-set of Q and denote by $(Q \setminus X, \mathcal{E}_0)$ the coarse space $Q \setminus X$ with the C_0-coarse structure associated to the metric $d\,|_{(Q\setminus X)\times(Q\setminus X)}$. It is obvious that if $X \neq \varnothing$ is a closed Z-set in Q, then the metric $d\,|_{(Q\setminus X)\times(Q\setminus X)}$ is never a proper metric (recall that a metric is proper when the compact subsets are precisely the closed and bounded sets).

On the other hand, there is a concept of *proper* coarse structure ([14, page 25]) which is important in coarse geometry. To establish that notion first let us recall some definitions from [14].

A *coarse structure* $\mathcal{E}$ on X is a family of subsets E (called *controlled sets*) of $X \times X$ satisfying the following properties:

(1) The diagonal $\Delta = \{(x, x)\}_{x\in X}$ belongs to $\mathcal{E}$.
(2) $E_1 \in \mathcal{E}$ implies $E_2 \in \mathcal{E}$ for every $E_2 \subset E_1$.

(3) $E \in \mathcal{E}$ implies $E^{-1} \in \mathcal{E}$, where $E^{-1} = \{(y,x)\}_{(x,y)\in E}$.
(4) $E_1, E_2 \in \mathcal{E}$ implies $E_1 \cup E_2 \in \mathcal{E}$.
(5) $E, F \in \mathcal{E}$ implies $E \circ F \in \mathcal{E}$, where $E \circ F$ consists of (x,y) such that there is $z \in X$ so that $(x,z) \in E$ and $(z,y) \in F$.

A subset $B \subset X$ is *bounded* for $\mathcal{E}$ (C_0-*bounded* in the case of $\mathcal{E}_0$) if $B \times B \in \mathcal{E}$ (see Proposition 2.16 in [14] for equivalent definitions of boundedness).

For every $E \subset X \times X$ and $K \subset X$ let

$$E[K] = \{x' \in X \mid \exists x \in K,\ (x', x) \in E\},$$
$$E^{-1}[K] = \{x' \in X \mid \exists x \in K,\ (x, x') \in E\}.$$

When X is a topological space, $E \subset X \times X$ is said to be *proper* (see Definition 2.1 in [14]) if $E[K]$ and $E^{-1}[K]$ are relatively compact whenever $K \subset X$ is relatively compact.

Given a paracompact Hausdorff topological space X, a coarse structure on X is *proper* (see Definition 2.22 in [14]) if:

(i) There is a controlled neighborhood of the diagonal Δ_X, and
(ii) Every bounded set (in the coarse sense) is relatively compact.

N. Wright [15] proved that the C_0-coarse structure of a proper metric space (Y,d) is proper. For the sake of completeness, we are going to prove that the C_0-coarse structure of $(Q \setminus X, d\,|_{(Q\setminus X)\times(Q\setminus X)})$ is proper in spite of $d\,|_{(Q\setminus X)\times(Q\setminus X)}$ not being proper (see exercise 2.25 in [14]).

Proposition 1. *Let (Q,d) be the Hilbert cube with a fixed metric d. The C_0-coarse structure $\mathcal{E}_0$ of $Q \setminus X$ is a proper coarse structure for any compact Z-set $X \subset Q$.*

Proof. For any point $x \in Q \setminus X$ put $\varepsilon_x = d(x,X) > 0$. Let

$$W = \{(x,y) \in (Q \setminus X) \times (Q \setminus X) \mid d(x,y) < \frac{\varepsilon_x}{3} \ \text{ and } \ d(x,y) < \frac{\varepsilon_y}{3}\}.$$

It is clear that the diagonal $\Delta_{Q\setminus X}$ is contained in W. We are going to prove that given $(x,x) \in \Delta_{Q\setminus X}$ there exists $n \in \mathbb{N}$ such that

$$A_{(x,x)} = B(x, \frac{\varepsilon_x}{n}) \times B(x, \frac{\varepsilon_x}{n})$$

is contained in W. In fact, if $(\alpha,\beta) \in A_{(x,x)}$, then $d(\alpha,x) < \frac{\varepsilon_x}{n}$ and $d(\beta,x) < \frac{\varepsilon_x}{n}$. Consider $z_\alpha \in X$ with $d(\alpha, z_\alpha) = d(\alpha, X) = \varepsilon_\alpha$. Then

$$\varepsilon_x = d(x,X) \le d(x,z_\alpha) \le d(x,\alpha) + d(\alpha,z_\alpha) < \frac{\varepsilon_x}{n} + \varepsilon_\alpha.$$

Consequently

$$\varepsilon_x - \frac{\varepsilon_x}{n} = \frac{n-1}{n}\,\varepsilon_x < \varepsilon_\alpha.$$

Hence

$$d(\alpha,\beta) \le d(\alpha,x) + d(x,\beta) < \frac{\varepsilon_x}{n} + \frac{\varepsilon_x}{n} = \frac{2}{n}\,\varepsilon_x < \frac{2}{n}\,\frac{n}{n-1}\,\varepsilon_\alpha = \frac{2}{n-1}\,\varepsilon_\alpha.$$

Analogously

$$d(\alpha,\beta) < \frac{2}{n-1}\,\varepsilon_\beta.$$

Now it is sufficient to consider $n = 7$ in order to obtain

$$d(\alpha,\beta) < \frac{\varepsilon_\alpha}{3} \ \text{ and } \ d(\alpha,\beta) < \frac{\varepsilon_\beta}{3},$$

which implies that $(\alpha, \beta) \in W$ and, in this manner, we have shown that W is a neighborhood of $\Delta_{Q \setminus X}$.

In order to establish that W is C_0-controlled, for each $\delta > 0$, take the compact subset $K = Q \setminus B(X, \delta)$. If $(x, x') \in W \setminus K \times K$, then $d(x, x') < \min\{\frac{\varepsilon_x}{3}, \frac{\varepsilon_{x'}}{3}\}$. Since $(x, x') \notin K \times K$ we can suppose that $x \notin K$. So $\varepsilon_x = d(x, X) < \delta$ and

$$d(x, x') < \frac{\varepsilon_x}{3} < \frac{\delta}{3} < \delta.$$

Therefore W is C_0-controlled.

Let $E \subset (Q \setminus X) \times (Q \setminus X)$ be a C_0-controlled set and $K \subset Q \setminus X$ be a relatively compact set in $Q \setminus X$. The subset

$$E[K] = \{x' \in Q \setminus X \mid \exists x \in K,\, (x', x) \in E\}$$

is relatively compact in $Q \setminus X$. Indeed, suppose that K is compact and $E[K]$ is not relatively compact in $Q \setminus X$. In this case there exist a point $x_0 \in X$ and a sequence $\{x_n\}_{n \in \mathbb{N}} \subset E[K]$ with $\lim\limits_{n \to \infty} x_n = x_0$ in Q. Fix a sequence $\{k_n\}_{n \in \mathbb{N}} \subset K$ with $(x_n, k_n) \in E$ for all $n \in \mathbb{N}$. From the compactness of K we assume that $\lim\limits_{n \to \infty} k_n = k_0 \in K$. Consequently, $\lim\limits_{n \to \infty} (x_n, k_n) = (x_0, k_0)$, but $d(x_0, k_0) > 0$; thus the distance function d, restricted to E, doesn't tend to zero at infinity and this contradicts that E is a C_0-controlled set. When $K \subset Q \setminus X$ is relatively compact, since $E[K] \subset E[Cl_{Q \setminus X}(K)]$, $E[K]$ is also relatively compact.

For every $E \in \mathcal{E}_0$ we have $E^{-1} \in \mathcal{E}_0$, so $E^{-1}[K]$ is relatively compact for each subset K relatively compact in $Q \setminus X$. Thus we proved that any $E \in \mathcal{E}_0$ is proper.

Finally, take a bounded subset (in the C_0-sense) $B \subset Q \setminus X$. From Proposition 2.16 in [14], $B = E[\{p\}]$ for some $E \in \mathcal{E}_0$ and certain $p \in Q \setminus X$. Since E is proper it follows that B is relatively compact and the proof is completed. $\square$

Some of the main concepts in coarse geometry are those of a coarse map and close coarse maps. We are going to recall herein the versions for the C_0-coarse geometry. See [14, pages 23-24] for the general concepts.

Let (Q, d) be the Hilbert cube, with a fixed metric d, and X, Y be compact Z-sets in Q. A function $f : Q \setminus X \longrightarrow Q \setminus Y$ (not necessarily continuous) is said to be a C_0-*coarse map* if the following conditions are satisfied:

a) $f^{-1}(B)$ is C_0-bounded in $Q \setminus X$ if B is C_0-bounded in $Q \setminus Y$.
b) If $E \subset (Q \setminus X) \times (Q \setminus X)$ is C_0-controlled, then $(f \times f)(E) = \{(f(x), f(y)) : (x, y) \in E\}$ is C_0-controlled in $Q \setminus Y$.

Recall also that, in our context, being C_0-bounded is equivalent to being relatively compact.

Two C_0-coarse maps, $f, g : Q \setminus X \longrightarrow Q \setminus Y$, are C_0-*close* if the subset $\{(f(x), g(x)) : x \in Q \setminus X\}$ is C_0-controlled in $Q \setminus Y$.

In order to establish an isomorphism of categories we need to introduce some notation. Given the Hilbert cube (Q, d) with a fixed metric, denote by $\mathcal{Z}$ the category whose objects $\mathrm{Ob}(\mathcal{Z})$ are compact Z-sets of Q and the morphisms are continuous functions between Z-sets. Denote also by $\mathcal{C}_0(\mathcal{Z})$ the category whose objects are the complements, in Q, of objects in $\mathcal{Z}$ with the C_0-coarse structure induced by the restriction of the metric d. The morphisms between objects in $\mathcal{C}_0(\mathcal{Z})$ are the equivalence classes, under the relation of C_0-closeness, of C_0-coarse maps. Note that, from the second paragraph in [14, page 25], the composition of classes

is well defined by means of the class of the composition of arbitrary corresponding representations.

Theorem 2. *There is an isomorphism of categories*

$$T : \mathcal{C}_0(\mathcal{Z}) \longrightarrow \mathcal{Z}$$

so that the action on objects is defined by

$$T\big((Q \setminus X, \mathcal{E}_0)\big) = X.$$

Proof. Define T on objects as stated above.

Consider a C_0-coarse map $f : Q \setminus X \longrightarrow Q \setminus Y$. Since X is a Z-set, and therefore nowhere dense in Q, given $x_0 \in X$ there exists a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset Q \setminus X$ with $\lim\limits_{n\to\infty} \alpha_n = x_0$. Obviously the set $E = \{(\alpha_n, \alpha_{n+1}) \ : \ n \in \mathbb{N}\}$ is C_0-controlled in $Q \setminus X$. Consequently $(f \times f)(E) = \{(f(\alpha_n), f(\alpha_{n+1})) \ : \ n \in \mathbb{N}\}$ is C_0-controlled in $Q \setminus Y$. Let $\beta \in Q$ be an accumulation point of the sequence $\{f(\alpha_n)\}_{n\in\mathbb{N}}$ and choose a subsequence $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ of $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $\lim_k f(\alpha_{n_k}) = \beta$. Assume that $\beta \in Q \setminus Y$. In that case, the compact $K = \{f(\alpha_{n_k})\}_{k\in\mathbb{N}} \cup \{\beta\}$ is a subset of $Q \setminus Y$. As f is a C_0-coarse map, $f^{-1}(K)$ should be a relatively compact subset in $Q \setminus X$, but this is impossible because $f^{-1}(K)$ contains a sequence converging to the point $x_0 \in X$. Thus we have proved that any accumulation point of $\{f(\alpha_n)\}_{n\in\mathbb{N}}$ must be in Y.

Consider $\beta_1, \beta_2 \in Y$ to be two accumulation points of $\{f(\alpha_n)\}_{n\in\mathbb{N}}$ and consider two subsequences $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$, $\{\alpha_{m_k}\}_{k\in\mathbb{N}}$ of $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $\lim_k f(\alpha_{n_k}) = \beta_1$ and $\lim_k f(\alpha_{m_k}) = \beta_2$. Take the sequence $\{\gamma_j\}_{j\in\mathbb{N}}$ defined by

$$\gamma_{2l} = \alpha_{n_l} \ , \ \gamma_{2l-1} = \alpha_{m_l} \quad \text{for } l \in \mathbb{N}.$$

Clearly $\lim\limits_{j\to\infty} \gamma_j = x_0 \in X$, $\{\gamma_j\}_{j\in\mathbb{N}} \subset Q \setminus X$, the set $L = \{(\gamma_j, \gamma_{j+1}) \ : \ j \in \mathbb{N}\}$ is C_0-controlled in $Q \setminus X$ and $(f \times f)(L)$ is C_0-controlled in $Q \setminus Y$. Hence when we fix a number $\delta > 0$ there exists $n \in \mathbb{N}$ such that if $(x, y) \in (f \times f)(L) \setminus (K_n \times K_n)$, then $d(x, y) < \delta$, where $K_n = Q \setminus B(Y, 1/n) = Q \setminus \{x \in Q \, ; \ d(x, Y) < 1/n\}$. By the construction of the sequence $\{\gamma_j\}_{j\in\mathbb{N}}$ it is obvious that, for each natural number $n \in \mathbb{N}$, there exists $j_n \in \mathbb{N}$ such that $f(\gamma_j) \in B(Y, 1/n)$ for any $j \geq j_n$. So for every $\delta > 0$ there is a number $j_\delta \in \mathbb{N}$ such that $d(f(\gamma_j), f(\gamma_{j+1})) < \delta$ for $j \geq j_\delta$ and consequently $d(\beta_1, \beta_2) = 0$, that is, $\beta_1 = \beta_2$. In this manner, the sequence $\{f(\alpha_n)\}_{n\in\mathbb{N}}$ has a unique accumulation point, denote it by y_{x_0}, that belongs to Y. Due to the compactness of Q, this means that, in fact, $\lim\limits_{n\to\infty} f(\alpha_n) = y_{x_0}$.

The point $y_{x_0} \in Y$ does not depend on the choice of the initial sequence because if $\{\alpha'_n\}_{n\in\mathbb{N}} \subset Q \setminus X$ is another sequence, with $\lim\limits_{n\to\infty} \alpha'_n = x_0$, applying the procedure described above to the new sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ defined as

$$\sigma_{2l} = \alpha_l \ , \ \sigma_{2l-1} = \alpha'_l \quad \text{for } l \in \mathbb{N},$$

then necessarily, $\lim\limits_{n\to\infty} f(\alpha'_n) = y_{x_0}$.

Through the assignment $x \mapsto y_x$ we obtain a map

$$z(f) : X \longrightarrow Y.$$

This is a continuous map. In fact, if $\lim\limits_{n\to\infty} x_n = x_0$, with $\{x_n\}_{n\in\mathbb{N}} \cup \{x_0\} \subset X$, for each $n \in \mathbb{N}$ consider a sequence $\{\alpha_{n,k}\}_{k\in\mathbb{N}}$ such that $\lim_k \alpha_{n,k} = x_n$ and $\{\alpha_{n,k}\}_{k\in\mathbb{N}} \subset Q \setminus X$. Using now a diagonal process one can construct a sequence of the form $\{\alpha_{n,k_n}\}_{n\in\mathbb{N}}$ such that $d(\alpha_{n,k_n}, x_n) < 1/n$ and $d(f(\alpha_{n,k_n}), z(f)(x_n)) < 1/n$.

Clearly $\{\alpha_{n,k_n}\}_{n\in\mathbb{N}}$ is a convergent sequence and $\lim_{n\to\infty}\alpha_{n,k_n}=x_0$. Moreover, given $\varepsilon>0$, by the definition of $z(f)$, there is $n_0\in\mathbb{N}$ such that

$$d(f(\alpha_{n,k_n}),z(f)(x_0))<\varepsilon/2 \ \text{ and } \ d(f(\alpha_{n,k_n}),z(f)(x_n))<\varepsilon/2 \ \text{ for all } n\geq n_0$$

and therefore

$$d(z(f)(x_n),z(f)(x_0))<\varepsilon \ \text{ for all } n\geq n_0.$$

Thus the function $z(f)$ is continuous.

In order to define T on the morphisms in $\mathcal{C}_0(\mathcal{Z})$, we need to state that for two C_0-close maps f and g, the equality $z(f)=z(g)$ holds. Suppose that the coarse maps $f,g:Q\setminus X\longrightarrow Q\setminus Y$ are C_0-close, that is to say, $E=\{(f(x),g(x)):x\in Q\setminus X\}$ is C_0-controlled in $Q\setminus Y$. Take a point $x_0\in X$ and a sequence $\{\alpha_n\}_{n\in\mathbb{N}}\subset Q\setminus X$ such that $\lim_{n\to\infty}\alpha_n=x_0$. Since $z(f)(x_0)=\lim_{n\to\infty}f(\alpha_n)\in Y$ and $z(g)(x_0)=\lim_{n\to\infty}g(\alpha_n)\in Y$ we deduce that for any compact $K\subset Q\setminus Y$ there is a number $n(K)\in\mathbb{N}$ such that $(f(\alpha_n),g(\alpha_n))\in E\setminus(K\times K)$ for $n\geq n(K)$. So, from the C_0-controllability of the set E, it follows that $z(f)(x_0)=z(g)(x_0)$.

For any C_0-coarse map f let us denote by $[f]$ the C_0-close class of f. Now we define the functor on the morphisms as

$$T([f])=z(f).$$

<u>T is faithful</u>: Assume that $T([f])=T([g])$ and consider representative C_0-coarse maps $f,g:Q\setminus X\longrightarrow Q\setminus Y$. Suppose now that f and g are not C_0-close. In that case, as $E=\{(f(x),g(x)):x\in Q\setminus X\}$ is not C_0-controlled, there is a number $\varepsilon_0>0$ such that for each compact $K_n=Q\setminus B(Y,1/n)$, $n\in\mathbb{N}$, there exists $(\lambda_n,\rho_n)=(f(\alpha_n),g(\alpha_n))\in E\setminus(K_n\times K_n)$ with $d(\lambda_n,\rho_n)\geq\varepsilon_0$. Refining the process if necessary, we assume that $\lambda_n=f(\alpha_n)\notin K_n$ for all $n\in\mathbb{N}$. By its definition, roughly speaking, a coarse map transforms controlled sets into controlled sets; furthermore, a C_0-coarse map also transforms relatively compact subsets (the C_0-bounded ones) into relatively compact subsets. Therefore, as the accumulation points of $\{\lambda_n\}_{n\in\mathbb{N}}$ lie in Y, necessarily the accumulation points of $\{\alpha_n\}_{n\in\mathbb{N}}$ belong to X. In this manner, without loss of generality, we can assume that $\lim_{n\to\infty}\alpha_n=x_0$ for some element $x_0\in X$. Thus

$$\lim_{n\to\infty}f(\alpha_n)=z(f)(x_0)=T([f])(x_0)=T([g])(x_0)=z(g)(x_0)=\lim_{n\to\infty}g(\alpha_n),$$

and this equality contradicts the fact that $d(f(\alpha_n),g(\alpha_n))\geq\varepsilon_0>0$ for all $n\in\mathbb{N}$. So we conclude that the set $E=\{(f(x),g(x)):x\in Q\setminus X\}$ is C_0-controlled in $Q\setminus Y$ and hence $[f]=[g]$.

<u>T is full</u>: Given two compact Z-sets of Q, $X,Y\in OB(\mathcal{Z})$, and a continuous function $f:X\longrightarrow Y$, consider a continuous extension $F:Q\longrightarrow Q$ of f. Since Y is a Z-set there exists a homotopy, see [3] or [4], $H:Q\times I_Q\longrightarrow Q$, where $I_Q=[0,\operatorname{diam}(Q)]$, with $H_0\equiv id_Q$ and $H_t(Q)\cap Y=\varnothing$ for any $t\in I_Q\setminus\{0\}$. The function

$$\begin{array}{rccl}\tilde{f}: & Q & \longrightarrow & Q\\ & x & \mapsto & \tilde{f}(x)=H(F(x),d(x,X))\end{array}$$

is continuous and satisfies

$$\tilde{f}\mid_X\equiv f \ \text{ and } \ \tilde{f}(Q\setminus X)\subset Q\setminus Y.$$

So the map

$$\hat{f}\equiv\tilde{f}\mid_{Q\setminus X}:(Q\setminus X,d)\longrightarrow(Q\setminus Y,d)$$

is a proper (in the topological sense) uniformly continuous function because $\hat{f}$ is the restriction of the uniformly continuous function $\tilde{f}$. Moreover, for any compact set $K \subset Q \setminus Y$, $\hat{f}^{-1}(K) = \tilde{f}^{-1}(K)$ is then compact. In order to establish that $\hat{f}$ is a C_0-coarse map, we only need to state that $(\hat{f} \times \hat{f})(E)$ is a C_0-controlled set in $(Q \setminus Y) \times (Q \setminus Y)$ for any C_0-controlled set $E \subset (Q \setminus X) \times (Q \setminus X)$. Indeed, for each $\varepsilon > 0$, from the uniform continuity of $\hat{f}$ there is $\delta > 0$ such that $d(\hat{f}(x), \hat{f}(x')) < \varepsilon$ whenever $x, x' \in Q \setminus X$ with $d(x, x') < \delta$. As E is C_0-controlled it is possible to choose a compact subset $K_\delta \subset (Q \setminus X) \times (Q \setminus X)$ with the property that if $(x, x') \in E \setminus (K_\delta \times K_\delta)$, then $d(x, x') < \delta$. For every pair $(\hat{f}(x), \hat{f}(x')) \in (\hat{f} \times \hat{f})(E) \setminus (\hat{f}(K_\delta) \times \hat{f}(K_\delta))$, we have $(x, x') \in E \setminus (K_\delta \times K_\delta)$; hence $d(x, x') < \delta$ and consequently $d(\hat{f}(x), \hat{f}(x')) < \varepsilon$. This shows that $\hat{f}$ is a C_0-coarse map and it is obvious that $T([\hat{f}]) = f$.

<u>Functorial behavior</u>: $T([g] \circ [f]) = T([g]) \circ T([f])$. Let $f : (Q \setminus X, \mathcal{E}_0) \longrightarrow (Q \setminus Y, \mathcal{E}_0)$ and $g : (Q \setminus Y, \mathcal{E}_0) \longrightarrow (Q \setminus Z, \mathcal{E}_0)$ be two C_0-coarse maps for $X, Y, Z \in \mathcal{Z}$. Take a sequence $\{x_n\}_{n \in \mathbb{N}} \subset Q \setminus X$ with $\lim_{n \to \infty} x_n = x_0 \in X$. Then $T([f])(x_0) = \lim_{n \to \infty} f(x_n)$ and

$$\begin{aligned} T([g] \circ [f])(x_0) &= \lim_{n \to \infty} (g \circ f)(x_n) = \lim_{n \to \infty} (g(f(x_n)) \\ &= T([g])\big(T([f])(x_0)\big) = \big(T([g]) \circ T([f])\big)(x_0). \end{aligned}$$

Note finally that

$$T([id_{Q \setminus X}]) = id_X. \qquad \square$$

Following the definition of *coarsely equivalent* coarse spaces given in [14, page 25] we have:

Corollary 3. *Given a metric Hilbert cube (Q, d), two compact spaces $X, Y \in OB(\mathcal{Z})$ are homeomorphic if and only if the coarse spaces $(Q \setminus X, \mathcal{E}_0)$ and $(Q \setminus Y, \mathcal{E}_0)$ are coarsely equivalent.*

Remark 3.1. Note that in our results we fix a metric d in the Hilbert cube. In fact if we consider another equivalent metric d' and we have two Z-sets $X \subset (Q, d)$ and $X' \subset (Q, d')$, then X and X' are homeomorphic if and only if the coarse spaces $Q \setminus X$ and $Q \setminus X'$ with the corresponding C_0-coarse structures are coarsely equivalent. On the other hand the concept of Z-set is independent of the choice of metric.

By a uniformly continuous homeomorphism between two metric spaces we mean a homeomorphism f such that both f and f^{-1} are uniformly continuous. Observe also that a homeomorphism $f : (Q \setminus X, d) \longrightarrow (Q \setminus Y, d)$ is a C_0-coarse map if and only if the map f, but perhaps not f^{-1}, is uniformly continuous.

Proposition 4. *Let $X, Y \in OB(\mathcal{Z})$ be two compact spaces in a metric Hilbert cube (Q, d).*

a) *If $f : (Q \setminus X, d) \longrightarrow (Q \setminus Y, d)$ is a uniformly continuous homeomorphism, then $[f]$ is a C_0-coarse equivalence. Conversely, if $[\alpha] : (Q \setminus X, \mathcal{E}_0) \longrightarrow (Q \setminus Y, \mathcal{E}_0)$ is a coarse equivalence, then there is a uniformly continuous homeomorphism $f : (Q \setminus X, d) \longrightarrow (Q \setminus Y, d)$ with $f \in [\alpha]$.*
b) *For any hereditary shape equivalence $h : X \longrightarrow Y$, there exists a C_0-coarse homeomorphism $f : (Q \setminus X, \mathcal{E}_0) \longrightarrow (Q \setminus Y, \mathcal{E}_0)$ such that $T([f]) = h$.*

Conversely, if $f : (Q\setminus X, \mathcal{E}_0) \longrightarrow (Q\setminus Y, \mathcal{E}_0)$ is a C_0-coarse homeomorphism, then $T([f]) : X \longrightarrow Y$ is a hereditary shape equivalence.

c) *For any C_0-coarse map $\alpha : (Q \setminus X, \mathcal{E}_0) \longrightarrow (Q \setminus Y, \mathcal{E}_0)$ there is a proper and uniformly continuous map $f : (Q \setminus X, d) \longrightarrow (Q \setminus Y, d)$ such that α and f are C_0-close.*

Proof. Note that the proof of c) was really done in the demonstration of Theorem 2.

Observe also that b) was in fact stated in [11].

To prove a) we only need to apply the Extension Homeomorphism Theorem ([4]) because if $[\alpha] : (Q \setminus X, \mathcal{E}_0) \longrightarrow (Q \setminus Y, \mathcal{E}_0)$ is an equivalence in $\mathcal{C}_0(\mathcal{Z})$, then $T([\alpha]) : X \longrightarrow Y$ is a surjective homeomorphism. Hence, there is a surjective homeomorphism of pairs $F : (Q, X) \longrightarrow (Q, Y)$ with $F\mid_X = T([\alpha])$ and, consequently, $[F\mid_{Q\setminus X}] = [\alpha]$ and $[F^{-1}\mid_{Q\setminus Y}] = [\alpha]^{-1}$. The converse is obvious. □

In order to strengthen our construction above with the development of coarse geometry we have the following result (for the concepts of Higson-Roe compactification and Higson-Roe corona, see [14]).

Proposition 5. *Given a metric Hilbert cube (Q, d) and a compact space $X \in OB(\mathcal{Z})$, the Higson-Roe compactification of the coarse space $(Q\setminus X, \mathcal{E}_0)$ is precisely Q and its Higson-Roe corona is X.*

Proof. Following [14], page 29, it is sufficient to prove that the C^*-algebra formed by the Higson functions on the coarse space $(Q\setminus X, \mathcal{E}_0)$ is *-isomorphic to $C(Q)$ (the C^*-algebra of all complex valued continuous functions on the Hilbert cube). To do so, recall that a Higson function on $(Q \setminus X, \mathcal{E}_0)$ is a bounded continuous function $f : Q \setminus X \longrightarrow \mathbb{C}$ such that the map

$$\begin{array}{rccc} df: & (Q \setminus X) \times (Q \setminus X) & \longrightarrow & \mathbb{C} \\ & (x, y) & \mapsto & f(y) - f(x), \end{array}$$

when restricted to any controlled set $E \in \mathcal{E}_0$, vanishes at infinity. Note that if $g \in C(Q)$, then $g\mid_{Q\setminus X}$ is a Higson function. Conversely, consider a Higson function $f : Q \setminus X \longrightarrow \mathbb{C}$ and fix $x \in X$. Take a sequence $\{\alpha_n\} \subset Q \setminus X$ converging to x. Consider the C_0-controlled set $E = \{(\alpha_n, \alpha_{n+1}) : n \in \mathbb{N}\}$. Using analogous arguments to those in Theorem 2 we obtain that $F(x) = \lim_{n\to\infty} f(\alpha_n)$ defines a continuous extension of f to Q. So we have proved that *f is a Higson function on $(Q \setminus X, \mathcal{E}_0)$ if and only if $f : (Q \setminus X, d) \longrightarrow \mathbb{C}$ is uniformly continuous (when considering the usual metric on $\mathbb{C}$).* Since (Q, d) is the metric completion of $(Q\setminus X, d)$, a continuous complex function f on $Q\setminus X$ is a Higson function if and only if f is continuously extendable to Q. This fact, jointly with the density of $Q \setminus X$ in Q, provides a *-isomorphism between the C^*-algebra of the Higson functions on the coarse space $(Q \setminus X, \mathcal{E}_0)$ and $C(Q)$. □

In fact more can be said. If we denote by $\mathcal{E}_1$ the *continuously coarse structure* induced by the compactification Q in $Q \setminus X$, see [14], we have

Proposition 6. $\mathcal{E}_0 = \mathcal{E}_1$.

Proof. First, take $E \in \mathcal{E}_0$. We are going to prove that (see [14], Theorem 2.27.a)

$$A = Cl_{Q\times Q}(E) \cap \big[(Q \times Q) \setminus (Q \setminus X) \times (Q \setminus X)\big] \subset \Delta_X.$$

Consider a point $(x_0, y_0) \in A$ and a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset E$ such that

$$\lim_{n\to\infty} (x_n, y_n) = (x_0, y_0).$$

If we assume $x_0 \in X$ and $x_0 \neq y_0$, we have $d(x_0, y_0) = \delta > 0$. Define $\varepsilon = \delta/2$ and let K be a compact subset of $Q \setminus X$. Since $Q \setminus K$ is an open neighborhood of X, there exists a natural number n_1 such that $x_n \in Q \setminus K$ for all $n \geq n_1$. Due to the convergence of $\{x_n\}_{n\in\mathbb{N}}$ to x_0 and $\{y_n\}_{n\in\mathbb{N}}$ to y_0, there exists $n_2 \in \mathbb{N}$, $n_2 \geq n_1$, such that

$$d(x_{n_2}, y_{n_2}) > \delta/2 \qquad \text{and} \qquad (x_{n_2}, y_{n_2}) \in E \setminus (K \times K).$$

Therefore $E \notin \mathcal{E}_0$, a contradiction.

On the other hand, take $E \in \mathcal{E}_1$ and assume that $E \notin \mathcal{E}_0$, that is to say, there exists $\varepsilon > 0$ such that for each compact $K_n = Q - B(X, 1/n) \subset Q \setminus X$, $n \in \mathbb{N}$, it is possible to choose $(x_n, y_n) \in E \setminus (K_n \times K_n)$ such that $d(x_n, y_n) \geq \varepsilon$. From the compactness of $Q \times Q$, the sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ has an accumulation point (x_0, y_0). Necessarily this accumulation point belongs to the closure of E in $Q \times Q$ and, by the construction of the sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$, $x_0 \in X$ or $y_0 \in X$. Therefore $(x_0, y_0) \notin (Q \setminus X) \times (Q \setminus X)$. As $d(x_n, y_n) \geq \varepsilon$ for all $n \in \mathbb{N}$, $d(x_0, y_0) \geq \varepsilon$. Hence $x_0 \neq y_0$ and $(x_0, y_0) \notin \Delta_X$. This contradicts the fact that $E \in \mathcal{E}_1$ (see [14], Theorem 2.27.a). □

3. Relating invariants of Z-sets and their complements

Now we can reinterpret some topological invariants of a Z-set X in terms of the coarse space $(Q \setminus X, \mathcal{E}_0)$. In particular

Corollary 7. *Let X be a Z-set in the Hilbert cube Q. X is finite dimensional if and only if the asymptotic dimension of $(Q \setminus X, \mathcal{E}_0)$ is finite. Moreover,*

$$\dim(X) = \operatorname{asdim}(Q \setminus X, \mathcal{E}_0) - 1.$$

Proof. Since X is the Higson-Roe corona of $(Q \setminus X, \mathcal{E}_0)$, the result follows from [7, Theorem 2.5.7 page 32]. □

Using any of the equivalent definitions of the asymptotic dimension of a coarse structure ([7, 14]), we have descriptions, from the outside, of the dimension of a Z-set in the Hilbert cube. That way we answer Question 1 in [1].

One of the many ways to describe the topological dimension of a compact metric space is that of using extensions of maps to spheres (see [10, chapter VI]). Part a) of the next result is an analog for the asymptotic dimension of $(Q \setminus X, \mathcal{E}_0)$. We use this to prove part b), that is related to a result in [11], stated there without proof.

Proposition 8. *Consider a metric Hilbert cube (Q, d) and let S^n be any Z-copy of the n-sphere. Suppose that $X \subset (Q, d)$ is a compact Z-set. Then the inequality* $\operatorname{asdim}(Q \setminus X, \mathcal{E}_0) \leq n + 1$, *or* $\dim(X) \leq n$, *is equivalent to any of the following facts:*

a) *Any uniformly continuous and proper map $f : A \longrightarrow (Q \setminus S^n, d)$ is extendable to a uniformly continuous proper map $F : (Q \setminus X, d) \longrightarrow (Q \setminus S^n, d)$, where A is any closed subset of $(Q \setminus X, d)$.*

b) *Any uniformly continuous and proper map $f : A \longrightarrow (\mathring{I}^{n+1}, d')$ is extendable to a uniformly continuous proper map $F : (Q \setminus X, d) \longrightarrow (\mathring{I}^{n+1}, d')$, where*

$\overset{\circ}{I}^{n+1}$ denotes the combinatorial interior of the $(n+1)$-cube, d' is any metric inducing the usual topology in the cube I^{n+1} and A is any closed subset of $(Q \setminus X, d)$.

Proof. To prove a) suppose $\operatorname{asdim}(Q \setminus X, \mathcal{E}_0) \leq n+1$. Using Corollary 7, it is equivalent to $\dim X \leq n$. For any Z-set Y, we denote by $H^Y : Q \times I_Q \longrightarrow Q$, where $I_Q = [0, \operatorname{diam}(Q)]$, a homotopy such that $H_0^Y \equiv id_Q$ and $H_t^Y(Q) \cap Y = \varnothing$ for any $t \in I_Q \setminus \{0\}$ (see [3] or [4]). Fix $A \subset Q \setminus X$ a closed subset and a uniformly continuous and proper map $f : A \longrightarrow (Q \setminus S^n, d)$. Define the trace of A as $Tr(A) = \overline{A}^Q \cap X$. Since f is a uniformly continuous map, there is a continuous extension $\overline{f} : A \cup Tr(A) \longrightarrow Q$ of f with $\overline{f}(Tr(A)) \subset S^n$ due to properness. Now $\overline{f}\mid_{Tr(A)}$ is extendable to a continuous map $\hat{f} : X \longrightarrow S^n$ (see [10, Theorem VI.4]). Clearly the function

$$\begin{array}{rccl} g: & X \cup A & \longrightarrow & Q \\ & y & \mapsto & g(y) = \begin{cases} \hat{f}(y) & \text{if } y \in X \\ \overline{f}(y) & \text{if } y \in A \end{cases} \end{array}$$

is continuous. Since Q is an AR, there is a continuous extension $G : Q \longrightarrow Q$ of g. Finally consider the function

$$\begin{array}{rccl} F: & Q & \longrightarrow & Q \\ & x & \mapsto & F(x) = H^{S^n}(G(x), d(x, X \cup A)). \end{array}$$

By the construction $F(Q \setminus X) \subset Q \setminus S^n$ and it is easy to see that $F\mid_{Q \setminus X}$ is an extension of f that satisfies all the required conditions.

To prove the converse take $C \subset X$ a closed subset and a continuous function $h : C \longrightarrow S^n$. Consider a homotopy H^X as above. $A = H^X(C \times I_Q) \setminus C$ is a closed subset of $Q \setminus X$ whose trace is simply C. Let $f : (A, d) \longrightarrow (Q \setminus S^n, d)$ be defined by

$$f(x) = H^{S^n}(\tilde{h}(x), d(x, C))$$

where $\tilde{h} : Q \longrightarrow Q$ is any continuous extension of h. f is proper and uniformly continuous. So there is a proper and uniformly continuous extension $F : (Q \setminus X, d) \longrightarrow (Q \setminus S^n, d)$; therefore F is a C_0-coarse map. Using Theorem 2 we obtain that $T([F]) : X \longrightarrow S^n$ is a continuous extension of h. Consequently $\dim X \leq n$ and now, from Corollary 7, $\operatorname{asdim}(Q \setminus X, \mathcal{E}_0) \leq n+1$.

To prove b) suppose that $\operatorname{asdim}(Q \setminus X, \mathcal{E}_0) \leq n+1$. Consider a uniformly continuous and proper map $f : A \longrightarrow (\overset{\circ}{I}^{n+1}, d')$, with $A \subset Q \setminus X$ a closed set. By compactness, we can suppose that I^{n+1} is a closed Z-set in Q and that the metric d' is inherited from Q. Note that the combinatorial boundary S^n of I^{n+1} is a Z-set in Q. Therefore $f(A) \subset Q \setminus S^n$. Then, by a), there is a proper uniformly continuous extension $\tilde{f} : Q \setminus X \longrightarrow Q \setminus S^n$. Take a retraction $r : Q \longrightarrow I^{n+1}$. Define $R : Q \longrightarrow I^{n+1}$ by $R(x) = H^{S^n}(r(x), d(x, I^{n+1}))$, where H^{S^n} is a homotopy realizing the fact that the combinatorial boundary is a Z-set in I^{n+1}. Observe that $R(Q \setminus S^n) \subset \overset{\circ}{I}^{n+1}$. Consequently, $R \circ \tilde{f} : Q \setminus X \longrightarrow \overset{\circ}{I}^{n+1}$ is a uniformly continuous proper extension of the initial map f.

To prove the converse take $C \subset X$ to be a closed subset and let $h : C \longrightarrow S^n$ be a continuous function. Consider S^n as the combinatorial boundary of I^{n+1}. Since I^{n+1} is an AR space there is a continuous extension $\tilde{h} : Q \longrightarrow I^{n+1}$. Let $G : Q \longrightarrow I^{n+1}$ be defined by $G(x) = H^{S^n}(\tilde{h}(x), d(x, C))$. Note that $G(Q \setminus C) \subset$

$\overset{\circ}{I}^{n+1}$. Consider $A = H^X(C \times I_Q) \setminus C$. A is closed in $Q \setminus X$. By the hypothesis, $G\mid_A$ can be extended to a proper uniformly continuous function $\tilde{G} : Q \setminus X \longrightarrow \overset{\circ}{I}^{n+1}$. So there is a continuous extension to the corresponding metric completions $\overline{G} : Q \longrightarrow I^{n+1}$ with $\overline{G}(X) \subset S^n$. By construction, $\overline{G}\mid_X$ is a continuous extension of h. Consequently $\dim(X) \leq n$; hence $\operatorname{asdim}(Q \setminus X, \mathcal{E}_0) \leq n+1$. □

From the isomorphism of categories in Theorem 2 another of the topological invariants of a metric compactum that can be clearly described from the outside is the algebraic type of the group of autohomeomorphisms. In [14, page 112] the *quasi-isometry group* of a coarse space Y, $\operatorname{Qis}(Y)$, is introduced as the group of closeness classes of coarse equivalences in Y. For each Z-set X in the Hilbert cube (Q, d), where d is any fixed metric, denote by $H_u(Q \setminus X)$ the group of uniform autohomeomorphisms of $(Q \setminus X, d)$. Let $H_{id}(Q \setminus X)$ be the normal subgroup of $H_u(Q \setminus X)$ defined by

$$H_{id}(Q \setminus X) = \{f \in H_u(Q \setminus X) \,:\, T([f]) = id_X\}.$$

Note that $H_{id}(Q \setminus X)$ is, up to the obvious identification, the subgroup of the group $H(Q)$ (of all autohomeomorphisms of the Hilbert cube) leaving fixed all points in X.

Using the Homeomorphism Extension Theorem for part b), one can prove (we leave it to the reader):

Corollary 9. *The group of autohomeomorphisms $H(X)$ of a compactum $X \in \mathcal{Z}$ is isomorphic to the following groups:*

a) $\operatorname{Qis}(Q \setminus X)$*, where we consider $Q \setminus X$ as an object in $\mathcal{C}_0(\mathcal{Z})$;*
b) $\dfrac{H_u(Q \setminus X)}{H_{id}(Q \setminus X)}$.

Using again the isomorphism T constructed in Theorem 2 one can prove the following:

Proposition 10. *Let (Q, d) be a fixed metric Hilbert cube. The homotopy category of compact Z-sets in Q (and the homotopy category of compact metrizable spaces) is isomorphic to the proper uniformly continuous homotopy category of their complements with the inherited metrics. This isomorphism transforms any compact Z-set into its complement.*

Proof. Denote by $\mathcal{C}_1$ the category whose objects are the complements, in Q, of compact Z-sets and the morphisms are the uniform and proper homotopy classes of uniformly continuous and proper functions. Let $\mathcal{C}_2$ be the category whose objects are the compact Z-sets in Q and the morphisms are the classes of homotopic continuous functions. We assert that there is an isomorphism of categories

$$[T] : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$$

whose action on objects is defined by

$$[T](Q \setminus X) = X$$

and on morphisms by

$$[T]([f]) = [T(f)]_H$$

where T is the isomorphism of categories constructed in Theorem 2 and $[T(f)]_H$ is the homotopy class of the map $T(f)$.

[T] is well-defined: Given two homotopic proper and uniformly continuous functions

$$f, g : Q \setminus X \longrightarrow Q \setminus Y$$

there exists a proper and uniformly continuous homotopy

$$H : (Q \setminus X) \times I \longrightarrow Q \setminus Y$$

from $f \equiv H(\cdot, 0)$ to $g \equiv H(\cdot, 1)$. As $(Q \setminus X) \times I = (Q \times I) \setminus (X \times I)$, $(Q \times I)$ is homeomorphic to Q and $X \times I$ is a compact Z-set in the Hilbert cube $(Q \times I)$, it is possible to extend H to another homotopy

$$\hat{H} : Q \times I \longrightarrow Q$$

such that

$$\hat{H} \mid_{X \times I} : X \times I \longrightarrow Y.$$

Since $\hat{H}(\cdot, 0)$ and $\hat{H}(\cdot, 1)$ are continuous extensions to Q of f and g respectively, necessarily the homotopy $\hat{H} \mid_{X \times I}$ connects $T(f)$ with $T(g)$. In this way, the definition of the functor $[T]$ on morphisms is correct.

[T] is faithful: Suppose that $[T]([f_1]) = [T]([f_2])$ for some maps $f_1, f_2 : Q \setminus X \longrightarrow Q \setminus Y$. As the maps $T(f_1), T(f_2) : X \longrightarrow Y$ are homotopic, there exists a homotopy $G : X \times I \longrightarrow Y$ such that $G(\cdot, 0) \equiv T(f_1)$ and $G(\cdot, 1) \equiv T(f_2)$. Recall that if we paste the functions f_i and $T(f_i)$ $(i \in \{1, 2\})$, they give rise to continuous functions $\hat{f}_1, \hat{f}_2 : Q \longrightarrow Q$.

Consider the closed subset of $Q \times I$,

$$C = (X \times I) \cup (Q \times \{0\}) \cup (Q \times \{1\}),$$

and define the continuous function

$$\begin{array}{llll} \tilde{H} : & C & \longrightarrow & Q \\ & (x, t) & \mapsto & \tilde{H}(x, t) = \begin{cases} \hat{f}_1(x) & \text{if } t = 0 \\ \hat{f}_2(x) & \text{if } t = 1 \\ G(x, t) & \text{if } x \in X. \end{cases} \end{array}$$

Since Q is an AR space, $\tilde{H}$ extends to a continuous map $\hat{H} : Q \times I \longrightarrow Q$. Denote by H^Y, as in previous results, a homotopy related to the compact Z-set Y and define the homotopy

$$\begin{array}{llll} H : & Q \times I & \longrightarrow & Q \\ & (x, t) & \mapsto & H(x, t) = H^Y(\tilde{H}(x, t), d((x, t), C)). \end{array}$$

Clearly, $H(\cdot, 0) \equiv \hat{f}_1$, $H(\cdot, 1) \equiv \hat{f}_2$ and $H(x, t) = G(x, t)$ for all $x \in X$ and $t \in I$. Due to the fact that $H^{-1}(Y) = X \times I$, the restriction

$$H \mid_{(Q \setminus X) \times I} : (Q \setminus X) \times I \longrightarrow Q \setminus Y$$

is a proper and uniformly continuous homotopy between f_1 and f_2.

[T] is full: It follows straightforwardly from the fullness of the functor T (see Theorem 2). □

In the proposition above the phrase *proper uniformly continuous homotopy category* means that the homotopy should also be proper and uniformly continuous if one considers the maximum metric when crossing by a closed bounded real interval.

In the remainder of the paper we are going to concentrate on the task of describing the ANR-property of a compact Z-set by means of its complement. The

main reason is that the ANR-property is a necessary one for being a manifold. The following results could be considered a modest contribution to the second question in [1]. We think they are of their own interest because the ANR-property is also important for other problems in topology.

The concepts of Absolute Neighborhood Retract (ANR) and Absolute Neighborhood Extensor (ANE) are equivalent for metrizable spaces [2, 9]. We are going to define our corresponding candidates from the outside. First of all fix any metric d in Q inducing its topology. We are going to consider $(Q\setminus X, d\,|_{(Q\setminus X)\times(Q\setminus X)})$ for any compact Z-set X in Q. From now on we omit the reference to the induced metrics but understanding that when we refer to uniform properties we mean related to those metrics. So, from now on, $f : Q\setminus X \longrightarrow Q\setminus Y$ is a *uniform embedding* if f is a topological embedding such that both f and f^{-1}, in the corresponding domain, are uniformly continuous with the corresponding relative metrics. By a *uniform neighborhood* of a subset we mean a neighborhood which contains an ε-neighborhood of the subset.

Let us consider the following.

Definition 1. $Q\setminus X$ is a C_0-ANR space if for every uniform embedding $f : Q\setminus X \longrightarrow Q\setminus Y$, as closed subset, there exists a uniform neighborhood U of $f(Q\setminus X)$ in $Q\setminus Y$ and a retraction $r : U \longrightarrow f(Q\setminus X)$ that is uniformly continuous and proper.

Definition 2. $Q\setminus X$ is a C_0-ANE space if every uniformly continuous and proper map from a closed subset $\tilde{A}$ of $Q\setminus Y$ to $Q\setminus X$ admits a uniformly continuous and proper extension to a uniform neighborhood of $\tilde{A}$ in $Q\setminus Y$.

The following result will be useful.

Proposition 11. *If X and Y are compact Z-sets in (Q, d), then $Q\setminus X$ admits a uniform embedding, as a closed subset, into $Q\setminus Y$ if and only if X can be topologically embedded into Y.*

Proof. If $f : Q\setminus X \longrightarrow Q\setminus Y$ is a closed uniform embedding and $F : X \longrightarrow Y$ is the natural map induced by f as in Theorem 2, then F is clearly a topological embedding.

To prove the remaining part we can suppose that the topological embedding of X into Y is just an inclusion. So consider

$$X \subset Y.$$

Finally consider $Q\times\{0\} \subset Q\times I$. Note that $Q\times\{0\}$ is an isometric copy of (Q, d) if we consider the maximum metric in $Q\times I$ which is again a Hilbert cube. Let $\mu : Q \longrightarrow [0, 1]$ be a continuous function such that $\mu^{-1}(0) = X$. Clearly

$$\begin{array}{rccc} h: & Q & \longrightarrow & Q\times I \\ & q & \mapsto & (q, \mu(q)) \end{array}$$

is a topological embedding such that $h(Q)\cap Q\times\{0\} = X\times\{0\}$. Moreover,

$$h(Q\setminus X) \subset Q\times(0, 1] \subset Q\times I\setminus Y\times\{0\}.$$

Furthermore $h(Q\setminus X)$ is closed in $Q\times I\setminus Y\times\{0\}$. Using now the Homeomorphism Extension Theorem and the fact that $Y\times\{0\}$ is a Z-set in $Q\times I$ one easily obtains that $(Q\setminus X, d)$ can be uniformly embedded as a closed subset in $(Q\setminus Y, d)$. $\square$

From now on we are going to denote by $\mathcal{Q}$ any Z-copy of the Hilbert cube inside (Q, d). Now we have

Proposition 12. *$Q \setminus X$ is a C_0-ANR space if and only if for every uniform embedding of $Q \setminus X$ in $Q \setminus \mathcal{Q}$, as a closed subset, there exists a uniformly continuous and proper retraction $r : U \longrightarrow Q \setminus X$, where U is a uniform neighborhood of $Q \setminus X$ in $Q \setminus \mathcal{Q}$.*

Proof. From the previous proposition, $Q \setminus X$ admits a uniform embedding

$$j : Q \setminus X \longrightarrow Q \setminus \mathcal{Q},$$

as a closed subset, into $Q \setminus \mathcal{Q}$.

Suppose that $Q \setminus X$ is a C_0-ANR space. From the hypothesis, there exists a uniform neighborhood U of $j(Q \setminus X)$ in $Q \setminus \mathcal{Q}$ and a retraction $r : U \longrightarrow j(Q \setminus X)$ that is uniformly continuous and proper.

Conversely, let $f : Q \setminus X \longrightarrow Q \setminus Y$ be a uniform embedding, as a closed subset. Take $i : Q \setminus Y \longrightarrow Q \setminus \mathcal{Q}$ to be a closed uniform embedding. The composition

$$i \circ f : Q \setminus X \longrightarrow Q \setminus \mathcal{Q}$$

is a uniform embedding of $Q \setminus X$ into $Q \setminus \mathcal{Q}$ as a closed subset. So there exists a uniformly continuous and proper retraction $r : U \longrightarrow (i \circ f)(Q \setminus X)$, where U is a uniform neighborhood of $(i \circ f)(Q \setminus X)$ in $Q \setminus \mathcal{Q}$. Since i is a uniformly continuous map, there exists a uniform and closed neighborhood of the subset $f(Q \setminus X)$ in $Q \setminus Y$, W, such that $i(W) \subset U$. In this manner, the function

$$\begin{array}{rccc} r' : & W & \longrightarrow & f(Q \setminus X) \\ & z & \mapsto & (i^{-1} \circ r \circ i)(z) \end{array}$$

is uniformly continuous and a proper retraction. □

We will also establish the following equivalence.

Proposition 13. *$Q \setminus X$ is a C_0-ANR space if and only if $Q \setminus X$ is a C_0-ANE space.*

Proof. Assume that $Q \setminus X$ is a C_0-ANR space and let $f : \tilde{A} \longrightarrow Q \setminus X$ be a uniformly continuous and proper map from a closed subset $\tilde{A}$ of $Q \setminus Y$. Since f is a proper map, Q is an AE space and X is a Z-set, we can extend it to a continuous function $F : Q \longrightarrow Q$ such that $F^{-1}(X) \subset Y$. Consider a uniform embedding

$$j : Q \setminus X \longrightarrow Q \setminus \mathcal{Q}$$

as a closed subset. As $Q \setminus X$ is a C_0-ANR, there exists a uniform neighborhood U of $j(Q \setminus X)$ in $Q \setminus \mathcal{Q}$ and a retraction $r : U \longrightarrow j(Q \setminus X)$ that is uniformly continuous and proper. Moreover, as in the proof of Proposition 11, it is possible to extend j to a continuous function $\tilde{j} : Q \longrightarrow Q$ such that $\tilde{j} \mid_X : X \longrightarrow \tilde{j}(X) \subset \mathcal{Q}$ is a homeomorphism. Since $\tilde{j} \circ F : Q \longrightarrow Q$ is a uniformly continuous map, there exists a uniform and closed neighborhood of $\tilde{A}$ in $Q \setminus Y$, W, such that $(\tilde{j} \circ F)(W) \subset U$. In this way, the function $\tilde{f} : W \longrightarrow Q \setminus X$ defined by

$$\tilde{f} \equiv \tilde{j}^{-1} \circ r \circ \tilde{j} \circ F \mid_W$$

is uniformly continuous and is a proper extension of f.

Now suppose that $Q \setminus X$ is a C_0-ANE space and let

$$i : Q \setminus X \longrightarrow Q \setminus \mathcal{Q}$$

be a uniform embedding, as a closed subset, of $Q \setminus X$ in $Q \setminus \mathcal{Q}$. Clearly

$$i^{-1} : i(Q \setminus X) \longrightarrow Q \setminus X$$

is a uniformly continuous and proper map from a closed subset $i(Q \setminus X)$ of $Q \setminus \mathcal{Q}$ to $Q \setminus X$. Therefore i^{-1} admits a uniformly continuous and proper extension to a uniform neighborhood U of $i(Q \setminus X)$ in $Q \setminus \mathcal{Q}$:

$$s : U \longrightarrow Q \setminus X.$$

Finally,

$$r \equiv i \circ s : U \longrightarrow i(Q \setminus X)$$

is uniformly continuous and a proper retraction. □

The following result, which is inspired by [11], is our characterization, from the outside, of the ANR-property of compacta. We prefer to use the extension version in the outside instead of the retraction version used in a related analogous result of [11].

Proposition 14. *Let X be a compact metric space embedded in the Hilbert cube (Q, d) as a Z-set. $Q \setminus X$ is a C_0-ANE space if and only if X is an ANR space.*

Proof. Let X be a compact metric space embedded in the Hilbert cube (Q, d) as a Z-set, such that $Q \setminus X$ is a C_0-ANE space. Take a closed subset A of another compact metric space Y and a continuous function $f : A \longrightarrow X$. From a homotopy $H^Y : Q \times I \longrightarrow Q$ associated to the Z-set Y, we construct a closed subset of $Q \setminus Y$ defined by $\tilde{A} = H^Y(A \times I) \setminus A$. As Q is an AR space there exists a continuous extension $\tilde{f}$ of f, $\tilde{f} : \tilde{A} \cup A \longrightarrow Q$. The function

$$\begin{array}{rccl} \hat{f} : & \tilde{A} \cup A & \longrightarrow & Q \\ & z & \mapsto & \hat{f}(z) = H^X\big(\tilde{f}(z), d(z, A)\big) \end{array}$$

is continuous between compacta. Since $\hat{f}^{-1}(X) = A$, the restriction

$$\hat{f}\mid_{\tilde{A}} : \tilde{A} \longrightarrow Q \setminus X$$

is a uniformly continuous and proper map. Hence, by hypothesis, it is possible to extend it, by means of another uniformly continuous and proper function h, to a uniform closed neighborhood of $\tilde{A}$ in $Q \setminus Y$,

$$h : \overline{B}(\tilde{A}, \varepsilon) \cap (Q \setminus Y) \longrightarrow Q \setminus X.$$

The properness of h allows us to extend it continuously from $Cl_Q(\overline{B}(\tilde{A}, \varepsilon) \cap (Q \setminus Y)) = \overline{B}(\tilde{A}, \varepsilon)$ to Q,

$$\tilde{h} : \overline{B}(\tilde{A}, \varepsilon) \longrightarrow Q.$$

Since $\overline{B}(A, \varepsilon) \cap Y \subset \overline{B}(\tilde{A}, \varepsilon) \cap Y$ and, necessarily, $\tilde{h}(\overline{B}(\tilde{A}, \varepsilon) \cap Y) \subset X$,

$$\tilde{h}\mid_{\overline{B}(A,\varepsilon)\cap Y} : \overline{B}(A, \varepsilon) \cap Y \longrightarrow X$$

is a continuous map which extends f to a closed neighborhood of A in Y.

Conversely, let X be an ANR compact metric space embedded in the Hilbert cube (Q, d) as a Z-set. Consider a closed subset $\tilde{A}$ of $Q \setminus Y$, where Y is another compact metric space embedded in the Hilbert cube (Q, d) as a Z-set, and a uniformly continuous and proper map $f : \tilde{A} \longrightarrow Q \setminus X$.

a) If $\mathrm{Tr}(\tilde{A}) = \varnothing$, then $\tilde{A}$ is compact and the function

$$\begin{array}{rccl} g: & \tilde{A} \cup Y & \longrightarrow & Q \\ & z & \mapsto & g(z) = \begin{cases} f(z)\,, & \text{if } z \in \tilde{A} \\ x_0\,, & \text{if } z \in Y \end{cases} \end{array}$$

for a fixed point $x_0 \in X$, is continuous. From a continuous extension $F : Q \longrightarrow Q$ of g define the map

$$\begin{array}{rccl} \tilde{F}: & Q & \longrightarrow & Q \\ & x & \mapsto & \tilde{F}(x) = H^X\big(F(x), d(x, \tilde{A} \cup Y)\big). \end{array}$$

Since $\tilde{F}$ is a uniformly continuous proper map and $\tilde{F}^{-1}(X) = Y$, the restriction

$$\tilde{F}\,|_{Q \setminus Y} \colon Q \setminus Y \longrightarrow Q \setminus X$$

is also uniformly continuous and a proper map which extends f.

b) If $\mathrm{Tr}(\tilde{A}) = \mathrm{Cl}_Q(\tilde{A}) \cap Y = A \neq \varnothing$, there exists a continuous extension of f, $\tilde{f} : \tilde{A} \cup A = \mathrm{Cl}_Q(\tilde{A}) \longrightarrow Q$ such that $\tilde{f}(A) \subset X$. As X is an ANR space, the map $\tilde{f}\,|_A \colon A \longrightarrow X$ admits a continuous extension, $\hat{f}$, to a closed neighborhood of A in Y. Since A is compact, we can assume that this closed neighborhood is $\overline{B}(A, \varepsilon) \cap Y$, for certain $\varepsilon > 0$,

$$\hat{f} : \overline{B}(A, \varepsilon) \cap Y \longrightarrow X.$$

From their coincidence in A, it is possible to paste the functions $\tilde{f}$ and $\hat{f}$. That way one obtains the continuous map

$$\begin{array}{rccl} \overline{f}: & \big(\overline{B}(A, \varepsilon) \cap Y\big) \cup \tilde{A} & \longrightarrow & Q \\ & x & \mapsto & \overline{f}(x) = \begin{cases} \tilde{f}(x)\,, & \text{if } x \in \tilde{A} \\ \hat{f}(x)\,, & \text{if } x \in \overline{B}(A, \varepsilon) \cap Y. \end{cases} \end{array}$$

Since the compact sets $\tilde{A} \cup A$ and $Y \setminus B(A, \varepsilon)$ are disjoint, there exists a number δ, $0 < \delta < \varepsilon$, such that

$$\overline{B}(\tilde{A}, \delta) \cap Y \subset \overline{B}(A, \varepsilon).$$

As Q is an AR space, the map $\overline{f}$ can be continuously extended to a function

$$g : \big(\overline{B}(A, \varepsilon) \cap Y\big) \cup \overline{B}(\tilde{A}, \delta) \longrightarrow Q.$$

Define

$$\begin{array}{rccl} \tilde{g}: & \big(\overline{B}(A, \varepsilon) \cap Y\big) \cup \overline{B}(\tilde{A}, \delta) & \longrightarrow & Q \\ & x & \mapsto & \tilde{g}(x) = H^X\big(g(x), d(x, \big(\overline{B}(A, \varepsilon) \cap Y\big) \cup \tilde{A})\big). \end{array}$$

Since $\tilde{g}$ is a uniformly continuous and proper map and $\tilde{g}^{-1}(X) = \overline{B}(A, \varepsilon) \cap Y$, the restriction

$$\tilde{g}\,|_{\left(\overline{B}(\tilde{A}, \delta)\right) \setminus Y} \colon \big(\overline{B}(\tilde{A}, \delta)\big) \setminus Y \longrightarrow Q \setminus X$$

is also uniformly continuous and a proper map which extends f to a uniform closed neighborhood of $\tilde{A}$ in $Q \setminus Y$.

$\square$

References

[1] M. Alonso-Morón and A. González Gómez, *The Hausdorff metric and classifications of compacta*, Bull. London Math. Soc. 38 (2006), 314-322. MR2215924 (2006m:54019)

[2] K. Borsuk, *Theory of retracts*, PWN-Polish Scientific Publishers, Warsaw, 1967. MR0216473 (35:7306)

[3] T.A. Chapman, *On some applications of infinite-dimensional manifolds to the theory of shape*, Fund. Math. 76 (1972), 181–193. MR0320997 (47:9530)

[4] T.A. Chapman, *Lectures on Hilbert cube manifolds*, American Mathematical Society, 1976. MR0423357 (54:11336)

[5] T.A. Chapman and L. Siebenmann, *Finding a boundary for a Hilbert cube manifold*, Acta Math. 137 (1976), 171–208. MR0425973 (54:13922)

[6] A. Dranishnikov, *Asymptotic Topology*, Russian Math. Surveys 55 (2000), no. 6, 71-116. MR1840358 (2002j:55002)

[7] B. Grave, *Asymptotic dimension of coarse spaces*, available at http://www.uni-math.gwdg.de/bgr/veroeffentl/, 2004.

[8] M. Gromov, *Asymptotic invariants of infinite groups*, London Math. Soc. Lecture Notes 182 (1993). MR1253544 (95m:20041)

[9] S-T. Hu, *Theory of retracts*, Wayne State University Press, Detroit, 1965. MR0181977 (31:6202)

[10] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, Princeton, NJ, 1941. MR0006493 (3:312b)

[11] P. Mrozik, *Hereditary shape equivalences and complement theorems*, Topology Appl. 22 (1986), 131–137. MR836320 (88a:57030)

[12] J. Roe, *Coarse cohomology and index theory on complete Riemannian manifolds*, Memoirs of the American Mathematical Society 497, 1993. MR1147350 (94a:58193)

[13] J. Roe, *Index Theory, Coarse Geometry, and Topology of Manifolds*, volume 90 of CBMS Conference Proceedings. American Mathematical Society, Providence, R.I., 1996. MR1399087 (97h:58155)

[14] J. Roe, *Lectures on Coarse Geometry*, American Mathematical Society, 2003. MR2007488 (2004g:53050)

[15] N. Wright, *C_0 coarse geometry*, Ph.D. thesis, Penn State, 2002.

[16] N. Wright, *C_0 coarse geometry and scalar curvature*, J. Funct. Anal. 197 (2003), 469–488. MR1960422 (2003m:58034)

[17] N. Wright, *The coarse Baum-Connes conjecture via C_0 coarse geometry*, J. Funct. Anal. 220 (2005), 265–303. MR2119281 (2006d:58023)

Departamento Matemática Aplicada, E.T.S.I. Montes, Universidad Politécnica, 28040 Madrid, Spain

E-mail address: eduardo.cuchillo@upm.es

Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996

E-mail address: dydak@math.utk.edu

Department of Mathematics, Shizuoka University, Shizuoka, Japan

E-mail address: sakoyam@ipc.shizuoka.ac.jp

Departamento Geometría y Topología, Facultad de Cc.Matemáticas, Universidad Complutense, 28040 Madrid, Spain

E-mail address: ma_moron@mat.ucm.es