TWISTED FIBER SUMS
OF FINTUSHEL-STERN’S KNOT SURGERY 4-MANIFOLDS

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Abstract. In the article, we study Fintushel-Stern’s knot surgery four-manifold \( E(n)_K \) and its monodromy factorization. For fibered knots we provide a smooth classification of knot surgery 4-manifolds up to twisted fiber sums. We then show that other constructions of 4-manifolds with the same Seiberg-Witten invariants are in fact diffeomorphic.

1. Introduction

In 1994, Seiberg-Witten invariants were introduced, and we have experienced significant progress in four dimensional topology. But a classification of smooth 4-manifolds is still out of reach. Several examples of four dimensional manifolds with infinitely many exotic smooth structures have been constructed by various authors. Underlying many of these constructions is Fintushel-Stern’s knot surgery \([FS98b]\).

Let \( X \) be a closed smooth 4-manifold which has a c-imbedded torus \( T \) and \( K \subset S^3 \) be a knot. Then Fintushel-Stern’s knot surgery 4-manifold is defined by

\[
X_K = X^*_T = T_m M_K \times S^1
\]

where \( M_K \) is the 3-manifold obtained by doing 0-surgery along \( K \), \( m \) is the meridian of \( K \) and \( T_m = m \times S^1 \). The Seiberg-Witten invariants of \( X_K \) are given by

\[
SW_{X_K} = SW_X \cdot \Delta_K(t)
\]

where \( t = \exp(2[T]) \) and \( \Delta_K(t) \) is the symmetrized Alexander polynomial of the knot \( K \) \([FS98a]\). By using this method, R. Fintushel and R. Stern could construct infinitely many nondiffeomorphic symplectic and nonsymplectic 4-manifolds in the same homeomorphism class. During the ICM 1998, they proposed the following conjecture:

Conjecture 1.1 \([FS98a]\). For \( X = E(2) \) and \( T \) the elliptic fiber of \( E(2) \), the manifolds \( X_{K_1} \) and \( X_{K_2} \) are diffeomorphic if and only if \( K_1 \) and \( K_2 \) are equivalent knots.

Here \( E(n) \) is the simply connected elliptic surface without multiple fibers and which has holomorphic Euler characteristic \( n \), or equivalently, Euler characteristic \( 12n \).

In \([AK02]\) S. Akbulut showed that for any knot \( K \), \( X_K \) is diffeomorphic to \( X_{K^*} \) where \( K^* \) is the mirror to \( K \). So the question remains whether \( X_K \) determines \( K \)

Received by the editors October 2, 2006.

2000 Mathematics Subject Classification. Primary 57N13, 57R17, 53D35.

This work was supported by Grant No. R14-2002-007-01002-0 from KOSEF.

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5853
up to its mirror. This is one of the most interesting unsolved problems in four
dimensional topology. The difficulty of this conjecture comes from the fact that
there are infinitely many inequivalent knots with the same Alexander polynomial.
Moreover for any given monic integral symmetric Laurent polynomial which has
Laurent degree at least 4 and evaluates to a unit at 1, we can construct infinitely
many inequivalent fibered knots which share the same Laurent polynomial as their
symmetrized Alexander polynomial [Mor83]. We cannot distinguish such \( E(n)_{K'} \)’s
in a smooth category by using Seiberg-Witten invariants only and we need a new
method to classify them.

If \( K \subset S^3 \) is a fibered knot and the torus \( T \) is symplectically embedded in a
symplectic 4-manifold \( X \), then Fintushel-Stern’s knot surgery 4-manifold \( X_K \) also
has a symplectic structure. Moreover a Lefschetz fibration structure on \( E(n)_K \) was
found by R. Fintushel and R. Stern [FS04] as a twisted fiber sum of two copies of \( M(n,g) \) (Definition 2.3) as a genus \((2g + n - 1)\) Lefschetz fibration over \( S^2 \). A
twisted fiber sum (Definition 2.6) of two genus \( g \) Lefschetz fibrations \( X_1 \) and \( X_2 \)
with monodromy factorization \( W_1 \) and \( W_2 \) with respect to a fixed generic fiber \( \Sigma_g \),
denoted \( X_1 \times_{\psi} X_2 \), is the genus \( g \) Lefschetz fibration \((X_1 \setminus \nu(\Sigma_g)) \cup r \times \nu (X_2 \setminus \nu(\Sigma_g))\)
with monodromy factorization \( \psi(W_2) \cdot W_1 \) where \( r : S^1 \to S^1 \) is an orientation
reversing diffeomorphism and \( \psi : \Sigma_g \to \Sigma_g \) is an orientation preserving diffeomor-
phism. An explicit monodromy factorization of \( M(n,g) \) [Gur, Yun06] together
with this Lefschetz fibration structure of \( E(n)_K \) gives an explicit monodromy
factorization of \( E(n)_K \). In the case of fibered knots we will use this factorization
to prove the following stabilization of any Fintushel-Stern’s knot surgery 4-manifold.

Theorem 1.2. Let \( K_1, K_2 \) be any two genus \( g \geq 2 \) fibered knots in \( S^3 \) and let \( K_0 \)
be the 2-bridge knot \( C(-2, -2, \cdots, -2, -2) \). Then

\[
E(n)_{K_1} \sharp_{t_{b_2}} E(n)_{K_0} \approx E(n)_{K_2} \sharp_{t_{b_2}} E(n)_{K_0}
\]

for each fixed integer \( n \geq 1 \).

In the statement \( t_{b_2} \) means the right-handed Dehn twist along a simple closed curve
\( b_2 \) as in Figure 3. This result can be compared to the stabilization of the Lefschetz
fibration which was obtained by D. Auroux [Aur05]. In our case, we obtain a
diffeomorphism by doing a twisted fiber sum with a Lefschetz fibration of genus
\((2g + n - 1)\) whose monodromy factorization has word length \( 4(4n + 2g - 2) \). It is
usually much shorter than the word length of the monodromy factorization which
was used by D. Auroux.

R. Fintushel and R. Stern have given other constructions of families of four-
manifolds which have the same Seiberg-Witten invariants [FS04]. One of them is

\[
Y(n; K_1, K_2) = E(n)_{K_1} \sharp \Sigma_{2g+n+1} \to E(n)_{K_2}
\]

where \( K_1, K_2 \subset S^3 \) are two genus \( g \) fibered knots and the surface \( \Sigma_{2g+n+1}, n \geq 1, \)
is the generic fiber of the Lefschetz fibrations on \( E(n)_{K_1} \) and \( E(n)_{K_2} \) mentioned
above. We are interested in whether the smooth type of \( Y(n; K_1, K_2) \) determines
the knot type of \( K_1 \) and \( K_2 \). In this article we show that the smooth type of
\( Y(n; K_1, K_2) \) does not always determine the knot type.
Theorem 1.3. Let $K_i, K_j$ be two 2-bridge knots of the form
\[ C(2\varepsilon_{i,1},2\varepsilon_{i,2},\ldots,2\varepsilon_{i,2g-1},2\varepsilon_{i,2g}) \]
where $g \geq 1$ and $\varepsilon_{i,k} = +1$ or $\varepsilon_{i,k} = -1$ for each $k = 1, 2, \ldots, 2g$. Let $K_0$ be the 2-bridge knot of the form $C((-2, -2, \ldots, -2, -2))$. Then for each $n \geq 1$,
\[ Y(n; K_i, K_0) \approx Y(n; K_j, K_0). \]

2. Monodromy factorization of knot surgery 4-manifold

If $K$ is a fibered knot in $S^3$, then $E(n)K$ has a symplectic structure as well as a naturally defined Lefschetz fibration [FS04]. Any Lefschetz fibration is characterized by its monodromy factorization. In this section we will find an explicit monodromy factorization of Fintushel-Stern’s knot surgery 4-manifold $E(n)K$ and as an application we will show that some of S. Akbulut’s examples [Akb02] can be obtained by using this monodromy factorization.

Definition 2.1. Let $X$ be a compact, oriented smooth 4-manifold. A (smooth) Lefschetz fibration is a proper smooth map $\pi : X \to B$ where $B$ is a compact connected oriented surface and $\pi^{-1}(\partial B) = \partial X$ such that

1. the set of critical points $C = \{p_1, p_2, \ldots, p_n\}$ of $\pi$ is nonempty and lies in $\text{int}(X)$ and $\pi$ is injective on $C$;
2. about each $p_i$ and $\pi(p_i)$, there are local complex coordinate charts agreeing with the orientations of $X$ and $B$ such that $\pi$ can be expressed as $\pi(z_1, z_2) = z_1^2 + z_2^2$.

Let $\Sigma_g$ be the closed connected orientable surface of genus $g$ and let $t_c$ be the right-handed Dehn twist along a simple closed curve $c$ on $\Sigma_g$. A genus $g$ Lefschetz fibration over $\mathbb{C}P^1$ with $n$ critical values gives a factorization of the identity in the mapping class group
\[ M_g = \pi_0(\text{Diff}^+(\Sigma_g)) \]
as a product of $n$ right-handed Dehn twists
\[ t_{c_n} \circ t_{c_{n-1}} \circ \cdots \circ t_{c_2} \circ t_{c_1} = \text{id} \]
along simple closed curves $c_1, \ldots, c_n$ in $\Sigma_g$. The ordered sequence of right-handed Dehn twists
\[ t_{c_n} \cdot t_{c_{n-1}} \cdot \cdots \cdot t_{c_2} \cdot t_{c_1} \]
is called monodromy factorization of the Lefschetz fibration. (We use $\circ$ when we consider it as an element of $M_g$ or $M^1_g = \pi_0(\text{Diff}^+(\Sigma^1_g))$ where $\Sigma^1_g$ is the connected orientable surface of genus $g$ with one boundary component. We use $\cdot$ when we consider it as a monodromy factorization.)

For any element $f \in M_g$ and a simple closed curve $c \subset \Sigma_g$,
\[ f(t_c) = f \circ t_c \circ f^{-1} = t_{f(c)}. \]

Definition 2.2. Two monodromy factorizations $W_1$ and $W_2$ are Hurwitz equivalent, denoted by $W_1 \sim W_2$, if $W_1$ can be changed to $W_2$ in finitely many steps by using the following two operations:

1. Hurwitz move: $t_{c_n} \cdots t_{c_{i+1}} t_{c_i} \cdots t_{c_1} \to t_{c_n} \cdots t_{c_{i+1}} (t_{c_i}) \cdot t_{c_{i+1}} \cdots t_{c_1}$,
2. inverse Hurwitz move: $t_{c_n} \cdots t_{c_{i+1}} t_{c_i} \cdots t_{c_1} \to t_{c_n} \cdots t_{c_{i+1}} t_{c_i}^{-1} (t_{c_{i+1}}) \cdots t_{c_1}$.
The simultaneous conjugation equivalence of two monodromy factorizations is given by
\[ t_{c_n} \cdot t_{c_{n-1}} \cdots t_{c_2} \cdot t_{c_1} \equiv f(t_{c_n}) \cdot f(t_{c_{n-1}}) \cdots f(t_{c_2}) \cdot f(t_{c_1}) \]
for some \( f \in \mathcal{M}_g \). We will consider \( f(w_k \cdots w_2 \cdot w_1) \) as \( f(w_k) \cdots f(w_2) \cdot f(w_1) \).

**Definition 2.3.** Two Lefschetz fibrations \( f : M \to B \), \( f' : M' \to B' \) are isomorphic if and only if \( W_1 \) can be changed to \( W_2 \) by a finite sequence of Hurwitz equivalences and simultaneous conjugation equivalences.

**Theorem 2.4** ([Kas80], [Mat96]). Let \( X_i \to \mathbb{CP}^1 \), \( i = 1, 2 \), be Lefschetz fibrations of genus \( g \) with monodromy factorization \( W_i \) corresponding to a fixed generic fiber \( F_i \). Then the two Lefschetz fibrations are isomorphic if and only if \( W_1 \) can be changed to \( W_2 \) by a finite sequence of Hurwitz equivalences and simultaneous conjugation equivalences.

**Remark 2.5.** If two monodromy factorizations \( W_1 \) and \( W_2 \) give the isomorphic Lefschetz fibration, then they are diffeomorphic as 4-manifolds and we will write \( W_1 \approx W_2 \). We will use the same notation \( X_1 \approx X_2 \) when two manifolds \( X_1 \) and \( X_2 \) are diffeomorphic.

**Definition 2.6.** Let \( X_i \), \( i = 1, 2 \), be Lefschetz fibrations over \( \mathbb{CP}^1 \) with generic fiber \( F_i \approx \Sigma_g \) and which have monodromy factorization \( W_i = w_{i,n_i} \cdots w_{i,2} \cdot w_{i,1} \) corresponding to a fixed generic fiber \( F_i \). Let \( r : S^1 \to S^1 \) be an orientation reversing diffeomorphism and \( \psi : F_2 \to F_1 \) be an orientation preserving diffeomorphism. Then the twisted fiber sum of two genus \( g \) Lefschetz fibrations, denoted \( X_1 \#_{\psi} X_2 \), is the genus \( g \) Lefschetz fibration
\[ (X_1 \setminus \nu(F_1)) \cup_{r \times \psi} (X_2 \setminus \nu(F_2)) \]
which has monodromy factorization \( \psi(W_2) \cdot W_1 \).

**Remark 2.7.** Consider the following diagram where \( \phi_i : F_i \to \Sigma_g \) is an orientation preserving diffeomorphism:

\[
\begin{array}{ccc}
F_2 & \xrightarrow{\psi} & F_1 \\
\phi_2 & \downarrow & \phi_1 \\
\Sigma_g & \xrightarrow{\phi_1 \circ \psi \circ \phi_2} & \Sigma_g.
\end{array}
\]

Observe that \( \psi(W_2) \cdot W_1 \in \mathcal{M}_{F_1} \) and \( (\phi_1 \circ \psi \circ \phi_2^{-1})(\phi_2(W_2)) \cdot \phi_1(W_1) \in \mathcal{M}_{\Sigma_g} \) and these two can be identified via the following relation:
\[ \mathcal{M}_{\Sigma_g} = \phi_1 \circ \mathcal{M}_{F_1} \circ \phi_1^{-1} \]
between two mapping class groups. So we may assume that \( W_1 \) and \( W_2 \) are elements of \( \mathcal{M}_{\Sigma_g} \) by choosing \( \Sigma_g \) as a generic fiber. Usually \( \psi_1(W_2) \cdot W_1 \) is not isomorphic to \( \psi_2(W_2) \cdot W_1 \) for different choices of diffeomorphisms \( \psi_i : \Sigma_g \to \Sigma_g \), \( i = 1, 2 \). Therefore to get a well-defined monodromy factorization of \( X_1 \#_{\psi} X_2 \), we
have to specify a fixed generic fiber \( F_i \) of the Lefschetz fibration \( X_i \) and specify the identifying diffeomorphism \( \psi : F_2 \to F_1 \).

**Definition 2.8** ([FS04]). Let \( M(n, g) \) be the desingularization of the double cover of \( \Sigma_g \times S^2 \) branched over \( 2n(\{pt\} \times S^2) \cup 2(\Sigma_g \times \{pt\}) \). Let \( D(n, g) \) be the desingularization of the double cover of \( \Sigma_g \times S^2 \) branched over \( 2n(\{pt\} \times S^2) \cup 4(\Sigma_g \times \{pt\}) \).

**Remark 2.9.** It is known that \( M(n, g) \) is diffeomorphic to \((\Sigma_g \times S^2)\#4\mathbb{CP}^2\) and \( D(n, g) \) is diffeomorphic to \((\Sigma_g \times S^2)\#8\mathbb{CP}^2\).

A Lefschetz fibration structure of a knot surgery 4-manifold \( E(n)_K \) was studied by R. Fintushel and R. Stern [FS04] where \( K \subset S^3 \) is a fibered knot. The complex manifold \( M(n, g) \) can be considered as a Lefschetz fibration with generic fiber \( \Sigma_{2g+n-1} \) because each of the two singular fibers of \( M(n, g) \) is perturbed to \((4n + 2g - 2)\) node type singularities by local deformation. They considered \( E(n)_K \) as a twisted fiber sum \( M(n, g)\#_{\Phi_K} M(n, g) \) by using the diffeomorphism \( \Phi_K = \varphi_K \oplus id \oplus id \) which is defined on \( \Sigma_{g}^{\#} \Sigma_{n-1}^{\#} \Sigma_{g} \) where \( \varphi_K \) is the geometric monodromy of the fibered knot \( K \).

Moreover we know an explicit monodromy factorization of \( M(n, g) \).

**Lemma 2.10** ([Yun06]). \( M(n, g) \) has monodromy factorization \( \eta_{n-1, g} \) where

\[
\eta_{n-1, g} = t_{A_{2n-2}} \cdot t_{A_{2n-3}} \cdot \ldots \cdot t_{A_2} \cdot t_{A_1}^2 \cdot t_{A_2} \cdot \ldots \cdot t_{A_{2n-2}} \cdot t_{B_0} \cdot t_{B_1} \cdot \ldots \cdot t_{B_{2g}} \cdot t_{A_{2n-1}}
\]

and \( A_i, B_j \) are simple closed curves on \( \Sigma_{n+2g-1} \) as in Figure [1].

This proposition combined with R. Fintushel and R. Stern’s result gives an explicit monodromy factorization of \( E(n)_K \).

**Theorem 2.11.** Let \( K \subset S^3 \) be a fibered knot of genus \( g \). Then \( E(n)_K \) has a monodromy factorization of the form

\[
\Phi_K(\eta_{n-1, g}) \cdot \Phi_K(\eta_{n-1, g}) \cdot \eta_{n-1, g} \cdot \eta_{n-1, g}
\]

where \( \eta_{n-1, g} \) is as in Proposition 2.10 and

\[
\Phi_K = \varphi_K \oplus id \oplus id : S\Sigma_g^{\#} \Sigma_{n-1}^{\#} \Sigma_g \to \Sigma_g^{\#} \Sigma_{n-1}^{\#} \Sigma_g
\]

by using a (geometric) monodromy \( \varphi_K \) of the fibered knot \( K \) such that

\[
S^3 \setminus \nu(K) = (I \times \Sigma_g^{1})/(\{(1, x) \sim (0, \varphi_K(x))\})
\]
where \( g = \text{genus}(K) \) and \( \Sigma_g^1 \) is the connected orientable surface of genus \( g \) with one boundary component.

Proof. We will first recall R. Fintushel and R. Stern’s construction in [FS04]. Let us consider \( E(n) \) as the desingularization of the double cover of \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) branched over \( 2n(\{pt.\} \times \mathbb{CP}^1) \cup 4(\mathbb{CP}^1 \times \{pt.\}) \) as in Figure 2. Then we can easily get two fibrations: one corresponds to \( \pi_1 \) which has generic fiber \( \Sigma_2g+n-1^1 \) and 4 singular fibers of type (b). The other corresponds to \( \pi_2 \) with generic fiber \( T^2 = \Sigma_1 \) and \( 2n \) singular fibers of type (a). Therefore we get two different Lefschetz fibration structures for \( E(n) \) by locally deforming each singular fiber to nodal type singular fibers. In particular, if we locally deform each singular fiber of type (b), then its monodromy is related to the hyperelliptic involution \( \omega \) of \( \Sigma_2g+n-1^1 \).

Now we will consider the Lefschetz fibration structure of \( E(n)_K \) for some fibered knot \( K \). Each \( \Sigma_{n-1} \) which comes from a 2-fold branched covering of the curve \( c \) in Figure 2 meets twice with the generic elliptic fiber \( T^2 \) of \( E(n) \), which corresponds to the double cover of curve \( d \), where we will do knot surgery. These two points are located in the same orbit of \( \omega \). When we do knot surgery, we remove two disks from \( \Sigma_{n-1} \) and replace them by two \( \Sigma_g^1 \), the fiber surface of

\[
S^3 \setminus \nu(K) = (I \times \Sigma_2g+n-1^1) / ((1, x) \sim (0, \varphi_K(x))).
\]

Therefore we may consider \( E(n)_K \) as a singular fibration with generic fiber \( \Sigma_{2g+n-1} \) and which has 4 singular fibers of the same type. By locally deforming these 4 singular fibers, each singular fiber becomes \((4n + 2g - 2)\) node type singularities and its monodromy corresponds to an involution of \( \Sigma_{2g+n-1} = \Sigma_g^1 \Sigma_{n-1}^1 \Sigma_g \) which is an extension of the hyperelliptic involution \( \omega \) on \( \Sigma_{n-1} \). One such monodromy factorization is given in Proposition 2.10. Now by closely investigating the monodromy map \( \pi_1(S^2 \setminus \{4 \text{ points}\}) \rightarrow \mathcal{M}_{2g+n-1} \), we can decompose \( E(n)_K \) as

\[
(M(n,g) \setminus \nu(\Sigma_{2g+n-1})) \cup_{r \times \psi_K} (M(n,g) \setminus \nu(\Sigma_{2g+n-1}))
\]

which depends on the choice of \( \varphi_K \).

If \( S^3 \setminus \nu(K) = (I \times F) / ((1, x) \sim (0, \varphi(x))) \) for some other homeomorphism \( \varphi : F \rightarrow F \), then there is a homeomorphism \( \phi : \Sigma_g^1 \rightarrow F \) such that \( \phi(\partial \Sigma_g^1) = \partial F \) respecting orientation and \( \varphi = \phi \circ \varphi_K \circ \phi^{-1} \) (Proposition 5.10 of [BZ03]).
If \( \eta^2 \) is another monodromy factorization of \( M(n, g) \) as a genus \((2g + n - 1)\) Lefschetz fibration which is isomorphic to \( \eta_{n-1, g}^2 \), then after Hurwitz moves and inverse Hurwitz moves we can find a diffeomorphism

\[
\phi = \phi_1 \oplus \phi_2 \oplus \phi_3 : \Sigma_g \Sigma_{g-1} \Sigma_g \to \Sigma_{2g+n-1}
\]
such that \( \eta' = \phi^{-1}(\eta_{n-1, g}) \). So

\[
\Phi_K(\eta^2) \cdot \eta^2 = \Phi_K(\phi^{-1}(\eta_{n-1, g}^2)) \cdot \phi^{-1}(\eta_{n-1, g}^2)
\]
\[
= \phi^{-1}((\phi \circ \Phi_K \circ \phi^{-1})(\eta_{n-1, g}^2) \cdot \eta_{n-1, g}^2)
\]
\[
= (\phi \circ \Phi_K \circ \phi^{-1})(\eta_{n-1, g}^2) \cdot \eta_{n-1, g}^2
\]

and \( \phi \circ \Phi_K \circ \phi^{-1} = (\phi_1 \circ \varphi_K \circ \phi_1^{-1}) \oplus \text{id} \oplus \text{id} \). It is also clear that \( \phi_1 \circ \varphi_K \circ \phi_1^{-1} \) is a monodromy map of the fibered knot \( K \).

Therefore \( E(n)_K \) has a monodromy factorization of the form

\[
\Phi_K(\eta_{n-1, g}) \cdot \Phi_K(\eta_{n-1, g}) \cdot \eta_{n-1, g} \cdot \eta_{n-1, g}
\]

where \( \eta_{n-1, g} \) is as in Proposition 2.10 and \( \Phi_K \) comes from a monodromy map \( \varphi_K \) of the given fibered knot \( K \).

**Corollary 2.12.** Let \( K_1, K_2 \) be two fibered knots such that \( g(K_1) = g(K_2) \) and \( \Phi_{K_2} = \Phi_{K_1}^{-1} \). Then \( E(n)_{K_1} \approx E(n)_{K_2} \).

**Proof.** \( E(n)_{K_2} \) has a monodromy factorization

\[
\Phi_{K_2}(\eta_{n-1, g}) \cdot \Phi_{K_2}(\eta_{n-1, g}) \cdot \eta_{n-1, g} \cdot \eta_{n-1, g} = \Phi_{K_1}^{-1}(\eta_{n-1, g}) \cdot \Phi_{K_1}^{-1}(\eta_{n-1, g}) \cdot \eta_{n-1, g} \cdot \eta_{n-1, g}
\]

and by using a sequence of Hurwitz moves and \( \lambda_{n-1, g}^2 = \text{id} \) we get

\[
\Phi_{K_1}(\eta_{n-1, g}) \cdot \Phi_{K_1}(\eta_{n-1, g}) \cdot \eta_{n-1, g} \cdot \eta_{n-1, g}
\]

which is a monodromy factorization of \( E(n)_{K_1} \). Therefore \( E(n)_{K_1} \approx E(n)_{K_2} \). 

**Remark 2.13.** Some of S. Akbulut’s examples [AkBu02] are easily obtained from this corollary. For example, if \( K = 3_1 \), then \( K \) is not equivalent to \( K^* \) and \( \Phi_{K^*} = \Phi_{K}^{-1} \).

### 3. Examples and applications

A construction of a fibered link was studied by J. Harer [Har82], J. Montesinos-Amilibia and H. Morton [MAM91] and it is well known how to find a geometric monodromy of a fibered knot or link.

The mapping class groups \( \mathcal{M}_g \) and \( \mathcal{M}_g^1 \) are extensively studied and their properties are somewhat well known. Among these properties, S. Humphries [Hum79] showed that \( \mathcal{M}_g \) and \( \mathcal{M}_g^1 \) are generated by \( 2g + 1 \) Dehn twists when \( g \geq 2 \) and B. Wajnryb [Waj96] proved that they can be generated by two elements.

**Lemma 3.1.** Suppose that \( g \geq 2 \) and let \( a_1, b_1, c_1 \) be simple closed curves on \( \Sigma_g^1 \) as in Figure 3. Let \( S_g := t_{a_1} \circ t_{c_1} \circ \cdots \circ t_{a_1} \circ t_{c_1} \circ t_{a_2} \circ t_{c_2} \circ t_{a_1} \circ t_{c_1} \). We can choose the following
system of generators for $M_g$ or $M^1_g$:

1. $t_{c_1}, t_{a_1}, t_{c_2}, t_{a_2}, \ldots, t_{c_g}, t_{a_g}$ and $t_{b_2}$ [Hum79],
2. $R := t_{b_2}^{-1} \circ t_{b_1}^{-1}$ and $S_g$ [Waj96],
3. $B := t_{b_2}$ and $S_g$ [Kor05].

**Definition 3.2.** For a given sequence of right-handed Dehn twists $W = w_n \cdot \ldots \cdot w_2 \cdot w_1$, the element $w_n \circ \ldots \circ w_2 \circ w_1$ in the mapping class group is denoted by $\lambda_W$ and the group $\langle w_1, w_2, \ldots, w_n \rangle$ which is generated by the Dehn twists in $W$ is denoted by $G(W)$.

**Lemma 3.3.** If $W$ and $W'$ are Hurwitz equivalent, i.e. $W \sim W'$, then $G(W) = G(W')$.

**Proof.** Let $W = w_n \cdot \ldots \cdot w_2 \cdot w_1$ and let $W'$ be obtained from $W$ by performing one Hurwitz move. Then $W'$ is of the form

$$w_n \cdot \ldots \cdot w_{i+2} \cdot w_{i+1}(w_i) \cdot w_{i+1} \cdot w_{i-1} \cdot \ldots \cdot w_1.$$ 

Therefore

$$G(W') = \{ w_1, \ldots, w_{i-1}, w_{i+1}(w_i), w_{i+1}, \ldots, w_n \}$$

and $w_{i+1}(w_i) = w_i \circ w_{i+1} \in G(W)$ because $w_i, w_{i+1}^{-1} \in G(W)$. So $G(W') \subseteq G(W)$. Similarly, $w_i = w_{i+1}^{-1} \circ w_{i+1}(w_i) \circ w_{i+1} \in G(W')$ and this implies $G(W) \subseteq G(W')$. So we have $G(W) = G(W')$ when $W'$ is obtained by a Hurwitz move. Observe that $W$ is obtainable from $W'$ by one inverse Hurwitz move.

Therefore $G(W) = G(W')$ for any word $W'$ in the same Hurwitz equivalence class of $W$. $\square$

**Lemma 3.4.** Let $W_i = w_{i,n_i} \cdot \ldots \cdot w_{i,2} \cdot w_{i,1}$ be a sequence of right-handed Dehn twists along a simple closed curve on $\Sigma_g$ such that $\lambda_{W_i} = \text{id}$ for $i = 1, 2$. Suppose $f \in G(W_2)$. Then

$$f(W_1) \cdot W_2 \sim W_1 \cdot W_2.$$
Proof. Since $f \in G(W_2)$, we can write $f = w_i^{k} \circ \cdots \circ w_2^{r_2} \circ w_1^{r_1}$ where each $w_i$ is a letter in $W_2$ which corresponds to a Dehn twist and $\varepsilon_i$ is $+1$ or $-1$. Let $f_i = w_i^{k} \circ \cdots \circ w_2^{r_2} \circ w_1^{r_1}$ for $i = 1, 2, \cdots , k$ and $f_0 = id$. Then $f_i = w_i^{r_i} \circ f_{i-1}$ and $W_2 = W_{2,1} \cdot w_1 \cdot W_{2,2}$ by decomposing $W_2$ as a product of two subsequences $W_{2,1}$, $W_{2,2}$ and a letter $w_1$.

We will prove $f_i(W_1) \cdot W_2 \sim f_{i-1}(W_1) \cdot W_2$ as follows:

If $\varepsilon_i = +1$, then

$$f_i(W_1) \cdot W_2 = \circ (w_i \circ f_{i-1})(W_1) \cdot W_{2,1} \cdot w_1 \cdot W_{2,2}$$

$$\sim (w_i \circ f_{i-1})(W_1) \cdot w_1 \cdot w_i^{-1}(W_{2,1}) \cdot W_{2,2}$$

$$\sim w_i \cdot f_{i-1}(W_1) \cdot w_i^{-1}(W_{2,1}) \cdot W_{2,2}$$

$$\sim f_{i-1}(W_1) \cdot w_1 \cdot w_i^{-1}(W_{2,1}) \cdot W_{2,2}$$

$$\sim \circ f_{i-1}(W_1) \cdot W_{2,1} \cdot w_1 \cdot W_{2,2} = f_{i-1}(W_1) \cdot W_2.$$}

We will explain these explicitly. We get 3.2 from 3.1 by using a sequence of inverse Hurwitz moves (Definition 2.2). For example, if $W_{2,1} = v_n \cdot v_{n-1} \cdots v_2 \cdot v_1$ where each $v_i$ is a right-handed or left-handed Dehn twist, then

$$W_{2,1} \cdot w_i = v_n \cdot v_{n-1} \cdots v_2 \cdot v_1 \cdot w_i$$

$$\sim v_n \cdot v_{n-1} \cdots v_2 \cdot w_i \cdot w_i^{-1}(v_1)$$

$$\sim v_n \cdot v_{n-1} \cdots w_i \cdot w_i^{-1}(v_2) \cdot w_i^{-1}(v_1)$$

$$\sim \cdots$$

$$\sim v_n \cdot w_i \cdot w_i^{-1}(v_{n-1}) \cdots w_i^{-1}(v_2) \cdot w_i^{-1}(v_1)$$

$$\sim w_i \cdot w_i^{-1}(v_n) \cdot w_i^{-1}(v_{n-1}) \cdots w_i^{-1}(v_2) \cdot w_i^{-1}(v_1)$$

$$= w_i \cdot w_i^{-1}(v_n \cdot v_{n-1} \cdots v_2 \cdot v_1)$$

$$= w_i \cdot w_i^{-1}(W_{2,1})$$

by using the notational convention

$$w_i^{-1}(v_n) \cdot w_i^{-1}(v_{n-1}) \cdots w_i^{-1}(v_2) \cdot w_i^{-1}(v_1) = w_i^{-1}(v_n \cdot v_{n-1} \cdots v_2 \cdot v_1)$$

as in Definition 2.2. From 3.2 to 3.3, we use a sequence of inverse Hurwitz moves as above to the word $(w_i \circ f_{i-1})(W_1) \cdot w_i$ so that we get $w_i \cdot w_i^{-1}((w_i \circ f_{i-1})(W_1))$ which is $w_i \cdot f_{i-1}(W_1)$. From 3.3 to 3.4, we apply a sequence of inverse Hurwitz moves to $w_i \cdot f_{i-1}(W_1)$ as follows:

$$w_i \cdot f_{i-1}(W_1) = w_i \cdot f_{i-1}(w_{1,n_1} \cdot w_{1,n_1-1} \cdots w_{1,1})$$

$$= w_i \cdot f_{i-1}(w_{1,n_1}) \cdot f_{i-1}(w_{1,n_1-1}) \cdots f_{i-1}(w_{1,1})$$

$$\sim f_{i-1}(w_{1,n_1}) \cdot (f_{i-1}(w_{1,n_1}))^{-1}(w_i) \cdot f_{i-1}(w_{1,n_1-1}) \cdots f_{i-1}(w_{1,1})$$

$$\sim f_{i-1}(w_{1,n_1}) \cdot f_{i-1}(w_{1,n_1-1}) \cdot (f_{i-1}(w_{1,n_1} \circ w_{1,n_1-1}))^{-1}(w_i) \cdot f_{i-1}(w_{1,n_1-2}) \cdots f_{i-1}(w_{1,1})$$

$$\sim \cdots$$

$$\sim f_{i-1}(w_{1,n_1}) \cdot f_{i-1}(w_{1,n_1-1}) \cdots f_{i-1}(w_{1,1}) \cdot (f_{i-1}(w_{1,n_1} \circ \cdots \circ w_{1,2} \circ w_{1,1}))^{-1}(w_i)$$

$$= f_{i-1}(W_1) \cdot \lambda_{f_{i-1}(W_1)}^{-1}(w_i).$$
For (3.5), we use \( \lambda_{f_{i-1}(W_1)}^{-1}(w_i) = w_i \) because

\[
\lambda_{f_{i-1}(W_1)}^{-1} = f_{i-1} \circ \lambda_W^{-1} \circ f_{i-1}^{-1} = f_{i-1} \circ f_{i-1}^{-1} = id.
\]

We obtain (3.6) from (3.5) by using the reversed process of (3.1) to (3.2). The proof of the \( \varepsilon_i = -1 \) case is very similar to the \( \varepsilon_i = 1 \) case. If \( \varepsilon_i = -1 \), then

\[
\begin{align*}
(3.7) \quad f_i(W_1) \cdot W_2 & \sim \lambda_{f_i(W_1)}(W_2) \cdot f_i(W_1) = W_2 \cdot f_i(W_1) \\
(3.8) & = W_{2,1} \cdot w_i \cdot W_{2,2} \cdot (w_i^{-1} \circ f_{i-1})(W_1) \sim W_{2,1} \cdot w_i \cdot W_{2,2} \cdot w_i \cdot (w_i^{-1} \circ f_{i-1})(W_1) \\
(3.9) & \sim W_{2,1} \cdot w_i \cdot W_{2,2} \cdot f_{i-1}(W_1) \cdot w_i \\
(3.10) & \sim W_{2,1} \cdot w_i \cdot W_{2,2} \cdot f_{i-1}(W_1) \cdot w_i \\
(3.11) & \sim W_{2,1} \cdot w_i \cdot W_{2,2} \cdot \lambda_{f_{i-1}(W_1)}(w_i) \cdot f_{i-1}(W_1) \\
(3.12) & = W_{2,1} \cdot w_i \cdot W_{2,2} \cdot w_i \cdot f_{i-1}(W_1) \\
(3.13) & \sim W_{2,1} \cdot w_i \cdot W_{2,2} \cdot f_{i-1}(W_1) = W_2 \cdot f_{i-1}(W_1) \\
(3.14) & \sim \lambda_{W_2}(f_{i-1}(W_1)) \cdot W_2 = f_{i-1}(W_1) \cdot W_2.
\end{align*}
\]

We have \( f_i(W_1) \cdot w \sim \lambda_{f_i(W_1)}(w) \cdot f_i(W_1) \) for each letter \( w \) by using a sequence of Hurwitz moves. So we apply this for each letter in \( W_2 \) one by one from the leftmost letter in \( W_2 \). Then we have \( \lambda_{f_i(W_1)}(W_2) \cdot f_i(W_1) \). Since \( \lambda_W = id \), we have \( \lambda_{f_i(W_1)} = id \) and this implies (3.7). (3.8) is just a rewriting of (3.7) by using \( W_2 = W_{2,1} \cdot w_i \cdot W_{2,2} \) and \( f_i = w_i^{-1} \circ f_{i-1} \). (3.9) is obtained from (3.8) by using \( w_i \cdot W_{2,2} \sim w_i(W_{2,2}) \cdot w_i \), which is a result of a sequence of Hurwitz moves. (3.10) is a result of the Hurwitz moves

\[
\begin{align*}
w_i \cdot (w_i^{-1} \circ f_{i-1})(W_1) & \sim w_i((w_i^{-1} \circ f_{i-1})(W_1)) \cdot w_i = f_{i-1}(W_1) \cdot w_i
\end{align*}
\]

from (3.9). (3.12) is a result of \( \lambda_W = id \) and the Hurwitz moves

\[
f_{i-1}(W_1) \cdot w_i \sim \lambda_{f_{i-1}(W_1)}(w_i) \cdot f_{i-1}(W_1) = w_i \cdot f_{i-1}(W_1)
\]

from (3.10). (3.12) to (3.13) is the reversed process of (3.8) to (3.9), and (3.13) to (3.14) is the reversed process of (3.7).

Therefore we have \( f_i(W_1) \cdot W_2 \sim f_{i-1}(W_1) \cdot W_2 \) for each \( i = 1, 2, \ldots, k \) and an induction process gives the conclusion. \( \square \)

**Remark 3.5.** In [Aur05], D. Auroux also gets a similar result.

**Definition 3.6 (BZ03).** The 2-bridge knot \( b(\alpha, \beta) \) is the knot

\[
C(n_1, -n_2, n_3, -n_4, \ldots, (-1)^{k-1}n_k)
\]

as in Figure 4 where

\[
\frac{\beta}{\alpha} = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{\ddots + \frac{1}{n_{k-1} + \frac{1}{n_k}}}}} = [n_1, n_2, \ldots, n_k].
\]

The 2-bridge knot \( b(\alpha, \beta) \) is characterized by its two-fold branched covering space \( L(\alpha, \beta) \), the lens space of type \((\alpha, \beta)\). So we may assume that \( \alpha > 0 \), \( \alpha \) and \( \beta \) are relatively prime and \( 0 \leq \beta < \alpha \).

**Definition 3.7 (PS04).** Let \( K_1, K_2 \) be two fibered knots in \( S^3 \) and \( n \geq 1 \). Then

\[
Y(n; K_1, K_2) = E(n)_{K_1} \sharp id: \Sigma_{2g+n-1} \rightarrow \Sigma_{2g+n-1} E(n)_{K_2}.\]
Lemma 3.8. Let $K$ be a fibered knot of genus $g \geq 1$ such that
\[ \Phi_K \in \langle t_{a_1}, t_{c_1}, t_{a_2}, t_{c_2}, \ldots, t_{a_g}, t_{c_g} \rangle \]
and let $K_0$ be the 2-bridge knot $C(-2, -2, \ldots, -2)$. Then
\[ E(n)K_0^* \cdot E(n)K_0 \approx D(n, g) \cdot E(n)K_0 \]
for each fixed $n \geq 1$.

Proof. $E(n)K_0^* \cdot E(n)K_0$ has a monodromy factorization of the form
\[ \Phi_{K_0}(\eta_{n-1,g}^2) \cdot \eta_{n-1,g} ^2 \cdot \Phi_K(\eta_{n-1,g}^2) \cdot \eta_{n-1,g}^2. \]
Now by applying a sequence of Hurwitz moves to move each letter in $\Phi_{K_0}(\eta_{n-1,g}^2) \cdot \eta_{n-1,g}^2$ to the right crossed over $\Phi_K(\eta_{n-1,g}^2)$, we get
\[ \Phi_K(\eta_{n-1,g}^2) \cdot \Phi_{K_0}(\eta_{n-1,g}^2) \cdot \eta_{n-1,g}^2 \cdot \eta_{n-1,g}^2 \]
because $\lambda_{\eta_{n-1,g}^2} = id$ and $\lambda_{\Phi_{K_0}(\eta_{n-1,g}^2)} = id$, which comes from the fact that $\lambda_{\eta_{n-1,g}}$ is an involution in $\mathcal{M}_{2g+n-1}$. 

Since $\Phi_K \in \langle t_{a_1}, t_{c_1}, t_{a_2}, t_{c_2}, \ldots , t_{a_g}, t_{c_g} \rangle$, by Proposition 3.3 it is enough to show that $t_{a_i}, t_{c_i} \in G(\Phi_K, (\eta^2_{n-1,g}) \cdot \eta_{n-1,g}^2)$ for each $i = 1, 2, \ldots , g$. By Proposition 3.3 we can do this by checking
\begin{align}
(3.15) \quad \Phi_K (\eta^2_{n-1,g}) \cdot \eta_{n-1,g}^2 \cdot \eta^2_{n-1,g} \sim \cdots t_{a_i} \cdots ,
(3.16) \quad \Phi_K (\eta^2_{n-1,g}) \cdot \eta_{n-1,g}^2 \cdot \eta^2_{n-1,g} \sim \cdots t_{c_i} \cdots
\end{align}
for $i = 1, 2, \ldots , g$ where a word is expressed as $\cdots t_{a_i} \cdots$ when the word contains the letter $t_{a_i}$ in some place.

Since $K_0$ can be constructed from an unknot by plumbing $2g$ negative Hopf bands (Figure 5), we may consider $\Phi$ the letter $t$ for $i$.

Similarly from (3.18) we get
\begin{align}
(3.17) \quad t_{a_i} = (t_{B_2i}^{-1} \circ \Phi_{K_0})(t_{B_2i}),
(3.18) \quad t_{c_i} = (t_{B_2i-1}^{-1} \circ \Phi_{K_0})(t_{B_2i-1})
\end{align}
for $i = 1, 2, \ldots , g$.

Since the word $\eta^2_{n-1,g}$ contains the letter $t_{B_2i}$ and $\Phi_{K_0}(\eta^2_{n-1,g})$ contains the letter $\Phi_{K_0}(t_{B_2i})$ in some place, we can write
$$\Phi_{K_0}(\eta^2_{n-1,g}) \cdot \eta_{n-1,g}^2 \cdot \eta^2_{n-1,g} = \cdots \Phi_{K_0}(t_{B_2i}) \cdots t_{B_2i} \cdots$$
and by performing Hurwitz moves we can move the letter $\Phi_{K_0}(t_{B_2i})$ to the left position of the letter $t_{B_2i}$ so that we obtain a word of the form
$$\cdots \Phi_{K_0}(t_{B_2i}) \cdots t_{B_2i} \cdots$$
and by doing one inverse Hurwitz move we get
$$\cdots t_{B_2i} \cdots t_{B_2i}^{-1} (\Phi_{K_0}(t_{B_2i})) \cdots = \cdots t_{B_2i} \cdots (t_{B_2i}^{-1} \circ \Phi_{K_0})(t_{B_2i}) \cdots$$
and by using (3.17) we get a word of the form
$$\cdots \cdots t_{a_i} \cdots$$
in the Hurwitz equivalence class of $\Phi_{K_0}(\eta^2_{n-1,g}) \cdot \eta^2_{n-1,g} \cdot \eta^2_{n-1,g}$.

Similarly from (3.18) we get
\begin{align*}
\Phi_{K_0}(\eta^2_{n-1,g}) \cdot \eta_{n-1,g}^2 \cdot \eta^2_{n-1,g} & = \cdots \Phi_{K_0}(t_{B_2i-1}) \cdots t_{B_2i-1} \cdots \\
& \sim \cdots \Phi_{K_0}(t_{B_2i-1}) \cdots t_{B_2i-1} \cdots \\
& \sim \cdots t_{B_2i-1} \cdots (t_{B_2i-1}^{-1} \circ \Phi_{K_0})(t_{B_2i-1}) \cdots \\
& = \cdots \cdots t_{c_i} \cdots .
\end{align*}
So (3.15) and (3.16) are proved. Therefore by using Proposition 3.4 we get
$$\Phi_K (\eta^2_{n-1,g}) \cdot \Phi_{K_0}(\eta^2_{n-1,g}) \cdot \eta_{n-1,g}^2 \cdot \eta^2_{n-1,g} \sim \eta^2_{n-1,g} \cdot \Phi_{K_0}(\eta^2_{n-1,g}) \cdot \eta^2_{n-1,g} \cdot \eta^2_{n-1,g}$$
and
$$\Phi_{K_0}(\eta^2_{n-1,g}) \cdot \eta_{n-1,g}^2 \cdot \eta^2_{n-1,g} \sim \Phi_{K_0}(\eta^2_{n-1,g}) \cdot \eta^2_{n-1,g} \cdot \eta^2_{n-1,g} \cdot \eta^2_{n-1,g},$$
which is a monodromy factorization of $D(n,g)_{id}E(n)_{K_0}$. \hfill $\square$

Remark 3.9. $C(-2, -2, \ldots , -2, -2)$ is $b(2g+1, -2g) = b(2g+1, 1)$, which is the torus knot $T(2g+1, 2)$.  

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Theorem 3.10. Let $K_1, K_2$ be two 2-bridge knots of the form

$$C(2\varepsilon_{i,1}, 2\varepsilon_{i,2}, \ldots, 2\varepsilon_{i,2g-1}, 2\varepsilon_{i,2g})$$

where $g \geq 1$ and $\varepsilon_{i,k} = +1$ or $\varepsilon_{i,k} = -1$ for each $k = 1, 2, \ldots, 2g$. Let $K_0$ be the 2-bridge knot of the form $C(-2, -2, \ldots, -2, -2)$. Then for each $n \geq 1$,

$$Y(n; K_1, K_0) \approx Y(n; K_2, K_0).$$

Proof. Since we can select $\varphi_{K_0}$ so that

$$\Phi_{K_1} = T_{a_g}^{\varepsilon_{i,2g}} \circ T_{c_g}^{\varepsilon_{i,2g-1}} \circ \cdots \circ T_{a_1}^{\varepsilon_{i,1}},$$

it is clear from Proposition 3.8.

Remark 3.11. In [FS99], R. Fintushel and R. Stern could construct (non-simply-connected) nondiffeomorphic 4-manifolds which have the same Seiberg-Witten invariants. Underlying this construction are the 2-bridge knots

$$K_1 = b(105, 64) = C(2, 2, -2, -2, -2, 2, 2)$$

and

$$K_2 = b(105, 76) = C(2, 2, 2, -2, -2, 2, 2, 2).$$

Even though we do not know whether $E(n)_{K_1}$ is diffeomorphic to $E(n)_{K_2}$, we know that they become diffeomorphic after doing a fiber sum with $E(n)_{K_0}$. On the other hand, for any fibered knot $K_1$, $K_2$ of genus $g$, R. Fintushel and R. Stern [FS04] showed that $SW(Y_{n}(K_1, K_2)) = t_K + (-1)^{n-1}$. Therefore we cannot distinguish $Y(n; K_1, K_2)$ by using Seiberg-Witten invariants. It is an interesting question whether the diffeomorphism type of $Y(n; K_1, K_2)$ determines the knot type of $K_1$ and $K_2$ or not. Theorem 3.10 combined with the above $K_1 = b(105, 64)$ and $K_2 = b(105, 76)$ gives an example such that $Y(n; K_1, K_0) \approx Y(n; K_2, K_0)$ although $K_1$ is not equivalent to $K_2$ as a knot. For each $g \geq 1$ there are inequivalent genus $g$ fibered knots $K, K'$ such that $Y(n; K, K_0) \approx Y(n; K', K_0)$. Moreover if $g \geq 2$, then we can construct infinitely many inequivalent genus $g$ fibered knots $\{K_i\}$ which share the same Alexander polynomial and $Y(n; K_i, K_0)$ are all diffeomorphic. Such examples are obtained by using Stallings' twist.

Theorem 3.12. Let $K_1$, $K_2$ be any two genus $g \geq 2$ fibered knots in $S^3$ and let $K_0$ be the 2-bridge knot $C(-2, -2, \ldots, -2, -2)$. Then

$$E(n)_{K_1} \# t_{a_2} E(n)_{K_0} \approx E(n)_{K_2} \# t_{a_2} E(n)_{K_0}$$

for each fixed integer $n \geq 1$.

Proof. $E(n)_{K_1} \# t_{a_2} E(n)_{K_0}$ has a monodromy factorization of the form

$$t_{a_2}(\Phi_{K_0}(\eta_{n-1,g}^2) \cdot \eta_{n-1,g}^2) \cdot \Phi_{K}(\eta_{n-1,g}^2) \cdot \Phi_{K}(\eta_{n-1,g}^2)$$

and by using inverse Hurwitz moves we have

$$t_{a_2}(\Phi_{K_0}(\eta_{n-1,g}^2) \cdot \eta_{n-1,g}^2) \cdot \eta_{n-1,g}^2 \cdot \lambda_{\eta_{n-1,g}}^{-1}(\Phi_{K}(\eta_{n-1,g}^2))$$

$$= t_{a_2}(\Phi_{K_0}(\eta_{n-1,g}^2) \cdot \eta_{n-1,g}^2 \cdot \lambda_{\eta_{n-1,g}}^{-1}(\Phi_{K}(\eta_{n-1,g}^2)))$$

because $\lambda_{\eta_{n-1,g}}^{-1} = id$.

We claim that

$$(3.19) \quad \langle t_{a_1}, t_{c_i}, t_{b_2} \mid i = 1, 2, \ldots, g \rangle \subseteq G(t_{b_2}(\Phi_{K_0}(\eta_{n-1,g}^2) \cdot \eta_{n-1,g}^2) \cdot \eta_{n-1,g}^2).$$
From this claim, we get the conclusion as follows: for any given genus \(g \geq 2\) fibered knot \(K \subset S^3\), we can select \(\Phi_K \in \langle t_{c_1}, t_{c_2}, \cdots, t_{c_n}, t_{a_g}, t_{b_2} \rangle\) (Proposition 3.1 and Theorem 2.11). Therefore

\[
(3.20) \quad t_{b_2}(\Phi_{K_0}(\eta_{n-1,g}) \cdot \eta_{n-1,g}) \cdot \Phi_{K}(\eta_{n-1,g}) \cdot \eta_{n-1,g} \\
(3.21) \quad \sim \quad \Phi_{K}(\eta_{n-1,g}) \cdot t_{b_2}(\Phi_{K_0}(\eta_{n-1,g}) \cdot \eta_{n-1,g}) \cdot \eta_{n-1,g} \\
(3.22) \quad \sim \quad \eta_{n-1,g}^2 \cdot t_{b_2}(\Phi_{K_0}(\eta_{n-1,g}) \cdot \eta_{n-1,g}) \cdot \eta_{n-1,g}^2 \\
(3.23) \quad \sim \quad t_{b_2}(\Phi_{K_0}(\eta_{n-1,g}) \cdot \eta_{n-1,g}) \cdot \eta_{n-1,g}^2 \cdot \eta_{n-1,g}^2,
\]

because \((3.21)\) is obtained from \((3.20)\) by a sequence of Hurwitz moves and the fact that \(\lambda_{t_{b_2}(\Phi_{K_0}(\eta_{n-1,g}) \cdot \eta_{n-1,g})} = id\), \((3.22)\) is obtained from \((3.21)\) by Proposition 3.2 and \(\Phi_K \in G(t_{b_2}(\Phi_{K_0}(\eta_{n-1,g}) \cdot \eta_{n-1,g}) \cdot \eta_{n-1,g}^2)\), and \((3.23)\) is obtained from \((3.22)\) by a sequence of Hurwitz moves. Since \((3.23)\) is a monodromy factorization of \(D(n,g) \Phi_{t_{b_2}} E(n)\), we get the conclusion.

Therefore we only need to check the above claim \((3.19)\), and it is enough to show

\[
t_{a_1}, \ t_{c_1}, \ t_{b_2} \in G(t_{b_2}(\Phi_{K_0}(\eta_{n-1,g}) \cdot \eta_{n-1,g}) \cdot \eta_{n-1,g}),
\]

for all \(i = 1, 2, \cdots, g\).

In Proposition 3.8 we already showed that \(t_{a_i}, t_{c_i} \in G(\Phi_{K_0}(\eta_{n-1,g}) \cdot \eta_{n-1,g})\) for \(i = 1, 2, \cdots, g\). Since \(t_{b_2}(t_{a_i}) = t_{a_i}\) for \(i = 1, 3, 4, \cdots, g\) and \(t_{b_2}(t_{c_i}) = t_{c_i}\) for all \(i = 1, 2, \cdots, g\), we have

\[
t_{a_1}, t_{c_1} \in G(t_{b_2}(\Phi_{K_0}(\eta_{n-1,g}) \cdot \eta_{n-1,g})) \subseteq G(t_{b_2}(\Phi_{K_0}(\eta_{n-1,g}) \cdot \eta_{n-1,g})),
\]

for all \(i = 1, 2, \cdots, g\) except \(t_{a_2}\). Therefore the remaining part is to show

\[
t_{a_2}, \ t_{b_2} \in G(t_{b_2}(\Phi_{K_0}(\eta_{n-1,g}) \cdot \eta_{n-1,g}) \cdot \eta_{n-1,g}),
\]

and it can be proved as follows:

\[
(3.24) \quad t_{b_2}(\Phi_{K_0}(\eta_{n-1,g}) \cdot \eta_{n-1,g}) = t_{b_2}(\Phi_{K_0}(\eta_{n-1,g})) \cdot t_{b_2}(\eta_{n-1,g}) \cdot \eta_{n-1,g} \\
(3.25) \quad \sim \quad t_{b_2}(\Phi_{K_0}(\eta_{n-1,g})) \cdot t_{b_2}(\eta_{n-1,g}) \cdot t_{b_2}(\eta_{n-1,g}) \\
(3.26) \quad = \quad \cdots \cdot t_{b_2}(\Phi_{K_0}(t_{B_4})) \cdot \cdots \cdot t_{b_2}(t_{B_3}) \cdot \cdots \cdot t_{b_2}(t_{B_3}) \cdot \cdots \\
(3.27) \quad \sim \quad \cdots \cdot t_{b_2}(\Phi_{K_0}(t_{B_4})) \cdot \cdots \cdot t_{b_2}(t_{B_3}) \cdot \cdots \cdot t_{b_2}(t_{B_3}) \cdot \cdots \\
(3.28) \quad \sim \quad \cdots \cdot t_{b_2}(t_{b_2}(t_{B_3})) \cdot \cdots \cdot t_{b_2}(t_{B_3}) \cdot \cdots \cdot t_{b_2}(t_{B_3}) \cdot \cdots \\
(3.29) \quad \sim \quad \cdots \cdot t_{b_2}(t_{B_3}) \cdot \cdots \cdot t_{b_2}(t_{B_3}) \cdot \cdots \cdot t_{b_2}(t_{B_3}) \cdot \cdots \\
(3.30) \quad = \quad \cdots \cdot t_{b_2}(t_{a_2}) \cdot \cdots \cdot t_{b_2} \cdot \cdots \\
(3.31) \quad \sim \quad \cdots \cdot t_{b_2}(t_{a_2}) \cdot \cdots \cdot t_{b_2} \cdot \cdots \\
(3.32) \quad \sim \quad \cdots \cdot t_{b_2} \cdot t_{a_2} \cdot \cdots
\]

where

- \((3.26)\) is from \((3.24)\) by a sequence of inverse Hurwitz moves,
- in \((3.27)\), we just write the letters of \((3.26)\) which we need in our proof,
- \((3.28)\) is from \((3.26)\) by a sequence of Hurwitz moves,
- \((3.29)\) is from \((3.27)\) by one Hurwitz move and one inverse Hurwitz move,
- in \((3.29)\), we just write the letters of \((3.28)\) which we need in our proof and we use \((t_{b_2}(t_{B_3}))^{-1}(t_{b_2}(\Phi_{K_0}(t_{B_4}))) = ((t_{b_2}(t_{B_3}))^{-1} \circ t_{b_2})(\Phi_{K_0}(t_{B_4})) = (t_{b_2} \circ t_{B_3}^{-1} \circ t_{b_2} \circ t_{b_2})(\Phi_{K_0}(t_{B_4})) = t_{b_2}(t_{B_3}^{-1} \circ \Phi_{K_0}(t_{B_4})),\)
• (3.30) is from (3.29) because $(t_{B_4}^{-1} \circ \Phi_K) (t_{B_4}) = t_{a_2}$ by equation (3.17) and $t_{B_4} (t_{B_4} (t_{B_4})) = t_b$ by the braid relation,
• (3.31) is from (3.30) by using a sequence of Hurwitz moves,
• (3.32) is from (3.31) by an inverse Hurwitz move.

\[ \square \]

Remark 3.13. Let $X_1$, $X_2$ and $X$ be genus $g$ Lefschetz fibrations over $\mathbb{C}P^1$ such that $X_1$ and $X_2$ are isomorphic as Lefschetz fibrations and let $\Phi$ be an element of $\mathcal{M}_g$. Then we can select a monodromy factorization $W_i$ of $X_i$, $i=1,2$, such that $W_1 \sim W_2$. Therefore $\Phi(W) \cdot W_1 \sim \Phi(W) \cdot W_2$ for any genus $g$ monodromy factorization $W$ of $X$ and this implies $X_1 \sharp \Phi X \approx X_2 \sharp \Phi X$. But in general $X_1 \sharp \Phi X \approx X_2 \sharp \Phi X$ does not imply $X_1 \approx X_2$.

It is also an intersting question whether a given symplectic 4-manifold has non-isomorphic monodromy factorization with the diffeomorphic generic fiber or not. We will discuss this problem in [PY07].

Acknowledgment

The author would like to thank Ronald Stern and Jongil Park for their comments and suggestions to improve this article. He also appreciates the anonymous referee for giving several suggestions and corrections.

References


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