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NON-ISOTOPIC SYMPLECTIC SURFACES IN PRODUCT 4-MANIFOLDS

CHRISTOPHER S. HAYS AND B. DOUG PARK

Abstract. Let Σ_g be a closed Riemann surface of genus g. Generalizing Ivan Smith's construction, we give the first examples of an infinite family of homotopic but pairwise non-isotopic symplectic surfaces of even genera inside the product symplectic 4-manifolds $\Sigma_g \times \Sigma_h$, where $g \geq 1$ and $h \geq 0$.

1. Introduction

Let M be a symplectic 4-manifold. Given a homology class $\eta \in H_2(M;\mathbb{Z})$, it is a fundamental problem in symplectic topology to classify all symplectic submanifolds of M representing η up to smooth isotopy. This classification problem is commonly called the "symplectic isotopy problem" for the pair (M, η). There has been steady progress on this problem in recent years. For results on the existence of a symplectic representative of η, we refer to [19] and [20]. It can turn out that there is a unique symplectic representative for certain η's. For example, Siebert and Tian showed in [27] that a symplectic surface in the complex projective plane $\mathbb{CP}^2$ that is homologous to an algebraic curve C of degree d is in fact smoothly isotopic to C when $d \leq 17$.

On the other hand, infinite families of homologous but non-isotopic symplectic surfaces in symplectic 4-manifolds were first constructed by Fintushel and Stern in their seminal paper [9]. All of their surfaces however had genus equal to 1. Higher genus families were first constructed in [28] and later in [25]. For a summary of results on the symplectic (and Lagrangian) isotopy problem, we refer to the recent survey [7]. Throughout this paper, Σ_g will denote a closed Riemann surface of genus g. We will usually denote Σ_0 and Σ_1 by S^2 and T^2, respectively. Our main result can be stated as follows.

Theorem 1. *Let $g \geq 1$, $h \geq 0$, and p be integers. Equip $\Sigma_g \times \Sigma_h$ with a standard product symplectic form. The homology class $p[\Sigma_g \times \{\mathrm{pt}\}] \in H_2(\Sigma_g \times \Sigma_h;\mathbb{Z})$ contains an infinite family of homotopic but pairwise non-isotopic symplectic surfaces of genus equal to $p(g-1)+1$ if $h = 0$, $p \geq 4$ or if $h \geq 1$, $p \geq 2$.*

More general homology classes are dealt with in Theorems 14, 16 and 19, and Corollary 20 in Section 4. An immediate application of Theorem 1 is the following.

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Corollary 2. *Let m be a non-negative integer that is not in the set $\{0, 2\}$. Then there exists a simply connected complex projective surface V_m that contains a connected symplectic surface of genus m which is not smoothly isotopic to any complex curve.*

Recall that I. Smith has already shown in [28] that Theorem 1 holds when $h = 0$ and p is even. In particular, [28] only exhibits symplectic surfaces of *odd* genus not equal to 3, whereas the genus of surfaces in our paper can be any strictly positive integer. For example, surfaces in Theorem 1 have genus equal to 1 when $g = 1$, and genus greater than or equal to 3 when $g = 2$. For genus 2 surfaces, set $g = 2$, $h \geq 2$ and $p = 1$ in Theorem 19 or Corollary 20 in Section 4.

As is the case with I. Smith's construction, our construction relies on Σ_g being a connected sum of tori, and so g is necessarily strictly positive. In particular, we have no result when the ambient symplectic 4-manifold is $S^2 \times S^2$. We also note that the methods in this paper still do not yield infinite families when the genus is 0 (also cf. Theorem 18 and Remark 21 in Section 4). If g and h are both strictly positive, then $\pi_2(\Sigma_g \times \Sigma_h) = 0$ and hence $\Sigma_g \times \Sigma_h$ does not contain any homologically essential sphere. Other infinite families of homologous but non-isotopic symplectic surfaces of any positive genus are found in [25]. We observe that the non-isotopic symplectic surfaces constructed in [9], [25], [28] and our paper all have self-intersection 0.

Here is how our paper is organized. In Section 2, we will study an isomorphism between certain mapping class groups which will play a crucial part in our proof of Theorem 1. In Section 3, we will use the techniques in [28] to construct infinite families of homologous symplectic surfaces in $\Sigma_g \times \Sigma_h$, and we will also present the proof of Corollary 2. In Section 4, we will prove that many of our surfaces constructed in Section 3 are indeed pairwise non-isotopic.

2. Mapping class groups

Throughout this section, $h \geq 0$, $k \geq 0$, $n \geq 2$ and $p \geq 2$ will denote integers. Also assume throughout this section that we have an n-fold cyclic branched covering $\pi : \Sigma_k \to \Sigma_h$ with branch locus $X = \{x_1, \ldots, x_p\} \subset \Sigma_h$. By "cyclic" we mean that π restricted to $\Sigma_k \setminus \pi^{-1}(X)$ is a regular covering determined by a surjective homomorphism $\pi_1(\Sigma_h \setminus X) \to \mathbb{Z}_n$. We will also require that each branch point x_i is totally ramified; i.e., the local model of π at each x_i is the map $z \mapsto z^n$. Without loss of generality, we may assume that X is contained in a disk $D^2 \subset \Sigma_h$.

Whether or not there exists a branched covering $\pi : \Sigma_k \to \Sigma_h$ between two Riemann surfaces with prescribed branch data is famously called the Hurwitz problem. For the current status of research on the Hurwitz problem, we refer to [26]. We will later need the following easy lemma guaranteeing the existence of a particular type of cyclic branched covering.

Lemma 3. *Let $h \geq 0$ and $p \geq n \geq 2$ be integers such that n divides p. Then there exists an n-fold cyclic branched covering $\pi : \Sigma_k \to \Sigma_h$, with $k = hn + \frac{1}{2}(n-1)(p-2)$, and the branch locus consisting of p distinct points $\{x_1, \ldots, x_p\} \subset \Sigma_h$.*

Proof. When $h \geq 1$, the existence of such a π follows immediately from Proposition 3.3 in [4] since the "total branching" number $\upsilon(\pi) = p(n-1)$ is always even under our hypothesis (also cf. Theorem 4 in [17]). When $h = 0$, the existence of π follows from Lemma 2.1(a) in [4] by associating the n-cycle $(1, 2, \ldots, n)$ to every branch

point x_i (also cf. Proposition 5.2 in [4]). The orientability of the cover and the formula for genus k follow from Propositions 2.3 and 2.4 in [4], respectively. $\square$

From now on, we fix a decomposition of $\Sigma_k \setminus \pi^{-1}(X)$ into n sheets of the branched covering π. The sheets are then a $\mathbb{Z}_n$ torsor. Moreover, $\pi_1(\Sigma_h \setminus X)$ acts on this torsor, where a closed curve acts on the sheets by mapping the i-th sheet to the sheet containing the endpoint of the lift of the curve starting at the i-th sheet. For simplicity, we will usually blur the distinction between an element in the fundamental group and a simple closed curve that represents it.

Around each branch point $x_i \in X$, we choose a generator $\alpha_i \in \pi_1(\Sigma_h \setminus X)$ such that the loop α_i lies in D^2 and partitions x_i from the rest of X. Furthermore, we require the actions of the α_i's on the sheets to be independent of i; i.e., they are all equal to the action of some fixed generator of $\mathbb{Z}_n$. If we use Seifert-Van Kampen theorem along the sets $\Sigma_h \setminus D^2$ and $D^2 \setminus X$, we obtain a presentation of $\pi_1(\Sigma_h \setminus X)$ as

$$\left\langle \alpha_1, \ldots, \alpha_p, \gamma_1, \ldots, \gamma_h, \delta_1, \ldots, \delta_h \mid \alpha_1 \cdots \alpha_p = \prod_{i=1}^{h} [\gamma_i, \delta_i] \right\rangle, \tag{1}$$

where the γ_i's and δ_i's are the usual generators of $\pi_1(\Sigma_h)$.

Next we define an "α-length" homomorphism $\ell_\alpha : \pi_1(\Sigma_h \setminus X) \to \mathbb{Z}$ by mapping the generators in (1) as follows:

$$\ell_\alpha(\xi) \;=\; \begin{cases} 1 & \text{if} \quad \xi \;=\; \alpha_1, \ldots, \alpha_{p-1}, \\ 1-p & \text{if} \quad \xi \;=\; \alpha_p, \\ 0 & \text{if} \quad \xi \;=\; \gamma_1, \ldots, \gamma_h, \delta_1, \ldots, \delta_h. \end{cases} \tag{2}$$

Lemma 4. *A closed curve representing $\xi \in \pi_1(\Sigma_h \setminus X)$ will lift to a closed curve in Σ_k under the cyclic branched covering π if and only if n divides $\ell_\alpha(\xi)$.*

Proof. First we will motivate our definition (2). For the purpose of deciding whether or not a curve will lift to a closed curve, the only interesting part of the curve is the part that intersects $D^2 \setminus X$. If we alter a curve in $\Sigma_h \setminus X$ away from $D^2 \setminus X$, we do not change whether or not the curve will lift to a closed curve, as this alteration only affects how the lift looks on a specific sheet in the covering space $\Sigma_k \setminus \pi^{-1}(X)$. Thus we might as well replace $\Sigma_h \setminus D^2$ with another disk D^2. This means that ℓ_α can be factored as the composition

$$\pi_1(\Sigma_h \setminus X) \;\to\; \pi_1(S^2 \setminus X) \;\cong\; \langle \alpha_1, \ldots, \alpha_p \mid \alpha_1 \cdots \alpha_p = 1 \rangle \;\to\; \mathbb{Z},$$

where the first homomorphism maps α_i to itself, and maps γ_i and δ_i to 1. We can also think of this as a "surgery" that changes our covering π with base space Σ_h to a new covering with base space S^2, which is obtained by taking $\pi^{-1}(D^2 \setminus X)$ and capping off the n copies of the boundary S^1 by gluing in a disk along each. Thus a curve in $\Sigma_h \setminus X$ will lift to a closed curve if and only if its image in $S^2 \setminus X$ does.

For the proof proper, we define two subgroups K and W of $\pi_1(\Sigma_h \setminus X)$ as follows. K consists of curves in $\pi_1(\Sigma_h \setminus X)$ that lift to closed curves under π. W consists of the curves ξ such that $n \mid \ell_\alpha(\xi)$.

Since the α_i's all generate the same automorphism of the n sheets, any word for which n divides its α-length will lift to a closed curve. Thus $W \subseteq K$. Since K and W are both normal in $\pi_1(\Sigma_h \setminus X)$, and both have quotients isomorphic to $\mathbb{Z}_n$, we

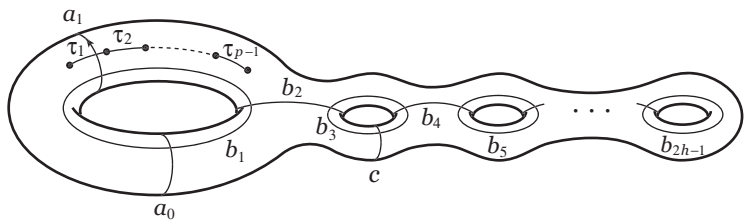

FIGURE 1. Generators of $\Gamma_p(\Sigma_h)$ when $h \geq 1$

have

$$\mathbb{Z}_n \;\cong\; \pi_1(\Sigma_h \setminus X)/K \;\cong\; \frac{\pi_1(\Sigma_h \setminus X)/W}{K/W} \;\cong\; \frac{\mathbb{Z}_n}{K/W}\,.$$

Thus we must have $W = K$. □

Let $\Gamma_p(\Sigma_h)$ denote the (punctured) mapping class group of the Riemann surface Σ_h that permutes the p points in $X = \{x_1, \ldots, x_p\}$. The x_i's are commonly called "marked points" or "punctures" in this context. The following lemma implies that the elements of $\Gamma_p(\Sigma_h)$ preserve the α-length.

Lemma 5. *Let $\varphi : \Sigma_h \setminus X \to \Sigma_h \setminus X$ be an orientation preserving homeomorphism. For any $\xi \in \pi_1(\Sigma_h \setminus X)$, we have $\ell_\alpha(\xi) = \ell_\alpha(\varphi_* \xi)$.*

Proof. We need to divide the proof into two cases, when $h = 0$ and when $h \geq 1$. First we consider the case when $h = 0$. From Theorem 4.5 in Birman's book [2] we have a presentation of $\Gamma_p(S^2)$, when $p \geq 2$, as

$$\left\langle \omega_1, \ldots, \omega_{p-1} \;\middle|\; \begin{array}{l} \omega_i\omega_j = \omega_j\omega_i \text{ for } |i-j| \geq 2,\ \omega_i\omega_{i+1}\omega_i = \omega_{i+1}\omega_i\omega_{i+1} \text{ for all } i \\ \omega_1 \cdots \omega_{p-2}\omega_{p-1}^2\omega_{p-2} \cdots \omega_1 \;=\; 1 \;=\; (\omega_1 \cdots \omega_{p-1})^p \end{array} \right\rangle.$$

Moreover, each ω_i can be geometrically realized as a "half" Dehn twist about a simple curve $\sigma_i : [0,1] \to S^2$ that satisfies $\sigma_i(0) = x_i$, $\sigma_i(1) = x_{i+1}$, $\operatorname{Im}\sigma_i \cap X = \{x_i, x_{i+1}\}$. By definition, ω_i interchanges x_i and x_{i+1}, and ω_i^2 is a full Dehn twist about the boundary circle of a neighborhood of $\operatorname{Im}\sigma_i$. One can directly compute that for each $i = 1, \ldots, p-1$,

$$(\omega_i)_*\alpha_j \;=\; \begin{cases} \alpha_j & \text{if} \quad j \neq i, i+1, \\ \alpha_{i+1} & \text{if} \quad j = i, \\ \alpha_{i+1}\alpha_i\alpha_{i+1}^{-1} & \text{if} \quad j = i+1. \end{cases}$$

Thus the ω_i's preserve ℓ_α. Now, since φ is isotopic to some word Ω in the ω_i's, $\varphi_*\xi$ is isotopic to $\Omega_*\xi$, and thus $\ell_\alpha(\varphi_*\xi) = \ell_\alpha(\Omega_*\xi) = \ell_\alpha(\xi)$. This completes the proof when $h = 0$.

Now suppose $h \geq 1$. Corollary 2.11 of [18] states that when $h \geq 1$ and $p \geq 0$, $\Gamma_p(\Sigma_h)$ is generated by the (full) Dehn twists along $a_0, a_1, b_1, \ldots, b_{2h-1}, c$ and the half Dehn twists along $\tau_1, \ldots, \tau_{p-1}$, where the a_i's, b_i's, τ_i's and c are the curves in Figure 1 (also cf. Figure 13 of [18]). Note that a half Dehn twist is called a "braid twist" in [18].

Without loss of generality, we may choose the disk D^2 containing X in such a way that the only generators of $\Gamma_p(\Sigma_h)$ that will affect the interior of D^2 are the

half Dehn twists along the τ_i's and the Dehn twist along a_1. We can then perform the following computations. For each $i = 1, \ldots, p-1$,

$$(\tau_i)_*\alpha_j = \begin{cases} \alpha_j & \text{if } j \neq i, i+1, \\ \alpha_i^{-1}\alpha_{i+1}\alpha_i & \text{if } j = i, \\ \alpha_i & \text{if } j = i+1; \end{cases}$$

$$(a_1)_*\alpha_j = \begin{cases} \alpha_j & \text{if } j \neq 1, \\ a_1\,\alpha_1 a_1^{-1} & \text{if } j = 1. \end{cases}$$

For the second computation, we are assuming that the base point of $\pi_1(\Sigma_h \setminus X)$ is to the right of a_1, and we have chosen to parameterize the curve a_1 in the direction shown in Figure 1. Now we can proceed exactly as in the previous case. □

By Lemmas 4 and 5, the property of lifting to a closed curve is preserved under orientation preserving self-homeomorphisms of $\Sigma_h \setminus X$. We now state an important definition.

Definition 6. For a fixed orientation preserving homeomorphism $\zeta : \Sigma_k \to \Sigma_k$, let $\mathrm{Aut}^\zeta(\Sigma_k)$ denote the space of all orientation preserving self-homeomorphisms of Σ_k that commute with ζ. Define the *ζ-mapping class group* of Σ_k to be $\Gamma^\zeta(\Sigma_k) = \pi_0(\mathrm{Aut}^\zeta(\Sigma_k))$.

We are interested in the cases where $\zeta^n = \mathrm{id}_{\Sigma_k}$ for some positive integer $n \geq 2$, and the induced quotient space $\Sigma_k/\langle\zeta\rangle$ is Σ_h. We can now present the main result of this section.

Theorem 7. *Let $\zeta : \Sigma_k \to \Sigma_k$ be an orientation preserving homeomorphism that induces an n-fold cyclic branched covering $\pi : \Sigma_k \to \Sigma_h$ with the branch locus $X = \{x_1, \ldots, x_p\}$. Then there exists an isomorphism $\Phi : \Gamma^\zeta(\Sigma_k)/\langle\zeta\rangle \to \Gamma_p(\Sigma_h)$.*

Proof. First, we define Φ as follows. For a given $\psi \in \mathrm{Aut}^\zeta(\Sigma_k)$ we can construct an element of $\Gamma_p(\Sigma_h)$ as follows. Note that the homeomorphisms in $\mathrm{Aut}^\zeta(\Sigma_k)$ are precisely the fiber preserving homeomorphisms of π. Thus, for any point $w \in \Sigma_h$, $\pi\psi\pi^{-1}(w)$ is well-defined. This pointwise definition extends to a homeomorphism of Σ_h. By definition, the elements in the subgroup $\langle\zeta\rangle$ induce the identity map on Σ_h, and we obtain the desired group homomorphism Φ. Note that, since ψ preserves the set $\pi^{-1}(X)$, we can make the restriction $\pi\psi\pi^{-1} : \Sigma_h \setminus X \to \Sigma_h \setminus X$.

Next, we have the following construction of Φ^{-1}. Let $f : \Sigma_h \setminus X \to \Sigma_h \setminus X$ be an orientation preserving homeomorphism, and choose two "base" points $z_0 \in \Sigma_k \setminus \pi^{-1}(X)$ and $f^*(z_0) \in \pi^{-1}f\pi(z_0)$. For any other point $y \in \Sigma_k \setminus \pi^{-1}(X)$, let λ be a path from z_0 to y. Then the path $f\pi\lambda$ lifts uniquely under π to a path starting at $f^*(z_0)$. By Lemmas 4 and 5, if a curve in $\Sigma_h \setminus X$ lifts to a closed curve in $\Sigma_k \setminus \pi^{-1}(X)$, then the image of this curve under f also lifts to a closed curve in $\Sigma_k \setminus \pi^{-1}(X)$. Because of this, the endpoint of the lift of $f\pi\lambda$ is well-defined and independent of our choice of the path λ. Call this endpoint $f^*(y)$. It is not hard to check that f^* extends to a homeomorphism of Σ_k that commutes with ζ (since f^* is fiber preserving). Moreover, if $f \simeq \mathrm{id}_{\Sigma_h \setminus X}$ via f_t, then f_t^* is a path in $\mathrm{Aut}^\zeta(\Sigma_k)$. Thus we have a set map $\Gamma_p(\Sigma_h) \to \Gamma^\zeta(\Sigma_k)$. However, depending on the initial choice of the base point $f^*(z_0)$, id_{Σ_h} can be mapped to any element of $\langle\zeta\rangle$. Thus we have a homomorphism $\Gamma_p(\Sigma_h) \to \Gamma^\zeta(\Sigma_k)/\langle\zeta\rangle$. This is readily checked to be the inverse of Φ. □

3. Generalization of Smith's construction

Throughout the section, let $h \geq 0$ and $p \geq 1$ be integers. Let $\mathrm{Br}_p(\Sigma_h)$ denote the braid group on p strings in the Riemann surface Σ_h. We refer the reader to [1] or [2] for the definition of $\mathrm{Br}_p(\Sigma_h)$. (Zariski seems to be the first to have studied this braid group. He called it the "topological discriminant group" in [29].) The image of a braid $\beta \in \mathrm{Br}_p(\Sigma_h)$ defines a disjoint union of simple closed curves $\beta : \bigsqcup_{i=1}^{\sharp\beta} S^1 \to S^1 \times \Sigma_h$, where $\sharp\beta$ is the number of components of $\mathrm{Im}\,\beta$.

Definition 8. We say that a braid $\beta \in \mathrm{Br}_p(\Sigma_h)$ is *connected* if $\sharp\beta = 1$, and *disconnected* if $\sharp\beta \geq 2$.

There exists a surjective homomorphism $\sigma : \mathrm{Br}_p(\Sigma_h) \to S_p$, where S_p is the symmetric group on p letters, defined by mapping a braid to the permutation it induces on its endpoints. $\sharp\beta$ is the number of disjoint cycles in the product decomposition of $\sigma(\beta)$ including cycles of length 1 if necessary. Note that the connected braids are exactly the inverse images of length p cycles under σ.

Given a braid $\beta \in \mathrm{Br}_p(\Sigma_h)$, the cartesian product $S^1 \times \mathrm{Im}\,\beta \subset S^1 \times (S^1 \times \Sigma_h)$ is easily seen to be a disjoint union of symplectic torus-submanifolds of $T^2 \times \Sigma_h$ with respect to a standard product symplectic form (cf. Lemma 1.3 in [28]). Let us write $T_\beta = S^1 \times \mathrm{Im}\,\beta$ for short. We will view the 4-manifold $T^2 \times \Sigma_h$ as the trivial fiber bundle with fiber Σ_h over base T^2. Then T_β is a "multi-section" of this trivial bundle that intersects each fiber $\{\mathrm{pt}\} \times \Sigma_h$ transversely p times.

Let $\gamma = S^1 \times \{\mathrm{pt}\}$ be a first factor of the base torus $T^2 = S^1 \times S^1$, and let $\mathrm{pr}_2 : S^1 \times \Sigma_h \to \Sigma_h$ be the projection map onto the second factor. Note that we have $\mathrm{pr}_2 \circ \beta : \bigsqcup_{i=1}^{\sharp\beta} S^1 \to \Sigma_h$. It is not hard to see that the union of tori T_β represent the homology class

$$p\,[T^2 \times \{\mathrm{pt}\}] + [\gamma \times \mathrm{Im}(\mathrm{pr}_2 \circ \beta)] \in H_2(T^2 \times \Sigma_h; \mathbb{Z}).$$

Definition 9. We say that a braid $\beta \in \mathrm{Br}_p(\Sigma_h)$ is *local* if $[\mathrm{Im}(\mathrm{pr}_2 \circ \beta)] = 1 \in \pi_1(\Sigma_h)$, and *non-local* otherwise.

In particular, every braid in $\mathrm{Br}_p(S^2)$ is local. If $\beta, \beta' \in \mathrm{Br}_p(\Sigma_h)$ are connected and $[\mathrm{Im}(\mathrm{pr}_2 \circ \beta)] = [\mathrm{Im}(\mathrm{pr}_2 \circ \beta')]$ in $\pi_1(\Sigma_h)$, then clearly T_β is homotopic to $T_{\beta'}$ inside $T^2 \times \Sigma_h$. In particular, T_β and $T_{\beta'}$ are homotopic if β and β' are both connected and local.

Let $g \geq 1$ be an integer. Now take g braids $\beta_1, \ldots, \beta_g \in \mathrm{Br}_p(\Sigma_h)$ and the corresponding multi-sections $T_{\beta_1}, \ldots, T_{\beta_g}$ in g copies of the fiber bundle $T^2 \times \Sigma_h$. By symplectically fiber-summing $g-1$ times along one or two Σ_h fibers from each copy of $T^2 \times \Sigma_h$, we obtain the trivial Σ_h-bundle $\Sigma_g \times \Sigma_h$ over base Σ_g. In forming this $(g-1)$-fold symplectic fiber sum, we take out the tubular neighborhood of one or two Σ_h fibers from each $T^2 \times \Sigma_h$, thereby taking out the disjoint union of p or $2p$ small disks from each multi-section T_{β_i}. After small perturbations in the gluing regions, if necessary, we may assume that these g punctured multi-sections $T_{\beta_i} \setminus \bigsqcup_{j=1}^{p \text{ or } 2p} D_j^2$ $(i = 1, \ldots, g)$ can be lined up and glued together to form a smooth multi-section of the trivial bundle $\Sigma_g \times \Sigma_h$. Let us denote this new multi-section by $\Theta(\beta_1, \ldots, \beta_g)$. Of course, when $g = 1$, we are just left with $\Theta(\beta_1) = T_{\beta_1} \subset T^2 \times \Sigma_h$. Since the symplectic fiber sum construction still leaves the punctured T_{β_i}'s symplectic (cf. Corollary 1.7 in [11]), we can conclude that $\Theta(\beta_1, \ldots, \beta_g)$ is a (possibly disconnected) symplectic submanifold of $\Sigma_g \times \Sigma_h$. Note that $\Theta(\beta_1, \ldots, \beta_g)$ is connected if at least one of $\beta_1, \ldots, \beta_g$ is a connected braid.

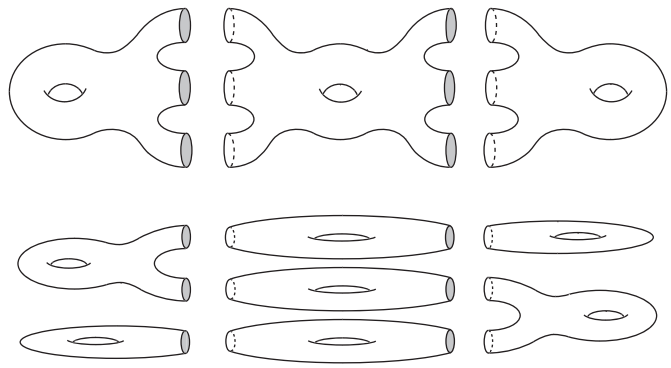

FIGURE 2. Two possible ways to obtain a genus 7 surface $\Theta(\beta_1, \beta_2, \beta_3)$

Let $\gamma_1, \ldots, \gamma_g$ be disjoint simple closed curves in the base Σ_g such that each γ_i was a first factor of the i-th base torus before the fiber sums. We can then easily see that $\Theta(\beta_1, \ldots, \beta_g)$ represents the homology class

$$p\,[\Sigma_g \times \{\mathrm{pt}\}] \,+\, \sum_{i=1}^{g} [\gamma_i \times \mathrm{Im}(\mathrm{pr}_2 \circ \beta_i)] \,\in\, H_2(\Sigma_g \times \Sigma_h; \mathbb{Z}). \tag{3}$$

In particular, note that $[\Theta(\beta_1, \ldots, \beta_g)] = p\,[\Sigma_g \times \{\mathrm{pt}\}]$ when every β_i is local. Recall that the canonical class of $\Sigma_g \times \Sigma_h$ is

$$(2h-2)[\Sigma_g \times \{\mathrm{pt}\}]_{\mathrm{PD}} \,+\, (2g-2)[\{\mathrm{pt}\} \times \Sigma_h]_{\mathrm{PD}} \,\in\, H^2(\Sigma_g \times \Sigma_h; \mathbb{Z}),$$

where the subscript PD means the Poincaré dual. If the surface $\Theta(\beta_1, \ldots, \beta_g)$ is connected, then the adjunction formula tells us that the genus of $\Theta(\beta_1, \ldots, \beta_g)$ is $p(g-1)+1$. See Figure 2 for the case when $g = 3$ and $p = 3$. It is not hard to see that $\Theta(\beta_1, \ldots, \beta_g)$ is homotopic to $\Theta(\beta'_1, \ldots, \beta'_g)$ inside $\Sigma_g \times \Sigma_h$ if β_i and β'_i are connected, local and satisfy $\sigma(\beta_i) = \sigma(\beta'_i)$ for each $i = 1, \ldots, g$.

Remark 10. We should point out here that our construction of $\Theta(\beta_1, \ldots, \beta_g)$ is exactly the same as in [28] except that we allow the braids to be disconnected, the fiber genus h to be positive (hence allowing non-local braids), and the number of braid strings p to be odd. As already remarked in the introduction, this enables us to consider surfaces of even genus which were left out in [28]. This also enables us to work with more general homology classes of the form (3).

Proof of Corollary 2. We proceed as in [28]. Throughout this proof, fix $h = 0$ and assume that at least one of $\beta_1, \ldots, \beta_g \in \mathrm{Br}_p(S^2)$ is connected so that the corresponding $\Theta(\beta_1, \ldots, \beta_g)$ is also connected. We now consider $\Sigma_g \times S^2$ as the trivial fiber bundle over S^2 with fiber Σ_g. Note that each symplectic surface $\Theta(\beta_1, \ldots, \beta_g)$ lies outside a tubular neighborhood $\nu F' \cong \Sigma_g \times D^2$ of some fiber $F' = \Sigma_g \times \{\mathrm{pt}\}$. Assume that V_m is a simply connected complex projective surface such that there exists a holomorphic Lefschetz fibration $V_m \to \mathbb{CP}^1$ with generic fiber $F \cong \Sigma_g$. If we fiber sum such V_m and $\Sigma_g \times S^2$ along F and F', we will just get V_m back. But after this fiber sum we can locate the symplectic surface $\Theta(\beta_1, \ldots, \beta_g)$ inside V_m, which is now viewed as the union

$$V_m \;=\; [V_m \setminus \nu F] \,\cup\, \left[(\Sigma_g \times S^2) \setminus \nu F'\right].$$

It follows that $\Theta(\beta_1, \ldots, \beta_g)$ represents the homology class $p[F] \in H_2(V_m; \mathbb{Z})$. Since V_m is simply connected, it is well known that the only complex curve that can represent $p[F]$ is the disjoint union of p fibers. Clearly, the connected surface $\Theta(\beta_1, \ldots, \beta_g)$ cannot be isotopic to such union when $p \geq 2$.

To complete our proof, it only remains to exhibit such V_m. For $m = 1$, we may take V_1 to be any simply connected elliptic surface, e.g. a rational surface $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ or a $K3$ surface. In this case, we will have $g = 1$, and any choice of $p \geq 2$ will work. For $m \geq 3$, we may take $V_m = (K3)\#2\overline{\mathbb{CP}}^2$, i.e. a $K3$ surface blown up twice. It is well known that the surface $(K3)\#2\overline{\mathbb{CP}}^2$ has a Lefschetz fibration structure with generic fiber Σ_2 (cf. Exercise 8.4.2(e) in [12]). Hence we can set $g = 2$ and $p = m - 1$. □

Remark 11. Recall from [3] that every symplectic 4-manifold, after some blowups if necessary, admits a Lefschetz fibration. The argument in the proof of Corollary 2 allows us to symplectically embed our surfaces $\Theta(\beta_1, \ldots, \beta_g) \subset \Sigma_g \times S^2$ into quite a large class of symplectic 4-manifolds, and in this sense, our construction can be deemed "universal". Our surfaces have the potential to be pairwise non-isotopic after the embedding, and so they could be "locally knotted" in the sense that the non-isotopic surfaces would all be contained in a tubular neighborhood of a fiber. Note that such locally knotted construction is impossible in the Lagrangian setting (cf. [5] and [15]).

4. Proof of non-isotopy

We start this section with a technical lemma regarding the sizes of the braid group $\mathrm{Br}_p(\Sigma_h)$ and some of its subsets.

Lemma 12. *Let $h \geq 0$ and $p \geq 1$ be integers.*

(i) *If $|\mathrm{Br}_p(\Sigma_h)|$ denotes the order of $\mathrm{Br}_p(\Sigma_h)$, then*

$$|\mathrm{Br}_p(\Sigma_h)| = \begin{cases} 1 & \text{if } h = 0,\ p = 1, \\ 2 & \text{if } h = 0,\ p = 2, \\ 12 & \text{if } h = 0,\ p = 3, \\ \infty & \text{if } h = 0,\ p \geq 4 \ \text{ or if } \ h \geq 1. \end{cases}$$

(ii) *Let $\Delta_{p,h}$ denote the center of $\mathrm{Br}_p(\Sigma_h)$. Then*

$$\Delta_{p,h} \cong \begin{cases} 1 & \text{if } h \geq 2 \ \text{ or if } \ h = 0,\ p = 1, \\ \mathbb{Z}_2 & \text{if } h = 0,\ p \geq 2, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } h = 1. \end{cases}$$

(iii) *Let $\Lambda_{p,h}$ be the subset of $\mathrm{Br}_p(\Sigma_h)$ consisting of connected local braids. Then*

$$|\Lambda_{p,h}| = \begin{cases} 1 & \text{if } p = 1 \ \text{ or if } \ h = 0,\ p = 2, \\ 4 & \text{if } h = 0,\ p = 3, \\ \infty & \text{if } h = 0,\ p \geq 4 \ \text{ or if } \ h \geq 1,\ p \geq 2. \end{cases}$$

(iv) *Let $q_\Delta : \mathrm{Br}_p(\Sigma_h) \to \mathrm{Br}_p(\Sigma_h)/\Delta_{p,h}$ be the quotient homomorphism. Then*

$$|q_\Delta(\Lambda_{p,h})| = \begin{cases} 1 & \text{if } p = 1 \ \text{ or if } \ h = 0,\ p = 2, \\ 2 & \text{if } h = 0,\ p = 3, \\ \infty & \text{if } h = 0,\ p \geq 4 \ \text{ or if } \ h \geq 1,\ p \geq 2. \end{cases}$$

Proof. For a proof of (i) when $h = 0$, we refer the reader to [8] or Theorem 1.11 (and its proof) in [2]. When $h \geq 1$, our statement follows immediately from the inclusion of the infinite group $\pi_1(\Sigma_h) = \mathrm{Br}_1(\Sigma_h) \hookrightarrow \mathrm{Br}_p(\Sigma_h)$.

For a proof of (ii) when $h = 0$ or $h \geq 2$, we refer the reader to Lemmas 4.2.2 and 4.2.3 in [2] (also cf. [10] when $h = 0$). A proof that $\Delta_{p,1} \cong \pi_1(T^2)$ when $h = 1$ is found in [23] (Proposition 4.2).

To prove (iii), recall that we have a surjective homomorphism $\sigma : \mathrm{Br}_p(\Sigma_h) \to S_p$, and that the connected braids are the inverse images of p-cycles under σ. Thus our statement follows easily from (i) when $h = 0$. If $h \geq 1$, there is a natural inclusion $\mathrm{Br}_p(D^2) \hookrightarrow \mathrm{Br}_p(\Sigma_h)$, where $D^2 \subset \Sigma_h$ is a disk that contains the set of base points $\{x_1, \ldots, x_p\}$ (cf. Proposition 2.2 in [23]). Since no element of $\mathrm{Br}_p(D^2)$ has finite order (cf. Proposition 1.5 in [23]), $|\mathrm{Br}_p(D^2)| = \infty$ when $p \geq 2$. From the commutative diagram

$$\begin{tikzcd} \mathrm{Br}_p(D^2) \arrow[r, hook] \arrow[dr] & \mathrm{Br}_p(\Sigma_h) \arrow[d] \\ & S_p \end{tikzcd}$$

we easily conclude that $|\mathrm{Br}_p(D^2) \cap \Lambda_{p,h}| = \infty$ when $h \geq 1$ and $p \geq 2$. Finally, it follows immediately from the definition that there is a unique local braid in $\mathrm{Br}_1(\Sigma_h) = \pi_1(\Sigma_h)$.

When $h = 0$ and $p = 3$, (iv) follows from the short exact sequence

$$1 \to \Delta_{3,0} \to \mathrm{Br}_3(S^2) \to S_3 \to 1.$$

To prove (iv) when $h = 1$ and $p \geq 2$, it is enough to show $|q_\Delta(\mathrm{Br}_p(D^2) \cap \Lambda_{p,1})| = \infty$. The description of $\Delta_{p,1} \subset \mathrm{Br}_p(T^2)$ in [23] easily implies that $\mathrm{Br}_p(D^2) \cap \Delta_{p,1} = 1$. Now we have

$$\mathrm{Br}_p(D^2) \cap \Lambda_{p,1} \subset \mathrm{Br}_p(D^2) \cong (\mathrm{Br}_p(D^2)\Delta_{p,1})/\Delta_{p,1} \subset \mathrm{Br}_p(T^2)/\Delta_{p,1}.$$

Since we already saw in the proof of (iii) that $|\mathrm{Br}_p(D^2) \cap \Lambda_{p,1}| = \infty$, our statement follows immediately. All other remaining cases of (iv) follow trivially from (ii) and (iii). □

To state our next two theorems, we will need the following.

Definition 13. Let $\mathbb{Z}_{\geq 0}$ be the set of non-negative integers. Given $\mu \in H_1(\Sigma_h; \mathbb{Z})$, we define the *divisibility* of μ to be the non-negative integer

$$\mathrm{div}(\mu) = \max\{m \in \mathbb{Z}_{\geq 0} \mid \mu = ma \text{ for some } a \in H_1(\Sigma_h; \mathbb{Z})\}.$$

Theorem 1 is just a special case ($\mu_i = 0$ for all $i \in I(j_0)$) of the following more general theorem.

Theorem 14. *Assume that either $h = 0$ and $p \geq 4$, or that $h \geq 1$ and $p \geq 2$. Let $g \geq 1$, and let $\gamma_1, \ldots, \gamma_g$ be simple closed curves in Σ_g as in* (3)*. Choose any $j_0 \in \{1, \ldots, g\}$ and let $I(j_0) = \{1, \ldots, g\} \setminus \{j_0\}$. For each $i \in I(j_0)$, choose $\mu_i \in H_1(\Sigma_h; \mathbb{Z})$, and let $d_i = \mathrm{div}(\mu_i)$. Let n be the greatest common divisor of $\{p\} \cup \{d_i \mid i \in I(j_0)\}$. If $n \geq 2$, then the homology class*

$$p[\Sigma_g \times \{\mathrm{pt}\}] + \sum_{i \in I(j_0)} [\gamma_i] \times \mu_i \in H_2(\Sigma_g \times \Sigma_h; \mathbb{Z}) \tag{4}$$

contains an infinite family of homotopic but pairwise non-isotopic symplectic surfaces of genus equal to $p(g-1)+1$.

Proof. For each $i \in I(j_0)$, we fix a 1-string braid $\beta_i \in \mathrm{Br}_1(\Sigma_h) = \pi_1(\Sigma_h)$ such that $[\mathrm{Im}(\mathrm{pr}_2 \circ \beta_i)] = \mu_i$; i.e., β_i maps to μ_i under the abelianization $\pi_1(\Sigma_h) \to H_1(\Sigma_h;\mathbb{Z})$. We will view β_i as a p-string braid via the canonical inclusion $\mathrm{Br}_1(\Sigma_h) \hookrightarrow \mathrm{Br}_p(\Sigma_h)$, which sends β_i to the p-string braid whose first string is β_i and the rest constant. Let $\Lambda_{p,h}$ denote the subset of $\mathrm{Br}_p(\Sigma_h)$ consisting of connected local braids. Let $\beta_{j_0} \in \Lambda_{p,h}$ be arbitrary.

Consider the symplectic surface $\Theta(\beta_1,\dots,\beta_g)$ constructed in Section 3. Since β_{j_0} is connected, $\Theta(\beta_1,\dots,\beta_g)$ is also connected and has genus $p(g-1)+1$. By our choice of β_i's, the homology class (3) simplifies to (4). It is not hard to see that $\Theta(\beta_1,\dots,\beta_{j_0},\dots,\beta_g)$ and $\Theta(\beta_1,\dots,\beta'_{j_0},\dots,\beta_g)$ are homotopic inside $\Sigma_g \times \Sigma_h$ whenever $\beta_{j_0}, \beta'_{j_0} \in \Lambda_{p,h}$ induce the same permutation (p-cycle) on their endpoints, i.e. whenever $\sigma(\beta_{j_0}) = \sigma(\beta'_{j_0})$ in the notation of Section 3.

Let $\langle\Theta(\beta_1,\dots,\beta_g)\rangle$ denote the smooth isotopy class of $\Theta(\beta_1,\dots,\beta_g)$ in $\Sigma_g \times \Sigma_h$. Since there are only finitely many p-cycles in the symmetric group S_p, our theorem will follow if we can show that the set

$$\{\langle\Theta(\beta_1,\dots,\beta_g)\rangle \mid \beta_{j_0} \in \Lambda_{p,h}\}$$

is infinite under the hypotheses of our theorem.

Assume that $n \geq 2$. Since the homology class of $\Theta(\beta_1,\dots,\beta_g)$ is divisible by n, there exists an n-fold cyclic branched covering $Z(\beta_1,\dots,\beta_g) \to \Sigma_g \times \Sigma_h$ branched over the surface $\Theta(\beta_1,\dots,\beta_g)$ (cf. [16]). From Lemma 3 and the commutative diagram (5), it is clear that the smooth 4-manifold $Z(\beta_1,\dots,\beta_g)$ is a fiber bundle over Σ_g with fiber diffeomorphic to a closed Riemann surface Σ_k of genus $k = hn + \frac{1}{2}(n-1)(p-2) > 0$:

$$\begin{tikzcd}
\Sigma_k \arrow[r, hook, "\mathrm{inc}"'] \arrow[d, "\pi"'] & Z(\beta_1,\dots,\beta_g) \arrow[d, "n:1"'] \arrow[dr, "\widetilde{\pi}"] & \\
\{\mathrm{pt}\} \times \Sigma_h \arrow[r, hook, "\mathrm{inc}"'] & \Sigma_g \times \Sigma_h \arrow[r, "\mathrm{pr}_1"'] & \Sigma_g.
\end{tikzcd} \tag{5}$$

Since the homeomorphism type of $Z(\beta_1,\dots,\beta_g)$ depends only on the isotopy class of the branch locus $\Theta(\beta_1,\dots,\beta_g)$, we will be done if we can show that the set

$$\{Z(\beta_1,\dots,\beta_g) \mid \beta_{j_0} \in \Lambda_{p,h}\}$$

contains infinitely many pairwise non-homeomorphic 4-manifolds when $h = 0$ and $p \geq 4$, or when $h \geq 1$ and $p \geq 2$. To show this, it is enough to prove that

$$\{\pi_1(Z(\beta_1,\dots,\beta_g)) \mid \beta_{j_0} \in \Lambda_{p,h}\}$$

contains infinitely many pairwise non-isomorphic groups. Since Σ_g is aspherical, we have a short exact sequence of groups

$$1 \;\to\; \pi_1(\Sigma_k) \;\to\; \pi_1(Z(\beta_1,\dots,\beta_g)) \;\to\; \pi_1(\Sigma_g) \;\to\; 1. \tag{6}$$

Hence $\pi_1(Z(\beta_1,\dots,\beta_g))$ is completely determined by the monodromy homomorphism $\rho_{\widetilde{\pi}} : \pi_1(\Sigma_g) \to \Gamma^{\zeta}(\Sigma_k)$ (cf. §5.1 in [14]). Here, $\zeta : \Sigma_k \to \Sigma_k$ is an orientation preserving homeomorphism inducing the branched covering map π in Lemma 3, and $\Gamma^{\zeta}(\Sigma_k)$ is as in Definition 6. Note that the image of $\rho_{\widetilde{\pi}}$ lies in the ζ-mapping class group $\Gamma^{\zeta}(\Sigma_k)$ since, according to diagram (5), the n-fold branched covering

map $Z(\beta_1, \ldots, \beta_g) \to \Sigma_g \times \Sigma_h$ is a bundle homomorphism which restricts to π on the fibers.

It now follows that we will be done if we can show that elements in the set

$$\left\{\rho_{\widetilde{\pi}} : \pi_1(\Sigma_g) \to \Gamma^{\zeta}(\Sigma_k) \mid \beta_{j_0} \in \Lambda_{p,h}\right\}$$

represent infinitely many distinct conjugacy classes of homomorphisms. Let $q_{\langle\zeta\rangle} : \Gamma^{\zeta}(\Sigma_k) \to \Gamma^{\zeta}(\Sigma_k)/\langle\zeta\rangle$ denote the quotient homomorphism. Recall from Theorem 7 that we have an isomorphism $\Phi : \Gamma^{\zeta}(\Sigma_k)/\langle\zeta\rangle \to \Gamma_p(\Sigma_h)$. Clearly, it will be enough to show that elements in the set

$$\left\{\Phi \circ q_{\langle\zeta\rangle} \circ \rho_{\widetilde{\pi}} : \pi_1(\Sigma_g) \to \Gamma_p(\Sigma_h) \mid \beta_{j_0} \in \Lambda_{p,h}\right\}$$

represent infinitely many distinct conjugacy classes of homomorphisms.

Let $\Delta_{p,h}$ denote the center of the group $\mathrm{Br}_p(\Sigma_h)$. From Theorem 4.3 in [2], we also have an *injective* homomorphism $\Psi : \mathrm{Br}_p(\Sigma_h)/\Delta_{p,h} \to \Gamma_p(\Sigma_h)$ when $h = 0$, $p \geq 3$ or when $h = 1$, $p \geq 2$ or when $h \geq 2$. It follows from the way we constructed $Z(\beta_1, \ldots, \beta_g)$ that there exists a canonical presentation

$$\pi_1(\Sigma_g) \;=\; \left\langle \gamma_1, \ldots, \gamma_g, \delta_1, \ldots, \delta_g \mid \prod_{i=1}^{g} [\gamma_i, \delta_i] = 1 \right\rangle \tag{7}$$

such that the corresponding "quotient" monodromy homomorphism $\Phi \circ q_{\langle\zeta\rangle} \circ \rho_{\widetilde{\pi}}$ maps

$$\gamma_i \mapsto 1, \quad \delta_i \mapsto \Psi(q_{\Delta}(\beta_i)) \quad \text{for all } \; i = 1, \ldots, g, \tag{8}$$

where $q_{\Delta}(\beta_i)$ denotes the image of the braid $\beta_i \in \mathrm{Br}_p(\Sigma_h)$ in the quotient group $\mathrm{Br}_p(\Sigma_h)/\Delta_{p,h}$.

If $h = 0$ and $p \geq 4$, or if $h \geq 1$ and $p \geq 2$, then there exist infinitely many elements within the set

$$\{\Psi(q_{\Delta}(\beta)) \in \Gamma_p(\Sigma_h) \mid \beta \in \Lambda_{p,h}\} \tag{9}$$

by Lemma 12(iv). We will be done if we can show that the elements of (9) represent infinitely many distinct conjugacy classes in the mapping class group $\Gamma_p(\Sigma_h)$.

First, we consider the case when $h \geq 1$ and $p \geq 2$. As before, let D^2 be a disk in Σ_h containing all of the marked points $\{x_1, \ldots, x_p\}$. It is shown in [24] (Corollary 4.2) that the inclusion $D^2 \hookrightarrow \Sigma_h$ induces an inclusion $\Gamma_p(D^2) \hookrightarrow \Gamma_p(\Sigma_h)$ when $h \geq 1$. (As usual, we define a mapping class of a surface with boundary to be a path-component of homeomorphisms which restrict to the identity map on the boundary components.) Moreover, this inclusion will identify a mapping class of D^2 with the mapping class of Σ_h that contains a homeomorphism which equals the identity map outside of the disk, and whose restriction to the disk lies in the original mapping class. It is well known (cf. [2]) that $\mathrm{Br}_p(D^2) \cong \Gamma_p(D^2)$, and that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Br}_p(D^2) & \hookrightarrow & \mathrm{Br}_p(\Sigma_h) \\ \cong \big\downarrow & & \big\downarrow \Psi \circ q_{\Delta} \\ \Gamma_p(D^2) & \hookrightarrow & \Gamma_p(\Sigma_h). \end{array}$$

We claim that elements of the infinite subset $\Psi(q_{\Delta}(\mathrm{Br}_p(D^2) \cap \Lambda_{p,h})) \subset \Gamma_p(D^2)$ represent infinitely many conjugacy classes of $\Gamma_p(\Sigma_h)$. To prove our claim, suppose we have $[f_1], [f_2] \in \Gamma_p(D^2)$ such that there exists $[\varphi] \in \Gamma_p(\Sigma_h)$ satisfying $[f_2] =$

$[\varphi^{-1}][f_1][\varphi]$. After an isotopy if necessary, we may choose a representative φ of $[\varphi]$ such that $\varphi(D^2) = D^2$ and $\varphi|_{\partial D^2} = \mathrm{id}_{\partial D^2}$. Note that for $y \in \Sigma_h \setminus D^2$, we have $\varphi^{-1} f_1 \varphi(y) = y = f_2(y)$. We can therefore define φ_0 to equal φ on D^2 and to equal the identity map on $\Sigma_h \setminus D^2$. Then $[\varphi_0] \in \Gamma_p(D^2)$, and $\varphi_0^{-1} f_1 \varphi_0 = f_2$. So $f_1, f_2 \in \Gamma_p(D^2)$ are conjugate in $\Gamma_p(\Sigma_h)$ if and only if they are conjugate in $\Gamma_p(D^2)$. Since the connected braids in $\mathrm{Br}_p(D^2)$ represent infinitely many knot types in $\mathbb{R}^3$ for fixed $p \geq 2$, and distinct knot types in $\mathbb{R}^3$ correspond to distinct conjugacy classes in $\mathrm{Br}_p(D^2) \cong \Gamma_p(D^2)$, we have an infinite number of the desired conjugacy classes.

Finally, we consider the case when $h = 0$ and $p \geq 4$. Let $\omega_1, \ldots, \omega_{p-1} \in \Gamma_p(S^2)$ be the half Dehn twist generators (cf. proof of Lemma 5) that are the images of the standard generators of $\mathrm{Br}_p(S^2)$. Each ω_i is of infinite order. Consider the infinite set

$$\left\{\omega_1^{2m}(\omega_1 \cdots \omega_{p-1}) \in \Gamma_p(S^2) \mid m \in \mathbb{Z}_{\geq 0}\right\}. \tag{10}$$

We know that the elements of (10) all lie in the image of connected braids in $\mathrm{Br}_p(S^2)$ since they all induce the same p-cycle $(1, 2, \ldots, p) \in S_p$ on the marked points $\{x_1, \ldots, x_p\}$. Thus we will be done if we can show that the elements of (10) represent infinitely many distinct conjugacy classes of $\Gamma_p(S^2)$.

For starters, let $p = 4$. Let $\tilde{G} = \mathrm{SL}(2, \mathbb{Z})/\{\pm I\}$; i.e., $\tilde{G}$ is the quotient group of $\mathrm{SL}(2, \mathbb{Z})$ obtained by identifying a matrix A with $-A$. From Lemmas 5.4.1 and 5.4.3 in [2], we obtain a surjective homomorphism $\tau : \Gamma_4(S^2) \to \tilde{G}$ such that

$$\tau(\omega_1) = \tau(\omega_3) = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \tau(\omega_2) = \pm \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Notice that τ maps $\omega_1^{2m}(\omega_1\omega_2\omega_3)$ to

$$\pm \begin{pmatrix} 1 & 2m+1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \pm \begin{pmatrix} -2m & 1 \\ -1 & 0 \end{pmatrix}$$

whose trace is $\pm 2m$. If $\pm A_1$ and $\pm A_2$ are conjugate in $\tilde{G}$, then $\mathrm{tr}(A_1) = \pm \mathrm{tr}(A_2)$. Hence no two distinct elements of (10) can lie in the same conjugacy class of $\Gamma_p(S^2)$ when $p = 4$.

Now let $p > 4$. The $p > 4$ case can be reduced to the $p = 4$ case. We will take the approach that was suggested to us in [21]. Let ω_p be the half Dehn twist about a simple curve σ_p starting at the p-th marked point x_p and ending at x_1, completing the chain of curves $\sigma_1, \ldots, \sigma_{p-1}$ which were used to define $\omega_1, \ldots, \omega_{p-1}$ to a closed loop. For each $i \in \{1, \ldots, p\}$, let $\eta_i = \omega_i^2$. Then η_i is a full Dehn twist about a circle enclosing the two marked points $\{x_i, x_{i+1}\}$ (with $x_{p+1} = x_1$) and the curve σ_i joining them. If we write $\xi = \omega_1 \cdots \omega_{p-1}$, then elements in (10) are of the form $\eta_1^m \xi$. From the presentation of $\Gamma_p(S^2)$ in the proof of Lemma 5, we have $\xi^p = 1$. Now $\xi \eta_i \xi^{-1} = \eta_{i+1}$ for each $i \in \{1, \ldots, p-1\}$. It follows that $(\eta_1^m \xi)^p = \eta_1^m \cdots \eta_p^m$.

Let $\mathrm{P}\Gamma_p(S^2)$ denote the kernel of the surjective homomorphism $\Gamma_p(S^2) \to S_p$, which sends a mapping class to the permutation it induces on the marked points. The subgroup $\mathrm{P}\Gamma_p(S^2)$ is usually called a "pure" mapping class group. Note that $(\eta_1^m \xi)^p \in \mathrm{P}\Gamma_p(S^2)$. Consider the set

$$\left\{(\eta_1^m \xi)^p \in \mathrm{P}\Gamma_p(S^2) \mid m \in \mathbb{Z}_{\geq 0}\right\}.$$

If we can show that these p-th powers represent infinitely many distinct conjugacy classes in $\mathrm{P}\Gamma_p(S^2)$, then it will follow from the fact that $\mathrm{P}\Gamma_p(S^2)$ is of finite index

in $\Gamma_p(S^2)$ that our original mapping classes, $\eta_1^m\xi$, represent infinitely many distinct conjugacy classes in $\Gamma_p(S^2)$.

The advantage of working in the pure mapping class group is that there are natural "forgetful" homomorphisms corresponding to forgetting one or more marked points. By forgetting $\{x_5,\dots,x_p\}$, we get a homomorphism $\mathrm{P}\Gamma_p(S^2)\to\mathrm{P}\Gamma_4(S^2)$. A Dehn twist about a circle bounding a disk containing 0 or 1 marked point is isotopic to the identity map. Hence the image of $(\eta_1^m\xi)^p=\eta_1^m\cdots\eta_p^m$ in $\mathrm{P}\Gamma_4(S^2)$ under the forgetful homomorphism is $\eta_1^m\eta_2^m\eta_3^m$. Therefore we will be done if we can show that elements in the set

$$\{\omega_1^{2m}\omega_2^{2m}\omega_3^{2m}\in\Gamma_4(S^2)\mid m\in\mathbb{Z}_{\geq 0}\} \tag{11}$$

represent infinitely many distinct conjugacy classes in $\Gamma_4(S^2)$. τ maps $\omega_1^{2m}\omega_2^{2m}\omega_3^{2m}$ to

$$\pm\begin{pmatrix}1&2m\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-2m&1\end{pmatrix}\begin{pmatrix}1&2m\\0&1\end{pmatrix}=\pm\begin{pmatrix}1-4m^2&4m-8m^3\\-2m&1-4m^2\end{pmatrix}$$

whose trace is $\pm(2-8m^2)$. Hence (11) is an infinite set and no two distinct elements of (11) represent the same conjugacy class in $\Gamma_4(S^2)$. □

When $h\geq 1$, we can generalize Theorem 14 slightly. To do so, we will need the following terminology.

Definition 15. Let $h\geq 1$ and $p\geq 2$. Let $\mathrm{Ab}:\mathrm{Br}_1(\Sigma_h)=\pi_1(\Sigma_h)\to H_1(\Sigma_h;\mathbb{Z})$ be the abelianization. Let $\Lambda_{p,h}$ and q_Δ be as in Lemma 12. Let $\Psi:\mathrm{Br}_p(\Sigma_h)/\Delta_{p,h}\to\Gamma_p(\Sigma_h)$ be the injective homomorphism obtained from Theorem 4.3 in [2]. We say that a homology class $\mu\in H_1(\Sigma_h;\mathbb{Z})$ is (p,h)-*admissible* if for some $\tilde\mu\in(\mathrm{Ab})^{-1}(\mu)\subset\mathrm{Br}_1(\Sigma_h)\subset\mathrm{Br}_p(\Sigma_h)$, elements in the set

$$\{\Psi(q_\Delta(\tilde\mu\lambda))\in\Gamma_p(\Sigma_h)\mid\lambda\in\Lambda_{p,h}\} \tag{12}$$

represent infinitely many distinct conjugacy classes of $\Gamma_p(\Sigma_h)$.

Theorem 16. *Let $g\geq 1$, $h\geq 1$ and $p\geq 2$. Let $\gamma_1,\dots,\gamma_g$ be simple closed curves in Σ_g as in* (3) *or* (7)*. For each $i\in\{1,\dots,g\}$, choose $\mu_i\in H_1(\Sigma_h;\mathbb{Z})$, and let $d_i=\mathrm{div}(\mu_i)$. Let n be the greatest common divisor of $\{p\}\cup\{d_1,\dots,d_g\}$. If $n\geq 2$ and μ_{j_0} is (p,h)-admissible for some $j_0\in\{1,\dots,g\}$, then the homology class*

$$p[\Sigma_g\times\{\mathrm{pt}\}]\ +\ \sum_{i=1}^{g}[\gamma_i]\times\mu_i\ \in\ H_2(\Sigma_g\times\Sigma_h;\mathbb{Z}) \tag{13}$$

contains an infinite family of homotopic but pairwise non-isotopic symplectic surfaces of genus equal to $p(g-1)+1$.

Proof. The proof of Theorem 14 can be transplanted here with very little change as follows. First fix $\tilde\mu_{j_0}\in\mathrm{Br}_1(\Sigma_h)$ satisfying $\mathrm{Ab}(\tilde\mu_{j_0})=\mu_{j_0}$ and the conjugacy condition in Definition 15. The main change we need to make is that, instead of varying $\beta_{j_0}\in\Lambda_{p,h}$, we vary

$$\beta_{j_0}\ \in\ \tilde\mu_{j_0}\Lambda_{p,h}\ =\ \{\tilde\mu_{j_0}\lambda\mid\lambda\in\Lambda_{p,h}\}.$$

It is not hard to see that $\sigma(\tilde\mu_{j_0}\lambda)=\sigma(\tilde\mu_{j_0}\lambda')$ implies that the corresponding Θ surfaces are homotopic, as long as the other β_i's remain fixed for all $i\in I(j_0)$. We will need to replace (9) with (12). Our hypothesis that μ_{j_0} is (p,h)-admissible presupposes that elements of (12) represent infinitely many distinct conjugacy classes, and hence we do not need the paragraphs after (9). □

Remark 17. In the proof of Theorem 14, we have already shown that $\mu = 0$ is (p,h)-admissible. It is reasonable to conjecture that every $\mu \in H_1(\Sigma_h;\mathbb{Z})$ is (p,h)-admissible (so that we may drop the hypothesis that μ_{j_0} is (p,h)-admissible from Theorem 16). Unfortunately, we are not aware of any general theorem that will imply this conjecture. For any fixed μ, however, one should be able to show that μ is (p,h)-admissible by comparing conjugacy invariants that can be computed from an invariant train track as outlined in [22].

When $h = 0$ and $p = 3$, we cannot construct any infinite family, but we can obtain the following lower bounds.

Theorem 18. *For each integer $g \geq 1$, let N_g be the number of pairwise non-isotopic connected symplectic surfaces of genus $3g-2$ representing the homology class $3[\Sigma_g \times \{\mathrm{pt}\}] \in H_2(\Sigma_g \times S^2;\mathbb{Z})$. Then we have*

$$N_g \;\geq \sum_{\substack{a+b+c=g-1\\ a,b,c\in\mathbb{Z},\ a,b,c\geq 0}} (a+1)P_3(b) \;\geq\; \binom{g+1}{2}, \tag{14}$$

where $P_3(b)$ is the number of partitions of b into at most three summands, i.e. the number of elements in the set

$$\left\{(x,y,z)\in\mathbb{Z}^3 \mid x+y+z=b,\ x\geq y\geq z\geq 0\right\}.$$

Proof. The argument in the proof of Theorem 14 can still be applied to our situation. In particular, we have the 3-fold branched cover $Z(\beta_1,\ldots,\beta_g)$ given by the commutative diagram (5) and the quotient monodromy homomorphism (8). Now Theorem 4.3 of [2] and Lemma 12 imply that $\mathrm{Br}_3(S^2)/\Delta_{3,0} \cong \Gamma_3(S^2) \cong S_3$ (also cf. the proof of Theorem 4.5 in [2]). Since we want the surface $\Theta(\beta_1,\ldots,\beta_g)$ to be connected, we will assume for simplicity that at least one of the β_i's is connected and hence the corresponding δ_i is mapped to a 3-cycle by the quotient monodromy homomorphism. Furthermore, up to conjugation in S_3 and an automorphism of Σ_g, we can assume that $\delta_1 \mapsto (1,2,3)$. Since we can randomly map the remaining $g-1$ generators $\delta_2,\ldots,\delta_g$ to three possible conjugacy classes of S_3, it is not hard to obtain the cruder lower bound $N_g \geq \binom{g}{2} + g = \binom{g+1}{2}$.

To obtain the better lower bound in (14), let $a+1$, b and c be the number of δ_i's that map to a 3-cycle, a 2-cycle and the identity, respectively. Also, let x, y and z be the number of δ_i's that map to $(1,2)$, $(2,3)$ and $(1,3)$, respectively. Note that the only elements $\psi \neq 1 \in S_3$ that satisfy $\psi^{-1}(1,2,3)\psi = (1,2,3)$ are the 3-cycles $(1,2,3)$ and $(1,3,2)$. Since conjugation by 3-cycles is transitive on the set of 2-cycles, our desired inequalities follow from a simple counting argument. □

Note that the surfaces in the proof of Theorem 18 are not necessarily homotopic to one another. Since we do not necessarily need any of the β_i's to be connected in order for $\Theta(\beta_1,\ldots,\beta_g)$ to be connected (cf. Figure 2), the lower bound (14) can be definitely improved for $g \geq 2$. It is well known (cf. §19.3 in [13]) that

$$\frac{1}{(1-x)(1-x^2)(1-x^3)} \;=\; \sum_{b=0}^{\infty} P_3(b)x^b.$$

It is shown in [6] that asymptotically

$$P_3(b) \;\sim\; \frac{1}{3!}\binom{b-1}{2} \;=\; \frac{1}{12}(b-1)(b-2).$$

It follows that we have an asymptotic lower bound for N_g:

$$\begin{aligned} N_g \;\geq\; \sum_{n=1}^{g}\sum_{k=1}^{n} k\cdot P_3(n-k) \;&\sim\; \frac{1}{12}\sum_{n=1}^{g}\sum_{k=1}^{n}[k^3-(2n-3)k^2+(n-1)(n-2)k] \\ &=\; \frac{1}{144}\sum_{n=1}^{g}(n^4-6n^3+11n^2+18n) \\ &=\; \frac{1}{720}(g^5-5g^4+5g^3+65g^2+54g). \end{aligned}$$

A table of lower bounds for N_g given by (14) for the first few values of g follows.

$g\ =$	1	2	3	4	5	6	7	8	9	10	11	12
$N_g \geq$	1	4	11	25	50	91	155	250	386	575	831	1170

$g\ =$	100	1,000	10,000
$N_g \geq$	16,473,833	1,413,345,296,663	139,132,094,946,299,963

When $p=1$, the branched covering argument in the proof of Theorem 14 cannot be used since we are dealing with the *primitive* homology class $[\Theta(\beta_1,\ldots,\beta_g)]\in H_2(\Sigma_g\times\Sigma_h;\mathbb{Z})$. When $h\geq 2$, we can still prove the following weaker result.

Theorem 19. *Assume that $g\geq 1$, $h\geq 2$ and $p\geq 1$ are integers. Let K_h be the kernel of the abelianization $\mathrm{Br}_1(\Sigma_h)=\pi_1(\Sigma_h)\to H_1(\Sigma_h;\mathbb{Z})$. Given any braids $\beta_1,\ldots,\beta_g\in\mathrm{Br}_1(\Sigma_h)$, if the set of finitely generated subgroups of $\pi_1(\Sigma_h)$,*

$$\{\langle\beta_1\kappa_1,\,\beta_2\kappa_2,\,\ldots\,,\beta_g\kappa_g\rangle\;\subset\;\pi_1(\Sigma_h)\mid\kappa_1,\ldots,\kappa_g\in K_h\}\,, \tag{15}$$

contains infinitely many distinct subgroups of $\pi_1(\Sigma_h)$, then the homology class (3) *contains an infinite family of pairwise non-homotopic* (*and hence automatically pairwise non-isotopic*) *symplectic surfaces of genus $p(g-1)+1$.*

Proof. Fix a connected local braid $\lambda\in\Lambda_{p,h}$. View each β_i and κ_i as a p-string braid via the inclusion $\mathrm{Br}_1(\Sigma_h)\hookrightarrow\mathrm{Br}_p(\Sigma_h)$. Define $\bar\beta_i\in\mathrm{Br}_p(\Sigma_h)$ by $\bar\beta_1=\lambda\beta_1\kappa_1$, and $\bar\beta_i=\beta_i\kappa_i$ for $i=2,\ldots,g$. The corresponding surface $\Theta(\bar\beta_1,\ldots,\bar\beta_g)$ is connected since $\bar\beta_1$ is connected, and it is clear from Section 3 that it represents the homology class (3).

Let $\iota:\Theta(\bar\beta_1,\ldots,\bar\beta_g)\hookrightarrow\Sigma_g\times\Sigma_h$ denote the inclusion. Let $\iota_*(\pi_1(\Theta(\bar\beta_1,\ldots,\bar\beta_g)))$ be the induced subgroup of $\pi_1(\Sigma_g\times\Sigma_h)$. Note that if $\Theta(\bar\beta_1',\ldots,\bar\beta_g')$ is homotopic to $\Theta(\bar\beta_1,\ldots,\bar\beta_g)$ inside $\Sigma_g\times\Sigma_h$, then we must have

$$\iota_*'(\pi_1(\Theta(\bar\beta_1',\ldots,\bar\beta_g')))\;=\;\iota_*(\pi_1(\Theta(\bar\beta_1,\ldots,\bar\beta_g))).$$

If $\mathrm{pr}_2:\Sigma_g\times\Sigma_h\to\Sigma_h$ denotes the projection map onto the second factor, then the induced subgroup

$$(\mathrm{pr}_2)_*(\iota_*(\pi_1(\Theta(\bar\beta_1,\ldots,\bar\beta_g))))\;\subset\;\pi_1(\Sigma_h)$$

is just $\langle\beta_1\kappa_1,\beta_2\kappa_2,\ldots,\beta_g\kappa_g\rangle$. Hence our hypothesis implies that there are infinitely many different subgroups of $\pi_1(\Sigma_g\times\Sigma_h)$ of the form $\iota_*(\pi_1(\Theta(\bar\beta_1,\ldots,\bar\beta_g)))$. $\square$

Distinguishing subgroups in (15) can turn out to be tedious, if not challenging. The following corollary deals with a few simple cases.

Corollary 20. *Assume that $g \geq 1$, $h \geq 2$ and $p \geq 1$ are integers. Choose any $\mu \in H_1(\Sigma_h;\mathbb{Z})$. Let $\gamma_1, \ldots, \gamma_g$ be simple closed curves in Σ_g as in* (3) *or* (7)*. If I and J are disjoint subsets of $\{1, \ldots, g\}$ satisfying $I \cup J \neq \emptyset$, then the homology class*

$$p[\Sigma_g \times \{\mathrm{pt}\}] + \sum_{i\in I} [\gamma_i] \times \mu - \sum_{j\in J} [\gamma_j] \times \mu \in H_2(\Sigma_g \times \Sigma_h;\mathbb{Z}) \tag{16}$$

contains an infinite family of pairwise non-homotopic (*and hence automatically pairwise non-isotopic*) *symplectic surfaces of genus $p(g-1)+1$.*

Proof. Choose a non-trivial (and hence of infinite order) element $\kappa \in K_h$. Choose a braid $\beta \in \mathrm{Br}_1(\Sigma_h)$ such that $[\mathrm{Im}(\mathrm{pr}_2 \circ \beta)] = \mu$. If $\mu = 0$, then we choose $\beta = 1$. Let $m > 0$ be an integer. Note that $\beta(\beta\kappa^m)^{-1} \in K_h$. Define 1-string braids

$$\beta_i = \begin{cases} \beta & \text{if} \quad i \in I, \\ \beta^{-1} & \text{if} \quad i \in J, \\ 1 & \text{if} \quad i \in \{1, \ldots, g\} \setminus (I \cup J), \end{cases}$$

$$\kappa_i = \begin{cases} \kappa^m & \text{if} \quad i \in I, \\ \beta(\beta\kappa^m)^{-1} & \text{if} \quad i \in J, \\ 1 & \text{if} \quad i \in \{1, \ldots, g\} \setminus (I \cup J). \end{cases}$$

For these β_i's, the homology class (3) simplifies to (16). It also follows that

$$\langle \beta_1\kappa_1,\, \beta_2\kappa_2,\, \ldots,\, \beta_g\kappa_g \rangle = \langle \beta\kappa^m \rangle,$$

the infinite cyclic subgroup of $\pi_1(\Sigma_h)$ generated by $\beta\kappa^m$.

Suppose $\langle \beta\kappa^{m_1} \rangle = \langle \beta\kappa^{m_2} \rangle$ for some positive integers $m_1 \neq m_2$. Then we must have $\beta\kappa^{m_2} = (\beta\kappa^{m_1})^{\pm 1}$. Since κ has infinite order, we must have $\beta\kappa^{m_2} = (\beta\kappa^{m_1})^{-1}$. But this implies that $\mu = -\mu$ in $H_1(\Sigma_h;\mathbb{Z})$. Thus $\mu = 0$ and we have chosen $\beta = 1$. Now $\kappa^{m_2} = \kappa^{-m_1}$, which is clearly impossible. Hence different m's give rise to different cyclic subgroups in (15). □

In particular, by setting $p = 1$ and $\mu = 0$ in Corollary 20, we obtain infinitely many pairwise non-homotopic symplectic surfaces of genus g in the primitive homology class $[\Sigma_g \times \{\mathrm{pt}\}] \in H_2(\Sigma_g \times \Sigma_h;\mathbb{Z})$ when $g \geq 1$ and $h \geq 2$.

Remark 21. From Lemma 12, it is obvious that our construction in Section 3 only yields a single connected symplectic surface in any homology class of the form (3) when $p = 1$ and h is either 0 or 1, or when $p = 2$, $h = 0$ and $g = 1$. When $p = 2$, $h = 0$ and $g \geq 2$, we can construct finitely many genus $2g-1$ symplectic surfaces $\Theta(\beta_1, \ldots, \beta_g)$ in $2[\Sigma_g \times \{\mathrm{pt}\}] \in H_2(\Sigma_g \times S^2;\mathbb{Z})$, and form the corresponding double branched covers $Z(\beta_1, \ldots, \beta_g)$. However, $\Sigma_k = S^2$ in this case, so (6) gives $\pi_1(Z(\beta_1, \ldots, \beta_g)) \cong \pi_1(\Sigma_g)$, independent of the β_i's. We do not know of any method that could possibly distinguish the isotopy types of these surfaces.

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Department of Mathematics, Michigan State University, East Lansing, Michigan 48824

E-mail address: cshays@math.msu.edu

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

E-mail address: bdpark@math.uwaterloo.ca