THE ZERO SET OF SEMI-INVARIANTS
FOR EXTENDED DYNKIN QUIVERS

CH. RIEDTMANN AND G. ZWARA

Abstract. We show that the set of common zeros \( Z_{Q,d} \) of all semi-invariants vanishing at 0 on the variety \( \text{rep}(Q,d) \) of all representations with dimension vector \( d \) of an extended Dynkin quiver \( Q \) under the group \( \text{GL}(d) \) is complete intersection if \( d \) is “big enough”. In case \( \text{rep}(Q,d) \) does not contain an open \( \text{GL}(d) \)-orbit, which is the case not considered so far, the number of irreducible components of \( Z_{Q,d} \) grows with \( d \), except if \( Q \) is an oriented cycle.

1. Introduction and main result

1.1. Let \( k \) be an algebraically closed field, and let \( Q = (Q_0, Q_1, t, h) \) be a quiver with \( n \) vertices and a finite set \( Q_1 \) of arrows \( \alpha : t\alpha \to h\alpha \), where \( t\alpha \) and \( h\alpha \) denote the tail and the head of \( \alpha \), respectively.

A representation of \( Q \) over \( k \) is a collection \( (X(i); i \in Q_0) \) of finite dimensional \( k \)-vector spaces together with a collection \( (X(\alpha) : X(t\alpha) \to X(h\alpha); \alpha \in Q_1) \) of \( k \)-linear maps. A morphism \( f : X \to Y \) between two representations is a collection \( (f(i) : X(i) \to Y(i)) \) of \( k \)-linear maps such that

\[
 f(h\alpha) \circ X(\alpha) = Y(\alpha) \circ f(t\alpha) \quad \text{for all } \alpha \in Q_1.
\]

By \( \sigma(X) \) we denote the number of pairwise non-isomorphic indecomposable direct summands occurring in a decomposition of \( X \) into indecomposables. According to the theorem of Krull-Schmidt, \( \sigma(X) \) is well-defined. The dimension vector of a representation \( X \) of \( Q \) is the vector

\[
 \dim X = (\dim X(i))_{i \in Q_0} \in \mathbb{N}^{Q_0}.
\]

We denote the category of representations of \( Q \) by \( \text{rep}(Q) \), and for any vector \( d = (d_i)_{i \in Q_0} \in \mathbb{N}^{Q_0} \),

\[
 \text{rep}(Q,d) = \prod_{\alpha \in Q_1} \text{Mat}(d_{h\alpha} \times d_{t\alpha}, k)
\]

is the vector space of representations \( X \) of \( Q \) with \( X(i) = k^{d_i}, i \in Q_0 \). The group

\[
 \text{GL}(d) = \prod_{i \in Q_0} \text{GL}(d_i, k)
\]

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acts on $\text{rep}(Q, d)$ by
\[(g_i)_{i \in Q_0} \ast X)(\alpha) = g_{\alpha} \circ X(\alpha) \circ g_{\alpha}^{-1}.
\]
Note that the $GL(d)$-orbit of $X$ consists of the representations $Y$ in $\text{rep}(Q, d)$ which are isomorphic to $X$.

A regular function $f \in k[\text{rep}(Q, d)]$ is called a semi-invariant if, for any $g \in GL(d)$, $g \ast f = \chi(g)f$ for some group homomorphism $\chi : GL(d) \to k^*$ which is a regular function on $GL(d)$, the so-called weight of $f$. Note that the $k$-algebra generated by all semi-invariants is just the algebra $k[\text{rep}(Q, d)]^{SL(d)}$ of polynomial functions which are invariant under the product
\[SL(d) = \prod_{i \in Q_0} SL(d_i, k).
\]
Indeed, the algebra $k[\text{rep}(Q, d)]^{SL(d)}$ is the direct sum of the spaces of $GL(d)$-semi-invariants of weight $\chi$, where $\chi$ ranges over all characters of $GL(d)$.

In case $d$ is a sincere prehomogeneous dimension vector, i.e. if $d_i > 0$ for all $i$ and if the orbit $GL(d) \cdot T$ of some representation $T$ is open and dense, the algebra $k[\text{rep}(Q, d)]^{SL(d)}$ is a polynomial algebra in $n - \sigma(T)$ generators. In fact, the generators correspond bijectively to the simple objects in the perpendicular category $T^\perp$, the full subcategory of $\text{rep}(Q)$ whose objects $Y$ satisfy $\text{Hom}_Q(T, Y) = \text{Ext}_Q^1(T, Y) = 0$ [12]. We showed in [8] that the variety of common zeros $Z_{Q, d}$ of all non-constant semi-invariants is a complete intersection if each of the pairwise non-isomorphic indecomposable direct summands $T_i$ of $T = \bigoplus_{i=1}^{\sigma(T)} T_i^{\lambda_i}$

arises with a sufficiently large multiplicity $\lambda_i$. Choosing $\lambda_i$ larger still, we obtain that $Z_{Q, d}$ is irreducible. By [9], $Z_{Q, d}$ is a complete intersection or irreducible, for $\lambda_i \geq 3$ or $\lambda_i \geq 4$, respectively, if $Q$ is a tame quiver, i.e. a disjoint union of Dynkin quivers and extended Dynkin quivers. Chang and Weyman, the first to consider this question, showed in [2] that $Z_{Q, d}$ is a complete intersection for any $d$ if $Q$ is a Dynkin quiver of type $A_n$.

The interest in knowing when $k[\text{rep}(Q, d)]^{SL(d)}$ is non-singular (or equivalently is a polynomial ring), and when $Z_{Q, d}$ is a complete intersection comes from the following fact: Assume a reductive algebraic group $G$ acts regularly and linearly on a finite dimensional $C$-vector space $V$. The action of $G$ on $V$ is called

(i) coregular if the algebraic quotient $[5] V//G$ has no singularities,
(ii) cofree if $C[V]$ is a free module over the invariant ring $C[V]^G$,
(iii) equidimensional if the dimension of $V//G$ equals the codimension in $V$ of the set of common zeros of all $G$-invariants which vanish at 0.

G. Schwarz proved in [16] that an action is cofree if and only if it is coregular and equidimensional. He classified all coregular and cofree representations of connected simple algebraic groups ([13], [15]). In [7], P. Littelmann classified all coregular and cofree irreducible representations of semisimple groups.

We have recalled above that in case $d$ is a prehomogeneous dimension vector the action of $SL(d)$ on $\text{rep}(Q, d)$ is always coregular, and it is equidimensional if $d$ is “big enough”.

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Unfortunately, most dimension vectors fail to be prehomogeneous, except for Dynkin quivers, in which case they all are. The algebra \( k[\text{rep}(Q, d)]^{\text{SL}(d)} \) for an arbitrary pair \((Q, d)\) is not known. If \( Q \) is an extended Dynkin quiver, however, Skowroński and Weyman have obtained in [17] a complete description of \( k[\text{rep}(Q, d)]^{\text{SL}(d)} \) by generators and relations for an arbitrary dimension vector \( d \). It turns out that in most cases the action of \( \text{SL}(d) \) on \( \text{rep}(Q, d) \) is coregular (compare 1.3).

Our purpose in the present paper is to study \( Z_{Q,d} \) for an extended Dynkin quiver and an arbitrary dimension vector. We find that, as in the prehomogeneous case, the action of \( \text{SL}(d) \) on \( \text{rep}(Q, d) \) is equidimensional if \( d \) is “big enough”. But \( Z_{Q,d} \) does not become irreducible with growing \( d \). In fact, except for the oriented cycle, the number of its irreducible components increases with \( d \).

1.2. From now on until the end of Section 4 we assume that \( Q \) does not contain oriented cycles. We will usually not repeat this assumption. We will treat the oriented cycle separately in Section 5.

We need to recall a few facts and definitions, mostly from [11], before we can state our results. For a quiver \( Q \), the Euler bilinear form \((-,-) : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}\) is defined by \( (x,y) = \sum_{i \in Q_0} x_iiy_i - \sum_{a \in Q_1} x_{a^-}yh_{a^+} \). The associated quadratic form \( q_Q : \mathbb{Z}^{Q_0} \to \mathbb{Z} \), given by \( q_Q(x) = \langle x, x \rangle \), is the Euler quadratic form. If \( X \) and \( Y \) are representations of \( Q \) and \( d \) is a dimension vector, we have

\[
\langle \text{dim} X, \text{dim} Y \rangle = [X,Y] - [1]X,Y,
\]

where we set \([X,Y] = \dim_k \text{Hom}_Q(X,Y)\) and \([1]X,Y] = \dim_k \text{Ext}^1_Q(X,Y)\).

Assume that \( Q \) is an extended Dynkin quiver. Then the Euler quadratic form is positive semi-definite, and its radical is \( \mathbb{Z}h \) for a unique vector \( h \in \mathbb{N}^{Q_0} \). The defect is the linear form \( \partial : \mathbb{Z}^{Q_0} \to \mathbb{Z} \) given by \( \partial(x) = \langle h, x \rangle = -\langle x, h \rangle \). A representation \( X \) is called preprojective, regular, or preinjective, if \( \partial(\text{dim} U) < 0 \), \( = 0 \), or \( > 0 \), respectively, for every indecomposable direct summand \( U \) of \( X \). Any representation \( X \) has a unique decomposition \( X = X_P \oplus X_R \oplus X_T \), where \( X_P \), \( X_R \), \( X_T \) are preprojective, regular, and preinjective, respectively.

The regular representations form an exact abelian subcategory \( R \) of \( \text{rep}(Q) \). The category \( R \) decomposes into a \( \mathbb{P}^1(k) \)-family \( \bigsqcup_{\mu \in \mathbb{P}^1(k)} R_{\mu} \) of uniserial categories. For each \( \mu \), the category \( R_{\mu} \) contains a finite number \( r_{\mu} \) of simple objects; their dimension vectors add up to \( h \). The set \( \mathcal{E} = \{ \mu \in \mathbb{P}^1(k) : r_{\mu} > 1 \} \) has at most three elements. For \( \mu \not\in \mathcal{E} \), we denote the unique simple representations of \( R_{\mu} \) by \( H_{\mu} \). A simple regular representation with dimension vector \( h \) is called homogeneous. We have \( \sum (r_{\mu} - 1) = \#Q_0 - 2 = n - 2 \).

If \( d \) is not a prehomogeneous dimension vector, then representations in \( \text{rep}(Q,d) \) are necessarily generically regular. In fact, \( \text{rep}(Q,d) \) contains an open subset consisting of representations \( H_{\mu_1} \oplus \cdots \oplus H_{\mu_p} \oplus V \), where \( p \geq 1 \) and \( \mu_1, \ldots, \mu_p \not\in \mathcal{E} \) are pairwise different, any indecomposable direct summand of \( V \) belongs to \( R_{\mu} \) for some \( \mu \in \mathcal{E} \), and \( \text{dim} V = e \) is prehomogeneous. We call the decomposition \( d = p \cdot h + e \) the canonical decomposition of \( d \).

1.3. We are now ready to state our results.

As a consequence of the theorem of Skowroński and Weyman [17 Thm.21], we know that the algebra \( k[\text{rep}(Q, d)]^{\text{SL}(d)} \) is a polynomial ring if \( Q \) is an extended Dynkin quiver, \( d = p \cdot h + e \) is not prehomogeneous and \( p \geq 2 \). If \( Q \) is of type
Theorem 1. Let \( Z \in Q, N \) be an extended Dynkin quiver different from the oriented cycle, \( d \in \mathbb{N}^{Q_0} \) a non-prehomogeneous dimension vector with canonical decomposition \( d = p \cdot h + e \), and \( V = \bigoplus_{i=1}^{n(V)} V_i^{\lambda_i} \) a representation in the open orbit of \( \text{rep}(Q,e) \) with \( V_i \) indecomposable and pairwise non-isomorphic. Assume that either \( Q \) is of type \( \tilde{A}_{n-1} \) or else \( p \geq 3 \) and \( \lambda_i \geq 3 \) for all \( i \). Then we have:

(i) The action of \( \text{SL}(d) \) on \( \text{rep}(Q,d) \) is equidimensional.

(ii) Each irreducible component of \( Z_{Q,d} \) is the closure of a \( \text{GL}(d) \)-orbit.

(iii) The number of irreducible components of \( Z_{Q,d} \) is at least \( p - 2 \).

Remark 1.1. Note that the behavior of the number of irreducible components of \( Z_{Q,d} \) is quite different from what happens for \( d \) prehomogeneous, since in that case \( Z_{Q,N,d} \) is irreducible for \( N \) large.

Remark 1.2. If \( Q \) is of type \( \tilde{D}_{n-1} \), it can be shown that the assumption on \( \lambda_i \) may be dropped if \( p \geq 4 \). We do not know if such a tradeoff is possible if \( Q \) is of type \( \tilde{E}_6, \tilde{E}_7, \) or \( \tilde{E}_8 \).

Remark 1.3. Our arguments do not extend to the case \( d = 2 \cdot h \). Indeed, for

\[
Q = \begin{array}{c}
\begin{array}{ccccc}
1 & 2 & 3 & 4 \\
\uparrow & \downarrow & \downarrow & \downarrow \\
5 & 1 & 1 & 1 \\
\end{array}
\end{array}
, \quad h = \begin{array}{cccc}
1 & 1 & 1 & 2
\end{array}
\]

the algebra \( k[\text{rep}(Q,d)]^{{\text{SL}(d)}} \) is a polynomial ring in 6 variables by [17, Thm. 17], but \( Z_{Q,d} \) contains the representation \( X = X_1 \oplus X_2 \), where \( X_1 \) is the one-dimensional representation supported at the vertex 5 and the orbit of the representation \( X_2 \) is open in \( \text{rep}(Q,2 \cdot h - \text{dim} X_1) \), and

\[
\text{codim GL}(2 \cdot h) \ast X = 1[X,X] = 1[X_2,X_1] = 5.
\]

The paper is organized as follows: In Section 2 we generalize Schofield’s result relating semi-invariants to objects in some perpendicular category [12] to non-prehomogeneous dimension vectors. In Section 3 we describe these generalized perpendicular categories, and we prove our main result in Section 4. The last section is devoted to the oriented cycle. As in that case there exist non-constant \( \text{GL}(d) \)-invariants on \( \text{rep}(Q,d) \), and we need to modify the description of \( Z_{Q,d} \) slightly. We obtain the following corollary.

Corollary 1.4. If \( Q \) is an \( \tilde{A}_{n-1} \)-quiver, then \( Z_{Q,d} \) is a complete intersection for any dimension vector \( d \).
2. Semi-invariants and perpendicular categories

2.1. Let $Q$ be a quiver, $d, e \in \mathbb{N}^{Q_0}$, and let $X, Y$ be representations of $Q$ with $\dim X = d$ and $\dim Y = e$. Consider the linear map

$$\mathcal{F}_{X,Y} : \bigoplus_{i \in Q_i} \text{Hom}_k(k^{d_i}, k^{e_i}) \to \bigoplus_{\alpha \in Q_1} \text{Hom}_k(k^{d_{\alpha}}, k^{e_{\alpha}})$$

which sends $(g; i \in Q_0)$ to $(h_\alpha; \alpha \in Q_1)$ with $h_\alpha = g_{h_\alpha} \circ X(\alpha) - Y(\alpha) \circ g_{\alpha}$. Note that $\text{Ker} \mathcal{F}_{X,Y} = \text{Hom}_Q(X, Y)$ and $\text{Coker} \mathcal{F}_{X,Y} = \text{Ext}^1_Q(X, Y)$, which implies that

$$\langle d, e \rangle = [X, Y] - 1 \langle X, Y \rangle.$$

If we assume that $\langle d, e \rangle = 0$, the linear map $\mathcal{F}_{X,Y}$ will be represented by a square matrix $H_{X,Y}$ (with respect to some bases), and the determinant $\det H_{X,Y}$ is a $\text{GL}(d) \times \text{GL}(e)$-semi-invariant on $\text{rep}(Q, d) \times \text{rep}(Q, e)$ by [12]. We denote by $f_Y \in k[\text{rep}(Q, d)]$ the semi-invariant associated to a representation $Y$. Note that, for a short exact sequence

$$0 \to Y' \to Y \to Y'' \to 0$$

with $\langle d, \dim Y' \rangle = \langle d, \dim Y \rangle = \langle d, \dim Y'' \rangle = 0$, we have that $f_Y$ is a non-zero multiple of $f_{Y'} \cdot f_{Y''}$ [3].

The semi-invariant $f_Y$ does not vanish identically on $\text{rep}(Q, d)$ if and only if there exists some $T \in \text{rep}(Q, d)$ with $[T, Y] = [1T, Y] = 0$. We define the perpendicular category $d^\perp$ to be the full subcategory of $\text{rep}(Q)$ whose objects are the representations $Y$ of $Q$ for which there is a $T \in \text{rep}(Q, d)$ with $[T, Y] = [1T, Y] = 0$. Note that in this case $[X, Y] = [1X, Y] = 0$ for $X$ in some dense open subset of $\text{rep}(Q, d)$, as $[-, Y] = 0$ and $[1-, Y] = 0$ are open conditions. In case $d$ is prehomogeneous, $d^\perp$ is just the category $T^\perp$ introduced by Schofield in [12], where $T \in \text{rep}(Q, d)$ lies in the open orbit.

The following result from [3], which in characteristic zero also follows from [13], relates $d^\perp$ to semi-invariants.

**Proposition 2.1.** If $Q$ does not contain oriented cycles and $d \in \mathbb{N}^{Q_0}$, the functions $f_Y, Y \in d^\perp$, span the space $k[\text{rep}(Q, d)]^{\text{SL}(d)}$.

2.2. We start with the following lemma.

**Lemma 2.2.** $d^\perp$ is an exact abelian subcategory of $\text{rep}(Q)$.

**Proof.** Clearly $d^\perp$ is closed under taking direct summands. Let $Y'$ and $Y''$ belong to $d^\perp$. Then there is a $T \in \text{rep}(Q, d)$ for which

$$[T, Y'] = [1T, Y'] = [T, Y''] = [1T, Y''] = 0.$$

Let $f : Y' \to Y''$ be a homomorphism. Then we get two induced long exact sequences

$$0 \to \text{Hom}_Q(T, \text{Ker}(f)) \to \text{Hom}_Q(T, Y') \to \text{Hom}_Q(T, \text{Im}(f)) \to$$

$$\to \text{Ext}^1_Q(T, \text{Ker}(f)) \to \text{Ext}^1_Q(T, Y') \to \text{Ext}^1_Q(T, \text{Im}(f)) \to 0,$$

$$0 \to \text{Hom}_Q(T, \text{Im}(f)) \to \text{Hom}_Q(T, Y'') \to \text{Hom}_Q(T, \text{Coker}(f)) \to$$

$$\to \text{Ext}^1_Q(T, \text{Im}(f)) \to \text{Ext}^1_Q(T, Y'') \to \text{Ext}^1_Q(T, \text{Coker}(f)) \to 0.$$
This implies that
\[ [T, \text{Ker}(f)] = [T, \text{Im}(f)] = 1[T, \text{Ker}(f)] = 1[T, \text{Im}(f)] \]
\[ = [T, \text{Coker}(f)] = 1[T, \text{Coker}(f)] = 0. \]

Hence the subcategory \( d^\perp \) is closed under kernels, images and cokernels. If \( Y \) is an extension
\[ 0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0 \]
of \( Y'' \) by \( Y' \), considering the long exact sequence obtained from mapping \( T \) to this short exact sequence yields \( Y \in d^\perp \). □

If \( d \) is prehomogeneous, \( d^\perp \) is equivalent to the category of representations of some quiver by [12]. For arbitrary \( d \), \( d^\perp \) may have infinitely many simple objects, however. We will compute \( d^\perp \) in case \( Q \) is an extended Dynkin quiver and \( d \) is not prehomogeneous in Section 3.

2.3. For further reference, we collect a few properties of \( d^\perp \).

**Proposition 2.3.** Let \( d = d' + d'' \) with \( d, d', d'' \in \mathbb{N}^Q \), and suppose that generically a representation \( T \in \text{rep}(Q,d) \) has a subrepresentation \( T' \in \text{rep}(Q,d') \). Then
\[ d^\perp \cap (d'^\perp) = (d'^\perp) \cap (d'\perp) = (d''\perp) \cap d^\perp. \]

**Proof.** We only prove the first equality. Suppose \( Y \in (d'^\perp) \cap (d'\perp) \), and choose \( T' \in \text{rep}(Q,d') \) with \( [T', Y] = 1[T', Y] = 0 \) and \( T'' \in \text{rep}(Q,d'') \) with \( [T'', Y] = 1[T'', Y] = 0 \). Then obviously \( [T' \oplus T'', Y] = 1[T' \oplus T'', Y] = 0 \), which implies \( Y \in d^\perp \). Conversely, if \( Y \in d^\perp \cap (d')\perp \), choose \( T \in \text{rep}(Q,d) \) with \( [T, Y] = 1[T, Y] = 0 \) and having a subrepresentation \( T' \subseteq T \) with \( \text{dim} T' = d' \). Note that \( 1[T, Y] = 0 \) implies \( 1[T', Y] = 0 \) as the map \( \text{Ext}^1_Q(T, Y) \rightarrow \text{Ext}^1_Q(T', Y) \) is surjective. But as \( Y \in (d')\perp \) we have
\[ \langle d', \text{dim} Y \rangle = [T', Y] = 1[T', Y] = 0 \]
and thus \( [T', Y] = 0 \). Applying the functor \( \text{Hom}_Q(\cdot, Y) \) we find \( [T/T', Y] = 1[T/T', Y] = 0 \) and thus \( Y \in (d')\perp \cap (d'\perp) \). □

**Corollary 2.4.** Let \( z \) be a sink of \( Q \), and denote by \( \varepsilon_z \in \mathbb{N}^Q \) the vector given by \( \varepsilon_z(y) = \delta_{z,y} \). For \( d \in \mathbb{N}^Q \), set \( \overline{d} = d - d_z \cdot \varepsilon_z \). Then
\[ \{ Y \in d^\perp ; Y(z) = 0 \} = \{ Y \in (\overline{d})^\perp ; Y(z) = 0 \}. \]

**Proof.** We may assume that \( d_z > 0 \). Apply Proposition 2.3 for \( d' = d_z \cdot \varepsilon_z \), and note that
\[ (d')^\perp = (d_z)^\perp = \{ Y \in \text{rep}(Q) ; Y(z) = 0 \}, \]
as the one dimensional representation \( E_z \) supported at \( z \) is simple projective, and \( \text{Hom}_Q(E_z, Y) \) is isomorphic to \( Y(z) \). □

**Proposition 2.5.** Let \( d = d' + d'' \), and suppose that any \( T \) in a dense open set \( \mathcal{U} \subseteq \text{rep}(Q,d) \) decomposes as \( T = T' \oplus T'' \) for some \( T' \in \text{rep}(Q,d') \) and some \( T'' \in \text{rep}(Q,d'') \). Then \( d^\perp \cap (d'^\perp) \subseteq d^\perp \).

**Proof.** The inclusion \( (d'^\perp) \cap (d'^\perp) \subseteq d^\perp \) follows from Proposition 2.3. Conversely, let \( Y \in d^\perp \), and choose \( T \in \mathcal{U} \) with \( [T, Y] = 1[T, Y] = 0 \). Then clearly
\[ \langle T', Y \rangle = 1[T', Y] = [T'', Y] = 1[T'', Y] = 0 \]
if \( T = T' \oplus T'' \) is a decomposition with \( \text{dim} T' = d' \) and \( \text{dim} T'' = d'' \). □
2.4. Finally, we wish to study the behavior of $d^1$ under reflection functors. Let $z$ be a sink of $Q$, and let $\alpha_j : y_j \to z$, $j = 1, \ldots, s$ be the arrows of $Q$ with head $z$. Define a new quiver $Q'$, obtained from $Q$ by deleting $z$ and $\alpha_1, \ldots, \alpha_s$ and by adding a new vertex $z'$ as well as arrows $\beta_j : z' \to y_j$, $j = 1, \ldots, s$. Let $E_z$ and $E'_z$ be the one-dimensional representations of $Q$ and $Q'$, supported at $z$ and $z'$, respectively. Note that $E_z$ is simple projective in $\text{rep}(Q)$ and $E'_z$ is simple injective in $\text{rep}(Q')$.

We consider the reflection functor

$$\mathcal{F} : \text{rep}(Q) \to \text{rep}(Q')$$

associated with $z$. Recall that

$$(\mathcal{F}X)(i) = \begin{cases} X(i) & i \neq z', \\
\text{Ker} \left( \bigoplus X(y_j) \xrightarrow{[X(\alpha_1), \ldots, X(\alpha_s)]} X(z) \right) & i = z',
\end{cases}$$

and that

$$(\mathcal{F}X)(\beta_i) : (\mathcal{F}X)(z') \to (\mathcal{F}X)(y_i) = X(y_i)$$

is the inclusion of $(\mathcal{F}X)(z')$ into $\bigoplus_{j=1}^{s} X(y_j)$ followed by the projection to $X(y_i)$ (see [1], [4]). The functor $\mathcal{F}$ restricts to an equivalence

$$\mathcal{F} : (\text{rep}(Q))' \to (\text{rep}(Q'))'$$

from the full subcategory $(\text{rep}(Q))'$ of $\text{rep}(Q)$ whose objects do not contain $E_z$ as a direct summand, or equivalently have no non-trivial morphisms to $E_z$, to the full subcategory $(\text{rep}(Q'))'$ of $\text{rep}(Q')$ whose objects do not contain $E'_z$ as a direct summand.

Suppose $d \in \mathbb{N}^{Q_0}$ satisfies $d_z < \sum_{x=z} d_{x} \alpha$, and define $d' \in \mathbb{Z}^{Q_0}$ by

$$d'_z = \begin{cases} d_z, & x \neq z', \\
\sum_{x=z} d_{x} - d_z, & x = z'.
\end{cases}$$

Note that $d'_z > 0$. For $T \in (\text{rep}(Q))'$ with $\text{dim} T = d$, we have $\mathcal{F}T \in (\text{rep}(Q'))'$ with $\text{dim} \mathcal{F}T = d'$.

**Proposition 2.6.** Let $Q$ be a quiver with a sink $z$, $d \in \mathbb{N}^{Q_0}$ with $d_z < \sum_{x=z} d_{x} \alpha$, and let $Q'$, $d'$ be defined as above. Then $\mathcal{F}(d^1) = (d')^1$.

**Proof.** We will prove that $\mathcal{F}Y \in (d')^1$ for $Y \in d^1$; the other inclusion is obtained from using the reflection functor $\mathcal{F}^{-1} : (\text{rep}(Q'))' \to (\text{rep}(Q))'$. Choose $T \in \text{rep}(Q, d)$ such that $[T, Y] = 1[T, Y] = 0$ and such that $E_z$ is not a direct summand of $T$. This is possible as generically $[T, E_z] = 0$ on $\text{rep}(Q, d)$. Note that $E_z$ is not a direct summand of $Y$ either as

$$\langle d, e_z \rangle = -d'_z = -1[T, E_z] < 0.$$

But then we have $T, Y \in (\text{rep}(Q))'$, and we know

$$[\mathcal{F}T, \mathcal{F}Y] = [T, Y] = 0 \quad \text{and} \quad 1[\mathcal{F}T, \mathcal{F}Y] = 1[T, Y] = 0.$$

So $\mathcal{F}Y$ belongs to $(d')^1$. \hfill $\square$
3. Extended Dynkin quivers

3.1. Throughout this section $Q$ is an extended Dynkin quiver. Remember that by $\mathcal{E}$ we denote the set $\mathcal{E} = \{ \mu \in \mathbb{P}^1(k); r_\mu > 1 \}$, where $r_\mu$ is the number of simple objects in the category $R_\mu$. We need to recall two more results from [11].

Lemma 3.1. Let $X_P, X_I, X_\mu$ be a non-zero preprojective, preinjective and regular representation in $R_\mu$, respectively, $\mu \in \mathbb{P}^1(k)$. Then we have

(i) $[X_{\mu}, X_P] = 0$ for all $\mu$,
(ii) $[X_I, X_\mu] = 0$ for all $\mu$,
(iii) $[X_I, X_P] = 0$,
(iv) $\mu \neq \nu$,
(v) $X_P, X_\mu > 0$ and $X_I, X_\mu > 0$ if $\mu \notin \mathcal{E}$.

3.2. For $r \geq 1$ we denote by $C_r$ the oriented cycle with $r$ vertices:

$$\begin{array}{c}
1 \x{\alpha_1} 2 \x{\alpha_2} \cdots \x{\alpha_{r-1}} r \\
\end{array}$$

We call a representation $X$ of $C_r$ nilpotent if there is a positive integer $N_X$ such that $X(\pi) = 0$ for any path $\pi$ of length greater than or equal to $N_X$.

Lemma 3.2. For $\mu \in \mathbb{P}^1(k)$, the category $R_\mu$ is equivalent to the category of nilpotent representations of the oriented cycle $C_{r_\mu}$.

Fix a non-prehomogeneous dimension vector $d$ with canonical decomposition $d = p \cdot h + e$, $p \geq 1$. We choose $V \in \text{rep}(Q, e)$ such that the $\text{GL}(e)$-orbit of $V$ is open, and we decompose $V = \bigoplus_{\mu \in \mathcal{E}} V_\mu$, $V_\mu \in R_\mu$. With this notation we have the following results.

Proposition 3.3. Let $e_\mu = \dim V_\mu$ for $\mu \in \mathcal{E}$. Then:

(i) $h^\perp = \prod_{\mu \in \mathbb{P}^1(k)} R_\mu$.
(ii) $d^\perp = h^\perp \cap e^\perp$.
(iii) An indecomposable representation $Y \in R_\mu$ belongs to $e^\perp$ if and only if either $\mu \notin \mathcal{E}$ or, for $\mu \in \mathcal{E}$, $Y \in (e_\mu)^\perp$.

Proof. (i) and (iii) follow directly from Lemma 3.1 and (ii) is a consequence of Proposition 2.3.5.

Our next goal is to describe $(e_\mu)^\perp$ for $\mu \in \mathcal{E}$. Fix $r \geq 1$, and set $C = C_r$. By $\mathcal{N}$ we denote the full subcategory of $\text{rep}(C)$ whose objects are the nilpotent representations. Note that $\mathcal{N}$ is an exact abelian subcategory of $\text{rep}(C)$. Let $T$ be a representation in $\mathcal{N}$ having a dense open orbit in $\text{rep}(C, d)$, where $d = \dim T$. Up to renumbering the vertices of $C$, we may suppose that $d_\iota \leq d_i$ for any vertex $i$ of $C$. Then the composition $T(\alpha_{r-1}) \circ \cdots \circ T(\alpha_1) \circ T(\alpha_\iota)$ is generically an automorphism. If $d_\iota$ were positive, then $T$ could not be nilpotent. So we see that $d_\iota = 0$.

An indecomposable representation $Y$ in $\mathcal{N}$ is uniquely determined by its socle, which is simple and thus corresponds to a vertex $i$ of $C$, and its dimension $l$. Let $\omega$ be the path in $C$ of length $l$ stopping at $i$; it is the shortest path stopping at $i$ with $Y(\omega) = 0$. Note that in this way we obtain a bijection from the set of indecomposable representations in $\mathcal{N}$, up to isomorphism, to the set of all paths of positive length in $C$. If $\omega$ is such a path, we let $Y_\omega$ be the corresponding indecomposable.
The following lemma is not difficult; its proof is left to the reader. By \(W^*\) we denote the dual of the vector space \(W\).

**Lemma 3.4.** Let \(\omega\) be a path of positive length in \(C\). Then we have for any representation \(X\) of \(C\):

\[
\text{Hom}_C(X, Y^\omega) \simeq (\text{Coker} X(\omega))^* \quad \text{and} \quad \text{Ext}_C^1(X, Y^\omega) \simeq (\text{Ker} X(\omega))^* .
\]

We obtain the following consequence.

**Corollary 3.5.** Let \(\omega\) be a path of positive length in \(C\), let \(T\) be a representation in \(\mathcal{N}\) with \(\frac{1}{2}[T, T] = 0\), and set \(d = \dim T\). Then we have:

(i) \(Y^\omega \in d^\perp\) if and only if \(d_x \geq d_{h\omega} = d_{t\omega}\) for all vertices of \(\omega\),

(ii) \(Y^\omega\) is a simple object in \(d^\perp\) if and only if \(d_x > d_{h\omega} = d_{t\omega}\) for all inner vertices of \(\omega\).

Here we set \(t\omega = t\beta_1\) and \(h\omega = h\beta_1\) for \(\omega = \beta_1 \cdots \beta_1\), and we call \(x\) a vertex of \(\omega\) if \(x\) is a tail or a head of some \(\beta_i\). An inner vertex of \(\omega\) is a vertex of the form \(t\beta_i\), \(i > 1\).

**Proof.** (i) From Lemma 3.4 we see that \(Y^\omega \in d^\perp\) if and only if \(T(\omega)\) is an isomorphism and therefore \(d_{h\omega} = d_{t\omega}\). As \(T(\omega)\) factors through \(T(x)\) for any vertex \(x\) of \(\omega\), we find \(d_x \geq d_{h\omega} = d_{t\omega}\). Conversely, this condition implies that \(T(\omega)\), which is a composition of generic maps, one for each arrow \(\beta_i\), is an isomorphism.

(ii) Assume \(Y^\omega\) is simple and there is an inner vertex \(x\) with \(d_{t\omega} = d_x = d_{h\omega}\). Denote by \(\omega'\) the subpath of \(\omega\) from \(x\) to \(h\omega\). By (i), \(Y^\omega' \in d^\perp\), and clearly \(Y^\omega'\) is a proper subrepresentation of \(Y^\omega\). Conversely, a representation \(Y^\omega\) with \(d_x > d_{h\omega} = d_{t\omega}\) for all inner vertices cannot have any proper subrepresentation in \(d^\perp\), again by (i). \(\square\)

**Proposition 3.6.** Let \(T\) be a representation in \(\mathcal{N}\) with \(\frac{1}{2}[T, T] = 0\), and set \(d = \dim T\). Then \(d^{1\perp}\) is an abelian category with \#(\(C\)) \(-\) \(\sigma(T) = r - \sigma(T)\) simple objects.

**Proof.** Let \(\hat{\mathcal{C}}\) be the quiver obtained from \(\mathcal{C} = C_r\) by deleting the arrow \(\alpha_r\); it is an \(\mathbb{A}_r\)-quiver. Recall that \(\mathcal{C}\) in fact lies in \(\text{rep}(\hat{\mathcal{C}}, \mathbf{d})\) as \(d_r = 0\). Our strategy is to show that \(d^{1\perp}\) and \(\hat{\mathcal{C}}\) have the same simple objects. Then our result follows from Schofield’s result [12, Thm. 2.5], as \(\hat{\mathcal{C}}\) is a quiver of finite representation type and thus \(\mathbf{d}\) is prehomogeneous when viewed as a dimension vector for \(\hat{\mathcal{C}}\).

For \(Y \in \text{rep}(\hat{\mathcal{C}})\), we have

\[
\text{Hom}_C(T, Y) = \text{Hom}_{\hat{\mathcal{C}}}(T, Y) \quad \text{and} \quad \langle \mathbf{d}, \dim Y \rangle_C = \langle \mathbf{d}, \dim Y \rangle_{\hat{\mathcal{C}}},
\]

as \(d_r = 0\). We conclude that \(d^{1\perp} = d^{1\perp} \cap \text{rep}(\hat{\mathcal{C}})\). Let \(Y^\omega\) be a simple object of \(d^{1\perp}\). By Corollary 3.5 the vertex \(r\) cannot be an inner vertex of \(\omega\), as \(d_r = 0\). Then \(Y^\omega(\alpha_r) = 0\) by the definition of \(Y^\omega\), and hence \(Y^\omega \in \text{rep}(\hat{\mathcal{C}})\). \(\square\)

**Proposition 3.7.** Let \(Q\) be an extended Dynkin quiver and \(\mathbf{d} \in \mathbb{N}^{Q_0}\) a non-prehomogeneous dimension vector with canonical decomposition \(\mathbf{d} = p\cdot \mathbf{h} + \mathbf{e}\), \(p \geq 1\). If either \(p \geq 2\) or \(Q\) is an \(\mathbb{A}_{n-1}\)-quiver, the algebra \(k[\text{rep}(Q, \mathbf{d})]^{\text{SL}(\mathbf{d})}\) is a polynomial ring in \(n + p - 1 - \sigma(V)\) variables, where \(V \in \text{rep}(Q, \mathbf{e})\) has an open orbit.
Proof. The main theorem of Skowroński and Weyman in [17] says that, under our assumptions, \( k[\text{rep}(Q, \mathbf{d})]^{\text{SL}(d)} \) is the quotient of a polynomial ring \( k[c_0, \ldots, c_p, f_Y] \) by an ideal generated by \#\( \mathcal{E} \) relations, each allowing for the cancellation of one of the \( c_i \)'s from the list of generators, where \( Y \) ranges over the simple non-homogeneous objects in \( \mathbf{d}^\perp \). Indeed, by Lemma 3.2 and Corollary 3.3 the simple objects in \( \mathcal{R}_\mu \) correspond bijectively to the "admissible arcs" of [17]. The number of simple objects of \( \mathbf{d}^\perp \) which belong to \( \mathcal{R}_\mu \) is \( r_\mu - \sigma(V_\mu) \), where \( V = \bigoplus_{\mu \in \mathcal{E}} V_\mu \), by Proposition 3.6.

So the number of simple non-homogeneous objects in \( \mathbf{d}^\perp \) equals

\[
\sum_{\mu \in \mathcal{E}} (r_\mu - \sigma(V_\mu)) = \sum_{\mu \in \mathcal{E}} r_\mu - \sigma(V) = n - 2 + \#\mathcal{E} - \sigma(V)
\]

as \( \sum_{\mu \in \mathcal{E}} (r_\mu - 1) = n - 2 \). Taking into account the \#\( \mathcal{E} \) relations, we conclude that \( k[\text{rep}(Q, \mathbf{d})]^{\text{SL}(d)} \) is a polynomial ring on

\[
(p + 1) + (n - 2 + \#\mathcal{E} - \sigma(V)) - \#\mathcal{E} = p + n - 1 - \sigma(V)
\]
generators.

4. Proof of the theorem

4.1. We recall the notation and assumptions for our theorem and keep them fixed throughout this section: \( Q \) is an extended Dynkin quiver with \#\( Q_0 = n \), not an oriented cycle, \( \mathbf{d} \in \mathbb{N}^{Q_0} \) is a non-prehomogeneous dimension vector with canonical decomposition \( d = p \cdot \mathbf{h} + \mathbf{e} \), \( V = \bigoplus_{i=1}^{\sigma(V)} V_i^{\lambda_i} \) is a representation in the open orbit of \( \text{rep}(Q, \mathbf{e}) \) with \( V_i \) indecomposable and pairwise non-isomorphic. We assume that either \( Q \) is of type \( \tilde{A}_{n-1} \) or else \( p \geq 3 \) and \( \lambda_i \geq 3 \) for \( i = 1, \ldots, \sigma(V) \). Note that by Proposition 2.1, the variety \( Z_{Q, \mathbf{d}} \) of common zeros of all non-constant semi-invariants has the following description:

\[
Z_{Q, \mathbf{d}} = \{ X \in \text{rep}(Q, \mathbf{d}) ; \ [X, Y] \neq 0 \text{ for all } Y \in \mathbf{d}^\perp, Y \neq 0 \} = \{ X \in \text{rep}(Q, \mathbf{d}) ; \ [1, X, Y] \neq 0 \text{ for all } Y \in \mathbf{d}^\perp, Y \neq 0 \}.
\]

The next result is an immediate consequence of Lemma 3.1 and Proposition 3.3.

**Proposition 4.1.** Any representation \( X \) in \( Z_{Q, \mathbf{d}} \) has a nonzero preprojective (and a preinjective) direct summand.

The following corollary implies part (ii) of our theorem.

**Corollary 4.2.** Any irreducible component \( \mathcal{C} \) of \( Z_{Q, \mathbf{d}} \) is the closure of some orbit \( \text{GL}(\mathbf{d}) \ast X \).

We call \( X \in Z_{Q, \mathbf{d}} \) generic if \( \text{GL}(\mathbf{d}) \ast X \) is an irreducible component of \( Z_{Q, \mathbf{d}} \).

**Proof.** Otherwise, \( \mathcal{C} \) contains an infinite number of distinct orbits of maximal dimension, none of which belong to any other irreducible component of \( Z_{Q, \mathbf{d}} \). Some must be given by representations having a direct summand from \( \mathcal{R}_\mu, \mu \not\in \mathcal{E} \). If \( X = X_1 \oplus X_2 \oplus X_3 \) is one of them with \( X_1 \neq 0 \) preprojective and \( X_2 \in \mathcal{R}_\mu, \mu \not\in \mathcal{E} \), there exists a non-split extension

\[
0 \rightarrow X_1 \rightarrow \bar{X}_1 \rightarrow X_2 \rightarrow 0,
\]

by Lemma 3.1. Note that \( \bar{X}_1 \) still contains a non-zero preprojective summand, as

\[
\partial(\text{dim } \bar{X}_1) = \partial(\text{dim } X_1) + \partial(\text{dim } X_2) < 0.
\]
Therefore $[\tilde{X}_1, H] \neq 0$ for any $\nu \notin \mathcal{E}$. For $Y \in d^\perp$ simple non-homogeneous, we have $[X_2, Y] = 0$ by Lemma 3.1. Mapping the short exact sequence above to $Y$, we conclude that

$$[\tilde{X}_1 \oplus X_3, Y] = [X_1 \oplus X_2 \oplus X_3, Y].$$

Hence $\tilde{X} = \tilde{X}_1 \oplus X_3$ still belongs to $Z_{Q,d}$, and even to $C$, as $X$ lies in the closure of $\text{GL}(d) \ast \tilde{X}$. This contradicts the maximality of the dimension of $\text{GL}(d) \ast X$. □

4.2. By $\tilde{d} = d|_{\tilde{Q}}$, we denote the full subquiver of $Q$ with vertex set $Q_0 \setminus \{z\}$, and we set $\bar{d} = d|_{\tilde{Q}}$. For $X \in \text{rep}(Q)$, we denote by $\tilde{X} \in \text{rep}(\tilde{Q})$ the restriction of $X$ to $\tilde{Q}$.

We fix a sink $z$ in $Q$, and we define $Q'$ as in Section 2.4. Note that generically a representation $T$ in $\text{rep}(Q, d)$ is regular and hence does not contain the simple projective $E_z$ as a direct summand and thus $[T, E_z] = 0$. But $E_z$ does not belong to $d^\perp$ either. We conclude that

$$\langle d, e_z \rangle = d_z - \sum_{j=1}^{s} d_{y_j} = [T, E_z] - 1[T, E_z] < 0,$$

and therefore we may apply Proposition 2.6. The same arguments yield that either $e_z = 0$ or else $e_z < \sum_{j=1}^{s} e_{y_j}$. In either case we have $e'_z = \sum_{j=1}^{s} e_{y_j} - e_z \geq 0$.

**Proposition 4.3.** If $X \in Z_{Q,d}$ does not contain the simple projective $E_z$ as a direct summand, we have that $X$ is generic in $Z_{Q,d}$ if and only if $FX$ is generic in $Z_{Q',d'}$. Moreover,

$$\text{codim}_{\text{rep}(Q, d)} \text{GL}(d) \ast \tilde{X} = \text{codim}_{\text{rep}(Q', d')} \text{GL}(d') \ast F\tilde{X}.$$

**Proof.** We know that $X \in (\text{rep}(Q))'$ and $FX \in (\text{rep}(Q'))'$. The sets

$$\text{rep}(Q, d)' = \text{rep}(Q, d) \cap (\text{rep}(Q))', \quad \text{rep}(Q', d')' = \text{rep}(Q', d') \cap (\text{rep}(Q'))'$$

are open as $[-, E_z] = 0$ and $[E_z, -] = 0$ are open conditions. Moreover, they are related by a fiber bundle construction [3]. In particular, there is a bijection compatible with $F$ between the $\text{GL}(d)$-orbits in $\text{rep}(Q, d)'$ and the $\text{GL}(d')$-orbits in $\text{rep}(Q', d')'$, preserving their codimensions, closures and inclusions between closures. Hence the claim follows from Proposition 2.6. □

4.2. By $\tilde{Q}$ we denote the full subquiver of $Q$ with vertex set $Q_0 \setminus \{z\}$, and we set $\bar{d} = d|_{\tilde{Q}}$. For $X \in \text{rep}(Q)$, we denote by $\tilde{X} \in \text{rep}(\tilde{Q})$ the restriction of $X$ to $\tilde{Q}$.

We set

$$Z''_{Q,d} = \{X \in Z_{Q,d}; \text{ E_z is a direct summand of } X\}.$$

As a generic $X \in Z_{Q,d}$ contains a non-zero preprojective direct summand and as any indecomposable preprojective representation becomes simple projective under a suitable series of reflection functors, part (i) of our theorem will follow if we show that

$$\text{codim}_{\text{rep}(Q, d)} Z''_{Q,d} = n + p - 1 - \sigma(V).$$

**Proposition 4.4.** The map

$$\text{rep}(Q, d) \rightarrow \text{rep}(\tilde{Q}, \bar{d}) \times \text{Mat}(d_z \times \sum_{j=1}^{s} d_{y_j}), k)$$

sending $X$ to $(\tilde{X}, (X(\alpha_1) \cdots X(\alpha_s)))$ restricts to an isomorphism

$$Z''_{Q,d} \cong Z_{\tilde{Q},\bar{d}} \times \mathcal{M},$$

where $\mathcal{M} = \{A \in \text{Mat}(d_z \times \sum_{j=1}^{s} d_{y_j}), k); \text{ rank } A < d_z\}$. 

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Proof. Indeed, \( X \in \text{rep}(Q, d) \) belongs to \( Z^0_{Q,d} \) if and only if \( X \simeq X' \oplus E_z \) and \([X', Y] \neq 0\) for all non-zero \( Y \in d^\perp \) with \( Y(z) = 0\). By Corollary \ref{cor:corollary2.4}, these are exactly the objects of \((\bar{Q})^1 \) in \( \text{rep}(\bar{Q}, \bar{d}) \), extended by 0 to \( Q \). But if \( Y(z) = 0\), we have \([X', Y] = [X, Y] \). The result follows.

Let \( \bar{H} \) be a representation in the open orbit of \( \text{rep}(\bar{Q}, \bar{d}) \), and note that the image \( \bar{H}_\mu \) of the simple homogeneous representation \( H_\mu \) is isomorphic to \( \bar{H} \) for any \( \mu \notin \mathcal{E} \). We will need the following lemma.

**Lemma 4.5.**

(i) \( \sigma(\bar{H}) \leq 3h_z - 2 = 3(h_z - 1) + 1 \).

(ii) If either \( Q \) is an \( \tilde{k}_{n-1} \)-quiver or else \( V = \bigoplus V^\lambda_i \) with \( \lambda_i \geq 3 \) for all \( i \), then \( \sigma(V) \leq \sigma(V) + e'_z \).

**Proof.** (i) Clearly \( \bar{H} \) has at most as many pairwise non-isomorphic direct summands as \( \sum_{j=1}^s \dim_k H_\mu(y_j), \mu \notin \mathcal{E} \), which implies that

\[
\sigma(\bar{H}) \leq \sum_{j=1}^s h_{y_j} = 2h_z.
\]

The last equality follows from

\[
0 = \langle e_z, h \rangle + \langle h, e_z \rangle = h_z + h_z - \sum_{j=1}^s h_{y_j}.
\]

This yields our claim except in the case \( h_z = 1 \). But in that case, \( \bar{Q} \) is a Dynkin quiver, and we have

\[
[\bar{H}, \bar{H}] = \langle \bar{h}, \bar{h} \rangle_Q = \langle \bar{h} - e_z, \bar{h} - e_z \rangle = \langle e_z, e_z \rangle = 1.
\]

Recall that \( 1[\bar{H}, \bar{H}] = 0 \) as the orbit of \( \bar{H} \) is open. In particular, \( \bar{H} \) is indecomposable and \( \sigma(\bar{H}) = 1 \).

(ii) We restrict \( V \) to the support of \( e \), which is a tame quiver \( K \) for which \( e \) is a sincere dimension vector. If \( e_z = 0 \), i.e. if \( z \) is not \( \sigma \)-exceptional, then \( K = K, V = V \) and \( e'_z \geq 0 \). Otherwise, \( z \) is a sink of \( K \), and we can apply our results from \( \mathbb{[9]} \) if \( \mathbb{K} \) is a complete intersection of codimension \( \#K_0 - \sigma(V) \), and \( Z_{\mathbb{K}}^{\mathbb{K},e} \) is a complete intersection of codimension \( \#K_0 - 1 - \sigma(V) \). Note that either every indecomposable \( V_i \mid K \) arising in \( V \mid K \) is at least 3. Set

\[
Z^{\mathbb{K},e}_{\mathbb{K},e} = \{ X \in \mathbb{Z}_{\mathbb{K},e}; [X, E_z] \neq 0 \}
\]

and remember that, as in Proposition \ref{prop:prop4.4}, \( Z^{\mathbb{K},e}_{\mathbb{K},e} \sim \mathbb{Z}_{\mathbb{K}}^{\mathbb{K},e} \times \mathcal{M}' \) with

\[
\mathcal{M}' = \{ B \in \text{Mat}(e_z \times \sum_{j=1}^s e_{y_j} , k); \text{rank}(B) < e_z \}.
\]

We conclude that

\[
\#K_0 - \sigma(V) = \text{codim}_{\text{rep}(\mathbb{K}, e)} Z_{\mathbb{K},e} \leq \text{codim}_{\text{rep}(\mathbb{K}, e)} Z^{\mathbb{K},e}_{\mathbb{K},e}
\]

\[
= \#K_0 - 1 - \sigma(V) + e'_z + 1,
\]

as \( \text{codim} \mathcal{M}' = \sum_{j=1}^s e_{y_j} - e_z + 1 = e'_z + 1 \). The result follows. \( \square \)
We are now ready to finish the proof of part (i) of our theorem. We need to show that
\[ \text{codim}_{\text{rep}(Q, \mathbf{d})} Z''_{Q, \mathbf{d}} \geq n + p - 1 - \sigma(V). \]
Note that \( Q \) is a disjoint union of Dynkin quivers; in fact it is an \( A_{n-1} \)-quiver if \( Q \) is an \( \tilde{A}_{n-1} \)-quiver. In the remaining cases, the multiplicity of any indecomposable direct summand in the representation \( \overline{Q} \odot \overline{V} \in \text{rep}(Q, \mathbf{d}) \), whose \( \text{GL}(\mathbf{d}) \)-orbit is open, is at least 3. Therefore we know from [9] that
\[ \text{codim} Z_{Q, \mathbf{d}} = n - 1 - \sigma(\overline{V} \oplus \overline{H}). \]

We compute, using Proposition 4.4, the preceding lemma, and [9] for \( (\overline{Q}, \mathbf{d}) \):
\[
\text{codim}_{\text{rep}(Q, \mathbf{d})} Z''_{Q, \mathbf{d}} - (n + p - 1 - \sigma(V))
= \text{codim}_{\text{rep}(Q, \mathbf{d})} Z_{Q, \mathbf{d}} + (d'_z + 1) - (n + p - 1 - \sigma(V))
= (n - 1 - \sigma(\overline{V} \oplus \overline{H})) + (p h_z + e'_z + 1) - (n + p - 1 - \sigma(V))
= (\sigma(\overline{V}) + \sigma(\overline{H}) - \sigma(\overline{V} \oplus \overline{H})) + (p h_z - 1) + 1 - \sigma(\overline{H})
+ (\sigma(\overline{V}) + e'_z - \sigma(\overline{V})) \geq 0.
\]

In the last sum, each summand is non-negative. This is obvious for the first one, and it follows from the preceding lemma for the second and the third. Note that either \( p \geq 3 \) or else \( Q \) is an \( \tilde{A}_{n-1} \)-quiver and \( h_z = 1 \).

4.3. Finally, we prove part (iii) of the theorem.

Lemma 4.6. For \( \mathbf{d} = p h + e, p \geq 3, Z_{Q, \mathbf{d}} \) contains a generic representation \( X = X_P \oplus X_R \oplus X_T \) with \( X_P \) preprojective, \( X_R \) regular, and \( X_T \) preinjective such that the preprojective part \( X_P \) has defect \( \partial(X_P) = -1 \).

Note that in particular \( X_P \) is indecomposable.

Proof. We assume there is a sink \( z \) with \( h_z = 1 \). If no such sink exists, there is a source \( y \) with \( h_y = 1 \), and applying the dual arguments we find that \( \partial(X_T) = 1 \), which implies \( \partial(X_P) = -1 \).

We start from a representation \( W \) with dimension vector \( p h \) which is a direct sum of all simple regular non-homogeneous representations and some simple homogeneous representations (if necessary). Obviously there is an exact sequence in \( \text{rep}(Q) \) of the form
\[ 0 \rightarrow E_z \rightarrow W \rightarrow W' \rightarrow 0 \]
for some \( W' \). Then no indecomposable direct summand of \( W' \) is preprojective, by Lemma 5.1. Let \( X' = E_z \oplus W' \). Thus the defect of the preprojective part \( E_z \) of \( X' \) equals \( (h, e_z) = -h_z = -1 \). Observe that \( [X', Y] \neq 0 \) for any regular representation \( Y \neq 0 \). Indeed, if \( Y \) is simple homogeneous, then \( [X', Y] \geq [E_z, Y] > 0 \), and if \( Y \) is simple non-homogeneous, then \( [X', Y] \geq [W, Y] > 0 \). Thus \( X' \) lies in \( Z_{Q, \mathbf{d}} \).

\( X' \) belongs to \( \text{GL}(\mathbf{d}) \cdot X \) for some generic \( X \), which we decompose as \( X = X_P \oplus X_R \oplus X_T \) with \( X_P \) preprojective, \( X_R \) regular, and \( X_T \) preinjective. By Proposition 4.1 we know that \( X_P \neq 0 \) and thus \( \partial(X_P) \leq -1 \). Choose \( \nu \in \mathbb{P}^1(k) \setminus \mathcal{E} \) in such a way that \( H_{\nu} \) is not a direct summand of \( X \) nor of \( X' \), and remember that \( [X, H_{\nu}] \leq [X', H_{\nu}] \). Using Lemma 3.1 we compute:
\[
1 \leq -\partial(X_P) = \langle \dim X_P, h \rangle = [X_P, H_{\nu}] = [X, H_{\nu}]
\leq [X', H_{\nu}] = -\partial(X'_P) = 1.
\]

\end{proof}
For each natural number \( r \), the vector \( \dim X_P + rh \) is the dimension vector of some indecomposable representation \( X_P[r] \), which is still preprojective with defect \(-1\) and thus has an open orbit in \( \text{rep}(Q, \dim X_P + rh) \). For \( s \in \mathbb{N} \) we let \( X_T[s] \) be an indecomposable representation with \( \dim X_T[s] = \dim X_T + sh \). Set \( X[r, s] = X_P[r] \oplus X_T \oplus X_T[s] \). The following result implies part (iii) of our theorem:

**Proposition 4.7.** The representations \( X[r, s] \) are pairwise non-isomorphic and generic in \( Z_{Q,d+(r+s)h} \).

**Proof.** Choose \( Y \in (d + (r+s)h)^\perp = d^\perp \), and remember that \( Y \) is regular and has defect \( 0 \). Therefore we have

\[
[X_P[r], Y] = \langle \dim X_P[r], \dim Y \rangle = \langle \dim X_P, \dim Y \rangle = [X_P, Y],
\]

and thus

\[
[X[r, s], Y] = [X_P, Y] + [X_T, Y] = [X, Y] > 0
\]

and \( X[r, s] \) belongs to \( Z_{Q,d+(r+s)h} \).

In order to show that \( X[r, s] \) is generic, it is enough to prove

\[
1[X[r, s], X[r, s]] = n + p + r + s - 1 - \sigma(V),
\]

which follows if

\[
1[X[r, s], X[r, s]] = 1[X, X] + r + s.
\]

As above we have

\[
1[X_T, X_P[r]] = 1[X_T, X_P] \quad \text{and} \quad 1[X_T, X_T] = 1[X_T, X_T].
\]

We compute

\[
1[X_T[s], X_P[r]] = -(\dim X_T[s], \dim X_P[r])
\]

\[
= -(\dim X_T + sh, \dim X_P + rh)
\]

\[
= 1[X_T, X_P] - s \partial(\dim X_P) + r \partial(\dim X_T)
\]

\[
= 1[X_T, X_P] + s + r.
\]

By Lemma 3.3 these are the only terms that do not vanish. \( \square \)

5. **The oriented cycle**

5.1. In this section we wish to generalize our results to the only extended Dynkin quiver not considered so far, the oriented cycle, i.e. the quiver \( Q \) with \( Q_0 = \{1, 2, \ldots, n\} \) and \( Q_1 = \{\alpha_i : i \mapsto (i+1); i \in Q_0\} \); we view the elements of \( Q_0 \) as representatives of \( \mathbb{Z}/n\mathbb{Z} \). The category \( \text{rep}(Q) \) of finite dimensional representations of \( Q \) decomposes into a family \( \prod_{\mu \in k} R_\mu \) of uniserial categories \( R_\mu \) parametrized by \( \mu \in k \). For \( \mu \neq 0 \), \( R_\mu \) contains a unique simple representation \( H_\mu \) with \( \dim H_\mu = h \), where \( h_i = 1 \) for \( i \in Q_0 \). For \( \mu = 0 \), \( R_0 \) consists of all nilpotent representations (compare Section 3.2). If \( n \geq 2 \), its simple objects are just the one-dimensional representations of \( Q \).

We recall the description of the semi-invariants for \( \text{rep}(Q, d) \) from [10] and [17]. For \( d \in \mathbb{N}Q_0 \), we may assume that \( d_1 = p \leq d_i, i \in Q_0 \), up to renumbering the vertices of \( Q \). For \( X \in \text{rep}(Q, d) \), the coefficients \( c_1(X), \ldots, c_p(X) \) of the characteristic polynomial

\[
\det(T - X(\alpha_n) \cdots X(\alpha_1)) = T^p + c_1(X)T^{p-1} + \cdots + c_p(X)
\]
are clearly invariant under $\text{GL}(d)$. For two integers $i < j \leq i + n$, the path $\alpha_{j-1} \cdots \alpha_i$ is called an admissible arc $A = [i, j]$ if $d_i = d_j < d_m$ for all $m$ with $i < m < j$. For any admissible arc $A = [i, j]$ the determinant

$$f_A(X) = \det(X(\alpha_{j-1}) \cdots \cdot X(\alpha_i))$$

is a semi-invariant. We call an admissible arc $B = [i, j]$ minimal if $d_i = d_j = p$.

**Proposition 5.1.** Let $Q$ be the oriented cycle, and let $d \in \mathbb{N}^{Q_0}$ with $d_1 = p \leq d_i$ for all $i \in Q_0$.

(i) The algebra of semi-invariants is the polynomial algebra

$$k[\text{rep}(Q, d)]^{SL(d)} = k[c_1, \ldots, c_p; \{f_A\}] / \left(\prod f_B - c_p\right),$$

where $A$ ranges over all admissible and $B$ over all minimal admissible arcs of $Q$.

(ii) The algebra $k[\text{rep}(Q, d)]^{GL(d)}$ of $\text{GL}(d)$-invariants is a polynomial ring in $c_1, \ldots, c_p$.

**Proof.** Both statements are essentially contained in [17]: (i) is stated explicitly, and (ii) is the fact that an invariant is a semi-invariant with trivial weight. \qed

**Theorem 2.** Let $Q$ be the oriented cycle, $d \in \mathbb{N}^{Q_0}$ with $d_1 = p \leq d_i$ for all $i \in Q_0$.

(i) The set $Z_{Q,d}$ of common zeros of $c_1, \ldots, c_p; \{f_A\}$ is a complete intersection, where $A$ ranges over all admissible arcs of $Q$.

(ii) The set $N_d$ of nilpotent representations in $\text{rep}(Q, d)$ is the set of common zeros of $c_1, \ldots, c_p$, it is a complete intersection.

As nilpotent representations do not depend on parameters, any irreducible component of $Z_{Q,d}$ or $N_d$ is the closure of an orbit. The number of irreducible components of $Z_{Q,d}$ can be shown to be bounded for the oriented cycle. Note that as a consequence of our two theorems, we obtain that $Z_{Q,d}$ is a complete intersection for any $A_{n-1}$-quiver.

5.2. Our strategy is to compare $\text{rep}(Q, d)$ with $\text{rep}(\tilde{Q}, \tilde{d})$, where

$$\tilde{Q} = \frac{\alpha_1 \cdots \alpha_n}{\beta}$$

and

$$\tilde{d}_i = \begin{cases} d_i & 1 \leq i \leq n, \\ p & d_i = d_1 \quad i = 0. \end{cases}$$

For $Y \in \text{rep}(\tilde{Q}, \tilde{d})$ the coefficients $\tilde{c}_0(Y), \ldots, \tilde{c}_p(Y)$ of the polynomial

$$\det(Y(\beta)T - Y(\alpha_n) \cdots \cdot Y(\alpha_1)) = \tilde{c}_0(Y)T^p + \tilde{c}_1(Y)T^{p-1} + \cdots + \tilde{c}_p(Y)$$

are semi-invariants; note that $\tilde{c}_0(Y) = \det(Y(\beta))$. For every admissible arc $A = [i, j]$ of $Q$ as defined in Section 5.1, there is a semi-invariant $\tilde{f}_A$ given by

$$\tilde{f}_A(Y) = \det(Y(\alpha_{j-1}) \cdots \cdot Y(\alpha_i)).$$

From [17] we know that

$$k[\text{rep}(\tilde{Q}, \tilde{d})]^{SL(\tilde{d})} = k[\tilde{c}_1, \ldots, \tilde{c}_p; \{\tilde{f}_A\}] / \left(\prod \tilde{f}_B - \tilde{c}_p\right),$$

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where $A$ ranges over all admissible and $B$ over all minimal admissible arcs for $Q$.

We leave the proof of the following lemma to the reader.

**Lemma 5.2.** The map 
$$\Phi : \text{GL}(p) \times \text{rep}(Q, d) \to \text{rep}(\tilde{Q}, \tilde{d})$$
given by
$$\Phi(s, X)(\gamma) = \begin{cases} 
X(\alpha_i) & \gamma = \alpha_i, \ i < n, \\
 s \cdot X(\alpha_n) & \gamma = \alpha_n, \\
 s & \gamma = \beta
\end{cases}$$
is a $\text{GL}(\tilde{d})$-equivariant open immersion onto the set
$$\text{rep}(\tilde{Q}, \tilde{d})' = \{ Y \in \text{rep}(\tilde{Q}, \tilde{d}); \det(Y(\beta)) \neq 0 \},$$
where $\text{GL}(\tilde{d})$ acts on $\text{GL}(p) \times \text{rep}(Q, d)$ by
$$h \ast (s, X) = (h_0sh_1^{-1}, h \ast X)$$
and $\tilde{h}_i = h_i$ for $i \in Q_0$.

Note that by definition,
$$\Phi^*(\tilde{c}_i)(s, X) = \begin{cases} 
\det(s) \cdot c_i(X) & i = 1, \ldots, p, \\
\det(s) & i = 0;
\end{cases}$$
$$\Phi^*(\tilde{f}_A)(s, X) = \begin{cases} 
\det(s) \cdot f_A(X) & \text{if } \alpha_n \text{ belongs to } A, \\
f_A(X) & \text{otherwise.}
\end{cases}$$

We conclude that the zero set $\mathcal{V}(c_1, \ldots, c_p, \{f_A\}) = Z_{Q, d} \subseteq \text{rep}(Q, d)$ has the same codimension as the zero set $\mathcal{V}(\tilde{c}_1, \ldots, \tilde{c}_p, \{\tilde{f}_A\}) = \mathcal{V}_{\tilde{Q}, \tilde{d}} \subseteq \text{rep}(\tilde{Q}, \tilde{d})$. As $Z_{\tilde{Q}, \tilde{d}} = \mathcal{V}(\tilde{c}_0) \cap \mathcal{V}_{\tilde{Q}, \tilde{d}}$ is a complete intersection by Theorem [1], $\mathcal{V}_{\tilde{Q}, \tilde{d}}$ is as well. This proves part (i).

As for part (ii), we need to study the set $\mathcal{V}(c_1, \ldots, c_p)$ of common zeros of $c_1, \ldots, c_{p-1}, c_p = \prod f_B$, where $B$ ranges over all minimal admissible arcs. As
$$\mathcal{V}(c_1, \ldots, c_p) = \bigcup_B \mathcal{V}(c_1, \ldots, c_{p-1}, f_B),$$
this is a complete intersection as well. Clearly, a representation $X$ is nilpotent if and only if the characteristic polynomial of $X(\alpha_n) \cdot \ldots \cdot X(\alpha_1)$ is $T^p$.

**References**


Mathematisches Institut, Universität Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland

E-mail address: christine.riedtmann@math-stat.unibe.ch

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

E-mail address: gzvara@mat.uni.torun.pl