TOEPLITZ OPERATORS AND LOCALIZATION OPERATORS

MIROSLAV ENGLIŠ

Abstract. We show that for any localization operator on the Fock space with polynomial window, there exists a constant coefficient linear partial differential operator $D$ such that the localization operator with symbol $f$ coincides with the Toeplitz operator with symbol $Df$. An analogous result also holds in the context of Bergman spaces on bounded symmetric domains. This verifies a recent conjecture of Coburn and simplifies and generalizes recent results of Lo.

1. Introduction

Let $\mathcal{F}$ be the Fock, or Segal-Bargmann, space of all entire functions on $\mathbb{C}^n$ square-integrable with respect to the Gaussian

$$d\mu(z) := e^{-\|z\|^2/2} \frac{dz}{(2\pi)^n},$$

dz being the Lebesgue volume measure on $\mathbb{C}^n$. It is well known (and easy to check) that the Weyl operators

$$W_a f(z) := e^{\langle z, a \rangle/2 - \|a\|^2/4} f(z - a), \quad a \in \mathbb{C}^n,$$

are unitary on $L^2(\mathbb{C}^n, d\mu)$ and on $\mathcal{F}$. For $w \in \mathcal{F}$ and $f \in L^\infty(\mathbb{C}^n)$, the Gabor-Daubechies localization operator $L_f^w$ with “window” $w$ and “symbol” $f$ is the operator on $\mathcal{F}$ defined by

$$\langle L_f^w u, v \rangle = (2\pi)^{-n} \int_{\mathbb{C}^n} f(a) \langle u, W_a w \rangle \langle W_a w, v \rangle \, da, \quad u, v \in \mathcal{F}.$$

On the other hand, for $f \in L^\infty(\mathbb{C}^n)$, the Toeplitz operator $T_f$ with symbol $f$ is the operator on $\mathcal{F}$ defined by

$$T_f u = P(fu), \quad u \in \mathcal{F},$$

where $P : L^2(\mathbb{C}^n, d\mu) \to \mathcal{F}$ is the orthogonal projection. Using the fact that the exponentials

$$K_y(z) := K(z, y) := e^{\langle z, y \rangle/2}$$

serve as the reproducing kernel for $\mathcal{F}$, in the sense that

$$f(x) = \langle f, K_x \rangle = \int_{\mathbb{C}^n} f(y) K(x, y) \, d\mu(y) \quad \forall f \in \mathcal{F}, \forall x \in \mathbb{C}^n,$$

Received by the editors July 31, 2006 and, in revised form, May 7, 2007.

2000 Mathematics Subject Classification. Primary 47B35; Secondary 42C40, 32M15, 81R30.

Key words and phrases. Toeplitz operator, localization operator, bounded symmetric domain, Segal-Bargmann space, Bergman space.

This research was supported by GA ČR grant no. 201/06/0128 and Ministry of Education research plan no. MSM4781305904.
we can also express $T_f$ as an integral operator
\begin{equation}
T_f u(x) = \int_{C^n} f(y) \, u(y) \, K(x, y) \, d\mu(y), \quad u \in \mathcal{F}, \ x \in \mathbb{C}^n. \tag{4}
\end{equation}

It is immediate from (3) that for $f \in L^\infty(\mathbb{C}^n)$, $T_f$ is bounded and
\begin{equation}
\|T_f\| \leq \|f\|_\infty. \tag{5}
\end{equation}

In principle, it is possible to define $T_f$ by the formula (3) or (4) even for some unbounded symbols $f$: for instance, for all $f$ such that $fK_y \in L^2(\mathbb{C}^n, d\mu)$ for all $y \in \mathbb{C}^n$. Then $T_f$ is a densely defined, closed operator on $\mathcal{F}$. Similarly, (2) can be extended also to some unbounded symbols $f$ as a densely defined operator.

It was observed by Coburn ([C2], [C3]) that for any polynomial $p$, $w \in \mathbb{C}$, and any polynomial $f$ for which (6) holds, $T_f$ was actually a constant coefficient linear differential operator and (6) held for all $f \in B_0(\mathbb{C}^n)$. This conjecture was verified by M.-L. Lo [Lo], who showed that (6) holds for any polynomials $p, w \in \mathcal{F}$ and any $f \in E(\mathbb{C}^n)$, where
\begin{equation}
E(\mathbb{C}^n) := \{ f \in C^\infty(\mathbb{C}^n) : \text{ for any multiindex } \alpha, \text{ there exist } M, \alpha > 0 \text{ such that } |D^\alpha f(z)| \leq M e^{\alpha \|z\|} \forall z \in \mathbb{C}^n \} \tag{7}
\end{equation}
contains both $B_0(\mathbb{C}^n)$ and all polynomials in $z$ and $\overline{z}$.

Lo’s proof went by a brute-force computation to establish the result for polynomials $f$ (in $z$ and $\overline{z}$), and then an approximation argument was used to extend it to all $f \in E(\mathbb{C}^n)$.

In this note, we present a simpler proof of these results, which also yields a bit more precise information for “nicer” symbols $f$.

**Theorem 1.** For any polynomial $w \in \mathcal{F}$, there exists a constant coefficient linear partial differential operator $D = D^{(w)}$ such that for any $f \in BC^\infty(\mathbb{C}^n)$ (the space of all $C^\infty$ functions on $\mathbb{C}^n$ whose partial derivatives of all orders are bounded),
\begin{equation}
L_{(w)} f = T_{Df} \quad \text{on } \mathcal{F}. \tag{8}
\end{equation}
Explicitly, the operator $D$ is given by

$$D(w) = \left[ e^{\Delta/2} |w(z)|^2 \right]_{\mathbb{C}^n},$$

where $e^{\Delta/2}$ should be understood as the infinite series

$$e^{\Delta/2} = \sum_{k=0}^{\infty} \frac{\Delta^k}{k!2^k}.$$

This infinite sum makes sense since, as $w$ is assumed to be a polynomial, $\Delta^k |w|^2$ vanishes as soon as $k > \deg w$; thus there are only finitely many nonzero terms.

Note also that for $f \in BC^\infty$ both sides of (8) are bounded operators, so the validity is not restricted to polynomials $p$ as in (6). In fact, the left-hand side in (8) is a bounded operator for any $f \in L^\infty$ (which are distributions at best).

One more virtue of our proof is that it uses solely harmonic analysis methods, and thus easily extends also to other situations than the Segal-Bargmann space on $\mathbb{C}^n$, for instance, to the standard weighted Bergman spaces on bounded symmetric domains, thus making contact with the work of Arazy and Upmeier [AU], de Mari and Nowak [MN], and others.

The paper is organized as follows. In Section 2, we review some preliminaries from Segal-Bargmann analysis. In Section 3, Theorem 1 is proved, and also extended to a wider class of functions $f$ (including the polynomials, the algebra $B_\alpha(\mathbb{C}^n)$, and the space $E(\mathbb{C}^n)$ from (7)). Generalizations to bounded symmetric domains are described in the final Section 4.

A substantial portion of this work was done during the Finnish Mathematical Society Visitor Program 2005–2006 in Helsinki and Joensuu; the author expresses his gratitude for the support and the nice time for research he had there.

2. BERELIN Symbals

In addition to $K_\alpha$, we also consider the normalized reproducing kernels

$$k_\alpha(z) := \frac{K_\alpha(z)}{\|K_\alpha\|} = e^{\langle z, \alpha \rangle / 2 - \|\alpha\|^2 / 4}.$$  

Note that the Weyl operators (11) can then be written simply as

$$W_\alpha f(z) = k_\alpha(z) f(z - \alpha).$$

In particular, as $k_\alpha = 1$ (the function constant one),

$$k_\alpha = W_\alpha 1, \quad \forall \alpha \in \mathbb{C}^n.$$  

One checks easily that $W_\alpha$ satisfy the composition law

$$W_\alpha W_\beta = e^{\langle \alpha - \beta \rangle / 4} W_{\alpha + \beta}, \quad \forall \alpha, \beta \in \mathbb{C}^n.$$  

Consequently, $W_\alpha^* = W_{-\alpha}$ and

$$W_\alpha k_\beta = e^{\langle \alpha - \beta \rangle / 4} k_{\alpha + \beta},$$

$$W_\alpha^* k_\beta = e^{\langle \alpha - \beta \rangle / 4} k_{\beta - \alpha}.$$
In particular, for \( w = 1 \), we get for any \( u, v \in \mathcal{F} \),
\[
\langle L_f^{(1)} u, v \rangle = (2\pi)^{-n} \int_{\mathbb{C}^n} f(a) \langle u, ka \rangle \langle ka, v \rangle \, da \\
= \int_{\mathbb{C}^n} f(a) \langle u, K_a \rangle \langle K_a, v \rangle \, d\mu(a) \\
= \int_{\mathbb{C}^n} f(a) u(a) \overline{v(a)} \, d\mu(a) \\
= \langle fu, v \rangle \\
= \langle Tu, v \rangle,
\]
so that indeed
\[
(13) \quad L_f^{(1)} = T_f.
\]

The next proposition is thus an analogue of \((5)\) for an arbitrary window \( w \). An analogous assertion is valid even in the much more general context of any square-integrable irreducible unitary representation of a unimodular group; see for instance Wong [Wo, Proposition 12.2], or [E] for an even further generalization; in the very special case that we have here, it is possible to give a simple direct proof based on the Fourier transform.

**Proposition 2.** For any \( w \in \mathcal{F} \) and \( f \in L^\infty(\mathbb{C}^n) \), the localization operator \( L_f^{(w)} \) is bounded, and
\[
\|L_f^{(w)}\| \leq \|f\| \|w\|^2.
\]

**Proof.** It is more convenient to pass from \( \mathcal{F} \) to \( L^2(\mathbb{R}^n) \), via the Bargmann transform
\[
\beta f(z) := c_n \int_{\mathbb{R}^n} f(x) \, e^{xz - x^2/2 - z^2/4} \, dx.
\]
With the proper choice of the constant \( c_n \), this is a unitary isomorphism of \( L^2(\mathbb{R}^n) \) onto \( \mathcal{F} \); see e.g. Folland [Fo]. (Here \( x^2 = x_1^2 + \cdots + x_n^2 \) for \( x \in \mathbb{R}^n \), and similarly for \( xz \) and \( z^2 \).) Its inverse is given by
\[
\beta^{-1} F(x) = c'_n \int_{\mathbb{C}^n} F(z) \, e^{z^2/2 - x^2/4} \, e^{-\|z\|^2/2} \, dz,
\]
and the Weyl operators \((1)\) satisfy \( W_{u+iv} = \beta U_{u,v} \beta^{-1} \), where the unitary operators \( U_{u,v} \) on \( L^2(\mathbb{R}^n) \) are given by
\[
U_{u,v} f(x) = e^{iu(x+y)/2-ixy} f(x-u), \quad x, u, v \in \mathbb{R}^n.
\]
It follows that
\[
\beta^{-1} L_f^{(w)} \beta = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(u + iv) \langle \cdot, U_{u,v} H \rangle \langle U_{u,v} H, \cdot \rangle \, du \, dv,
\]
where \( H = \beta^{-1} w \). To prove the proposition, it therefore suffices to show that
\[
\left| (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(u + iv) \langle F, U_{u,v} H \rangle \langle U_{u,v} H, G \rangle \, du \, dv \right| \leq \|f\| \|H\|^2 \|F\| \|G\|
\]
for all \( F, G \in L^2(\mathbb{R}^n) \).
By the Cauchy-Schwarz inequality, the left-hand side is bounded by

\[(2\pi)^{-n}\|f\|_{\infty} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\langle F, U_{u,v} H \rangle|^2 \, du \, dv \right)^{1/2} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\langle G, U_{u,v} H \rangle|^2 \, du \, dv \right)^{1/2}.\]

It is therefore enough to prove that

\[(14) \quad (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\langle F, U_{u,v} H \rangle|^2 \, du \, dv \leq \|F\|^2 \|H\|^2\]

for any \(F, H \in L^2(\mathbb{R}^n)\). However,

\[\langle F, U_{u,v} H \rangle = \int_{\mathbb{R}^n} F(x) e^{-iu\cdot x} e^{iv\cdot x} \overline{H(x-u)} \, dx = (2\pi)^{-n/2} e^{-iuv/2} \hat{h}_u(v),\]

where \(\hat{h}_u\) is the Fourier transform of the function \(h_u(x) = F(x)\overline{H(x-u)}\). Thus, by Parseval,

\[(2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\langle F, U_{u,v} H \rangle|^2 \, du \, dv = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{h}_u(v)|^2 \, du \, dv = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |h_u(x)|^2 \, du \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(x)|^2 |H(x-u)|^2 \, du \, dx = \int_{\mathbb{R}^n} |F(x)|^2 \left[ \int_{\mathbb{R}^n} |H(x-u)|^2 \, du \right] dx = \int_{\mathbb{R}^n} |F(x)|^2 \left[ \int_{\mathbb{R}^n} |H(y)|^2 \, dy \right] dx = \|F\|^2 \|H\|^2.\]

\[\square\]

Remark. We see that we have in fact an equality in (14). On the general level of square-integrable irreducible representations of an arbitrary unimodular group, this is of course just an immediate consequence of the Schur orthogonality relations.

Recall that for a bounded linear operator \(T\) on \(\mathcal{F}\), the Berezin symbol of \(T\) is the function \(\tilde{T}\) on \(\mathbb{C}^n\) defined by

\[\tilde{T}(x) := \langle Tk_x, k_x \rangle.\]

Again, the definition makes sense even for unbounded operators, as long as the reproducing kernels \(k_x\) are in the domain of \(T\), for all \(x\). The following proposition records some properties of the Berezin symbol which we will need.

Proposition 3. (a) The function \(\tilde{T}\) is real-analytic;
(b) \(\tilde{T}\) vanishes identically only if \(T = 0\);
(c) \( \| \widetilde{T} \|_\infty \leq \| T \| \);
(d) for any \( a \in \mathbb{C}^n \),

\[
(W^*_aTW_a)^- = \widetilde{T}(\cdot + a).
\]

Proof. All this is well known, but here is the proof for completeness. Note that \( \widetilde{T}(x) \) is the restriction to the diagonal \( x = y \) of the function

\[
\frac{\langle TK_y, K_x \rangle}{\langle K_y, K_x \rangle} = e^{-\langle x,y \rangle/2} \frac{\langle Te^{(\cdot,y)/2}, e^{(\cdot,x)/2} \rangle(x)}{\langle e^{(\cdot,y)/2}, e^{(\cdot,x)/2} \rangle(y)},
\]

which is holomorphic in \( x \) and \( y \); in particular, \( \widetilde{T} \) is a real-analytic function. Further, it is known that such functions are uniquely determined by their restriction to the diagonal (see e.g. Folland [Fo], Proposition 1.69); hence \( \widetilde{T} \equiv 0 \) only if \( \langle TK_y, K_x \rangle = TK_y(x) = 0 \) \( \forall x, y \), which implies that \( T = 0 \) since the linear combinations of \( K_y, y \in \mathbb{C}^n \), are dense in \( \mathcal{F} \). Finally, (c) is immediate from the Schwarz inequality, and the covariance property (15) is immediate from (12). \( \square \)

3. Main results

Proof of Theorem 1. From the definition of the localization operators \( L^{(w)}_f \), we have for any \( c \in \mathbb{C}^n \),

\[
L^{(w)}_{f(\cdot + c)} = (2\pi)^{-n} \int f(a + c) \langle \cdot, W_a w \rangle W_a w \, da
\]

\[
= (2\pi)^{-n} \int f(x) \langle \cdot, W_{x-c} w \rangle W_{x-c} w \, dx
\]

\[
= W^*_cL^{(w)}_f W_c,
\]

by (11). In particular, for \( w = 1 \), we get by (13),

\[
T_{f(\cdot + c)} = W^*_cT_f W_c.
\]

By Proposition 2 and parts (a), (c) and (d) of Proposition 3 we thus see that the two maps

\[
f \mapsto L^{(w)}_f, \quad f \mapsto T_f
\]

both map \( L^\infty(\mathbb{C}^n) \) continuously into bounded real-analytic functions on \( \mathbb{C}^n \), and commute with translations. Recall now (see e.g. [Ru], Theorem 6.33) that for any continuous linear map \( V \) from \( \mathcal{D}(\mathbb{C}^n) \) into \( C(\mathbb{C}^n) \) which commutes with translations there is a unique distribution \( v \in \mathcal{D}'(\mathbb{C}^n) \) such that \( Vf = v * f \) for all \( f \in \mathcal{D} \). Thus there exist distributions \( k = k^{(w)} \) and \( h = k^{(1)} \) on \( \mathbb{C}^n \) such that

\[
L^{(w)}_f = k * f, \quad T_f = h * f,
\]

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
for all \( f \in \mathcal{D}(\mathbb{C}^n) \). To find what \( k \) and \( h \) are, note that for any \( f \in L^\infty(\mathbb{C}^n) \) and \( z \in \mathbb{C}^n \),

\[
\tilde{L}_f^w(z) = (L_f^w)_{k_z, k_z}
\]

\[
= (2\pi)^{-n} \int f(a) \langle k_z, W_a w \rangle \langle W_a w, k_z \rangle \, da
\]

\[
= (2\pi)^{-n} \int f(a) |\langle W_a k_z, w \rangle|^2 \, da
\]

\[
= (2\pi)^{-n} \int f(a) |\langle k_{z-a}, w \rangle|^2 \, da \quad \text{by (12)}
\]

\[
= (2\pi)^{-n} \int f(z-y) |\langle k_y, w \rangle|^2 \, dy
\]

\[
= (2\pi)^{-n} \int f(z-y) |\langle K_y, w \rangle|^2 e^{-\|y\|^2/2} \, dy
\]

\[
= (2\pi)^{-n} \int f(z-y) |w(y)|^2 e^{-\|y\|^2/2} \, dy
\]

\[
= (f * (2\pi)^{-n}|w|^2 e^{-\|\cdot\|^2/2})(z).
\]

Thus \( k \) is not only a distribution but a function, given by

\[
k(z) = (2\pi)^{-n}|w(z)|^2 e^{-\|z\|^2/2},
\]

and, taking \( w = 1 \),

\[
h(z) = (2\pi)^{-n} e^{-\|z\|^2/2}.
\]

It also follows from the last computation that (17) holds not only for \( f \in \mathcal{D}(\mathbb{C}^n) \), but for any \( f \in L^\infty(\mathbb{C}^n) \).

Observe now that for any multiindices \( j, k \), the Leibniz formula implies that

\[
\partial^j \partial^k e^{-\|\cdot\|^2/2} = e^{-\|\cdot\|^2/2} \left[ \left( -\frac{1}{2} \right)^{|j+k|} |z|^{j+k} + \text{lower order terms} \right].
\]

By a straightforward induction argument, it follows that there exists a unique differential operator \( D = D^w \) with constant coefficients such that

\[
D e^{-\|\cdot\|^2/2} = |w|^2 e^{-\|\cdot\|^2/2},
\]

i.e. \( Dh = k \). By the properties of convolution,

\[
h * Df = Dh * f = k * f
\]

for any reasonable \( f \) (for instance, whenever all derivatives of \( f \) up to the order of \( D \) are bounded). Consequently,

\[
\tilde{T}Df = h * Df = k * f = L_f^w
\]

for any \( f \in BC^\infty(\mathbb{C}^n) \). By part (b) of Proposition 3, this implies that

\[
T_Df = L_f^w,
\]

thus completing the proof of (8).
It remains to show that the operator $D$ is given by the formula (19). To this end, write out the “lower order terms” in (20) explicitly:

$$\partial^j \partial^k e^{-\|z\|^2/2} = \partial^j \left[ \left( -\frac{z}{2} \right)^k e^{-\|z\|^2/2} \right]$$

$$= \sum_{l \leq j} \binom{j}{l} \left( -\frac{1}{2} \right)^l \frac{[k]}{[l]} \frac{k!}{(k-l)!} z^{k-l} \left( -\frac{z}{2} \right)^{j-l} e^{-\|z\|^2/2}$$

$$= \sum_{l \leq j} \frac{1}{(j-l)!} \frac{k!}{(k-l)!} z^{k-l} \left( -\frac{1}{2} \right)^{j-l} e^{-\|z\|^2/2}$$

$$= \left( -\frac{1}{2} \right)^{j+k} e^{-\|z\|^2/2} \sum_{l} \left( \partial^j \partial^l \right) \left( \partial^l \partial^k \right) \frac{(-2)^{[l]}}{l!}$$

$$= \left( -\frac{1}{2} \right)^{j+k} e^{-\|z\|^2/2} \sum_{L=0}^{\infty} \frac{(-2)^{L}}{L!} \sum_{|l|=L} \binom{L}{l} \partial^j \partial^l \partial^k e^{-\|z\|^2/2}$$

$$= \left( -\frac{1}{2} \right)^{j+k} e^{-\|z\|^2/2} \sum_{L=0}^{\infty} \frac{(-2)^{L}}{L!} \frac{\Delta^L}{4} \partial^j \partial^k e^{-\|z\|^2/2}$$

$$= \left( -\frac{1}{2} \right)^{j+k} e^{-\|z\|^2/2} e^{-\Delta/2} \partial^j \partial^k.$$

It follows that for any polynomial $p$ in two variables with complex coefficients,

$$p(-2\partial, -2\overline{\partial}) e^{-\|z\|^2/2} = e^{-\|z\|^2/2} e^{-\Delta/2} p(\partial, \overline{\partial}).$$

Thus if we choose

$$p(\partial, \overline{\partial}) = e^{\Delta/2} |w(z)|^2,$$

then $p(-2\partial, -2\overline{\partial}) = D$. This completes the proof of Theorem 1. \qed

**Corollary 4.** Let $w_1, w_2 \in \mathcal{F}$ be polynomials. Then the following two assertions are equivalent:

(a) There exists a constant coefficient linear differential operator $D$ such that

$$L_f^{(w_2)} = L_{Df}^{(w_1)}$$

for all $f \in \mathcal{D}(\mathbb{C}^n)$.

(b) The polynomial $e^{\Delta/2} |w_2|^2$ is divisible by the polynomial $e^{\Delta/2} |w_1|^2$.

Further, if (a) or (b) are fulfilled, then $D$ is of order $2(\deg w_2 - \deg w_1)$ and (22) holds for all $f \in BC^\infty(\mathbb{C}^n)$.

**Proof.** This is immediate from (3) and (19). \qed

Note that we have proved (3) not only for $f \in BC^\infty$, but in fact for any $f \in L^\infty$ whose derivatives up to the order of $D$ are bounded. Going through the above arguments with some care, it is not difficult to extend this even further. Let $r$ be the degree of $w$ and denote

$$M_r := \left\{ f \in C^{2r}(\mathbb{C}^n) : \text{for any multiindices } j, k \text{ with } |j|, |k| \leq r \right\}$$

and any $a > 0$, $e^{a \|z\|^2} |\partial^j \partial^k f| e^{-\|z\|^2/2} \in L^\infty(\mathbb{C}^n)$.

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Observe that the condition implies that for any $m \geq 0$ and $|j|, |k| \leq r$, $\|z\|^{m} |\partial^{j} \partial^{k} f| e^{-\|x\|^{2}/2}$ belongs to $L^{1}$ and vanishes at infinity. Integrating by parts in
\[
\int f(z - x) \, D e^{-\|x\|^{2}/2} \, dx,
\]
it therefore follows that
\[
f * Dh = Df * h \quad \forall f \in M_{r};
\]
i.e., (21) still holds for $f \in M_{r}$. Thus again
\[
\tilde{T}_{Df} = \tilde{L}_{f}^{(w)}.
\]
Since now $T_{Df}$ and $L_{f}^{(w)}$ need no longer be bounded in general, it is not clear whether this implies $T_{Df} = L_{f}^{(w)}$; however, from the proof of part (b) of Proposition 3 it is clear at least that $T_{Df}K_{z} = L_{f}^{(w)}K_{z}$ for any $z \in C^{n}$. Thus we arrive at the following strengthening of Theorem 1.

**Theorem 5.** Let $w \in F$ be a polynomial of degree $r$, and let $M_{r}$ be as in (23). Then for any $f \in M_{r}$, $T_{Df}$ and $L_{f}^{(w)}$ coincide on the linear span of $K_{z}$, $z \in C^{n}$.

Note that $E(C^{n}) \subset M_{r}$ for any $r$; thus, in particular, the last theorem covers completely the main result of [Lo] (except that the polynomials $p$ are replaced by linear combinations of $K_{z}$).

We conclude this section by a generalization in a different direction. It may seem a little artificial at first sight, but becomes very natural after we pass to the bounded symmetric domains in the next section. For any bounded linear operator $A$ on $F$, we may define a “localization operator” with symbol $f$ and “window” $A$ by

\[
L_{f}^{(A)} := (2\pi)^{-n} \int_{C^{n}} f(a) \, W_{a} A W_{a}^{*} \, da.
\]

The localization operators $L_{f}^{(w)}$ considered so far are recovered upon choosing $A = \langle f, w \rangle w$. We then have the following generalizations of Proposition 2 and Theorem 1.

**Proposition 6.** If $A$ is trace-class, then the integral (24) converges in the weak operator topology for any $f \in L^{\infty}(C^{n})$, and

\[
\|L_{f}^{(A)}\| \leq \|f\|_{\infty} \|A\|_{tr},
\]

where $\|\cdot\|_{tr}$ denotes the trace norm.

**Theorem 7.** Let $A$ be a finite sum
\[
A = \sum_{j} \langle \cdot, u_{j} \rangle v_{j}
\]
where $u_{j}, v_{j} \in F$ are polynomials. Then there exists a unique linear partial differential operator $D = D^{(A)}$ such that
\[
L_{f}^{(A)} = T_{Df} \quad \forall f \in BC^{\infty}(C^{n}).
\]
The proof of Proposition \[6\] can (again in a much more general setup) be found in \[E\], or carried out directly along the lines of the proof of Proposition\[2\]. Similarly, Theorem \[7\] can be proved either by mimicking the proof of Theorem \[1\] or from Theorem \[1\] directly using the linearity in \(A\) and the familiar polarization identity

\[
\langle \cdot, w_1 \rangle w_2 = \sum_{k=0}^{3} i^{-k} \langle \cdot, w_1 + i^k w_2 \rangle (w_1 + i^k w_2).
\]

4. Bounded symmetric domains

Throughout this section we let \(\Omega\) be an irreducible bounded symmetric domain in \(\mathbb{C}^n\) (i.e. a Cartan domain) in its Harish-Chandra realization (so \(\Omega\) is circular with respect to the origin and convex). Let \(G\) be the group of all biholomorphic self-maps of \(\Omega\); then \(G\) acts transitively on \(\Omega\), so denoting by \(K\) the stabilizer of the origin \(0 \in \Omega\) in \(G\), \(\Omega\) can be identified with the coset space \(G/K\). For each \(z \in \Omega\), there exists a unique so-called geodesic symmetry \(g_x \in G\) interchanging \(x\) and the origin, i.e. \(g_x\) is an involution (that is, \(g_x = g_x^{-1}\)), \(g_x(0) = x\), \(g_x(x) = 0\), and \(g_x\) has only isolated fixed-points. We refer e.g. to \[Ar\], \[Ko\] or \[Up\] for an overview of bounded symmetric domains.

Let \(dz\) be the Lebesgue measure on \(\Omega\) normalized so that \(\Omega\) has total mass one. Abusing the notation a little, we will denote by the same letter \(K\) kernel \(K_y(x) = K(x, y)\) of \(\Omega\), i.e. the reproducing kernel of the subspace \(\mathcal{H} = L^2_{\text{hol}}(\Omega, dz)\) of all holomorphic functions in \(L^2(\Omega, dz)\). We will also use the same notation \(k_z = K_z/\|K_z\|\) as before for the normalized reproducing kernels.

From the familiar formula for the change of variables, it is immediate that the operators

\[
U_g : f \mapsto j_{g^{-1}} \cdot (f \circ g^{-1}), \quad g \in G,
\]

are unitary on \(L^2(\Omega)\) and \(\mathcal{H}\); here \(j_g\) denotes the complex Jacobian of the mapping \(g\).

From the chain rule for derivatives it follows that

\[
U_{g_1} U_{g_2} = U_{g_1 g_2}, \quad \forall g_1, g_2 \in G,
\]

so that \(g \mapsto U_g\) is a unitary representation of \(G\) in \(\mathcal{H}\). In particular, \(U_g^* = U_{g^{-1}}\).

From the computation

\[
\langle f, U_g k_z \rangle = \langle U_{g^{-1}} f, k_z \rangle = K(z, z)^{-1/2} (U_{g^{-1}} f)(z)
\]

\[
= K(z, z)^{-1/2} j_g(z) f(g(z))
\]

\[
= K(g(z), g(z))^{1/2} K(z, z)^{-1/2} j_g(z) (f, k_g(z)), \quad \forall f \in \mathcal{H},
\]

it follows that \(U_g k_z = \text{const} \cdot k_{g(z)}\); since \(U_g\) is unitary and \(k_z, k_{g(z)}\) are both unit vectors, the constant must be unimodular, i.e.

\[
U_g k_z = \epsilon_{g,z} k_{g(z)}, \quad |\epsilon_{g,z}| = 1,
\]

which is an analogue of \[12\].

Yet another consequence of the change-of-variable formula is the equality

\[
K(x, y) = j_{g^{-1}}(x) K(g^{-1}(x), g^{-1}(y)) j_{g^{-1}}(y),
\]

from which it follows that the measure

\[
d\mu(z) := K(z, z) dz, \quad z \in \Omega,
\]

is \(G\)-invariant.
Denoting by \( dg \) the Haar measure on \( G \), we may now define for any bounded linear operator ("window") \( A \) on \( \mathcal{H} \) and any function ("symbol") \( f \) on \( G \) the "localization operator"

\[
L_f^{(A)} := \int_G f(g) U_g A U_g^* \, dg.
\]

Comparing this with (24), we immediately see the drawback that our symbols \( f \) now live on \( G \), not on \( \Omega \). As shown in [AU] and [E], this can be resolved by restricting to operators \( A \) which are \( K \)-invariant, in the sense that

\[
AU_k = U_k A \quad \forall k \in K.
\]

Indeed, then for any \( g \in G \) we have

\[
U_gkA U_g^* = U_gk A \quad \forall k \in K.
\]

Thus \( U_g AU_g^* \) depends only on the coset \( gK \) of \( g \) in \( G/K \), i.e. only on \( g(0) \in \Omega \). We can therefore define unambiguously the operator \( A_z \), for any \( z \in \Omega \), by

\[
A_z := U_g AU_g^* \quad \text{for any } g \in G \text{ such that } g(0) = z,
\]

and the localization operator

\[
L_f^{(A)} := \int_\Omega f(z) A_z \, d\mu(z).
\]

Such operator calculi were studied in [E]. It was shown there, for instance, that (27) converges in the weak operator topology whenever \( f \) is bounded and \( A \) is trace-class, and

\[
\|L_f^{(A)}\| \leq \|f\|_\infty \|A\|_\text{tr},
\]

an analogue of Propositions 2 and 6. Our goal in the rest of this section will be to establish also an analogue of Theorems 1 and 7. Before stating the latter, we need to review some facts about the structure of \( K \)-invariant operators.

It is known that under the action \( U_k \) of the group \( K \), the space \( \mathcal{H} \) decomposes into an orthogonal direct sum of irreducible subspaces (Peter-Weyl decomposition)

\[
\mathcal{H} = \bigoplus_m \mathcal{P}_m.
\]

Here \( m \) ranges over all signatures, i.e. \( r \)-tuples \( m = (m_1, \ldots, m_r) \) of integers satisfying \( m_1 \geq m_2 \geq \cdots \geq m_r \geq 0 \); the number \( r \) is the rank of \( \Omega \). One has \( \mathcal{P}_{(0,\ldots,0)} = \{ \text{the constant functions} \} \), \( \mathcal{P}_{(1,0,\ldots,0)} = \{ \text{the linear functions} \} \), and, in general, the elements of \( \mathcal{P}_m \) are homogeneous polynomials of degree \( |m| := m_1 + \cdots + m_r \). Let \( \mathcal{P}_m \) be the orthogonal projection in \( \mathcal{H} \) onto \( \mathcal{P}_m \). By construction, \( \mathcal{P}_m \) is a \( K \)-invariant operator. Conversely, if \( A \) is any \( K \)-invariant operator, then it follows from Schur’s lemma that the restriction of \( A \) to each \( \mathcal{P}_m \) is a multiple of the identity. Thus, the \( K \)-invariant operators on \( \mathcal{H} \) are precisely the operators of the form

\[
A = \sum_m c_m \mathcal{P}_m, \quad c_m \in \mathbb{C}.
\]

Clearly \( A \) is bounded if and only if \( \{c_m\} \) is a bounded sequence, and \( A \) is trace-class if and only if \( \sum_m c_m \dim \mathcal{P}_m < \infty \).

The simplest \( K \)-invariant operator is thus

\[
A = \mathcal{P}_{(0,\ldots,0)} = \langle \cdot, 1 \rangle 1,
\]
Theorem 8. Let \( g \in \Omega \) and the projection onto the constants. By (26), in that case,
\[
A_z = \langle \cdot, k_z \rangle k_z
\]
and
\[
L_f^{(A)} = \int_{\Omega} f(z) \langle \cdot, k_z \rangle k_z \, d\mu(z) = \int_{\Omega} f(z) \langle \cdot, K_z \rangle K_z \, dz = T_f,
\]
the Toeplitz operator with symbol \( f \).

We now have the following analogue of Theorems 1 and 7.

**Theorem 8.** Let \( A \) be a \( K \)-invariant operator on \( \mathcal{H} \) of the form
\[
A = \sum_{\text{finite}} c_m P_m.
\]

Then there exists a unique \( G \)-invariant linear partial differential operator \( D = D^{(A)} \) on \( \Omega \) such that
\[
L_f^{(A)} = T_{Df} \quad \forall f \in \mathcal{D}(\Omega).
\]

**Proof.** The proof is completely parallel to that of Theorem 1 so we will be brief. Using linearity, it is enough to prove the theorem for \( A = P_m \), which we will assume from now on. For any bounded linear operator \( T \) on \( \mathcal{H} \), we again define its Berezin symbol \( \tilde{T} \) by
\[
\tilde{T}(z) = \langle Tk_z, k_z \rangle, \quad z \in \Omega.
\]
The proof of Proposition 3 extends to the present setting without any changes, so that again \( \|\tilde{T}\|_{\infty} \leq \|T\| \), \( \tilde{T} \) is real-analytic, and \( \tilde{T} \equiv 0 \) only if \( T = 0 \). By a similar computation as for the Fock space, for any \( f \in L^\infty(\Omega) \),
\[
\tilde{L}_f^{(A)}(z) = \langle L_f^{(A)}k_z, k_z \rangle = \int_{\Omega} f(x) \langle A_z k_z, k_z \rangle d\mu(x).
\]

Let \( g_z \in G \) be the geodesic symmetry interchanging \( x \) and the origin, so that \( g_z = g_x^{-1} \), \( g_z(0) = x \) and \( g_z(x) = 0 \). Then \( \langle A_z k_z, k_z \rangle = \langle A U_g^* k_z, U_g^* k_z \rangle = \langle A k_{g_z(x)}, k_{g_z(x)} \rangle \) by (20). Since \( g_z(g_z(0)) = g_z(z) = g_{g_z(x)}(0) \), there exists \( k \in K \) such that \( g_z g_x = g_{g_z(x)} k \); taking inverses gives \( k g_x g_z = g_{g_z(x)} \), whence \( g_z(z) = g_{g_z(x)}(0) = k(g_z(g_z(0))) = k(g_z(x)) \). As \( A \) is \( K \)-invariant, \( \langle A k_{g_z(x)}, k_{g_z(x)} \rangle = \langle A U_k k_{g_z(x)}, U_k k_{g_z(x)} \rangle = \langle A k_{g_z(x)}, k_{g_z(x)} \rangle = \tilde{A}(g_z(x)) \). Thus
\[
\tilde{L}_f^{(A)}(z) = \int_{\Omega} f(x) \tilde{A}(g_z(x)) \, d\mu(x).
\]
The last integral is the definition of convolution (in \( G \)) of \( f \) and \( \tilde{A} \):\[
\tilde{L}_f^{(A)} = f * \tilde{A}.
\]
As \( A = P_m \) we have
\[
\tilde{A}(z) = \langle P_m k_z, k_z \rangle = K(z, z)^{-1} (P_m K_z)(z) = K(z, z)^{-1} K_m(z, z),
\]
where $K_m(x, y)$ is the reproducing kernel of the subspace $P_m \subset \mathcal{H}$. In particular, for $m = (0, \ldots , 0)$, we have $\tilde{P}_{(0,\ldots,0)}(z) = K(z, z)^{-1}$.

Now it was shown by Ørsted and Zhang [OZ], Proposition 3.15, that there exists a unique $G$-invariant linear partial differential operator $D = D^m$ on $\Omega$ such that
\[
DK(z, z)^{-1} = K_m(z, z)K(z, z)^{-1}.
\]

Arguing as in the Fock-space case, it follows that
\[
\tilde{L}^{(A)}_f = f \ast \tilde{P}_m = f \ast D\tilde{P}_{(0,\ldots,0)} = Df \ast \tilde{P}_{(0,\ldots,0)} = T_Df,
\]
whence $L^{(A)}_f = T_Df$ by part (b) of Proposition 3. This completes the proof. \[\Box\]

Remark. Again, it is evident from the proof that (28) holds not only for $f \in D(\Omega)$, but for any $f \in C^\infty(\Omega)$ whose derivatives do not grow too fast at the boundary, so that the partial integration implicit in the third equality in (28) is legitimate.

References


Mathematics Institute, Silesian University at Opava, Na Rybníčku 1, 74601 Opava, Czech Republic – and – Mathematics Institute, Žitná 25, 11567 Prague 1, Czech Republic

E-mail address: englis@math.cas.cz