TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 361, Number 2, February 2009, Pages 1053–1067
S 0002-9947(08)04596-0
Article electronically published on September 29, 2008

H^p-BOUNDS FOR SPECTRAL MULTIPLIERS ON GRAPHS

IOANNA KYREZI AND MICHEL MARIAS

Dedicated to the memory of Nikos Danikas

Abstract. We study the boundedness on the Hardy spaces H^p of spectral multiplier operators associated with the discrete Laplacian on a weighted graph. We assume that the graph satisfies the doubling volume property and a Poincaré inequality. We prove that there is $p_0 \in (0,1)$, depending on the geometry of the graph, such that if the multiplier satisfies a condition similar to the one we have in the classical Hörmander multiplier theorem, then the corresponding operator is bounded on H^p, $p \in (p_0, 1]$.

1. Introduction

Let $m(\xi)$ be a bounded measurable function in $\mathbb{R}^n$ and let T_m be the operator defined by $\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi)$ (where $\widehat{f}$ denotes the Fourier transform of the function f). Then, by the Plancherel formula, T_m is an operator bounded on L^2. The Hörmander multiplier theorem (cf. [13, 18]) asserts that if

$$\sup_{\xi\in\mathbb{R}^n} |\xi|^{|\alpha|}|\partial^\alpha m(\xi)| < +\infty, \tag{1.1}$$

for any multi-index α with $|\alpha| \le [n/2]+1$, then T_m extends to an operator bounded on L^p, $1 < p < \infty$, and from L^1 to weak-L^1.

This result was extended to the Hardy spaces H^p, $0 < p \le 1$, by Calderón and Torchinsky [4], who proved that if for example $m(\xi)$ satisfies (1.1) for any multi-index α with $|\alpha| \le \left[n\left(\frac{1}{p}-\frac{1}{2}\right)\right]+1$, then T_m is bounded on H^p.

The theorem of Hörmander has many generalizations to abstract contexts (see for example [1, 2, 3, 6, 10, 11, 14, 15, 17, 19]).

The goal of this article is to obtain a generalization of the result of Calderón and Torchinsky in the context of weighted graphs. More precisely, let Γ be a countable infinite set and let $\sigma(x,y)$ be a weight on Γ satisfying $\sigma(x,y) = \sigma(y,x) \ge 0$ and $\sigma(x,x) > 0$, $x, y \in \Gamma$. This weight induces a graph structure on Γ. We call the vertices x and y neighbors and we write $x \sim y$ when $\sigma(x,y) \ne 0$. We assume that Γ is connected.

Received by the editors November 14, 2005 and, in revised form, May 15, 2007.
2000 *Mathematics Subject Classification.* Primary 42B15, 42B20, 42B30.
Key words and phrases. Graph, multiplier, Hardy space, discrete Laplacian.
The first author was partially supported by a NATO (Greece) fellowship and the second author by the EPEAK program Pythagoras II (Greece) and the European TMR Network "Harmonic Analysis".

We consider the discrete measure μ defined by

$$\mu(\{x\}) = \sum_{y\sim x} \sigma(x,y), \quad x \in \Gamma.$$

For simplicity, we set $\mu(x) = \mu(\{x\})$, $|A| = \mu(A)$ and $L^p = L^p(\Gamma, \mu)$.

We consider the transition kernel

$$p(x,y) = \frac{\sigma(x,y)}{\mu(x)\mu(y)}, \quad x, y \in \Gamma.$$

This is a symmetric Markov kernel since $p(y,x) = p(x,y) \geq 0$ and

$$\sum_y p(x,y)\,\mu(y) = 1, \quad x, y \in \Gamma.$$

We consider the operator

$$Pf(x) = \sum_y p(x,y)\, f(y)\, \mu(y). \tag{1.2}$$

The operator $L = I - P$, called discrete Laplacian, is symmetric and we have

$$\|Lf\|_2 \leq \|f\|_2 + \|Pf\|_2 \leq 2\|f\|_2.$$

So, it admits the spectral resolution $L = \int_0^2 \lambda dE_\lambda$ (cf. [21]).

Let $m(\lambda)$ be a bounded Borel measurable function. Then, by the spectral theorem, we can define the operator

$$m(L) = \int_0^2 m(\lambda)\, dE_\lambda.$$

The operator $m(L)$ is bounded on L^2. The function $m(\lambda)$ is called a multiplier.

In this article, we study the boundedness of the operator $m(L)$ on the spaces H^p. Before stating our result we shall present the geometry of the graph and state the Gaussian estimates satisfied by the iterates $p_n(x,y)$ of the kernel $p(x,y)$.

1.1. **The geometry of the graph.** We assume that there is a positive integer N such that every vertex has at most N neighbors.

A path of length n joining the vertices x and y is a sequence of vertices $x = x_1, x_2, \ldots, x_n = y$ such that $x_i \sim x_{i-1}$, $i = 1, \ldots, n-1$. The distance $d(x,y)$ is defined as the infimum of the lengths of the paths joining x and y.

Let $B(x,r) = \{y \in \Gamma : d(x,y) \leq r\}$ denote the ball of center x and radius r. We assume that Γ satisfies the doubling volume property, i.e. there is $c > 0$, such that

$$|B(x,2r)| \leq c|B(x,r)|, \quad x \in \Gamma, \quad r > 0. \tag{1.3}$$

Note that this implies that there are $c, D > 0$ such that

$$|B(x,r)| \leq c\left(\frac{r}{s}\right)^D |B(x,s)|, \quad \text{for } r > s > 0. \tag{1.4}$$

The assumption (1.3) makes the graph Γ a space of homogeneous type in the sense of Coifman and Weiss [7]. Thus, we can define the atomic Hardy spaces H^p, $p > 0$, and the space of functions of bounded mean oscillation BMO, in the standard way (cf. Section 2 below for precise definitions).

Finally we assume that Γ satisfies a Poincaré inequality, i.e. there is a constant $c > 0$ such that for every function f and every $x_0 \in \Gamma$ and $r > 0$

$$\sum_{x \in B(x_0,r)} |f(x) - f_B|^2 \mu(x) \leq cr^2 \sum_{x,y \in B(x_0,2r)} |f(y) - f(x)|^2 \sigma(x,y), \tag{1.5}$$

where we have set

$$f_B = \frac{1}{|B(x_0,r)|} \sum_{x \in B(x_0,r)} f(x)\mu(x).$$

1.2. **Kernel estimates.** Let us set

$$p_0(x,y) = \frac{\delta_x(y)}{\mu(y)}, \quad p_1(x,y) = p(x,y),$$

where δ_x is the Dirac mass at x and denote by $p_n(x,y)$, $n \in \mathbb{N}$, the nth convolution power of $p(x,y)$, defined by

$$p_n(x,y) = \sum_z p(x,z) p_{n-1}(z,y)\mu(z). \tag{1.6}$$

Then we have

$$P^n f(x) = \sum_y p_n(x,y) f(y)\mu(y), \quad x \in \Gamma, n \in \mathbb{N}. \tag{1.7}$$

The assumptions (1.3) and (1.5) imply the following estimates (cf. [9, 8]):

(1) There are $C, c > 0$ such that

$$p_n(x,y) \leq \frac{C}{|B(x,\sqrt{n})|} e^{-cd(x,y)^2/n}, \quad x,y \in \Gamma, n \in \mathbb{N}. \tag{1.8}$$

(2) There is $\beta \in (0,1)$ and $C, c > 0$ such that

$$|p_n(y,x) - p_n(z,x)| \leq \frac{C}{|B(x,\sqrt{n})|} \left[\frac{d(y,z)}{\sqrt{n}}\right]^\beta e^{-cd(x,y)^2/n}, \tag{1.9}$$

for all $n \in \mathbb{N}$ and $x,y,z \in \Gamma$ satisfying $d(y,z) \leq \sqrt{n}$.

Note that in the case of left invariant random walks on discrete groups of polynomial volume growth, the above estimates are satisfied with $\beta = 1$ (cf. [12]).

1.3. **Statement of the result.** Let the constants β and D be as in (1.9) and (1.4) and let

$$p_0 = \frac{D}{D+\beta}.$$

Let also $p \in (p_0, 1]$, $\varepsilon > 0$ and let us set

$$A = D\left(\frac{1}{p} - \frac{1}{2}\right) + \varepsilon.$$

Let us fix a function $\varphi \in C_0^\infty(0,\infty)$, not identically zero, and let us consider the following condition on the multiplier $m(\lambda)$:

$$\sup_{0<t<\infty} \|\varphi(\lambda) m(t\lambda)\|_{C^A(\mathbb{R})} < \infty. \tag{1.10}$$

In this article we shall prove the following result.

Theorem 1. *Let p and A be as above and let us assume that* (1.10) *is satisfied. Then, the operator $m(L)$ is bounded on H^p.*

Note that by interpolation and duality, the above result implies that $m(L)$ is bounded on L^p, $1 < p < \infty$, and on BMO.

As we mentioned earlier, when Γ is a discrete group of polynomial volume growth and the kernel $p(x,y)$ is left invariant, then we have $\beta = 1$ and thus the above result holds with $p_0 = D/(D+1)$. Also, in the case of $\mathbb{R}^n$, when we define the atomic Hardy spaces H^p, for $p \in (0, n/(n+1)]$, a certain number of higher order moments of the atoms should vanish (while for $p \in (n/(n+1), 1]$, it is enough to have zero mean value). So, in order to study the operator $m(L)$ on H^p for $p \in (0, D/(D+1)]$, we should impose some structure on the graph Γ.

Throughout this article the different constants will always be denoted by the same letter c. When their dependence or independence is significant, it will be clearly stated.

2. Hardy spaces

In this section we shall recall the definition and the basic properties of the atomic Hardy spaces H^p, introduced in the context of spaces of homogeneous type by Coifman and Weiss [7].

Let $p \in (p_0, 1]$. We say that a function a is a (p,∞) atom, or simply a p-atom, if there is a ball $B(y,r)$ such that

$$\operatorname{supp}(a) \subseteq B(y,r), \quad \|a\|_\infty \le |B(y,r)|^{-1/p}, \qquad \sum_x a(x)\,\mu(x) = 0.$$

Note that these imply that

$$\|a\|_q \le |B(y,r)|^{(1/q)-(1/p)}, \qquad q \ge 1.$$

In particular, we have

$$\|a\|_1 \le |B(y,r)|^{1-(1/p)}, \qquad \|a\|_2 \le |B(y,r)|^{(1/2)-(1/p)}. \tag{2.1}$$

The Hardy space H^1 is defined as follows. We say that a function $f \in H^1$, if $f \in L^1$ and there is a sequence $(\lambda_n) \in \ell^1$ and a sequence of 1-atoms (a_n) such that $f = \sum_n \lambda_n a_n$. We set

$$\|f\|_{H^1} = \inf\left\{\sum_n |\lambda_n| : f = \sum_n \lambda_n a_n\right\}.$$

With this norm H^1 is a Banach space.

The space BMO is the dual of H^1 and it is defined as follows. Let us first denote by f_B the mean value of the function f on the ball B. We say that a function $f \in BMO$ if there is a constant $c > 0$, such that for every ball B

$$\|f - f_B\|^2_{L^2(B)} < c|B|. \tag{2.2}$$

The norm $\|f\|_{BMO}$ is defined as the smallest of those constants c.

Note that it follows from (2.2) that there is $c > 0$ such that for all $f \in BMO$, all $k \in \mathbb{N}$ and every ball $B = B(y,r)$

$$\|f - f_B\|^2_{L^2(B(y,2^k r))} < ck^2 |B(y,2^k r)|\,\|f\|^2_{BMO}. \tag{2.3}$$

To define the Hardy space H^p for $p \in (p_0, 1)$, we need the notion of the Lipschitz space $\mathcal{L}_\alpha$, $\alpha > 0$. We say that a function $f \in \mathcal{L}_\alpha$ if there is a constant $c > 0$, such that for every ball B and $x, y \in B$, we have

$$|f(x) - f(y)| \le c|B|^\alpha. \tag{2.4}$$

The norm $\|f\|_{\mathcal{L}_\alpha}$ is defined as the smallest of those constants c. With this norm, $\mathcal{L}_\alpha$ is a Banach space.

Now, let $p \in (p_0, 1)$ and let $\alpha = (1/p) - 1$. Then we define the Hardy space H^p as the space of those functionals $f \in \mathcal{L}'_\alpha$ which can be written as $f = \sum_n \lambda_n a_n$, where the a_n's are p-atoms and $(\lambda_n) \in \ell^p$. As in the case of H^1, we set

$$\|f\|_{H^p} = \inf \left\{ \left(\sum_n |\lambda_n|^p \right)^{1/p} : f = \sum_n \lambda_n a_n \right\}.$$

If $p \in (p_0, 1)$, then $\|f\|_{H^p}$ is not a norm, but $d(f, g) = \|f - g\|_{H^p}$ is a distance.

Finally, we note that the dual of H^p, $p \in (p_0, 1)$, is the space $\mathcal{L}_\alpha$.

3. Compactly supported multipliers

3.1. An approximation lemma.

We shall need the following lemma.

Lemma 1 (cf. [20, 16, 1]). *Let $n \in \mathbb{N}$, $0 < \alpha \leq 1$ and $A = n + \alpha$. Given a function $f \in C^n(\mathbb{R})$ with $\operatorname{supp}(f) \subset (-4, 4)$, let us set*

$$M_A(f) = \sup \left\{ \frac{\left| f^{(n)}(x+t) - f^{(n)}(x) \right|}{t^\alpha},\ t > 0,\ x \in \mathbb{R} \right\}.$$

Then, there is $c > 0$ such that for all $k \in \mathbb{N}$, there is a polynomial Q with $\deg(Q) \leq k$ such that

$$\sup \{ |f - Q| ;\ [-4, 4] \} \leq c \frac{M_A(f)}{k^A}. \tag{3.1}$$

Note that if $f \in C^A(\mathbb{R})$, then $\|f\|_{C^A} = \sum_{k=0}^n \left\| f^{(k)} \right\|_\infty + M_A(f)$, and hence we can replace $M_A(f)$ by $\|f\|_{C^A}$ in (3.1) above.

Let us now assume that the function f is even. Then we can assume that Q is also even and hence it has only terms of even order. If $\deg(Q) \leq 2k$, $k \in \mathbb{N}$, then $Q(\sqrt{\lambda})$ is a polynomial in λ with degree k. Hence the operator $Q(\sqrt{L})$ can be written as $Q(\sqrt{L}) = a_0 I + a_1 P + \cdots + a_k P^k$. It follows that if $\operatorname{supp}(\phi) \subseteq B(x, m)$, then

$$\operatorname{supp}(Q(\sqrt{L})\phi) \subseteq B(x, m + k). \tag{3.2}$$

We shall exploit this observation in order to obtain off diagonal estimates for the kernel of the operator $m(L)$. This observation has been exploited first by Carne [5] and later in [1].

3.2. L^2-estimates for $p_n(x, y)$.

By integrating the estimates (1.8) and (1.9) and by using the doubling volume property (1.4), we have the following estimates:

(1) There is a $c > 0$ such that, for all $j \in \mathbb{N}$ and all $y \in \Gamma$,

$$\|p_{2^j}(., y)\|_2^2 \leq \frac{c}{|B(y, 2^{j/2})|}. \tag{3.3}$$

(2) There is a $c > 0$ such that, for all $j, q \in \mathbb{N}$ and all $y \in \Gamma$,

$$\|p_{2^j}(., y)\|^2_{L^2(\{x : d(x,y) > 2^{q/2}\})} \leq \frac{c e^{-c 2^{q-j}}}{|B(y, 2^{j/2})|}. \tag{3.4}$$

(3) There is a $c>0$ such that, for all $j\in\mathbb{N}$ and all $y,z\in\Gamma$ with $d(y,z)\le 2^{j/2}$,

$$\|p_{2^j}(.,y)-p_{2^j}(.,z)\|_2^2\le c\left(\frac{d(y,z)}{2^{j/2}}\right)^{2\beta}\frac{1}{|B\left(y,2^{j/2}\right)|}. \tag{3.5}$$

(4) There is a $c>0$ such that, for all $q,j\in\mathbb{N}$ and all $y,z\in\Gamma$ with $d(y,z)\le 2^{j/2}$,

$$\|p_{2^j}(.,y)-p_{2^j}(.,z)\|^2_{L^2(\{x:d(x,y)>2^{q/2}\})}\le c\left(\frac{d(y,z)}{2^{j/2}}\right)^{2\beta}\frac{e^{-c2^{q-j}}}{|B\left(y,2^{j/2}\right)|}. \tag{3.6}$$

3.3. **Estimates for kernels of compactly supported multipliers.** Let $A>0$, let $f\in C_0^A((0,\infty))$ and let $K(x,y)$ denote the kernel of the operator $f(L)$.

Let $M_A(f)$ be as in (3.1), let D be as in (1.4) and let us set

$$A_q(y)=B\left(y,2^{(q+1)/2}\right)\backslash B\left(y,2^{q/2}\right),\quad y\in\Gamma,\ q\in\mathbb{N}. \tag{3.7}$$

Lemma 2. *There is a constant $c>0$ such that for all $y\in\Gamma$, $q\in\mathbb{N}$*

$$\|K(.,y)\|_{L^1\left(B\left(y,2^{q/2}\right)\right)}\le c\|f\|_{C^A}2^{Dq/4}, \tag{3.8}$$

$$\|K(.,y)\|_{L^1(A_q(y))}\le c\|f\|_{C^A}2^{((D/2)-A)q/2}. \tag{3.9}$$

Proof. Since $K(x,y)=(f(L)p_0(.,y))(x)$, we have

$$\begin{aligned}\|K(.,y)\|_{L^1(B(y,2^{q/2}))}&\le|B(y,2^{q/2})|^{1/2}\|K(.,y)\|_2\\&\le|B(y,2^{q/2})|^{1/2}\|f\|_\infty\|p_0(.,y)\|_2\\&\le|B(y,2^{q/2})|^{1/2}\|f\|_\infty\frac{1}{|B(y,1)|^{1/2}}\le c\|f\|_{C^A}2^{qD/4}.\end{aligned}$$

In order to prove (3.9), making use of Lemma 1, we consider a polynomial Q satisfying

$$\deg Q\le 2^{q/2},\qquad \|f-Q\|_\infty\le cM_A(f)2^{-Aq/2}.$$

Then, making use of (3.2), we have

$$\begin{aligned}\|K(.,y)\|_{L^1(A_q(y))}&\le|A_q(y)|^{1/2}\|K(.,y)\|_{L^2(A_q(y))}\\&\le|A_q(y)|^{1/2}\|(f-Q)(L)p_0(.,y)\|_{L^2(A_q(y))}\\&\le|B(y,2^{(q+1)/2})|^{1/2}\|f-Q\|_\infty\|p_0(.,y)\|_2\\&\le c|B(y,2^{(q+1)/2})|^{1/2}M_A(f)2^{-Aq/2}\frac{1}{|B(y,1)|^{1/2}}\\&\le c\|f\|_{C^A}2^{((D/2)-A)q/2}.\end{aligned}$$

□

Corollary 1. *If $A>D/2$, then there is $c>0$, such that for all $y\in\Gamma$*

$$\|K(.,y)\|_1\le c.$$

Proof. Making use of (3.8) and (3.9), we have

$$\begin{aligned}\|K(.,y)\|_1 &= \|K(.,y)\|_{L^1(B(y,1))} + \sum_{q\geq 2} \|K(.,y)\|_{L^1(A_q(y))} \\ &\leq c\,\|f\|_\infty\, 2^{D/4} + c\,\|f\|_{C^A} \sum_{q\geq 2} 2^{((D/2)-A)q/2} \leq c.\end{aligned}$$

□

Corollary 2. *If $A > D/2$, then for all $y \in \Gamma$,*

$$\sum_x K(x,y)\mu(x) = 0. \tag{3.10}$$

Proof. Let us first observe that if $\psi = (I-P)\phi$, $\phi \in L^1$, then $\sum_x \psi(x)\mu(x) = 0$. Now, let $h(\lambda) = \lambda^{-1} f(\lambda)$ and let $S(x,y)$ be the kernel of the operator $h(L)$. Then, by Corollary 1, we have $S(.,y) \in L^1$ and $K(x,y) = ((I-P)S(.,y))(x)$, $x,y \in \Gamma$, which by the previous observation proves (3.10). □

Now let $f_j \in C_0^A((0,\infty))$, $j \in \mathbb{N}$, such that $\mathrm{supp}(f_0) \subseteq [1/2,2]$, $\mathrm{supp}(f_j) \subseteq [2^{-(j+1)}, 2^{-j}]$, $j \geq 1$, and let $K_j(x,y)$ denote the kernel of the operator $f_j(L)$.

We set

$$h_0(\lambda) = f_0(\lambda^2), \quad h_j(\lambda) = f_j(\lambda^2)(1-\lambda^2)^{-2^j}, \quad j \geq 1.$$

Note that if $\lambda \in \mathrm{supp}(f_j)$, $j \geq 1$, then $(1-\lambda^2)^{2^j} \to 1$, as $j \to \infty$. So, there is a $c > 0$ such that

$$\|h_j\|_\infty \leq c\,\|f_j\|_\infty, \quad j \in \mathbb{N}. \tag{3.11}$$

Also,

$$M_A(h_j) \leq c M_A(f_j)\, 2^{jA/2}. \tag{3.12}$$

Since $f_j(\lambda) = h_j(\sqrt{\lambda})(1-\lambda)^{2^j}$, we have

$$f_j(L) = h_j(\sqrt{L})P^{2^j}.$$

Hence

$$K_j(x,y) = \Big(h_j(\sqrt{L})p_{2^j}(.,y)\Big)(x), \qquad x,y \in \Gamma. \tag{3.13}$$

Lemma 3. *There is $c > 0$, such that for all $q, j \in \mathbb{N}$ and $y \in \Gamma$,*

(i)

$$\|K_j(.,y)\|_2 \leq c\frac{\|f_j\|_{C^A}}{|B(y,2^{j/2})|^{1/2}}. \tag{3.14}$$

(ii) *If $q \geq j$, then*

$$\|K_j(.,y)\|_{L^2(\{x:d(x,y)\geq 2^{q/2}\})} \leq c\,\|f_j\|_{C^A}\,\frac{2^{-A(q-j)/2}}{|B(y,2^{j/2})|^{1/2}}. \tag{3.15}$$

Proof. (i) By (3.13) and (3.3)

$$\|K_j(.,y)\|_2 \leq \|h_j\|_\infty \|p_{2^j}(.,y)\|_2 \leq c\frac{\|f_j\|_{C^A}}{|B(y,2^{j/2})|^{1/2}}.$$

(ii) By making use of Lemma 1, let us choose a polynomial Q which has only terms of even order and such that

$$\deg Q \leq 2^{(q-2)/2}, \qquad \|h_j - Q\|_\infty \leq c M_A(h_j) 2^{-(q-2)A/2}.$$

Then by (3.12),

$$\|h_j - Q\|_\infty \le cM_A\left(f_j\right)2^{(j-q)A/2} \le c\left\|f_j\right\|_{C^A} 2^{-(q-2)A/2},$$

and by making use of (3.2), we have

$$\begin{aligned}
&\|K_j(.,y)\|_{L^2(\{x:d(x,y)\ge 2^{q/2}\})}\\
&\quad\le \|h_j(\sqrt{L})\left(\mathbf{1}_{B\left(y,2^{(q-2)/2}\right)}p_{2^j}(.,y)\right)\|_{L^2(\{x:d(x,y)\ge 2^{q/2}\})}\\
&\qquad + \|h_j(\sqrt{L})\left(\mathbf{1}_{\{x:d(x,y)\ge 2^{(q-2)/2}\}}p_{2^j}(.,y)\right)\|_2\\
&\quad\le \|(h_j-Q)(\sqrt{L})\left(\mathbf{1}_{B\left(y,2^{(q-2)/2}\right)}p_{2^j}(.,y)\right)\|_{L^2(\{x:d(x,y)\ge 2^{q/2}\})}\\
&\qquad + c\|f_j\|_\infty\|\mathbf{1}_{\{x:d(x,y)\ge 2^{(q-2)/2}\}}p_{2^j}(.,y)\|_2\\
&\quad\le c\|h_j-Q\|_\infty\|p_{2^j}(.,y)\|_2 + c\|f_j\|_\infty\|\mathbf{1}_{\{x:d(x,y)\ge 2^{(q-2)/2}\}}p_{2^j}(.,y)\|_2\\
&\quad\le c\left\|f_j\right\|_{C^A}\frac{2^{A(j-q)/2}}{|B(y,2^{j/2})|^{1/2}} + c\frac{\|f_j\|_\infty}{|B(y,2^{j/2})|^{1/2}}e^{-c2^{(q-j)}}.
\end{aligned}$$

□

Lemma 4. *Let β be as in* (1.9)*. Then, there is $c>0$, such that for all $q,j\in\mathbb{N}$ and $y,z\in\Gamma$,*

(i) *If $d(y,z)\le 2^{j/2}$, then*

$$\|K_j(.,y)-K_j(.,z)\|_2 \le c\frac{\|f_j\|_{C^A}}{\left|B\left(y,2^{j/2}\right)\right|^{1/2}}\left(\frac{d\left(y,z\right)}{2^{j/2}}\right)^\beta. \tag{3.16}$$

(ii) *If $d(y,z)\le 2^{j/2}$ and $j\le q$, then*

$$\begin{aligned}
&\|K_j\left(.,y\right)-K_j\left(.,z\right)\|_{L^2\left(\left\{x:d(x,y)\ge 2^{q/2}\right\}\right)}\\
&\qquad\le c\left\|f_j\right\|_{C^A}\frac{2^{-A(q-j)/2}}{\left|B\left(y,2^{j/2}\right)\right|^{1/2}}\left(\frac{d\left(y,z\right)}{2^{j/2}}\right)^\beta.
\end{aligned}\tag{3.17}$$

Proof. To prove the lemma we observe that

$$K_j(.,y)-K_j(.,z) = h_j(\sqrt{L})\left(p_{2^j}(.,y)-p_{2^j}(.,z)\right),$$

and then we argue in the same way as in the previous lemma and we use (3.5) and (3.6) instead of (3.3) and (3.4) respectively. We omit the details. □

Corollary 3. *If $A>D/2$, then there is $c>0$, such that for all $j\in\mathbb{N}$ and $y,z\in\Gamma$, with $d(y,z)\le 2^{j/2}$*

$$\|K_j(.,y)-K_j(.,z)\|_{L^1(\{x:d(x,y)\ge 2d(y,z)\})} \le c\left\|f_j\right\|_{C^A}\left(\frac{d(y,z)}{2^{j/2}}\right)^\beta. \tag{3.18}$$

Proof. By (3.16) we have

(3.19)

$$\begin{aligned}
\|K_j(.,y)-K_j(.,z)\|_{L^1(\{x:d(x,y)\le 2^{j/2}\})} &\le |B(y,2^{j/2})|^{1/2}\|K_j(.,y)-K_j(.,z)\|_2\\
&\le c\left\|f_j\right\|_{C^A}\left(\frac{d(y,z)}{2^{j/2}}\right)^\beta.
\end{aligned}$$

Also, if $q \geq j$, then, by (3.17),

$$\begin{aligned}\|K_j(.,y)-K_j(.,z)\|_{L^1(A_q(y))} &\leq |A_q(y)|^{1/2}\|K_j(.,y)-K_j(.,z)\|_{L^2(A_q(y))}\\ &\leq c\,\|f_j\|_{C^A}\,\frac{|B(y,2^{(q+1)/2})|^{1/2}}{|B(y,2^{j/2})|^{1/2}}2^{-A(q-j)/2}\left(\frac{d(y,z)}{2^{j/2}}\right)^{\beta}\\ &\leq c\,\|f_j\|_{C^A}\,\frac{2^{(A-D/2)j/2}}{2^{(A-D/2)q/2}}\left(\frac{d(y,z)}{2^{j/2}}\right)^{\beta}.\end{aligned} \tag{3.20}$$

Now to prove (3.18), we write

$$\begin{aligned}&|K_j(.,y)-K_j(.,z)\|_{L^1(\{x:d(x,y)\geq 2d(y,z)\})}\\ &\qquad\leq\|K_j(.,y)-K_j(.,z)\|_{L^1(\{x:d(x,y)\leq 2^{j/2}\})}\\ &\qquad\quad+\sum_{q\geq j}\|K_j(.,y)-K_j(.,z)\|_{L^1(A_q(y))}\end{aligned}$$

and apply (3.19) and (3.20). □

Lemma 5. *If $A > D/2$, then there is $c > 0$, such that for all $q, j \in \mathbb{N}$ with $q \geq j$ and all $y \in \Gamma$,*

$$\|K_j(.,y)\|_{L^1(\{x:d(x,y)\geq 2^{q/2}\})} \leq c\,\|f_j\|_{C^A}\,\frac{2^{(A-D/2)j/2}}{2^{(A-D/2)q/2}}. \tag{3.21}$$

Proof. If $q \geq j$, then, by (3.15),

$$\begin{aligned}\|K_j\,(.,j)\|_{L^1(A_q(y))} &\leq |A_q\,(y)|^{1/2}\,\|K_j\,(.,j)\|_{L^2(A_q(y))}\\ &\leq c\,\|f_j\|_{C^A}\,\frac{\left|B\left(y,2^{(q+1)/2}\right)\right|^{1/2}}{\left|B\left(y,2^{j/2}\right)\right|^{1/2}}2^{-A(q-j)/2}\\ &\leq c\,\|f_j\|_{C^A}\,\frac{2^{(A-D/2)j/2}}{2^{(A-D/2)q/2}}.\end{aligned} \tag{3.22}$$

To prove (3.21) we write

$$\|K_j\,(.,y)\|_{L^1(\{x:d(x,y)\geq 2^{q/2}\})} \leq \sum_{k\geq q}\|K_j(.,y)\|_{L^1(A_k(y))}$$

and apply (3.22). □

Corollary 4. *If $A > D/2$, then there is $c > 0$, such that for all $j \in \mathbb{N}$ and all $y, z \in \Gamma$ with $d(y,z) \geq 2^{j/2}$,*

$$\begin{aligned}&\|K_j(.,y)-K_j(.,z)\|_{L^1(\{x:d(x,y)\geq 4d(y,z)\})}\\ &\qquad\leq c\,\|f_j\|_{C^A}\,\frac{2^{(A-D/2)j/2}}{d(y,z)^{(A-D/2)}}.\end{aligned} \tag{3.23}$$

Proof. To prove (3.23) we observe that

$$\begin{aligned}&\|K_j(.,y)-K_j(.,z)\|_{L^1(\{x:d(x,y)\geq 4d(y,z)\})}\\ &\qquad\leq\|K_j(.,y)\|_{L^1(\{x:d(x,y)\geq d(y,z)\})}+\|K_j(.,z)\|_{L^1(\{x:d(x,z)\geq d(y,z)\})}\end{aligned}$$

and then we apply (3.21). □

Let us now assume that the multiplier $m(\lambda)$ is not necessarily compactly supported and let us set

$$f_n(\lambda) = m(\lambda)\mathbf{1}_{[0,1/n]}(\lambda), \quad h_n(\lambda) = f_n(\lambda)(1-\lambda)^{-n}, \quad n \geq 2,$$

and denote by $K_n(x,y)$ the kernel of the operator $f_n(L)$. Then, we have

$$K_n(x,y) = \left(h_n(L)p_n(.,y)\right)(x), \quad x,y \in \Gamma,$$

and so the argument given in the proof of Lemma 3 (i) gives that there is $c > 0$ such that

$$\|K_n(.,y)\|_2 \leq \frac{c}{|B\left(y,\sqrt{n}\right)|^{1/2}}, \quad y \in \Gamma. \tag{3.24}$$

This implies that, if the function ϕ is finitely supported, then

$$\|f_n(L)\phi\|_2 \to 0 \quad (n \to \infty). \tag{3.25}$$

4. Proof of Theorem 1

Let us consider a function $0 \leq \theta \in C_0^\infty(\mathbb{R})$ and let us assume that $\theta(\lambda) = 1$ for $\lambda \in [-1/4, 1/4]$ and that $\theta(\lambda) = 0$ for $\lambda \notin [-1/2, 1/2]$. We write

$$m = \theta m + (1-\theta)\, m = M + m_0.$$

Let us also consider a function $\phi \in C_0^\infty(\mathbb{R})$ satisfying

$$\operatorname{supp}\phi \subset \left(\tfrac{1}{2}, 1\right), \quad \sum_{j\geq 1} \phi\left(2^j t\right) = 1, \ \ t \in (0,2].$$

Let

$$m_j(\lambda) = m(\lambda)\,\phi\left(2^j\lambda\right), \quad j \geq 1.$$

Then, we have $\operatorname{supp}(m_j) \subseteq \left[2^{-(j+1)}, 2^{-j}\right]$. Further, by (1.10), there is a $c > 0$, such that

$$\|m_j\|_{C^A} \leq c, \ j \in \mathbb{N}. \tag{4.1}$$

Let $K(x,y)$ and $K_j(x,y)$ denote the kernels of the operators $m(L)$ and $m_j(L)$ respectively. Then,

$$m(L) = M(L) + m_0(L) = \sum_{j\geq 0} m_j(L), \quad K(x,y) = \sum_{j\geq 0} K_j(x,y).$$

Note that by (3.25), the point $\lambda = 0$ of the spectrum of L can be ignored and we can assume that $m(0) = 0$.

It follows from (3.18) and (3.23) that the kernel $K(x,y)$ satisfies the Hörmander integral condition, i.e. there is $c > 0$ such that for all $y, z \in \Gamma$

$$\|K(x,y) - K(x,z)\|_{L^1(\{x: d(x,y) \geq 4d(y,z)\})} \leq c. \tag{4.2}$$

This shows that the operator $m(L)$ is bounded from H^1 to L^1. This means in particular that if a is an atom, then $m(L)a \in L^1$. Furthermore, we have the following.

Lemma 6. *If a is an atom, then*

$$\sum_{x\in\Gamma} (m(L)a)(x)\mu(x) = 0. \tag{4.3}$$

Proof. Let us fix an atom a. Then, it follows from Corollary 2 that for every $j \geq 0$, $\sum_{x\in\Gamma}(m_j(L))a(x)\mu(x) = 0$. So, to prove the lemma, it is enough to show that the series $\sum_{j\geq 0} m_j(L)a$ converges in L^1.

Since a is an atom, we have that $\operatorname{supp}(a) \subseteq B(z,r)$, for some $r > 0$ and $z \in \Gamma$.

Now, on the one hand, it follows from (3.24) and (3.25) that $\sum_{j\geq 0} \mathbf{1}_{B(z,4r)} m_j(L)a$ converges in L^1.

On the other hand, since $\sum_{x\in\Gamma} a(x)\mu(x) = 0$, we have

$$(m_j(L))\, a(x) = \sum_{x\in\Gamma} \left(K_j(x,y) - K_j(x,z)\right) a(y)\mu(y).$$

This and Corollaries 3 and 4 imply that the series $\sum_{j\geq 0} \mathbf{1}_{\{x:d(x,z)\geq 4r\}} m_j(L)a$ converges in L^1. □

4.1. **Proof of Theorem 1 for $p \in (p_0, 1)$.** Let $p \in (p_0, 1)$ and let $\alpha = (1/p) - 1$. Since the dual of H^p is the Lipschitz space $\mathcal{L}_\alpha$ (cf. Section 2), in order to prove that the operator $m(L)$ is bounded on H^p, it is enough to show that there is a constant $c > 0$ such that for every atom $a \in H^p$ and every function ψ with finite support

$$|\langle m(L)\, a, \psi\rangle| \leq c \|\psi\|_{\mathcal{L}_\alpha}. \tag{4.4}$$

So, let us fix an atom $a \in H^p$ and let us assume that $\operatorname{supp}(a) \subseteq B(y,r)$, $r > 0$, $y \in \Gamma$.

Let $N \in \mathbb{N}$, such that $2^{N/2} < r \leq 2^{(N+1)/2}$.

By (4.3), we have

$$\langle m(L)\, a, \psi\rangle = \langle m(L)\, a, (\psi - \psi(y))\rangle. \tag{4.5}$$

We set

$$\psi_1 = (\psi - \psi(y))\, \mathbf{1}_{B(y,4r)}, \qquad \psi_2 = (\psi - \psi(y))\, \mathbf{1}_{\{x:d(x,y)>4r\}}.$$

Then, we have

$$\langle m(L)\, a, \psi\rangle = \langle m(L)\, a, \psi_1\rangle + \langle m(L)\, a, \psi_2\rangle.$$

4.1.1. *Estimation of the term $\langle m(L)\, a, \psi_1\rangle$.* Making use of (2.1) and (2.4), we have

$$\begin{aligned} |\langle m(L)a, \psi_1\rangle| &\leq \|m(L)\, a\|_2 \|\psi_1\|_2 \\ &\leq c\|a\|_2 \left\|(\psi - \psi(y))\, \mathbf{1}_{B(y,2^{(N+5)/2})}\right\|_2 \\ &\leq c|B(y,r)|^{(1/2)-(1/p)} |B(y,2^{(N+5)/2})|^{1/2} \|\psi\|_{\mathcal{L}_\alpha} |B(y,2^{(N+5)/2})|^\alpha \\ &\leq c\|\psi\|_{\mathcal{L}_\alpha}. \end{aligned} \tag{4.6}$$

4.1.2. *Estimation of the term $\langle m(L)\, a, \psi_2\rangle$.* We have

$$|\langle m(L)a, \psi_2\rangle| \leq \sum_{j\geq 0} |\langle m_j(L)a, \psi_2\rangle|.$$

So, we must estimate the terms $\langle m_j(L)a, \psi_2\rangle$. We shall distinguish two cases.

Case I : $N + 4 \leq j$. Then, we have

$$\begin{aligned} |\langle m_j(L)a, \psi_2\rangle| =& \|(m_j(L)a)\psi_2\|_{L^1(\{x:2^{(N+4)/2}\leq d(x,y)<2^{j/2}\})} \\ &+ \sum_{q\geq j} \|(m_j(L)a)\psi_2\|_{L^1(A_q(y))}. \end{aligned} \tag{4.7}$$

Let us recall again that

$$(m_j(L)\,a)(x) = \sum_{z\in B(y,r)} (K_j(x,z) - K_j(x,y))\,a(z)\,\mu(z).$$

So, by (2.1), (2.4), (3.16) and (4.1)

$$\begin{aligned}
&\|(m_j(L)a)\psi_2\|_{L^1(\{x:2^{(N+4)/2}\le d(x,y)<2^{j/2}\})}\\
&\quad\le \|m_j(L)a\|_{L^2(\{x:2^{(N+4)/2}\le d(x,y)<2^{j/2}\})}\|\psi_2\|_{L^2(\{x:2^{(N+4)/2}\le d(x,y)<2^{j/2}\})}\\
&\quad\le c\|a\|_1\big(\sup_{d(z,y)\le r}\|K_j(.,z)-K_j(.,y)\|_2\big)\|\psi-\psi(y)\|_{L^2(B(y,2^{j/2}))}\\
&\quad\le c|B(y,r)|^{1-(1/p)}\frac{\|m_j\|_{C^A}}{|B(y,2^{j/2})|^{1/2}}\left(\frac{d(y,z)}{2^{j/2}}\right)^{\beta}\|\psi\|_{\mathcal{L}_\alpha}|B(y,2^{j/2})|^{(1/p)-(1/2)}\\
&\quad\le c\|\psi\|_{\mathcal{L}_\alpha}\left(\frac{2^{N/2}}{2^{j/2}}\right)^{\beta}\frac{|B(y,2^{j/2})|^{(1/p)-1}}{|B(y,2^{N/2})|^{(1/p)-1}}\\
&\quad\le c\|\psi\|_{\mathcal{L}_\alpha}2^{\beta(N-j)/2}2^{(j-N)((1/p)-1)D/2}\\
&\quad= c\|\psi\|_{\mathcal{L}_\alpha}2^{\frac{D+\beta}{2p}\left(p-\frac{D}{D+\beta}\right)N}2^{-\frac{D+\beta}{2p}\left(p-\frac{D}{D+\beta}\right)j}.
\end{aligned}\tag{4.8}$$

Also, if $q\ge j$, then by (2.1), (2.4), (3.17) and (4.1)

$$\begin{aligned}
&\|(m_j(L)a)\psi_2\|_{L^1(A_q(y))}\\
&\quad\le \|m_j(L)a\|_{L^2(A_q(y))}\|\psi_2\|_{L^2(A_q(y))}\\
&\quad\le c\|a\|_1\big(\sup_{d(z,y)\le r}\|K_j(.,z)-K_j(.,y)\|_{L^2(A_q(y))}\big)\|\psi-\psi(y)\|_{L^2(B(y,2^{(q+1)/2}))}\\
&\quad\le c|B(y,r)|^{1-\frac{1}{p}}\frac{\|m_j\|_{C^A}\,2^{-A(q-j)/2}}{|B(y,2^{j/2})|^{1/2}}\left(\frac{d(y,z)}{2^{j/2}}\right)^{\beta}\|\psi\|_{\mathcal{L}_\alpha}|B(y,2^{q/2})|^{\frac{1}{p}-\frac{1}{2}}\\
&\quad= c\|\psi\|_{\mathcal{L}_\alpha}2^{-A(q-j)/2}\frac{|B(y,2^{q/2})|^{1/2}}{|B(y,2^{j/2})|^{1/2}}\left(\frac{d(y,z)}{2^{j/2}}\right)^{\beta}\frac{|B(y,2^{q/2})|^{(1/p)-1}}{|B(y,r)|^{(1/p)-1}}\\
&\quad\le c\|\psi\|_{\mathcal{L}_\alpha}2^{-A(q-j)/2}2^{(q-j)D/4}2^{(N-j)\beta/2}2^{(q-N)((1/p)-1)D/2}\\
&\quad= c\,\|\psi\|_{\mathcal{L}_\alpha}\,2^{-\frac{1}{2}\left(A-D\left(\frac{1}{p}-\frac{1}{2}\right)\right)q}2^{\frac{1}{2}\left(A-\frac{D}{2}-\beta\right)j}2^{\frac{1}{2}\left(D+\beta-\frac{D}{p}\right)N}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{q\ge j}\|(m_j(L)a)\psi_2\|_{L^1(A_q(y))}\\
&\qquad\le c\|\psi\|_{\mathcal{L}_\alpha}2^{\frac{1}{2}\left(A-\frac{D}{2}-\beta\right)j}2^{\frac{1}{2}\left(D+\beta-\frac{D}{p}\right)N}\sum_{q\ge j}2^{-\frac{1}{2}\left(A-D\left(\frac{1}{p}-\frac{1}{2}\right)\right)q}\\
&\qquad\le c\,\|\psi\|_{\mathcal{L}_\alpha}\,2^{\frac{1}{2}\left(A-\frac{D}{2}-\beta\right)j}2^{\frac{1}{2}\left(D+\beta-\frac{D}{p}\right)N}2^{-\frac{1}{2}\left(A-D\left(\frac{1}{p}-\frac{1}{2}\right)\right)j}\\
&\qquad= c\|\psi\|_{\mathcal{L}_\alpha}2^{\frac{1}{2}\left(D+\beta-\frac{D}{p}\right)N}2^{-\frac{1}{2}\left(D+\beta-\frac{D}{p}\right)j}\\
&\qquad= c\|\psi\|_{\mathcal{L}_\alpha}2^{\frac{D+\beta}{2p}\left(p-\frac{D}{D+\beta}\right)N}2^{-\frac{D+\beta}{2p}\left(p-\frac{D}{D+\beta}\right)j}.
\end{aligned}\tag{4.9}$$

It follows from (4.7), (4.8) and (4.9) that, if $N+4\le j$, then

$$|\langle m_j(L)\,a,\psi_2\rangle| \le c\,\|\psi\|_{\mathcal{L}_\alpha}\,2^{\frac{D+\beta}{2p}\left(p-\frac{D}{D+\beta}\right)N}2^{-\frac{D+\beta}{2p}\left(p-\frac{D}{D+\beta}\right)j}$$

and hence

$$\sum_{j\geq N+4} |\langle m_j(L)\,a, \psi_2\rangle| \leq c\,\|\psi\|_{\mathcal{L}_\alpha}. \tag{4.10}$$

Case II : $N+4 \geq j$. Then, we have

$$|\langle m_j(L)\,a, \psi_2\rangle| = \sum_{q\geq N+4} \|(m_j(L)\,a)\,\psi_2\|_{L^1(A_q(y))}.$$

If $q \geq N+4 \geq j$, then we have

$$\begin{aligned} \|(m_j(L)a)\psi_2\|_{L^1(A_q(y))} &\leq \|m_j(L)a\|_{L^2(A_q(y))}\|\psi_2\|_{L^2(A_q(y))} \\ &\leq \|a\|_1\big(\sup_{d(z,y)\leq r} \|K_j(.,z)\|_{L^2(A_q(y))}\big)\|\psi-\psi(y)\|_{L^2(B(y,2^{(q+1)/2}))}. \end{aligned} \tag{4.11}$$

If $d(z,y)\leq r$, then making use of (3.15) and (4.1), we have

$$\begin{aligned} \|K_j(.,z)\|_{L^2(A_q(y))} &\leq \|K_j(.,z)\|_{L^2(\{x:d(x,z)\geq 2^{(q-4)/2}\})} \\ &\leq c\,\|m_j\|_{C^A}\,\frac{2^{-A(q-j)/2}}{|B(z,2^{j/2})|^{1/2}} \\ &\leq c2^{-A(q-j)/2}\frac{|B(z,2^{(q+4)/2})|^{1/2}}{|B(z,2^{j/2})|^{1/2}}\frac{1}{|B(z,2^{(q+4)/2})|^{1/2}} \\ &\leq c2^{-A(q-j)/2}2^{(q-j)D/4}\frac{1}{|B(y,2^{q/2})|^{1/2}}. \end{aligned} \tag{4.12}$$

It follows from (4.11) and (4.12), by making use of (2.1) and (2.4) that

$$\begin{aligned} &\|(m_j(L)a)\psi_2\|_{L^1(A_q(y))} \leq c|B(y,r)|^{1-(1/p)} \\ &\quad\times 2^{-A(q-j)/2}2^{(q-j)D/4}\frac{1}{|B(y,2^{q/2})|^{1/2}}\|\psi\|_{\mathcal{L}_\alpha}|B(y,2^{q/2})|^{(1/p)-(1/2)} \\ &\quad\leq c\|\psi\|_{\mathcal{L}_\alpha}2^{-A(q-j)/2}2^{(q-j)D/4}\frac{|B(y,2^{q/2})|^{(1/p)-1}}{|B(y,r)|^{(1/p)-1}} \\ &\quad\leq c\|\psi\|_{\mathcal{L}_\alpha}2^{-A(q-j)/2}2^{(q-j)D/4}2^{(q-N)((1/p)-1)D/2} \\ &\quad= c\,\|\psi\|_{\mathcal{L}_\alpha}\,2^{-\frac{1}{2}\left(A-D\left(\frac{1}{p}-\frac{1}{2}\right)\right)q}2^{\frac{1}{2}\left(A-\frac{D}{2}\right)j}2^{\frac{1}{2}\left(D-\frac{D}{p}\right)N}. \end{aligned}$$

Hence

$$\begin{aligned} |\langle m_j(L)a,\psi_2\rangle| &\leq c\|\psi\|_{\mathcal{L}_\alpha}2^{\frac{1}{2}\left(A-\frac{D}{2}\right)j}2^{\frac{1}{2}\left(D-\frac{D}{p}\right)N}\sum_{q\geq N+4}2^{-\frac{1}{2}\left(A-D\left(\frac{1}{p}-\frac{1}{2}\right)\right)q} \\ &\leq c\,\|\psi\|_{\mathcal{L}_\alpha}\,2^{\frac{1}{2}\left(A-\frac{D}{2}\right)j}2^{\frac{1}{2}\left(D-\frac{D}{p}\right)N}2^{-\frac{1}{2}\left(A-D\left(\frac{1}{p}-\frac{1}{2}\right)\right)N} \\ &\leq c\,\|\psi\|_{\mathcal{L}_\alpha}\,2^{\frac{1}{2}\left(A-\frac{D}{2}\right)j}2^{\frac{1}{2}\left(\frac{D}{2}-A\right)N}. \end{aligned}$$

Since, by assumption

$$A > D\left(\frac{1}{p}-\frac{1}{2}\right) > \frac{D}{2},$$

we have

$$\sum_{0\leq j\leq N+4} |\langle m_j(L)\,a, \psi_2\rangle| \leq c\,\|\psi\|_{\mathcal{L}_\alpha}. \tag{4.13}$$

The estimates (4.10), (4.13) and (4.6) prove (4.4) and this ends the proof of Theorem 1, in the case $p \in (p_0, 1)$.

4.2. **Proof of Theorem 1 for $p = 1$.** Since the dual of H^1 is the space BMO (cf. Section 2), in order to prove that the operator $m(L)$ is bounded on H^1, it is enough to prove that there is a constant $c > 0$ such that for every atom $a \in H^1$ and every function ψ with finite support

$$|\langle m(L)a, \psi\rangle| \leq c\|\psi\|_{BMO}.$$

This is proved by arguing in the same way as in the case $p \in (p_0, 1)$. We just have to replace the term $\psi(y)$ in (4.5) by the mean value ψ_B of the function ψ on the ball $B(y, r)$ and make use of (2.3) instead of (2.4). We omit the details.

References

[1] G. Alexopoulos, Spectral multipliers on discrete groups, *Bull. London Math. Soc.*, **33**, (2001), 417-424. MR1832553 (2002d:22010)

[2] G. Alexopoulos, L^p bounds for spectral multipliers from Gaussian estimates of the transition kernels, *J. Math. Soc. Japan*, **56**, (2004), 833-852. MR2071675 (2005i:43021)

[3] J.-Ph. Anker, L^p Fourier multipliers on Riemannian symmetric spaces of noncompact type, *Ann. of Math.*, **132**, (1990), 597-628. MR1078270 (92e:43006)

[4] A.P. Calderón, A. Torchinsky, Parabolic maximal functions associated with a distribution II, *Adv. Math.*, **24**, (1977), 101-171. MR0450888 (56:9180)

[5] T.K. Carne, A transmutation formula for Markov chains, *Bull. Sci. Math.,* **106**, (1985), 399-405. MR837740 (87m:60142)

[6] M. Christ, L^p bounds for spectral multipliers on nilpotent groups, *Trans. Amer. Math. Soc.,* **328**, (1991), 73-81. MR1104196 (92k:42017)

[7] R.R. Coifman, G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.*, **83**, (1977), 569-645. MR0447954 (56:6264)

[8] T. Coulhon, Random walks and geometry on infinite graphs, *Lectures notes on analysis on metric spaces, Trento, C.I.R.M., 1999*, Luigi Ambrosio, Francesco Serra Cassano, ed., Scuola Normale Superiore di Pisa, (2000), 5-23. MR2023121 (2005f:60102)

[9] T. Delmotte, Parabolic Harnack inequality and estimates of Markov chains on graphs, *Rev. Mat. Iberoamericana*, **15**, (1999), 181-232. MR1681641 (2000b:35103)

[10] L. De-Michele, G. Mauceri, H^p multipliers on stratified groups, *Ann. Mat. Pura. Appl.*, **148**, (1987), 353-366. MR932770 (89e:43009)

[11] W. Hebisch, Multiplier theorem on generalized Heisenberg groups, *Coll. Math.*, **65**, (1993), 231-239. MR1240169 (94m:43013)

[12] W. Hebisch, L. Saloff-Coste, Gaussian estimates for Markov chains and random walks on groups, *Ann. Probab.*, **21**, (1993), 673-709. MR1217561 (94m:60144)

[13] L. Hörmander, Estimates for translation invariant operators in L^p spaces, *Acta Math.*, **104**, (1960), 93-139. MR0121655 (22:12389)

[14] J. Kyrezi, M. Marias, H^1-bounds for spectral multipliers on graphs, *Proc. Amer. Math. Soc.*, **132**, (2004), 1311-1320. MR2053335 (2005h:42026)

[15] C.-C. Lin, Hörmander's H^p multiplier theorem for the Heisenberg group, *J. London Math. Soc.*, **67**, (2003), 686-700. MR1967700 (2003k:42029)

[16] G.G. Lorentz, *Approximation of functions*, Holt, Rinehart and Wilson, New York, 1966. MR0213785 (35:4642)

[17] M. Marias, L^p-boundedness of oscillating spectral multipliers on Riemannian manifolds, *Ann. Math. Blaise Pascal*, **10**, (2003), 133-160. MR1990014 (2005f:58020)

[18] S.G. Mikhlin, *Multidimensional singular integral and integral equations*, Pergamon Press, Oxford, 1965. MR0185399 (32:2866)

[19] D. Müller, E.M. Stein, On spectral multipliers for Heisenberg and related groups, *J. Math. Pures Appl.*, **73**, (1994), 413-440. MR1290494 (96m:43004)

[20] I.P. Natanson, *Constructive function theory, vol. 1, Uniform approximation*, Frederick Ungar, New York, 1964. MR0196340 (33:4529a)
[21] K. Yoshida, *Functional analysis*, Springer-Verlag, Berlin, 1978. MR0500055 (58:17765)

Department of Applied Mathematics, University of Crete, Iraklion 714.09, Crete, Greece
E-mail address: kyrezi@tem.uoc.gr

Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki 54.124, Greece
E-mail address: marias@math.auth.gr