**Abstract.** We study the boundedness on the Hardy spaces $H^p$ of spectral multiplier operators associated with the discrete Laplacian on a weighted graph. We assume that the graph satisfies the doubling volume property and a Poincaré inequality. We prove that there is $p_0 \in (0, 1)$, depending on the geometry of the graph, such that if the multiplier satisfies a condition similar to the one we have in the classical Hörmander multiplier theorem, then the corresponding operator is bounded on $H^p$, $p \in (p_0, 1]$.

**1. Introduction**

Let $m(\xi)$ be a bounded measurable function in $\mathbb{R}^n$ and let $T_m$ be the operator defined by $\hat{T_m}f(\xi) = m(\xi) \hat{f}(\xi)$ (where $\hat{f}$ denotes the Fourier transform of the function $f$). Then, by the Plancherel formula, $T_m$ is an operator bounded on $L^2$. The Hörmander multiplier theorem (cf. [13, 18]) asserts that if

$$\sup_{\xi \in \mathbb{R}^n} |\xi|^{|\alpha|} |\partial^\alpha m(\xi)| < +\infty,$$

for any multi-index $\alpha$ with $|\alpha| \leq [n/2] + 1$, then $T_m$ extends to an operator bounded on $L^p$, $1 < p < \infty$, and from $L^1$ to weak-$L^1$.

This result was extended to the Hardy spaces $H^p$, $0 < p \leq 1$, by Calderón and Torchinsky [4], who proved that if for example $m(\xi)$ satisfies (1.1) for any multi-index $\alpha$ with $|\alpha| \leq \left[ n \left( \frac{1}{p} - \frac{1}{2} \right) \right] + 1$, then $T_m$ is bounded on $H^p$.

The theorem of Hörmander has many generalizations to abstract contexts (see for example [1, 2, 3, 6, 10, 11, 14, 15, 17, 19]).

The goal of this article is to obtain a generalization of the result of Calderón and Torchinsky in the context of weighted graphs. More precisely, let $\Gamma$ be a countable infinite set and let $\sigma(x, y)$ be a weight on $\Gamma$ satisfying $\sigma(x, y) = \sigma(y, x) \geq 0$ and $\sigma(x, x) > 0$, $x, y \in \Gamma$. This weight induces a graph structure on $\Gamma$. We call the vertices $x$ and $y$ neighbors and we write $x \sim y$ when $\sigma(x, y) \neq 0$. We assume that $\Gamma$ is connected.
We consider the discrete measure \( \mu \) defined by
\[
\mu(\{x\}) = \sum_{y \sim x} \sigma(x, y), \quad x \in \Gamma.
\]
For simplicity, we set \( \mu(x) = \mu(\{x\}) \), \( |A| = \mu(A) \) and \( L^p = L^p(\Gamma, \mu) \).

We consider the transition kernel
\[
p(x, y) = \frac{\sigma(x, y)}{\mu(x)\mu(y)}, \quad x, y \in \Gamma.
\]
This is a symmetric Markov kernel since \( p(y, x) = p(x, y) \geq 0 \) and
\[
\sum_y p(x, y) \mu(y) = 1, \quad x, y \in \Gamma.
\]

We consider the operator
\[(1.2) \quad Pf(x) = \sum_y p(x, y) f(y) \mu(y).\]

The operator \( L = I - P \), called discrete Laplacian, is symmetric and we have
\[
\|Lf\|_2 \leq \|f\|_2 + \|Pf\|_2 \leq 2\|f\|_2.
\]
So, it admits the spectral resolution \( L = \int_0^2 \lambda dE_\lambda \) (cf. [21]).

Let \( m(\lambda) \) be a bounded Borel measurable function. Then, by the spectral theorem, we can define the operator
\[
m(L) = \int_0^2 m(\lambda) dE_\lambda.
\]
The operator \( m(L) \) is bounded on \( L^2 \). The function \( m(\lambda) \) is called a multiplier.

In this article, we study the boundedness of the operator \( m(L) \) on the spaces \( H^p \). Before stating our result we shall present the geometry of the graph and state the Gaussian estimates satisfied by the iterates \( p_n(x, y) \) of the kernel \( p(x, y) \).

1.1. The geometry of the graph. We assume that there is a positive integer \( N \) such that every vertex has at most \( N \) neighbors.

A path of length \( n \) joining the vertices \( x \) and \( y \) is a sequence of vertices \( x = x_1, x_2, \ldots, x_n = y \) such that \( x_i \sim x_{i-1}, i = 1, \ldots, n - 1 \). The distance \( d(x, y) \) is defined as the infimum of the lengths of the paths joining \( x \) and \( y \).

Let \( B(x, r) = \{y \in \Gamma : d(x, y) \leq r\} \) denote the ball of center \( x \) and radius \( r \). We assume that \( \Gamma \) satisfies the doubling volume property, i.e. there is \( c > 0 \), such that
\[(1.3) \quad |B(x, 2r)| \leq c|B(x, r)|, \quad x \in \Gamma, \quad r > 0.
\]
Note that this implies that there are \( c, D > 0 \) such that
\[(1.4) \quad |B(x, r)| \leq c \left( \frac{r}{s} \right)^D |B(x, s)|, \quad \text{for } r > s > 0.
\]
The assumption (1.3) makes the graph \( \Gamma \) a space of homogeneous type in the sense of Coifman and Weiss [7]. Thus, we can define the atomic Hardy spaces \( H^p \), \( p > 0 \), and the space of functions of bounded mean oscillation \( BMO \), in the standard way (cf. Section 2 below for precise definitions).
Finally we assume that $\Gamma$ satisfies a Poincaré inequality, i.e. there is a constant $c > 0$ such that for every function $f$ and every $x_0 \in \Gamma$ and $r > 0$
\begin{equation}
  \sum_{x \in B(x_0, r)} |f(x) - f_B|^2 \mu(x) \leq cr^2 \sum_{x, y \in B(x_0, 2r)} |f(y) - f(x)|^2 \sigma(x, y),
\end{equation}
where we have set
\[
f_B = \frac{1}{|B(x_0, r)|} \sum_{x \in B(x_0, r)} f(x) \mu(x).
\]

1.2. **Kernel estimates.** Let us set
\[
p_0(x, y) = \frac{\delta_x(y)}{\mu(y)}, \quad p_1(x, y) = p(x, y),
\]
where $\delta_x$ is the Dirac mass at $x$ and denote by $p_n(x, y), \; n \in \mathbb{N}$, the $n$th convolution power of $p(x, y)$, defined by
\begin{equation}
  p_n(x, y) = \sum_z p(x, z)p_{n-1}(z, y)\mu(z).
\end{equation}
Then we have
\begin{equation}
  P^n f(x) = \sum_y p_n(x, y)f(y) \mu(y), \quad x \in \Gamma, n \in \mathbb{N}.
\end{equation}

The assumptions (1.3) and (1.4) imply the following estimates (cf. [9] [8]):

1. There are $C, c > 0$ such that
\begin{equation}
  p_n(x, y) \leq \frac{C}{|B(x, \sqrt{n})|} e^{-c d(x, y)^2/n}, \quad x, y \in \Gamma, n \in \mathbb{N}.
\end{equation}

2. There is $\beta \in (0, 1)$ and $C, c > 0$ such that
\begin{equation}
  |p_n(y, x) - p_n(z, x)| \leq \frac{C}{|B(x, \sqrt{n})|} \left[ \frac{d(y, z)}{\sqrt{n}} \right]^\beta e^{-c d(x, y)^2/n},
\end{equation}
for all $n \in \mathbb{N}$ and $x, y, z \in \Gamma$ satisfying $d(y, z) \leq \sqrt{n}$.

Note that in the case of left invariant random walks on discrete groups of polynomial volume growth, the above estimates are satisfied with $\beta = 1$ (cf. [12]).

1.3. **Statement of the result.** Let the constants $\beta$ and $D$ be as in (1.9) and (1.4) and let
\[
p_0 = \frac{D}{D + \beta}.
\]
Let also $p \in (p_0, 1], \; \varepsilon > 0$ and let us set
\[
A = D \left( \frac{1}{p} - \frac{1}{2} \right) + \varepsilon.
\]
Let us fix a function $\varphi \in C^\infty(0, \infty)$, not identically zero, and let us consider the following condition on the multiplier $m(\lambda)$:
\begin{equation}
  \sup_{0 < t < \infty} \|\varphi(t) m(t)\|_{C^A(\mathbb{R})} < \infty.
\end{equation}

In this article we shall prove the following result.

**Theorem 1.** Let $p$ and $A$ be as above and let us assume that (1.10) is satisfied. Then, the operator $m(L)$ is bounded on $H^p$. 


Note that by interpolation and duality, the above result implies that $m(L)$ is bounded on $L^p$, $1 < p < \infty$, and on $BMO$.

As we mentioned earlier, when $\Gamma$ is a discrete group of polynomial volume growth and the kernel $p(x, y)$ is left invariant, then we have $\beta = 1$ and thus the above result holds with $p_0 = D/(D + 1)$. Also, in the case of $\mathbb{R}^n$, when we define the atomic Hardy spaces $H^p$, for $p \in (0, n/(n + 1))$, a certain number of higher order moments of the atoms should vanish (while for $p \in (n/(n + 1), 1]$, it is enough to have zero mean value). So, in order to study the operator $m(L)$ on $H^p$ for $p \in (0, D/(D + 1)]$, we should impose some structure on the graph $\Gamma$.

Throughout this article the different constants will always be denoted by the same letter $c$. When their dependence or independence is significant, it will be clearly stated.

2. HARDY SPACES

In this section we shall recall the definition and the basic properties of the atomic Hardy spaces $H^p$, introduced in the context of spaces of homogeneous type by Coifman and Weiss [7].

Let $p \in (p_0, 1]$. We say that a function $a$ is a $(p, \infty)$ atom, or simply a $p$-atom, if there is a ball $B(y, r)$ such that

$$\text{supp}(a) \subseteq B(y, r), \quad \|a\|_\infty \leq |B(y, r)|^{-1/p}, \quad \sum_x a(x) \mu(x) = 0.$$ 

Note that these imply that

$$\|a\|_q \leq |B(y, r)|^{(1/q) - (1/p)}, \quad q \geq 1.$$ 

In particular, we have

$$\|a\|_1 \leq |B(y, r)|^{1 - (1/p)}, \quad \|a\|_2 \leq |B(y, r)|^{(1/2) - (1/p)}.$$ 

The Hardy space $H^1$ is defined as follows. We say that a function $f \in H^1$, if $f \in L^1$ and there is a sequence $(\lambda_n) \in l^1$ and a sequence of 1-atoms $(a_n)$ such that $f = \sum_n \lambda_n a_n$. We set

$$\|f\|_{H^1} = \inf \left\{ \sum_n |\lambda_n| : f = \sum_n \lambda_n a_n \right\}.$$ 

With this norm $H^1$ is a Banach space.

The space $BMO$ is the dual of $H^1$ and it is defined as follows. Let us first denote by $f_B$ the mean value of the function $f$ on the ball $B$. We say that a function $f \in BMO$ if there is a constant $c > 0$, such that for every ball $B$

$$\|f - f_B\|_{L^2(B)}^2 < c|B|.$$ 

The norm $\|f\|_{BMO}$ is defined as the smallest of those constants $c$.

Note that it follows from (2.2) that there is $c > 0$ such that for all $f \in BMO$, all $k \in \mathbb{N}$ and every ball $B = B(y, r)$

$$\|f - f_B\|_{L^2(B(y, 2^k r))}^2 < c k^2 |B(y, 2^k r)| \|f\|_{BMO}^2.$$ 

To define the Hardy space $H^p$ for $p \in (p_0, 1)$, we need the notion of the Lipschitz space $L_\alpha$, $\alpha > 0$. We say that a function $f \in L_\alpha$ if there is a constant $c > 0$, such that for every ball $B$ and $x, y \in B$, we have

$$|f(x) - f(y)| \leq c|B|^{\alpha}.$$
The norm \( \|f\|_{\mathcal{L}_p} \) is defined as the smallest of those constants \( c \). With this norm, \( \mathcal{L}_\alpha \) is a Banach space.

Now, let \( p \in (p_0, 1) \) and let \( \alpha = (1/p) - 1 \). Then we define the Hardy space \( H^p \) as the space of those functionals \( f \in \mathcal{L}_\alpha \) which can be written as \( f = \sum_n \lambda_na_n \), where the \( a_n \)'s are \( p \)-atoms and \( \lambda_n \) \( \in \mathcal{O}^\ell \). As in the case of \( H^1 \), we set

\[
\|f\|_{H^p} = \inf \left\{ \left( \sum_n |\lambda_n|^p \right)^{1/p} : f = \sum_n \lambda_na_n \right\}.
\]

If \( p \in (p_0, 1) \), then \( \|f\|_{H^p} \) is not a norm, but \( d(f, g) = \|f-g\|_{H^p} \) is a distance.

Finally, we note that the dual of \( H^p \), \( p \in (p_0, 1) \), is the space \( \mathcal{L}_\alpha \).

3. Compactly supported multipliers

3.1. An approximation lemma. We shall need the following lemma.

**Lemma 1** (cf. [20, 16, 1]). Let \( n \in \mathbb{N} \), \( 0 < \alpha \leq 1 \) and \( A = n + \alpha \). Given a function \( f \in C^a(\mathbb{R}) \) with \( \text{supp}(f) \subset (-4, 4) \), let us set

\[
M_A(f) = \sup \left\{ \frac{|f^{(n)}(x + t) - f^{(n)}(x)|}{t^\alpha} , \ t > 0, \ x \in \mathbb{R} \right\}.
\]

Then, there is \( c > 0 \) such that for all \( k \in \mathbb{N} \), there is a polynomial \( Q \) with \( \deg(Q) \leq k \) such that

\[
\sup \{|f - Q|; \ -4, 4\} \leq c\frac{M_A(f)}{k^\alpha}.
\]

Note that if \( f \in C^A(\mathbb{R}) \), then \( \|f\|_{C^A} = \sum_{k=0}^n \|f^{(k)}\|_{\infty} + M_A(f) \), and hence we can replace \( M_A(f) \) by \( \|f\|_{C^A} \) in (3.1) above.

Let us now assume that the function \( f \) is even. Then we can assume that \( Q \) is also even and hence it has only terms of even order. If \( \deg(Q) \leq 2k \), \( k \in \mathbb{N} \), then \( Q(\sqrt{L}) \) is a polynomial in \( \lambda \) with degree \( k \). Hence the operator \( Q(\sqrt{L}) \) can be written as \( Q(\sqrt{L}) = a_0I + a_1P + \cdots + a_kP_k \). It follows that if \( \text{supp}(\phi) \subseteq B(x, m) \), then

\[
\text{supp}(Q(\sqrt{L})\phi) \subseteq B(x, m + k).
\]

We shall exploit this observation in order to obtain off diagonal estimates for the kernel of the operator \( m(L) \). This observation has been exploited first by Carne [5] and later in [1].

3.2. \( L^2 \)-estimates for \( p_n(x, y) \). By integrating the estimates (1.8) and (1.9) and by using the doubling volume property (1.4), we have the following estimates:

(1) There is a \( c > 0 \) such that, for all \( j \in \mathbb{N} \) and all \( y \in \Gamma \),

\[
\|p_{2^j}(:, y)\|_{L^2}^2 \leq \frac{c}{|B(y, 2j/2)|}.
\]

(2) There is a \( c > 0 \) such that, for all \( j, q \in \mathbb{N} \) and all \( y \in \Gamma \),

\[
\|p_{2^j}(:, y)\|_{L^2(\{x:d(x, y) > 2^{q/2}\})}^2 \leq \frac{ce^{-c2^{q-j}}}{|B(y, 2j/2)|}.
\]
Lemma 2. There is a constant $c > 0$ such that for all $j \in \mathbb{N}$ and all $y, z \in \Gamma$ with $d(y, z) \leq 2^{j/2}$,

$$\|p_{2j}(., y) - p_{2j}(., z)\|^2_2 \leq c \left( \frac{d(y, z)}{2^{j/2}} \right)^{2\beta} \frac{1}{|B(y, 2^{j/2})|}. \quad (3.5)$$

(4) There is a $c > 0$ such that, for all $q, j \in \mathbb{N}$ and all $y, z \in \Gamma$ with $d(y, z) \leq 2^{j/2}$,

$$\|p_{2j}(., y) - p_{2j}(., z)\|^2_{L^2(\{x : d(x, y) > 2^{q/2}\})} \leq c \left( \frac{d(y, z)}{2^{j/2}} \right)^{2\beta} \frac{e^{-c2^{q-j}}}{|B(y, 2^{j/2})|}. \quad (3.6)$$

3.3. Estimates for kernels of compactly supported multipliers. Let $A > 0$, let $f \in C_0^1((0, \infty))$ and let $K(x, y)$ denote the kernel of the operator $f(L)$.

Let $M_A(f)$ be as in (3.4), let $D$ be as in (1.4) and let us set

$$A_q(y) = B\left(y, 2^{(q+1)/2}\right) \setminus B\left(y, 2^{q/2}\right), \quad y \in \Gamma, \quad q \in \mathbb{N}. \quad (3.7)$$

Lemma 2. There is a constant $c > 0$ such that for all $y \in \Gamma, q \in \mathbb{N}$

$$\|K(., y)\|_{L^1(B(y, 2^{q/2}))} \leq c \|f\|_{C^4} 2^{Dq/4}. \quad (3.8)$$

$$\|K(., y)\|_{L^1(A_q(y))} \leq c \|f\|_{C^4} 2^{(D/2 - A)q/2}. \quad (3.9)$$

Proof. Since $K(x, y) = (f(L)p_0(., y))(x)$, we have

$$\|K(., y)\|_{L^1(B(y, 2^{q/2}))} \leq |B(y, 2^{q/2})|^{1/2} \|K(., y)\|_2 \leq |B(y, 2^{q/2})|^{1/2} \|f\|_\infty \|p_0(., y)\|_2 \leq \frac{1}{|B(y, 1)|^{1/2}} \leq c \|f\|_{C^4} 2^{2qD/4}. \quad (3.10)$$

In order to prove (3.9), making use of Lemma 1 we consider a polynomial $Q$ satisfying

$$\text{deg} Q \leq 2^{q/2}, \quad \|f - Q\|_\infty \leq c M_A(f) 2^{-Aq/2}. \quad (3.11)$$

Then, making use of (3.2), we have

$$\|K(., y)\|_{L^1(A_q(y))} \leq |A_q(y)|^{1/2} \|K(., y)\|_{L^2(A_q(y))} \leq |A_q(y)|^{1/2} \|(f - Q)(L)p_0(., y)\|_{L^2(A_q(y))} \leq |B(y, 2^{(q+1)/2})|^{1/2} \|f - Q\|_\infty \|p_0(., y)\|_2 \leq c |B(y, 2^{(q+1)/2})|^{1/2} M_A(f) 2^{-Aq/2} \frac{1}{|B(y, 1)|^{1/2}} \leq c \|f\|_{C^4} 2^{(D/2 - A)q/2}. \quad (3.12)$$

\[\square\]

Corollary 1. If $A > D/2$, then there is $c > 0$, such that for all $y \in \Gamma$

$$\|K(., y)\|_1 \leq c.$$
Proof. Making use of (3.9) and (3.10), we have
\[ \|K(\cdot,y)\|_1 = \|K(\cdot,y)\|_{L^1(B(y,1))} + \sum_{q \geq 2} \|K(\cdot,y)\|_{L^1(A_q(y))} \]
\[ \leq c\|f\|_\infty 2^{D/4} + c\|f\|_{C^A} \sum_{q \geq 2} 2^{((D/2)-A)q/2} \leq c. \]
\[ \square \]

Corollary 2. If \( A > D/2 \), then for all \( y \in \Gamma \),
\[ (3.10) \]
\[ \sum_x K(x,y)\mu(x) = 0. \]

Proof. Let us first observe that if \( \psi = (I - P)\phi , \phi \in L^1 \), then \( \sum_x \psi(x)\mu(x) = 0 \). Now, let \( h(\lambda) = \lambda^{-1}f(\lambda) \) and let \( S(x,y) \) be the kernel of the operator \( h(L) \). Then, by Corollary 1, we have \( S(\cdot,y) \in L^1 \) and \( K(x,y) = ((I - P)S(\cdot,y))(x) \), \( x,y \in \Gamma \), which by the previous observation proves (3.10).

Now let \( f_j \in C^A_0((0,\infty)), j \in \mathbb{N} \), such that \( \text{supp}(f_0) \subseteq [1/2,2], \text{supp}(f_j) \subseteq [2^{-(j+1)},2^{-j}], j \geq 1 \), and let \( K_j(x,y) \) denote the kernel of the operator \( f_j(L) \).

We set \( h_0(\lambda) = f_0(\lambda^2), h_j(\lambda) = f_j(\lambda^2)(1-\lambda^2)^{-2^j}, j \geq 1 \). Note that if \( \lambda \in \text{supp}(f_j), j \geq 1 \), then \( (1-\lambda^2)^{2^j} \to 1 \), as \( j \to \infty \). So, there is a \( c > 0 \) such that
\[ (3.11) \]
\[ \|h_j\|_\infty \leq c\|f_j\|_\infty, j \in \mathbb{N}. \]

Also,
\[ (3.12) \]
\[ M_A(h_j) \leq cM_A(f_j)2^{jA/2}. \]

Since \( f_j(\lambda) = h_j(\sqrt{\lambda})(1-\lambda)^{2^j} \), we have
\[ f_j(L) = h_j(\sqrt{L})P^{2^j}. \]

Hence
\[ (3.13) \]
\[ K_j(x,y) = \left( h_j(\sqrt{L})p_{2^j}(\cdot,y) \right) (x), \quad x,y \in \Gamma. \]

Lemma 3. There is \( c > 0 \), such that for all \( j, q \in \mathbb{N} \) and \( y \in \Gamma \),
(i)
\[ (3.14) \]
\[ \|K_j(\cdot,y)\|_2 \leq c\|f_j\|_{C^A}\frac{\|f_j\|_{C^A}}{|B(y,2^{q/2})|^{1/2}}. \]

(ii) If \( q \geq j \), then
\[ (3.15) \]
\[ \|K_j(\cdot,y)\|_{L^2(|x:d(x,y)\geq 2^{q/2})} \leq c\|f_j\|_{C^A}\frac{2^{-A(q-j)/2}}{|B(y,2^{q/2})|^{1/2}}. \]

Proof. (i) By (3.13) and (3.3)
\[ \|K_j(\cdot,y)\|_2 \leq \|h_j\|_\infty \|p_{2^j}(\cdot,y)\|_2 \leq c\|f_j\|_{C^A}\frac{\|f_j\|_{C^A}}{|B(y,2^{q/2})|^{1/2}}. \]

(ii) By making use of Lemma 1, let us choose a polynomial \( Q \) which has only terms of even order and such that
\[ \deg Q \leq 2^{(q-2)/2}, \quad \|h_j - Q\|_\infty \leq cM_A(h_j)2^{-(q-2)A/2}. \]
Then by (3.12),
\[ \|h_j - Q\|_\infty \leq cM_A (f_j) 2^{(j-\epsilon) A/2} \leq c \|f_j\|_{C^A} 2^{-(q-2) A/2}, \]
and by making use of (3.2), we have
\[ (3.16) \] instead of (3.3) and (3.4) respectively. We omit the details. □

Lemma 4. Let β be as in (1.9). Then, there is c > 0, such that for all q, j ∈ \mathbb{N} and y, z ∈ Γ,
(i) If \( d(y, z) \leq 2^{j/2} \), then
\[ ||K_j(\cdot, y) - K_j(\cdot, z)||_2 \leq c \frac{\|f_j\|_{C^A}}{|B(y, 2^{j/2})|^{1/2}} \left( \frac{d(y, z)}{2^{j/2}} \right)^\beta. \]

(ii) If \( d(y, z) \leq 2^{j/2} \) and \( j \leq q \), then
\[ ||K_j(\cdot, y) - K_j(\cdot, z)||_{L^2(\{x:d(x,y)\leq 2^j\})} \leq c \frac{\|f_j\|_{C^A}}{|B(y, 2^{j/2})|^{1/2}} \left( \frac{d(y, z)}{2^{j/2}} \right)^\beta. \]

Proof. To prove the lemma we observe that
\[ K_j(\cdot, y) - K_j(\cdot, z) = h_j(\sqrt L) (p_{2j}(\cdot, y) - p_{2j}(\cdot, z)), \]
and then we argue in the same way as in the previous lemma and we use (3.5) and (3.6) instead of (3.3) and (3.4) respectively. We omit the details. □

Corollary 3. If \( A > D/2 \), then there is c > 0, such that for all j ∈ \mathbb{N} and y, z ∈ Γ, with \( d(y, z) \leq 2^{j/2} \)
\[ ||K_j(\cdot, y) - K_j(\cdot, z)||_{L^1(\{x:d(x,y)\geq 2d(y,z)\})} \leq c \frac{\|f_j\|_{C^A}}{|B(y, 2^{j/2})|^{1/2}} \left( \frac{d(y, z)}{2^{j/2}} \right)^\beta. \]

Proof. By (3.16) we have
\[ ||K_j(\cdot, y) - K_j(\cdot, z)||_{L^1(\{x:d(x,y)\leq 2^{j/2}\})} \leq |B(y, 2^{j/2})|^{1/2} ||K_j(\cdot, y) - K_j(\cdot, z)||_2 \leq c \frac{\|f_j\|_{C^A}}{|B(y, 2^{j/2})|^{1/2}} \left( \frac{d(y, z)}{2^{j/2}} \right)^\beta. \]
Also, if \( q \geq j \), then, by (3.17),
\[
\|K_j(\cdot, y) - K_j(\cdot, z)\|_{L^1(A_q(y))} \leq |A_q(y)|^{1/2} \|K_j(\cdot, y) - K_j(\cdot, z)\|_{L^2(A_q(y))}
\]
\[
\leq c \|f_j\|_{C^A} \frac{|B(y, 2^{(q+1)/2})|^{1/2}}{|B(y, 2^{j/2})|^{1/2}} 2^{-A(q-j)/2} \left( \frac{d(y, z)}{2^{j/2}} \right)^\beta \]
\[
\leq c \|f_j\|_{C^A} \frac{2^{(A-D)/2)}j/2}{2^{(A-D)/2)q/2}} \frac{d(y, z)}{2^{j/2}} \beta.
\]  
(3.20)

Now to prove (3.18), we write
\[
|K_j(\cdot, y) - K_j(\cdot, z)|_{L^1(\{x:d(x,y)\geq 2d(y,z)\})} \leq |K_j(\cdot, y) - K_j(\cdot, z)|_{L^1(\{x:d(x,y)\leq 2^{j/2}\})} + \sum_{q \geq j} |K_j(\cdot, y) - K_j(\cdot, z)|_{L^1(A_q(y))}
\]
and apply (3.19) and (3.20).

\[\text{Lemma 5. If } A > D/2, \text{ then there is } c > 0, \text{ such that for all } q, j \in \mathbb{N} \text{ with } q \geq j \text{ and all } y \in \Gamma, \]
\[
\|K_j(\cdot, y)\|_{L^1(\{x:d(x,y)\geq 2^{j/2}\})} \leq c \|f_j\|_{C^A} \frac{2^{(A-D)/2)}j/2}{2^{(A-D)/2)q/2}}.
\]
(3.21)

\[\text{Proof. If } q \geq j, \text{ then, by (3.15),} \]
\[
\|K_j(\cdot, y)\|_{L^1(A_q(y))} \leq |A_q(y)|^{1/2} \|K_j(\cdot, j)\|_{L^2(A_q(y))}
\]
\[
\leq c \|f_j\|_{C^A} \frac{|B(y, 2^{(q+1)/2})|^{1/2}}{|B(y, 2^{j/2})|^{1/2}} 2^{-A(q-j)/2} \left( \frac{d(y, z)}{2^{j/2}} \right)^\beta \]
\[
\leq c \|f_j\|_{C^A} \frac{2^{(A-D)/2)}j/2}{2^{(A-D)/2)q/2}} \frac{d(y, z)}{2^{j/2}} \beta.
\]  
(3.22)

To prove (3.21) we write
\[
\|K_j(\cdot, y)\|_{L^1(\{x:d(x,y)\geq 2^{j/2}\})} \leq \sum_{k \geq q} \|K_j(\cdot, y)\|_{L^1(A_k(y))}
\]
and apply (3.22).

\[\text{Corollary 4. If } A > D/2, \text{ then there is } c > 0, \text{ such that for all } j \in \mathbb{N} \text{ and all } y, z \in \Gamma \text{ with } d(y, z) \geq 2^{j/2}, \]
\[
\|K_j(\cdot, y) - K_j(\cdot, z)|_{L^1(\{x:d(x,y)\geq 2^{j}d(y,z)\})} \]
\[
\leq c \|f_j\|_{C^A} \frac{2^{(A-D)/2)}j/2}{d(y, z)(A-D)/2}.
\]  
(3.23)

\[\text{Proof. To prove (3.23) we observe that} \]
\[
\|K_j(\cdot, y) - K_j(\cdot, z)|_{L^1(\{x:d(x,y)\geq 2^{j}d(y,z)\})} \]
\[
\leq \|K_j(\cdot, y)|_{L^1(\{x:d(x,y)\geq d(y,z)\})} + \|K_j(\cdot, z)|_{L^1(\{x:d(x,z)\geq d(y,z)\})}
\]
and then we apply (3.21).
Let us now assume that the multiplier \( m(\lambda) \) is not necessarily compactly supported and let us set
\[
  f_n(\lambda) = m(\lambda)1_{[0, 1/n]}(\lambda), \quad h_n(\lambda) = f_n(\lambda)(1 - \lambda)^{-n}, \quad n \geq 2,
\]
and denote by \( K_n(x, y) \) the kernel of the operator \( f_n(L) \). Then, we have
\[
  K_n(x, y) = (h_n(L)p_n(\cdot, y))(x), \quad x, y \in \Gamma,
\]
and so the argument given in the proof of Lemma 3 (i) gives that there is \( c > 0 \) such that
\[
  \|K_n(\cdot, y)\|_2 \leq \frac{c}{|B(y, \sqrt{n})|^{1/2}}, \quad y \in \Gamma.
\]
This implies that, if the function \( \phi \) is finitely supported, then
\[
  \|f_n(L)\phi\|_2 \to 0 \quad (n \to \infty).
\]

4. PROOF OF THEOREM 1

Let us consider a function \( 0 \leq \theta \in C_0^\infty(\mathbb{R}) \) and let us assume that \( \theta(\lambda) = 1 \) for \( \lambda \in [-1/4, 1/4] \) and that \( \theta(\lambda) = 0 \) for \( \lambda \notin [-1/2, 1/2] \). We write
\[
m = \theta m + (1 - \theta) m = M + m_0.
\]

Let us also consider a function \( \phi \in C_0^\infty(\mathbb{R}) \) satisfying
\[
supp \phi \subset \left( \frac{1}{2}, 1 \right), \quad \sum_{j \geq 1} \phi(2^j t) = 1, \quad t \in (0, 2].
\]

Let
\[
m_j(\lambda) = m(\lambda) \phi(2^j \lambda), \quad j \geq 1.
\]
Then, we have \( supp(m_j) \subseteq [2^{-(j+1)}, 2^{-j}] \). Further, by (1.10), there is a \( c > 0 \), such that
\[
  \|m_j\|_{C^\alpha} \leq c, \quad j \in \mathbb{N}.
\]

Let \( K(x, y) \) and \( K_j(x, y) \) denote the kernels of the operators \( m(L) \) and \( m_j(L) \) respectively. Then,
\[
m(L) = M(L) + m_0(L) = \sum_{j \geq 0} m_j(L), \quad K(x, y) = \sum_{j \geq 0} K_j(x, y).
\]

Note that by (3.26), the point \( \lambda = 0 \) of the spectrum of \( L \) can be ignored and we can assume that \( m(0) = 0 \).

It follows from (3.18) and (3.25) that the kernel \( K(x, y) \) satisfies the Hörmander integral condition, i.e. there is \( c > 0 \) such that for all \( y, z \in \Gamma \)
\[
  \|K(x, y) - K(x, z)\|_{L^1((x:d(x,y)\geq 4d(y,z)))} \leq c.
\]

This shows that the operator \( m(L) \) is bounded from \( H^1 \) to \( L^1 \). This means in particular that if \( a \) is an atom, then \( m(L)a \in L^1 \). Furthermore, we have the following.

**Lemma 6.** If \( a \) is an atom, then
\[
  \sum_{x \in \Gamma} (m(L)a)(x)\mu(x) = 0.
\]
Proof. Let us fix an atom $a$. Then, it follows from Corollary 2 that for every $j \geq 0$, $\sum_{x \in \Gamma} (m_j(L)) a(x) \mu(x) = 0$. So, to prove the lemma, it is enough to show that the series $\sum_{j \geq 0} m_j(L)a$ converges in $L^1$.

Since $a$ is an atom, we have that $\text{supp}(a) \subseteq B(z, r)$, for some $r > 0$ and $z \in \Gamma$.
Now, on the one hand, it follows from (3.24) and (3.25) that $\sum_{j \geq 0} 1_{B(z, 4r)} m_j(L)a$ converges in $L^1$.

On the other hand, since $\sum_{x \in \Gamma} a(x) \mu(x) = 0$, we have

$$\langle m_j(L)a(x) = \sum_{x \in \Gamma} (K_j(x, y) - K_j(x, z))a(y) \mu(y).$$

This and Corollaries 3 and 4 imply that the series $\sum_{j \geq 0} 1_{\{x:d(x,z) \geq 4r\}} m_j(L)a$ converges in $L^1$. $\square$

4.1. Proof of Theorem 1 for $p \in (p_0, 1)$. Let $p \in (p_0, 1)$ and let $\alpha = (1/p) - 1$.
Since the dual of $H^p$ is the Lipschitz space $L_\alpha$ (cf. Section 2), in order to prove that the operator $m(L)$ is bounded on $H^p$, it is enough to show that there is a constant $c > 0$ such that for every atom $a \in H^p$ and every function $\psi$ with finite support

$$\langle m(L)a, \psi \rangle \leq c \|\psi\|_{L_\alpha}.$$

So, let us fix an atom $a \in H^p$ and let us assume that $\text{supp}(a) \subseteq B(y, r)$, $r > 0$, $y \in \Gamma$.
Let $N \in \mathbb{N}$, such that $2^{N/2} < r \leq 2^{(N+1)/2}$.
By (1.3), we have

$$\langle m(L)a, \psi \rangle = \langle m(L)a, (\psi - \psi(y)) \rangle.$$

We set

$$\psi_1 = (\psi - \psi(y)) 1_{B(y, 4r)}, \quad \psi_2 = (\psi - \psi(y)) 1_{\{x:d(x,y) > 4r\}}.$$

Then, we have

$$\langle m(L)a, \psi \rangle = \langle m(L)a, \psi_1 \rangle + \langle m(L)a, \psi_2 \rangle.$$

4.1.1. Estimation of the term $\langle m(L)a, \psi_1 \rangle$. Making use of (2.1) and (2.3), we have

$$\langle m(L)a, \psi_1 \rangle \leq \|m(L)a\|_2 \|\psi_1\|_2$$

$$\leq c \|a\|_2 \|\langle \psi - \psi(y) \rangle 1_{B(y, 2^{(N+5)/2})}\|_2$$

$$\leq c |B(y, r)|^{1/2 - (1/p)} |B(y, 2^{(N+5)/2})|^{1/2} \|\psi\|_{L_\alpha} |B(y, 2^{(N+5)/2})|^\alpha$$

$$\leq c \|\psi\|_{L_\alpha}.$$

4.1.2. Estimation of the term $\langle m(L)a, \psi_2 \rangle$. We have

$$\langle m(L)a, \psi_2 \rangle \leq \sum_{j \geq 0} \langle m_j(L)a, \psi_2 \rangle.$$

So, we must estimate the terms $\langle m_j(L)a, \psi_2 \rangle$. We shall distinguish two cases.

CASE I : $N + 4 \leq j$. Then, we have

$$\langle m_j(L)a, \psi_2 \rangle = \|(m_j(L)a)\psi_2\|_{L^1(\{x:2^{N+4}/2 \leq d(x,y) < 2^{N+2}/2\})}$$

$$+ \sum_{q \geq j} \|(m_j(L)a)\psi_2\|_{L^1(A_q(y))}.$$

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Let us recall again that
\[(m_j(L)a)(x) = \sum_{z \in B(y,r)} (K_j(x,z) - K_j(x,y)) a(z) \mu(z).\]

So, by (2.1), (2.4), (3.16) and (4.1)
\[
\| (m_j(L)a)\psi_2 \|_{L^1((x, \epsilon^{N+4}/2 \leq d(x,y) < 2^{j/2}))} \\
\leq \| m_j(L)a \|_{L^1((x, \epsilon^{N+4}/2 \leq d(x,y) < 2^{j/2}))} \| \psi_2 \|_{L^2((x, \epsilon^{N+4}/2 \leq d(x,y) < 2^{j/2}))} \\
\leq c \| a \|_1 \left( \sup_{d(z,y) \leq r} \| K_j(.,z) - K_j(.,y) \|_2 \right) \| \psi - \psi(y) \|_{L^2(B(y,2^{j/2}))}
\]
\[
(4.8)
\leq c |B(y,r)|^{-1/(p-1)} \frac{m_j \| C.A \|_{L^1(B(y,2^{j/2}))}}{2^{j/2}} \left( \frac{d(y,z)}{2^{j/2}} \right)^{\beta} \| \psi \|_{L^1(B(y,2^{j/2}))}^{1/(p-1)}
\]
\[
\leq c \| \psi \|_{L^1_n} \left( \frac{2^{N/2}}{2^{j/2}} \right)^{\beta} \frac{|B(y,2^{j/2})|^{1/(p-1)}}{|B(y,2^{j/2})|^{1/(p-1)}}
\]
\[
\leq c \| \psi \|_{L^1_n} 2^{(N-j)/2} \frac{|B(y,2^{j/2})|^{1/(p-1)}}{|B(y,2^{j/2})|^{1/(p-1)}}
\]
\[
=c \| \psi \|_{L^1_n} 2^{N/2} \left( p - \frac{2}{2p} \right) N 2^{(N-j)/2} \left( p - \frac{2}{2p} \right).
\]

Also, if \( q \geq j \), then by (2.1), (2.4), (3.17) and (4.1)
\[
\| (m_j(L)a)\psi_2 \|_{L^1(A_q(y))} \\
\leq \| m_j(L)a \|_{L^1(A_q(y))} \| \psi \|_{L^2(A_q(y))} \\
\leq c \| a \|_1 \left( \sup_{d(z,y) \leq r} \| K_j(.,z) - K_j(.,y) \|_{L^2(A_q(y))} \right) \| \psi - \psi(y) \|_{L^2(B(y,2^{j/2}))} \\
\leq c |B(y,r)|^{-1/(p-1)} \frac{m_j \| C.A \|_{L^1(B(y,2^{j/2}))}}{2^{j/2}} \left( \frac{d(y,z)}{2^{j/2}} \right)^{\beta} \| \psi \|_{L^1(B(y,2^{j/2}))}^{1/(p-1)}
\]
\[
(4.9)
\leq c \| \psi \|_{L^1_n} 2^{N/2} \left( p - \frac{2}{2p} \right) N 2^{(N-j)/2} \left( p - \frac{2}{2p} \right).
\]

It follows from (4.7), (4.8) and (4.9) that, if \( N + 4 \leq j \), then
\[
\| (m_j(L)a, \psi_2) \| \leq c \| \psi \|_{L^1_n} 2^{N/2} \left( p - \frac{2}{2p} \right) N 2^{(N-j)/2} \left( p - \frac{2}{2p} \right).
\]
and hence
\begin{equation}
\sum_{j \geq N + 4} |\langle m_j (L) a, \psi_2 \rangle| \leq c \|\psi\|_{L^\alpha}.
\end{equation}

CASE II : $N + 4 \geq j$. Then, we have
\[
|\langle m_j (L) a, \psi_2 \rangle| = \sum_{q \geq N + 4} \|m_j (L) a\|_{L^1(A_q(y))} \|\psi_2\|_{L^2(A_q(y))}.
\]

If $q \geq N + 4 \geq j$, then we have
\[
\|\langle m_j (L) a\|_{L^1(A_q(y))} \leq \|m_j (L) a\|_{L^2(A_q(y))} \|\psi_2\|_{L^2(A_q(y))} \leq c \|m_j\|_{C^4} \frac{2^{-A(q-j)/2}}{|B(z, 2^{(q+4)/2})|^{1/2} \|\psi\|_{L^\alpha} |B(y, 2^{q/2})|^{(1/p) - 1}}.
\]

(4.11)

If $d(z, y) \leq r$, then making use of (3.14) and (4.1), we have
\[
\|K_j(z, z)\|_{L^2(A_q(y))} \leq \|K_j(z, z)\|_{L^2(A_q(y))} \leq \frac{2^{-A(q-j)/2}}{|B(z, 2^{(q+4)/2})|^{1/2} \|\psi\|_{L^\alpha} |B(y, 2^{q/2})|^{(1/p) - 1}}.
\]

(4.12)

It follows from (4.11) and (4.12), by making use of (2.4) and (2.4) that
\[
\|\langle m_j (L) a\|_{L^1(A_q(y))} \leq c |B(y, r)|^{(1/p) - 1} \|\psi\|_{L^\alpha} |B(y, 2^{q/2})|^{(1/p) - 1}.
\]

Hence
\[
|\langle m_j (L) a, \psi_2 \rangle| \leq c \|\psi\|_{L^\alpha} \left( \sum_{q \geq N + 4} 2^{-A-D\left(\frac{1}{p} - \frac{1}{2}\right)} \right) \sum_{q \geq N + 4} 2^{-A-D\left(\frac{1}{p} - \frac{1}{2}\right)} \|\psi\|_{L^\alpha} \left( \sum_{q \geq N + 4} 2^{-A-D\left(\frac{1}{p} - \frac{1}{2}\right)} \right).
\]

(4.13)

Since, by assumption
\[
A > D \left( \frac{1}{p} - \frac{1}{2} \right) > \frac{D}{2},
\]
we have
\[
\sum_{0 \leq j \leq N + 4} |\langle m_j (L) a, \psi_2 \rangle| \leq c \|\psi\|_{L^\alpha}.
\]
The estimates (4.10), (4.13) and (4.9) prove (4.4) and this ends the proof of Theorem 1, in the case $p \in (p_0, 1)$.

4.2. **Proof of Theorem 1** for $p = 1$. Since the dual of $H^1$ is the space $BMO$ (cf. Section 2), in order to prove that the operator $m(L)$ is bounded on $H^1$, it is enough to prove that there is a constant $c > 0$ such that for every atom $a \in H^1$ and every function $\psi$ with finite support

$$|\langle m(L)a, \psi \rangle| \leq c \|\psi\|_{BMO}.$$

This is proved by arguing in the same way as in the case $p \in (p_0, 1)$. We just have to replace the term $\psi(y)$ in (4.5) by the mean value $\psi_B$ of the function $\psi$ on the ball $B(y, r)$ and make use of (2.3) instead of (2.4). We omit the details.

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