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$I[\omega_2]$ CAN BE THE NONSTATIONARY IDEAL ON $\mathrm{Cof}(\omega_1)$

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Abstract. We answer a question of Shelah by showing that it is consistent that every member of $I[\omega_2] \cap \mathrm{Cof}(\omega_1)$ is nonstationary if and only if it is consistent that there is a κ^+-Mahlo cardinal κ.

1. Introduction

In [She91, definition 2.1] Shelah defined the following ideal $I[\kappa^+]$:

Definition 1.1. Define, for any sequence $A = \langle a_\alpha : \alpha < \kappa^+ \rangle$, the set $B(A)$ to be the set of ordinals $\nu < \kappa^+$ such that there is a set $c \subset \nu$ with $\mathrm{otp}(c) = \mathrm{cf}(\nu)$, $\bigcup c = \nu$, and $\{ c \cap \xi : \xi < \nu \} \subset \{ a_\alpha : \alpha < \nu \}$. Then $I[\kappa^+]$ is the set of subsets of κ^+ which are contained, up to a nonstationary set, in some set $B(A)$.

He proved in [She91, theorem 4.4] that $\kappa^+ \cap \{ \nu : \mathrm{cf}(\nu) < \kappa \} \in I[\kappa^+]$ for all regular cardinals κ, and observed that it is consistent that the restriction of $I[\kappa^+]$ to $\{ \nu < \kappa^+ : \mathrm{cf}(\nu) = \kappa \}$ is generated by a single stationary, costationary set. He asked whether it is consistent that every subset of $\{ \nu < \kappa^+ : \mathrm{cf}(\nu) = \kappa \}$ in $I[\kappa^+]$ is nonstationary. We answer this in the affirmative for $\kappa = \omega_1$.

Theorem 1.2. *If it is consistent that there is a cardinal κ which is κ^+-Mahlo, then it is consistent that $I[\omega_2]$ does not contain any stationary subset of $\{ \nu < \omega_2 : \mathrm{cf}(\nu) = \omega_1 \}$.*

The fact that a κ^+-Mahlo cardinal κ is necessary is due to Shelah; a proof is given in [Mit04, theorem 13].

Our proof of Theorem 1.2 uses forcing to add a sequence $\langle D_\alpha : \alpha < \kappa^+ \rangle$ of closed unbounded subsets of κ, in the process collapsing the cardinals between ω_1 and κ onto ω_1 so that κ becomes ω_2. In the resulting model there is, for every set of the form $B(A)$, some ordinal $\alpha < \kappa^+$ such that $B(A) \cap D_\alpha$ does not contain any ordinal of cofinality ω_1. Thus every set in $I[\omega_2] \restriction \{ \nu < \omega_2 : \mathrm{cf}(\nu) = \omega_1 \}$ is nonstationary.

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Section 2 introduces the basic construction by showing how to add a single new closed unbounded set. This serves as a warm-up for section 3, introducing ideas of the forcing in a simpler context, and also is used in section 3 as the basic component of the forcing used to prove Theorem 1.2.

Most of our notation is standard. We write $\lim(X)$ for the set of limit ordinals α such that $\alpha \cap X$ is cofinal in α, and $\overline{X}$ for the topological closure, $X \cup \lim(X)$, of X, and we write $\mathrm{Cof}(\lambda)$ for $\{\, \nu : \mathrm{cf}(\nu) = \lambda \,\}$.

The reader of this paper may find it helpful to also consult the expository paper [Mit05], which discusses some of the material covered in this paper along with related topics.

A basic ingredient of the forcing in this paper is the idea of forcing with models as side conditions. This idea, in the form used in this paper, was discovered independently by the author, but the general technique method was originally introduced and has been extensively investigated by Todorcevic. His original applications concerned properties of ω_1 and used forcing notions which collapsed ω_2, but in later applications such as [Tod85] he used a form, related to that used in this paper, which did not collapse ω_2. Koszmider [Kos00] has developed a modification of Todorcevic's technique which uses a previously given morass to simplify the actual forcing. Koszmider's method is arguably simpler, but it is not suitable for the present construction: it would require a morass on ω_2 of the generic extension, which is the inaccessible cardinal κ of the ground model.

A forcing essentially identical to that described in section 2 was discovered independently by Sy Friedman [Fri06]. The presentation in [Fri06] does not collapse any cardinals, instead adding a closed unbounded subset of the ω_2 of the ground model; however this difference is due to the difference in the intended applications of the forcing rather than any intrinsic difference in the forcing itself.

2. Adding a single closed unbounded set

2.1. **The forcing notion.** In this section we define a new forcing P_B which adds a closed unbounded subset D of the set $B^* := B \cup \mathrm{Cof}(\omega)$, where B is a stationary subset of $\{\, \lambda < \kappa : \mathrm{cf}(\lambda) > \omega \,\}$ for a regular cardinal κ. The forcing preserves ω_1, while (if $\kappa > \omega_2$) collapsing the intervening cardinals to make $\kappa = \omega_2$. This forcing serves both as a warm-up for and as the basic building block of the forcing in section 3 which adds κ^+ many closed unbounded sets to construct a model in which $I[\omega_2] \cap \mathrm{Cof}(\omega_1)$ is the nonstationary ideal. As another application, it will be observed later in this section that this forcing gives a new construction of a model with no special $\aleph_2$-Aronszajn trees (or, starting from a weakly compact cardinal, no $\aleph_2$-Aronszajn trees), and it is shown in [Mit05] that stripping this forcing down to its basic technique yields a construction of such a model which is much simpler than any of those which were previously known.

The forcing P_B is based on the standard finite forcing P_{ω_1}, introduced by Baumgartner in [Bau84, page 926], for adding a closed unbounded subset of ω_1. In order to motivate the definition of P_B we give a brief description of this forcing P_{ω_1}, show how a straightforward attempt to apply it to ω_2 fails, and describe the new technique which we use to make it succeed.

The presentation of P_{ω_1} which we will give is a variant of a version, discovered by U. Abraham [AS83], of Baumgartner's forcing. The set D constructed by this forcing is not generic for Baumgartner's forcing as originally described in [Bau84],

since D has the property that $\liminf_{\alpha<\nu} \mathrm{otp}(D \cap (\nu \setminus \alpha))$ is as large as possible for any limit ordinal $\nu \in D$; however Zapletal [Zap96] has shown that the two forcings are equivalent.

The conditions in the forcing P_{ω_1} are finite sets of symbols which we call *requirements*. There are two types of these requirements: I_λ for ordinals $\lambda < \omega_1$, and $O_{(\eta',\eta]}$ for pairs of ordinals $\eta' < \eta < \omega_1$. Two requirements I_λ and $O_{(\eta',\eta]}$ are *incompatible* if $\eta' < \lambda \leq \eta$; any other two requirements are *compatible*. A condition in P_{ω_1} is a finite set of requirements, any two of which are compatible, and the ordering of P_{ω_1} is by superset: $p' \leq p$ if $p' \supseteq p$.

If G is a generic subset of P_{ω_1}, then we define $D := \{\, \lambda < \omega_1 : I_\lambda \in \bigcup G \,\}$. A little thought shows that

$$\forall \lambda < \omega_1 \ \Big(\lambda \notin D \iff \exists \eta', \eta \ \big(O_{(\eta',\eta]} \in \bigcup G \ \ \& \ \ \eta' < \lambda \leq \eta\big)\Big), \tag{2.1}$$

and it follows that D is a closed and unbounded subset of ω_1.

The cardinal ω_1 is preserved by the forcing P_{ω_1} because the forcing is proper; indeed it has the stronger property that if M is any countable elementary substructure of H_{ω_1} and $\lambda = \sup(M)$, then the condition $\{I_\lambda\}$ not only forces that $G \cap M$ is M-generic, but actually forces that $G \cap M$ is a V-generic subset of $P_{\omega_1} \cap M$. Note that for this property it is sufficient to take $M \prec H_{\omega_1}$ rather than H_{ω_2}: since the relevant dense sets are taken from V, rather than from M, it is not necessary that $P \in M$.

In order to define a similar forcing P_{ω_2} adding a new closed unbounded subset of ω_2, one could naively try to use the same definition, but with requirements I_λ for any $\lambda < \omega_2$ and $O_{(\eta',\eta]}$ for any $\eta' < \eta < \omega_2$; however this forcing is not proper and does collapse ω_1. To simplify notation we will show why this is true below the condition $\{I_{\omega_1 \cdot \omega}\}$, which forces $\omega_1 \cdot \omega \in D$. For each $n < \omega$ let $\xi_n = \sup\{\, \xi < \omega_1 : \omega_1 \cdot n + \xi \in D \,\}$, so that $0 \leq \xi_n \leq \omega_1$. If $p \leq \{I_{\omega_1 \cdot \omega}\}$ and $\xi < \omega_1$, then for any sufficiently large $n < \omega$ the set $p \cup \{I_{\omega_1 \cdot n + \xi}, O_{(\omega_1 \cdot n + \xi, \omega_1 \cdot (n+1)]}\}$ is a condition extending p which forces that $\xi_n = \xi$. It follows that $\omega_1 = \{\, \xi_n : n < \omega \ \ \& \ \ \xi_n < \omega_1 \,\}$; thus ω_1 is collapsed in $V[D]$.

In order to avoid this problem we will use a third type of requirement in the definition of P_{ω_2}. This new requirement, which we write as C_M for any countable $M \prec H_{\omega_2}$, is intended to play the same role as the requirement I_λ plays in the proof that P_{ω_1} is proper: the condition $\{C_M\}$ will force that $G \cap M$ is a V-generic subset of $P_{\omega_2} \cap M$. This will be accomplished by finding, for each condition $p \leq \{C_M\}$, a condition $p|M \in P_{\omega_2} \cap M$ with the property that every condition $q \leq p|M$ in $P_{\omega_2} \cap M$ is compatible with p: thus the condition $p|M \in M$ will capture all of the influence which p has on the forcing $P_{\omega_2} \cap M$.

To see how this works, consider a set $p = \{C_M, O_{(\eta',\eta]}\}$. If $(\eta', \eta] \cap M = \varnothing$, then $O_{(\eta',\eta]}$ will have no effect on the forcing inside M, and we will take $p|M = \varnothing$. If η' and η are in M, then $(\eta', \eta]$ is a member of M, and we will take $p|M = \{O_{(\eta',\eta]}\}$. In either case p will be a condition, but if neither of these holds—if $(\eta', \eta]$ intersects M but is not a member of M—then there is no requirement inside M which will have the same effect on $G \cap M$ as the requirement $O_{(\eta',\eta]}$ does, and in this case we will say that $O_{(\eta',\eta]}$ and C_M are not compatible, and hence p is not a condition.

To see how this will block the collapse of ω_1 described earlier for the naive version of the forcing, let M be any countable elementary substructure of H_{ω_2} with $\omega_1 \cdot \omega \in M$. The pair $\{C_M, I_{\omega_1 \cdot \omega}\}$ will be a a condition, and as with P_{ω_1} it will force

$\omega_1 \cdot \omega \in D$. Now suppose that $p \leq \{C_M, I_{\omega_1 \cdot \omega}\}$ is a condition which forces, for some $n < \omega$, that $\xi_n < \omega_1$, that is, that $D \cap \omega_1 \cdot (n+1)$ is bounded in $\omega_1 \cdot (n+1)$. By using (2.1) (which we will show to hold for P_{ω_2}) we can see that this implies that there is a requirement $O_{(\eta',\eta]} \in p$ with $\eta' < \omega_1 \cdot (n+1) \leq \eta$. Now $(\eta', \eta] \cap M \neq \varnothing$, since $\omega_1 \cdot (n+1)$ is in the intersection, so the compatibility of C_M with $O_{(\eta',\eta]}$ implies that $O_{(\eta',\eta]} \in M$ and in particular that $\eta' < \sup(M \cap \omega_1 \cdot (n+1))$. Hence $p \Vdash \dot{\xi}_n \leq \sup(M \cap \omega_1)$, and since n was arbitrary it follows that $\{C_M, I_{\omega_1 \cdot \omega}\} \Vdash \{\, \dot{\xi}_n : n < \omega \ \ \& \ \ \dot{\xi}_n < \omega_1 \,\} = M \cap \omega_1$.

We are now ready to give the definition of the forcing P_B. We assume that B is a stationary subset of an inaccessible cardinal κ and that every member λ of B is a cardinal with uncountable cofinality such that $H_\lambda \prec H_\kappa$. This definition can be easily adapted (assuming that $2^\omega = \omega_1$) to the case $\kappa = \omega_2$, discussed previously as P_{ω_2}, by replacing the models H_λ in the definition with structures $L_\lambda[A]$, where $A \subset \omega_2$ enumerates $[\omega_2]^\omega$. Friedman [Fri06] has pointed out that the assumption $2^\omega = \omega_1$ can be weakened, provided that there exists a stationary set $S \subset [\omega_2]^\omega$ such that $|\{\, x \cap \nu : x \in S \,\}| = \omega_1$ for all $\nu < \omega_2$.

We also assume that H_κ has definable Skolem functions, so that $M \cap N \prec H_\kappa$ whenever $M \prec H_\kappa$ and $N \prec H_\kappa$. This assumption can be avoided by replacing H_κ with a structure which does have Skolem functions.

We write $B^* = B \cup \{\, \lambda < \kappa : \mathrm{cf}(\lambda) = \omega \,\}$. The forcing will add a new closed unbounded subset of B^*.

The forcing P_B uses three types of *requirements*:

(1) I_λ, for any $\lambda \in B^*$,
(2) $O_{(\eta,\eta']}$, for any interval with $\eta < \eta' < \kappa$, and
(3) C_M, for any countable set $M \prec H_\kappa$.

These symbols I_λ, $O_{(\eta',\eta]}$ and C_M are used for convenience; since the subscripts are distinct we can take each requirement to be equal to its subscript, that is, $I_\lambda = \lambda$, $O_{(\eta',\eta]} = (\eta', \eta]$ and $C_M = M$.

We first specify which pairs of requirements are compatible. The first clause is the same as for P_{ω_1}, and an explanation of the second clause has already been given. Clauses 3 and 4 similarly assert that C_M is compatible with I_λ or C_N if and only if there there is a condition $I_\lambda|M$ or $C_N|M$ which is a member of M and reflects the effect which the requirement I_λ or C_N in P_B has on the forcing $P_B \cap M$. This will be made precise in Lemma 2.25.

Definition 2.1. (1) Two requirements $O_{(\eta',\eta]}$ and I_λ are *incompatible* if $\eta' < \lambda \leq \eta$; otherwise they are *compatible*.
(2) Two requirements $O_{(\eta',\eta]}$ and C_M are *compatible* if either $O_{(\eta',\eta]} \in M$ or every requirement $I_\lambda \in M$ is compatible with $O_{(\eta',\eta]}$.
(3) (a) An *M-fence for a requirement* I_λ is a requirement $I_{\lambda'} \in M$ such that any requirement $O_{(\eta',\eta]}$ in M incompatible with I_λ is also incompatible with $I_{\lambda'}$.
(b) Two requirements C_M and I_λ are *compatible* if either $\lambda \geq \sup(M)$ or there exists an M-fence for I_λ.
(4) (a) An *M-fence for a requirement* C_N is a finite set $x \in M$ of requirements I_λ, with $\lambda \in B$, with the following property: Let $O_{(\eta',\eta]} \in M$ be any requirement which is incompatible with N, and which has $\eta' \geq$

$\sup(M \cap N)$ if $M \cap N \in M$. Then there is some $I_\lambda \in x$ which is incompatible with $O_{(\eta',\eta]}$.

(b) Two requirements C_M and C_N are *compatible* if the following clauses hold both as stated and with M and N switched:
   (i) Either $M \cap N \in M$ or $M \cap N = M \cap H_{\sup(M\cap N)}$.
   (ii) There is a M-fence for C_N.

Definition 2.2. A condition p in the forcing P_B is a finite set of requirements such that each pair of requirements in p is compatible. The set P_B is ordered by reverse inclusion: $p' \leq p$ if $p' \supseteq p$.

Proposition 2.3. *If C_M is a requirement and $p \in M \cap P_B$, then $p \cup \{C_M\}$ is a condition.*

Proof. In verifying that C_M is compatible with any requirement in p, notice that any requirement $I_\lambda \in M$ is its own M-fence.

For each requirement $C_N \in p$, the model N is a member of M and hence $\varnothing$ is an M-fence for C_N. □

Notice that Clauses (4(a)) and (4(b)i) of Definition 2.1 imply that $\sup(M \cap N) \notin M$ unless $M \cap N \in M$. Suppose to the contrary that $M \cap N \notin M$ but $\lambda := \sup(M \cap N) \in M$. Then for any $\eta' \in M \cap \lambda$ the requirement $O_{(\eta',\lambda]}$ is in M and is incompatible with N. The only way that a fence $x \in M$ could be incompatible with all such requirements $O_{(\eta',\lambda]}$ would be if $I_\lambda \in x$, but $\lambda \notin B$ since $\mathrm{cf}(\lambda) = \omega$.

In the case $M \cap N \in M$, the requirement $C_{M\cap N}$ will be used in section 2.2 to augment the M-fence for C_N: Any requirement $O_{(\eta',\eta]} \in M$ with $\eta' < \sup(M \cap N)$ which is incompatible with C_N will be incompatible with $C_{M\cap N} \in M$.

Definition 2.1 described M-fences in terms of their function. We now give alternate structural characterizations and note that the fences are unique:

Proposition 2.4. *The requirements I_λ and C_M are compatible if either $\lambda \geq \sup(M)$ or else $\min(M \setminus \lambda) \in B^*$; in the latter case $I_{\min(M\setminus\lambda)}$ is the unique M-fence for I_λ.*

Proof. Set $\lambda' = \min(M \setminus \lambda)$. If $\lambda' \in B^*$, then $I_{\lambda'}$ is a requirement, and it is easy to see that it is an M-fence for I_λ.

To see that it is the only possible M-fence for I_λ, note that if $\eta \in M \cap \lambda'$, then the requirement $O_{(\eta,\lambda']}$ is a member of M and is incompatible with I_λ. However any requirement $I_{\lambda''} \in M$ with $\lambda'' \neq \lambda'$ will be compatible with $O_{(\eta,\lambda']}$, provided that $\eta > \lambda''$ in the case that $\lambda'' < \lambda'$. □

The structural characterization of an M-fence for C_N is slightly more complicated:

Proposition 2.5. *Suppose that C_M and C_N are requirements, and let y be the set of ordinals $\lambda \in M$ such that $\lambda > \sup(M \cap N)$ and $\lambda = \min(M \setminus \eta)$ for some $\eta \in N$.*

Then there is an M-fence for C_N if and only if y is finite, $y \subset B$, and if $M \cap N \notin M$ and $M \nsubseteq N$, then $\min(M \setminus \sup(M \cap N)) \in y$. In this case $x := \{ I_\lambda : \lambda \in y \}$ is an M-fence for C_N, and x is minimal in the sense that it is a subset of any other M-fence for C_N.

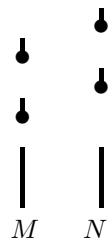

FIGURE 1. Fences for C_M and C_N when $M \cap N \notin M$ and $M \cap N \notin N$.

Proposition 2.5 asserts that two compatible requirements C_M and C_N divide each other into finitely many blocks: a common block below $\sup(M \cap N)$, followed by finitely many disjoint blocks alternating between M and N. Each block lies inside a gap in the other model, the upper end of which is delineated by a member of the M-fence for C_N or the N-fence for C_M. This is illustrated by Figure 1, where the solid dots show the required fences for compatibility of C_M and C_N in the case where $M \cap N = M \cap H_{\sup(M\cap N)} = N \cap H_{\sup(M\cap N)}$.

The cases $M \cap N \in M$ and $M \cap N \in N$ are similar, except that if, say $M \cap N \in M$, then $\sup(M \cap N) \in M$; that is, the bar in M is longer, and the smallest fence is in N.

Proof of Lemma 2.5. To see that any M-fence x' for C_N must contain x, and that therefore the existence of such a fence implies that y is finite and $y \subset B$, suppose that $\lambda \in y$ and $I_\lambda \notin x'$ and consider a requirement $O_{(\eta,\lambda]}$ where $\eta \in M \cap \lambda$, $\eta \geq \sup(M \cap N)$ if $M \cap N \in M$, and $\eta > \max(\{\, \tau < \lambda : I_\tau \in x' \,\})$. Then $O_{(\eta,\lambda]}$ is a member of M which is compatible with x'; however it is incompatible with C_N since $(\eta, \lambda] \cap N \neq \varnothing$ because $\sup(N \cap \lambda) \geq \sup(M \cap \lambda) > \eta$ and $O_{(\eta,\lambda]} \notin N$ because $\lambda \notin N$.

To see that the stated conditions imply that x is an M-fence for C_N, suppose that $O_{(\eta',\eta]} \in M$ is incompatible with C_N. If there is any ordinal $\gamma \in (\eta', \eta] \cap N$ with $\gamma \geq \sup(M \cap N)$, then $O_{(\eta',\eta]}$ is incompatible with $I_{\min(M\setminus\gamma)} \in x$, so we can assume that $(\eta', \eta] \cap N \subseteq \sup(M \cap N)$. Then $\eta' < \sup(M \cap N)$, so according to Definition 2.1(4(a)) we need only consider the case $M \cap N \notin M$. In this case $M \cap \sup(M \cap N) \subseteq N$, so $\eta < \sup(M \cap N)$ would imply $\{\eta', \eta\} \subset N$, contradicting the assumption that $O_{(\eta',\eta]}$ is incompatible with C_N. Thus we have $\eta' < \sup(M \cap N) < \eta$. Then the statement of the lemma requires $\lambda := \min(M \setminus \sup(M \cap N)) \in y$ so $\eta' < \lambda \leq \eta$, and $O_{(\eta',\eta]}$ is incompatible with $I_\lambda \in x$. $\square$

Whenever we refer to an M-fence for C_N we will mean the minimal fence x described in Proposition 2.5. We will also refer to any of the individual requirements in this minimal fence as an M-fence for C_N.

The fact that any superset of the minimal M-fence for C_N is, according to Definition 2.1(4(a)), also an M-fence for C_N is something of an anomaly; however alternate definitions which avoid this seem, at least in the forcing of section 3, to be significantly more complicated.

Corollary 2.6. *If the requirements C_M and $C_{M'}$ are compatible, then* $\lim(M \cap M') = \lim(M) \cap \lim(M')$. $\square$

Since the forcing P_B is not separative, it will be convenient to define notation for the equivalent separative forcing: if $\dot{G}$ is a name for the generic set, then we

will say that $p' \leq^* p$ if $p' \Vdash p \in \dot{G}$ and $p' =^* p$ if $p' \leq^* p$ and $p \leq^* p'$. The goal in this subsection is to prove the following lemma:

Lemma 2.7. *Suppose that p is a condition, and let X be the finite set of ordinals λ such that either* (i) *I_λ is one of the fences required for compatibility of two requirements in p,* (ii) *$\lambda = \sup(M \cap \lambda')$ for some $C_M \in p$ and some $I_{\lambda'}$ which is either in p or included in X by clause* (i)*, or* (iii) *$\lambda = \sup(M)$ for some $C_M \in p$. Then $p' = p \cup \{I_\lambda : \lambda \in X\}$ is a condition and $p' =^* p$. Furthermore, $\forall \lambda < \kappa\ ((p \Vdash I_\lambda \in \bigcup \dot{G}) \iff I_\lambda \in p')$.*

We first consider the fences:

Lemma 2.8. *Suppose that $p \in P_B$, and p' is the set obtained by adding to p each of the fences required for compatibility of requirements in p. Then $p' \in P_B$ and $p' =^* p$, and every fence required for compatibility of members of p' is a member of p'.*

Proof. We must show that each of the fences $I_\lambda \in p' \setminus p$ is compatible with any requirement in p, and that any fence required for this compatibility is already a member of p'. Suppose that $C_M \in p$ and I_λ is an M-fence for one of the requirements I_τ or C_N in p.

First we show that I_λ is compatible with any requirement $O_{(\eta',\eta]} \in p$. Suppose to the contrary that $\lambda \in (\eta', \eta]$. Since $I_\lambda \in M$, the compatibility of $O_{(\eta',\eta]}$ with C_M implies that $O_{(\eta',\eta]} \in M$, so that $\eta' < \sup(M \cap \lambda) < \lambda \leq \eta$. If I_λ is an M-fence for $I_\tau \in p$, then $\eta' \in M$ implies that $\eta' < \tau < \eta$, contradicting the compatibility of I_τ and $O_{(\eta',\eta]}$. If I_λ is an M-fence for $C_N \in p$, then $(\eta', \eta] \cap N \neq \varnothing$, so $O_{(\eta',\eta]}$ is a member of N as well as of M; however this is impossible since $\eta \geq \lambda > \sup(M \cap N)$.

It remains to show that I_λ is compatible with any requirement $C_{M'} \in p$. Let $\lambda' = \min(M' \setminus \lambda)$, so that $I_{\lambda'}$ is the M'-fence for I_λ required for compatibility of I_λ and $C_{M'}$. We need to show that $I_{\lambda'} \in p'$.

If $\lambda \in M'$, then $\lambda' = \lambda$ and so $I_{\lambda'} = I_\lambda \in p'$. If $\lambda \notin M'$ and $\lambda \geq \sup(M \cap M')$, then $I_{\lambda'}$ is a member of p' because it is an M'-fence for C_M. Thus we can assume that $\lambda < \sup(M \cap M')$ and $\lambda \notin M'$, so that $\lambda < \lambda' < \sup(M \cap N)$. Since $\lambda \in M \setminus M'$ it follows that $M \cap M' \in M$.

If I_λ is an M-fence for $I_\tau \in p$, then $(\tau, \lambda] \cap M' \subseteq (\tau, \lambda] \cap M = \varnothing$, so $\lambda' = \min(M' \setminus \lambda) = \min(M' \setminus \tau)$ and hence $I_{\lambda'}$ is in p' as the M'-fence for I_τ.

Now suppose that I_λ is an M-fence for $C_N \in p$; that is, $\lambda > \sup(M \cap N)$ and $\lambda = \min(M \setminus \eta)$ for some $\eta < \lambda$ in N. In the case that $\lambda \geq \sup(M' \cap N)$ we claim that $\sup(N \cap \lambda') > \sup(M' \cap \lambda')$, so that $I_{\lambda'}$ is in p' as an M'-fence for C_N. We have $\sup(N \cap \lambda') \geq \eta \geq \sup(M \cap \lambda)$, since $\lambda' > \lambda$. However $\sup(M \cap \lambda) > \sup(M' \cap \lambda)$ since $\lambda < \sup(M \cap M')$, $\mathrm{cf}(\lambda) > \omega = \mathrm{cf}(\sup(M' \cap \lambda))$, and $M \cap M' = M' \cap H_{\sup(M \cap M')} \in M$. Finally, $M' \cap \lambda = M' \cap \lambda'$ since $\lambda' = \min(M' \setminus \lambda)$. Hence $\sup(N \cap \lambda') > \sup(M' \cap \lambda')$, as claimed.

We will now show that the remaining case, $\lambda < \sup(M' \cap N)$, is not possible. If it did hold, then we would have $\sup(M' \cap N) > \lambda'$. Now $\lambda' \in M \cap M'$, since $M \cap M'$ is an initial segment of M'. It follows that $N \cap M'$ is not an initial segment of M', as this would imply that $\lambda' \in N$, contradicting the fact that $\lambda' \geq \lambda > \sup(M \cap N)$. Hence $N \cap M' = N \cap H_{\sup(N \cap M')} \in M'$, and it follows that $N \cap \lambda \in M$. Then $\sup(N \cap \lambda) < \sup(M \cap \lambda)$, but this contradicts the assumption that λ is an M-fence for C_N.

This completes the proof that every fence required for the compatibility of requirements of p' is already a member of p', and hence that $p' \in P_B$. To see that $p' =^* p$, let $q \leq p$ be arbitrary and let $q' \leq q$ be obtained from q as in the lemma by adding to q all of the fences required for the compatibility of requirements in q. Then $q' \supseteq p'$, and hence $q' \leq q$ forces that $p' \in \dot{G}$. □

Lemma 2.9. *Suppose that $p \in P_B$, and p' is obtained from p by adding those requirements I_λ such that there is some $C_M \in p$ such that either $\lambda = \sup(M)$ or $\lambda = \sup(M \cap \lambda')$ for some $I_{\lambda'} \in p$. Then $p' \in P_B$ and $p' =^* p$.*

Furthermore if $C_N \in p$ and $I_\lambda \in p' \setminus p$ with $\lambda < \sup(N)$, then the N-fence for I_λ either is equal to I_λ or else is an N-fence for some $I_{\lambda'} \in p$.

Proof. Again we need to show that every requirement $I_\lambda \in p'$ is compatible with every requirement $O_{(\eta',\eta]}$ or $C_{M'}$ in p.

In order to show that any requirement I_λ as specified in the lemma is compatible with any requirement $O_{(\eta',\eta]} \in p$, we will assume that I_λ is incompatible with $O_{(\eta',\eta]}$ and show that $O_{(\eta',\eta]}$ is incompatible with C_M or $I_{\lambda'}$, contradicting the assumption that it is in p. Now $(\eta', \eta] \cap M \neq \varnothing$ since $\lambda \in \lim(M)$, so $O_{(\eta',\eta]}$ is incompatible with C_M unless $O_{(\eta',\eta]} \in M$. Since $\eta \geq \lambda$, this is impossible if $\lambda = \sup(M)$. If $\lambda = \sup(M \cap \lambda')$, then $O_{(\eta',\eta]} \in M$ implies that $\eta \geq \lambda'$, so $O_{(\eta',\eta]}$ is incompatible with $I_{\lambda'}$.

Now we show that I_λ is compatible with every requirement $C_{M'} \in p$. This is immediate if $\lambda \geq \sup(M')$. If $\sup(M \cap M') \leq \lambda < \sup(M')$, then by Proposition 2.5 the fence $I_{\min(M' \setminus \lambda)}$ is a required M'-fence for C_M. Finally suppose that $\lambda < \sup(M' \cap M)$. If $M' \cap M \in M'$, then $\lambda \in M'$ and hence I_λ is its own M'-fence. Otherwise $M' \cap \sup(M' \cap M) \subset M$ so $\lambda \leq \lambda' \leq \min(M \setminus \lambda) \leq \min(M' \setminus \lambda)$, so the M'-fence $I_{\min(M' \setminus \lambda)}$ for I_λ is the same as the M'-fence $I_{\min(M' \setminus \lambda')}$ for $I_{\lambda'}$.

This completes the proof that $p' \in P_B$ and that any nontrivial fences for members of $p' \setminus p$ are already fences for members of p. To see that $p \leq^* p'$, notice that for any condition $q \leq p$ we have $q' \supseteq p'$ and hence $q' \leq p'$. Thus $p \Vdash p' \in \dot{G}$. □

Let us call a condition $p \in P_B$ *complete* if every fence required for compatibility of requirements in p is a member of p, and if $I_\lambda \in p$ whenever there is $C_M \in p$ such that $\lambda = \sup(M)$ or $\lambda = \sup(M \cap \lambda')$ for some $I_{\lambda'} \in p$.

Corollary 2.10. *For any condition p there is a complete condition $p' =^* p$.*

Proof. Begin by using Lemma 2.8 to add to p all fences required for compatibility of p, and then use Lemma 2.9 to add requirements of the form $I_{\sup(M)}$ or $I_{\sup(M \cap \lambda)}$. □

Lemma 2.11. *Suppose that p is a complete condition and $\lambda < \kappa$ is an ordinal such that $I_\lambda \notin p$. Then there is a requirement $O_{(\eta',\eta]}$ incompatible with I_λ such that $p \cup \{O_{(\eta',\eta]}\} \in P_B$.*

Proof. We may assume that there is $C_M \in p$ with $\sup(M) > \lambda$, for otherwise we could take $O_{(\eta',\eta]} = O_{(\eta',\lambda]}$ where η' is any sufficiently large ordinal less than λ. Since $I_{\sup(M)} \in p$ for each $C_M \in p$, it follows that there is some ordinal $\tau > \lambda$ with $I_\tau \in p$. Let τ be the least such.

If $C_M \in p$, then either $\sup(M \cap \tau) < \lambda$ or $\tau \in \lim(M)$, for otherwise we would have $\lambda \leq \sup(M \cap \tau) < \tau$ and $I_{\sup(M \cap \tau)} \in p$, contradicting the choice of either λ or τ. Let $Y = \{\, M : C_M \in p \ \ \& \ \ \tau \in \lim(M) \,\}$. Then $Y \neq \varnothing$, since otherwise we could take $O_{(\eta',\eta]} = O_{(\eta',\lambda]}$ for any sufficiently large $\eta' < \lambda$.

We claim that $\{ M \cap \tau : M \in Y \}$ is linearly ordered by $\subseteq$. To see this, note that $\tau \in \lim(M) \cap \lim(M') = \lim(M \cap M')$, so that $\sup(M \cap M') \geq \tau$. The claim then follows from the condition Definition 2.1(4(b)i) for compatibility of C_M and $C_{M'}$.

Now pick $M \in Y$ so that $M \cap \tau$ is as small as possible, and set $\eta = \min(M \setminus \lambda)$. If η' is any member of $M \cap \lambda$, then $O_{(\eta',\eta]} \in M'$ for all $M' \in Y$. We claim that there is $\eta' \in M \cap \lambda$ such that

$$\eta' > \max\bigl(\{ \xi < \lambda : I_\xi \in p \} \cup \{ \sup(M' \cap \tau) : C_{M'} \in p \ \ \& \ \ M' \notin Y \}\bigr).$$

It will follow that $O_{(\eta',\eta]}$ is compatible with every requirement in p, so that $p \cup \{O_{(\eta',\eta]}\} \in P_B$.

To prove the claim we need to show that $\sup(M \cap \lambda) > \xi$ for all $I_\xi \in p$ with $\xi < \lambda$, and $\sup(M \cap \lambda) > \sup(M' \cap \lambda)$ for all $C_{M'} \in p$ with $M' \notin Y$.

If $\xi < \lambda$ and $I_\xi \in p$, then $I_{\min(M\setminus\xi)} \in p$. Since $I_\eta \notin p$ and $\eta = \min(M \setminus \lambda)$ it follows that $\xi \leq \min(M \setminus \xi) < \lambda$. Hence $\sup(M \cap \lambda) > \xi$.

Now suppose that $C_{M'} \in p$ but $M' \notin Y$. If $\sup(M' \cap M) \geq \tau$, then $\eta \in M \setminus M'$ implies that $M \cap M' \in M$, so $\sup(M' \cap \lambda) = \sup((M \cap M') \cap \lambda) \in M$ and hence $\sup(M \cap \lambda) > \sup(M' \cap \lambda)$. Thus we can assume that $\sup(M' \cap M) < \tau$, so that $\sup(M' \cap M) < \lambda$. If ξ is any member of $M' \cap \lambda$ with $\xi \geq \sup(M \cap M')$, then $I_{\min(M\setminus\xi)}$ is an M-fence for M' and hence is in p. Since $I_\eta \notin p$ it follows that $\min(M \setminus \xi) < \lambda$. Thus we can assume that $M' \cap \lambda \subset \sup(M \cap M')$. If $M' \cap M \in M$ this implies $\sup(M' \cap \lambda) = \sup(M \cap M') \in M$, while if $M' \cap M \notin M$, then $I_{\min(M\setminus\sup(M\cap M'))} \in p$ as an M-fence for M', and as before this implies $\min(M \setminus \sup(M \cap N)) < \lambda$. Thus in any case we have $\sup(M \cap \lambda) > \sup(M' \cap \lambda)$. □

Proof of Lemma 2.7. We already know that p can be extended to a complete condition p' so that $p' =^* p$. By Lemma 2.11, if $I_\lambda \notin p'$, then there is $q \leq p'$ so that $q \Vdash I_\lambda \notin \bigcup \dot{G}$. □

Definition 2.12. If G is a generic subset of P_B, then we write D for the set of $\lambda < \kappa$ such that $I_\lambda \in \bigcup G$.

Corollary 2.13. *The set D is a closed and unbounded subset of B^*.*

Proof. That D is a subset of B^* follows from the fact that I_λ is a requirement only if $\lambda \in B^*$. To see that D is unbounded, suppose that $p \in P_B$ and $\eta < \kappa$. Let λ be any member of $\kappa \setminus \eta$ such that $\mathrm{cf}(\lambda) = \omega$ and $p \in H_\lambda$; then I_λ is a requirement which is compatible with p and which forces that $\lambda \in D \setminus \eta$.

Finally, let $\lambda < \kappa$ and $p \in P_B$ be arbitrary such that $p \Vdash \lambda \in \lim(\dot{D})$. Then p is incompatible with any requirement $O_{(\eta',\eta]}$ with $\eta' < \lambda \leq \eta$, and it follows by Lemma 2.11 that $p \Vdash \lambda \in \dot{D}$. Hence D is closed. □

2.2. **Strongly generic conditions.** It was pointed out in the discussion preceding the definition of P_B that the forcing P_{ω_1} satisfies a property stronger than that of being proper, and it was stated as part of the motivation for P_{ω_2} and hence for P_B that these forcings would satisfy the same property. We now make this notion precise:

Definition 2.14. If P is a forcing notion and X is a set, then we say that p is *strongly X, P-generic* if $p \Vdash_P$ "$\dot{G} \cap X$ is a V-generic subset of $P \cap X$" where $\dot{G}$ is a name for the generic set.[1]

[1]It should be noted that strong genericity as defined here is not related to the notion which Foreman, Magidor and Shelah [FMS88] call strong genericity.

Being strongly X, P-generic is stronger than Shelah's notion of a P, X-generic condition p, which only needs to force that $\dot{G} \cap X$ is an X-generic subset of $P \cap X$. Also, the existence of a strongly X, P-generic condition does not require that $P \in X$, as does the existence of an X, P-generic condition in Shelah's sense.

Definition 2.14 can be restated: p_0 is strongly X, P-generic if, below the condition p_0, the forcing P can be written as a two stage iteration. If we write P/p_0 for the forcing P below the condition p_0, Definition 2.14 implies that (with some abuse of notation) there is a $(P \cap X)$-term $\dot{R}$ such that $P/p_0 \equiv ((P/p_0) \cap X) * \dot{R}$. The following equivalent definition of strong genericity clarifies the meaning of the notation $(P/p_0) \cap X$:

Proposition 2.15. *A condition $p_0 \in P$ is strongly X, P-generic if and only if* (i) *if p, q_0 and q_1 are any conditions such that $p \leq p_0$, $p \leq q_0, q_1$, and $\{q_0, q_1\} \subset X$, then q_0 and q_1 are compatible in $P \cap X$, and* (ii) *for every $p \leq p_0$ in P there is a condition $p|X \in P \cap X$ such that any condition $q \leq p|X$ in X is compatible with p.*

Proof. First assume that p_0 is strongly X, P-generic. If p, q_0 and q_1 are as in clause (i), then $p \Vdash q_0, q_1 \subset \dot{G} \cap X$, and since p also forces that $\dot{G} \cap X$ is a generic subset of $P \cap X$ it follows that q_0 and q_1 are compatible in $P \cap X$. For clause (ii), suppose that $p \leq p_0$ and let D be the set of $q \in P \cap X$ such that either q is incompatible with p or every $q' \leq q$ in $P \cap X$ is compatible with p. Then p forces that $\dot{G} \cap D \neq \varnothing$, so there is some $q \in D$ which is compatible with p. This condition q is a suitable choice for $p|X$.

Now suppose that p_0 satisfies clauses (i) and (ii). First suppose that q_0, q_1 are members of $P \cap X$ and $p \leq p_0$ forces that $\{q_0, q_1\} \subseteq \dot{G}$. We can assume, by extending p if necessary, that p, q_0 and q_1 satisfy the hypothesis of clause (i), which implies that q_0 and q_1 are compatible in $P \cap X$. Hence p_0 forces that $\dot{G} \cap X$ is a pairwise compatible subset of $P \cap X$. Now suppose that D is a dense subset of $P \cap X$ and $p \leq p_0$. Then there is some $q \leq p|X$ such that $q \in D$. It follows that p and q are compatible in P, and if p' is any common extension of p and q, then $p' \Vdash q \in D \cap (\dot{G} \cap X)$. Thus p_0 forces that $\dot{G} \cap X$ is a generic subset of $P \cap X$. $\square$

All of the forcing notions P used in this paper will satisfy that $q_0 \cup q_1 = q_0 \wedge q_1$ for all compatible conditions $q_0, q_1 \in P$, and hence clause (i) will be satisfied by any set X which is closed under finite unions. Thus we will only need to consider clause (ii). A function $p \mapsto p|X$ satisfying clause (ii) will be called a *witness to the strong P, X-genericity of p_0*.

We will usually omit P, writing "strongly X-generic" instead of "strongly X, P-generic", when it is clear which forcing notion is meant.

We will say that a model X has strongly generic conditions if for every $p \in P \cap X$ there is a strongly X-generic condition $p' \leq p$. In many applications, including all the examples in this paper, there is a single strongly X-generic condition p_0 which is compatible with every condition $q \in P \cap X$.

The next two definitions are standard:

Definition 2.16. A set $Y \subset \mathcal{P}(I)$ is *stationary* if for every structure $\mathcal{A}$ with universe I and in a countable language, there is a set $M \in Y$ with $\mathcal{A}|M \prec \mathcal{A}$.

We will not normally specify the index set I. Notice that the property of being a strongly P, M-generic condition (unlike the property of being a P, M-generic condition) depends only on $M \cap P$; hence the set I can be taken to be the set P

of conditions. However we will also take advantage of the well-known fact that if $I' \supset I$ and Y is a stationary subset of $\mathcal{P}(I)$, then $\{\, x \subset I' : x \cap I \in Y \,\}$ is a stationary subset of $\mathcal{P}(I')$. This observation makes it possible to apply the stationarity of a given class $Y \subset \mathcal{P}(P)$ to obtain an elementary substructure $M \prec \mathcal{A}$ where $\mathcal{A}$ is a model with universe properly containing P.

Definition 2.17. A forcing notion P is said to be *δ-presaturated* if for any set $A \subset V$ in $V[G]$ with $|A|^{V[G]} \leq \delta$, there is a set $A' \supset A$ in V such that $|A|^V < \delta$.

We use δ-presaturation as a local version of the δ-chain condition: it is equivalent to the statement that for every collection $\mathcal{A}$ of fewer than δ antichains in P there is a dense set of conditions p such that the set of conditions in $\bigcup \mathcal{A}$ which are compatible with p has size less than δ. This ensures that forcing with P does not collapse δ.

The following is a well-known observation.

Lemma 2.18. *Suppose that P is a forcing notion such that for stationarily many models M of size less than δ there is, for each $q \in M$, an M-generic condition $p \leq q$. Then P is δ-presaturated.*

Proof. Assume that $\dot{A}$ is a P-name for a subset A of V in $V[G]$ such that $\mu := |A|^{V[G]} < \delta$, and let $\dot{k}$ be a P-name such that $p \Vdash \dot{k} \colon \mu \xrightarrow{\text{onto}} \dot{A}$. For any sufficiently large cardinal θ, pick a model $M \prec H_\theta$ of size less than δ such that $\{\dot{k}, \dot{A}, p, P\} \cup \mu \subset M$ and such that there is an M-generic condition $p_0 < p$. Then p_0 forces that for every $\xi < \mu$ there is $q \in M \cap \dot{G}$ and $x \in M$ such that $q \Vdash \dot{k}(\xi) = x$, and hence p_0 forces that $\dot{A} \subset M$. $\square$

Corollary 2.19. *If P is a forcing notion such that the trivial condition $\mathbf{1}^P$ is M-generic for stationarily many sets M of size less than δ, then P has the δ-chain condition.*

Proof. Let $\mathcal{A}$ be a maximal antichain in P, and apply the proof of the lemma with the singleton $G \cap \mathcal{A}$ as A and with $p_0 = \mathbf{1}^P$. $\square$

We say that a forcing notion P *has meets* if any compatible pair p, q of conditions has a greatest lower bound $p \wedge q$. The following definition states another property shared by all strongly generic conditions in this paper:

Definition 2.20. If P is a notion of forcing with meets and X is a set, then we say that a strongly X, P-generic condition p is *tidy* if there is a function $q \mapsto q|X$ witnessing the strong X-genericity of p such that $(q \wedge q')|X = q|X \wedge q'|X$ whenever $q, q' \leq p$ are compatible.

Proposition 2.21. *Suppose that a strongly X-generic condition is tidy with witnessing function $q \mapsto q|X$. Then* (i) *$q'|X \leq q|X$ for all $q' \leq q \leq p$, and* (ii) *$q \leq^* q|X$ for all $q \leq p$.*

Proof. For clause (i), we have $q'|X = (q' \wedge q)|X = (q'|X) \wedge (q|X) \leq q|X$. For clause (ii), if $q \not\leq^* q|X$, then there is $q' \leq q$ such that $q' \Vdash q|X \notin \dot{G}$; however $q' \wedge (q'|X) \leq q'$ and by clause (i) $q'|X \leq q|X$. $\square$

The next lemma states the critical fact which makes the existence of a strongly generic condition necessary to the constructions in this paper.

Lemma 2.22. *Suppose that p is a tidy strongly X, P-generic condition, and that stationarily many models M of size δ have strongly generic conditions for P. Let G be a generic subset of P with $p \in G$, and suppose $k \in V[G]$ is a function with domain $\mu \in V$ such that $k{\restriction}x \in V[G \cap X]$ for each $x \in ([\mu]^\delta)^V$. Then $k \in V[G \cap X]$.*

Recall that the strong X, P-genericity of p forces that $\dot{G} \cap X$ is a V-generic subset of $P \cap X$. Thus any two conditions $q, q' \leq p|X$ are compatible in $P \cap X$ if and only if the conditions $p \wedge q$ and $p \wedge q'$ are compatible in P, and if $q \leq p|X$ is in X and ϕ is any formula, then $q \Vdash_{P \cap X} \phi(\dot{G} \cap X)$ if and only if $p \wedge q \Vdash_P \phi(\dot{G} \cap X)$.

Proof. Let $\dot{k}$ be a name for k and let $p_0 \leq p$ be a condition which forces that $\dot{k}$ satisfies the hypothesis of the lemma. Let θ be a cardinal larger than κ and pick a model $M \prec H_\theta$ of size δ such that $\{ P, p_0, X, \dot{k} \} \subset M$, the function $q \mapsto q|X$ is in M, and there is a strongly M-generic condition $p_1 \leq p_0$. Let $p_2 \leq p_1$ be a condition such that $p_2 \Vdash \dot{k}{\restriction}(\mu \cap M) = \dot{s}$ for some $(P \cap X)$-term $\dot{s}$. Note that if $r \leq p_2$ is any condition such that $r \Vdash \dot{s}(\nu) = x$ for some $\nu \in \mu \cap M$, then $r|X \Vdash \dot{s}(\nu) = x$, as otherwise there would be $r' \leq r|X$ in X such that $r' \Vdash \dot{s}(\nu) \neq x$, which is impossible since r' and r are compatible.

We will show that

$$M \models \forall q \leq p \wedge (p_2|M)\ \forall \nu \in \mu\ \big(q \parallel \dot{k}(\nu) \implies (p \wedge (p_2|M) \wedge (q|X)) \parallel \dot{k}(\nu)\big). \tag{2.2}$$

Here $p \wedge (p_2|M)$ and $q|X$ are compatible since q and $q|X$ are compatible, and $q \wedge q|X \leq q \leq p \wedge (p_2|M)$. Furthermore, since the three conditions q, $p_2|M$ and $q|X$ are compatible the condition $p \wedge (p_2|M) \wedge (q|X)$ in formula (2.2) must decide $\dot{k}(\nu)$ in the same way as q does. It may also be noted that in the forcings used in this paper, and in most likely applications of Lemma 2.22, the inclusion of p in formula (2.2) is unnecessary, as $p_2 \leq p \in M$ implies $p_2|M \leq p$.

Suppose to the contrary that formula (2.2) is not valid, so that there are $q \leq p \wedge (p_2|M)$ in M, $\nu \in M \cap \mu$ and $x \in M$ such that $q \Vdash \dot{k}(\nu) = x$ but for some $r \leq p \wedge (p_2|M) \wedge (q|X)$ in M we have $r \Vdash \dot{k}(\nu) \neq x$. Then $q \wedge p_2 \Vdash \dot{s}(\nu) = \dot{k}(\nu) = x$, so $(q \wedge p_2)|X \Vdash \dot{s}(\nu) = x$. Now $r \leq p_2|M$ implies that $r \wedge p_2$ is a condition, and $r \wedge p_2 \Vdash \dot{s}(\nu) = \dot{k}(\nu) \neq x$, so $(r \wedge p_2)|X \Vdash \dot{s}(\nu) \neq x$. Thus $(r \wedge p_2)|X$ is incompatible with $(q \wedge p_2)|X$.

Now $r|X \leq^* q|X$ in $P \cap X$, since otherwise there is some $r' \leq r|X$ in X which is incompatible with $q|X$, but then r' is compatible with $r \leq q|X$ and hence with $q|X$. Thus $(r \wedge p_2)|X = r|X \wedge p_2|X \leq^* q|X \wedge p_2|X = (q \wedge p_2)|X$, again in $P \cap X$. Hence $(r \wedge p_2)|X$ and $(q \wedge p_2)|X$ are compatible, and this contradiction completes the proof of formula (2.2).

By elementarity V also satisfies the right side of formula (2.2). Since $q \Vdash q|X \in \dot{G}$ for any $q \leq p$ it follows that $p \wedge (p_2|M)$ forces that $\dot{k} \in V[\dot{G} \cap X]$:

$$\forall \nu < \mu \forall x\ \big(k(\nu) = x \iff \exists q' \in (G \cap X) \quad (p \wedge (p_2|M) \wedge q') \Vdash \dot{k}(\nu) = x\big).$$

To see this, suppose $V[G] \models k(\nu) = x$. Then there is $q \leq p \wedge (p_2|M)$ in G such that $q \Vdash \dot{k}(\nu) = x$, but then $q' = q|X \in G \cap X$ is as required. $\square$

Another application of the idea of this proof is given in [Mit06], where it is used to give an easier proof of the main lemma of [Mit73] and of a related lemma of Hamkins [Ham03].

2.3. **Strongly generic conditions in P_B.**

Lemma 2.23. *If $\lambda \in B$, then the condition $\{I_\lambda\}$ is tidily strongly H_λ-generic.*

Proof. Define $p|H_\lambda$ for $p \leq \{I_\lambda\}$ to be $p|H_\lambda := (p \cap H_\lambda) \cup \{\, C_{M\cap H_\lambda} : C_M \in p \,\}$.

It is straightforward to verify that $p|H_\lambda$ is a condition, and it is clearly tidy since each member of $(p|H_\lambda) \setminus p$ is determined by a single member of p other than I_λ.

To see that the function $p \mapsto p|H_\lambda$ witnesses the strong H_λ-genericity of $\{I_\lambda\}$, suppose that $q \leq p|H_\lambda$ is in H_λ. We need to show that the requirements in $p \cup q$ are compatible. We will show that any requirement in p is compatible with each requirement in q.

Any requirement $I_\tau \in p$ with $\tau \geq \lambda$ is compatible with any requirement in H_λ and in particular with any requirement in q, while any requirement $I_\tau \in p$ with $\tau < \lambda$ is in H_λ and hence is a member of q. Similarly, the assumption that $p \leq \{I_\lambda\}$ ensures that any requirement of the form $O_{(\eta',\eta]} \in p$ either satisfies $\lambda \leq \eta'$, in which case it is compatible with any condition in H_λ, or else it satisfies $\eta < \lambda$, in which case $O_{(\eta',\eta]} \in H_\lambda$ and hence $O_{(\eta',\eta]} \in p|H_\lambda \subset q$.

In the case of a requirement $C_N \in p$ we have $C_{N\cap H_\lambda} \in p|H_\lambda \subseteq q$. Any requirement $O_{(\eta',\eta]} \in H_\lambda$ which is compatible with $C_{N\cap H_\lambda}$ is also compatible with C_N. A requirement $I_\tau \in H_\lambda$ which is compatible with $C_{N\cap H_\lambda} \in p|H_\lambda$ must be compatible with C_N, using the same fences, unless $\sup(N) > \lambda > \tau \geq \sup(N \cap \lambda)$, and in that case the required N-fence is $I_{\min(N\setminus\tau)} = I_{\min(N\setminus\lambda)}$, which is required by the compatibility of I_λ and C_N.

Finally, if $C_{N'} \in q$, then $N' \cap N = N' \cap (N \cap H_\lambda)$, and so C_N and $C_{N'}$ satisfy clause 4(b)i of Definition 2.1 in the same way that $C_{N'}$ and $C_{N\cap H_\lambda}$ do. The N'-fence for $C_{N\cap H_\lambda}$ is also an N'-fence for C_N, and an N-fence for $C_{N'}$ is given by the $(N \cap H_\lambda)$-fence for $C_{N'}$ together with $I_{\min(N\cap\lambda)}$. □

Corollary 2.24. *If κ is inaccessible and B is stationary in κ, then P_B is κ-presaturated and hence preserves all cardinals greater than or equal to κ.*

Proof. This is immediate from Lemmas 2.23 and 2.18. □

As was pointed out earlier, Lemma 2.25 below, like Lemma 2.23 above, is a variation of the proof of properness for P_{ω_1}. Lemma 2.25 replaces the countable set $M \prec H_{\omega_1}$ with a countable set $M \prec H_\kappa$.

Lemma 2.25. *If C_M is a requirement, then $\{C_M\}$ is tidily strongly M-generic.*

Proof. The proof is similar to that of Lemma 2.23, but is more complicated because M is not transitive. For a condition $p \leq \{C_M\}$, let $\mathrm{cp}^M(p)$ be the set of all M-fences required for the compatibility of C_M with other members of p. We define the map $p \mapsto p|M$ by

$$p|M = (p \cap M) \cup \mathrm{cp}^M(p) \cup \{\, C_{N\cap M} : C_N \in p \ \ \&\ \ N \cap M \in M \,\}.$$

To see that $p|M$ is a condition, note that $(p \cap M) \cup \mathrm{cp}^M(p)$ is a condition because it is a subset of the condition $p' \supset p$, given by Lemma 2.8, which contains all C_N-fences for all $C_N \in p'$. Since $N \cap M$ is an initial segment of N for all $C_N \in p$ with $N \cap M \in M$, it is easy to see that the result of adding the requirements $C_{N\cap M}$ is still a requirement.

The function $p \mapsto \mathrm{cp}^M(p)$ is tidy, since each member of $\mathrm{cp}^M(p) \setminus p$ is determined by a single member of p other than C_M. Each member $C_{N\cap M}$ of $(p|M) \setminus \mathrm{cp}^M(p)$ is also determined by the single member C_N of p, and hence the full map $p \mapsto p|M$ is tidy.

In order to show that the function $p \mapsto p|M$ witnesses the strong M genericity of $\{C_M\}$, we need to show that if $q \leq p|M$ is in M, then any requirement in p is compatible with every requirement in q.

First consider a requirement $I_\tau \in p$. If $\tau \geq \sup(M)$, then I_τ is compatible with any requirement in M, and if $\tau \in M$, then $I_\tau \subset p \cap M \subset q$. Hence we can assume that $\tau \in \sup(M) \setminus M$. Set $\tau' = \min(M \setminus \tau)$. Then any requirement $O_{(\eta',\eta]}$ or C_N in M which is incompatible with I_τ would also be incompatible with $I_{\tau'} \in \mathrm{cp}^M(p) \subset q$, so I_τ is compatible with every requirement in q.

Any requirement $O_{(\eta',\eta]} \in p$ is compatible with C_M, and thus either $(\eta',\eta] \cap M = \varnothing$, in which case $O_{(\eta',\eta]}$ is compatible with any requirement in M, or else $O_{(\eta',\eta]} \in M$, in which case $O_{(\eta',\eta]} \in p \cap M \subseteq p|M \subseteq q$.

The case of a requirement $C_N \in p$ is somewhat more complicated than the previous two. We first show that every requirement $I_\tau \in q$ is compatible with C_N. If $\tau \geq \sup(N)$, then I_τ is compatible with C_N, and if $\sup(N) > \tau \geq \sup(M \cap N)$, then the required N-fence for I_τ is a member of the N-fence for C_M. Thus we can suppose that $\tau < \sup(M \cap N)$. If $M \cap N$ is an initial segment of N, then it follows that $\tau \in N$, so we can also suppose that $M \cap N \in M$. Then $C_{M\cap N} \in q$, and the required N-fence for I_τ is the same as the $(M \cap N)$-fence for I_τ required for the compatibility of $\{C_{M\cap N}, I_\tau\} \subseteq q$.

Now we show that any requirement $O_{(\eta',\eta]} \in q$ is compatible with C_N. If $(\eta',\eta] \cap N = \varnothing$, then this is immediate, so we can assume that there is some $\xi \in (\eta',\eta] \cap N$. We cannot have $\xi > \sup(M \cap N)$, since in that case $O_{(\eta',\eta]}$ would be incompatible with a member of the M-fence for C_N, which is contained in $\mathrm{cp}^M(p) \subseteq q$. Thus $\eta' < \delta := \sup(M \cap N)$. If $M \cap N \notin M$, then $\eta < \delta$ as well, as otherwise $O_{(\eta',\eta]}$ would be incompatible with $I_{\min(M\setminus\delta)}$, which is a member of the M-fence for C_N. But $M \cap N \notin M$ implies $M \cap \delta \subseteq N$, so $O_{(\eta',\eta]} \in N$ and thus $O_{(\eta',\eta]}$ is compatible with C_N. If, on the other hand, $M \cap N \in M$, then $C_{M\cap N} \in q$, and the compatibility of $O_{(\eta',\eta]}$ with C_N follows from its compatibility with $C_{M\cap N}$.

Finally we show that C_N is compatible with any requirement $C_{N'} \in q$. We verify clause 4(b)i first. In the case that $M \cap N$ is an initial segment of M, the set $N \cap N'$ is also an initial segment of N'. On the other hand $N \cap N'$ is a countable subset of H_δ in M, and since the cardinal $\delta' := \min(M \setminus \delta)$ is in B it follows that $N \cap N' \in M \cap H_{\delta'} = M \cap H_\delta = M \cap N$. Thus $N \cap N' \in N$.

In the other case, when $M \cap N \in M$, we have $N \cap N' = (N \cap M) \cap N'$. Since q is a condition this is an initial segment of one of N' and $N \cap M$, and either an initial segment or a member of the other. Now $N \cap N'$ will stand in the same relation to N as it does to $N \cap M$. Thus C_N and $C_{N'}$ satisfy clause 4(b)i.

It remains to verify that the necessary fences exist. If $M \cap N$ is an initial segment of M, then any N-fence for C_M is also an N-fence for $C_{N'}$. Otherwise the union of an N-fence for C_M with an $(M \cap N)$-fence for $C_{N'}$ gives an N-fence for $C_{N'}$.

If $M \cap N$ is an initial segment of M, then an N'-fence for C_N can be obtained by taking the set of all N'-fences for members of the M-fence for C_N, and otherwise the N'-fence for C_N can be obtained by taking the union of this set with an N'-fence for $C_{N\cap M}$.

This concludes the proof that any requirement in M which is compatible with $p|M$ is compatible with a requirement $C_N \in p$, and hence finishes the proof of Lemma 2.25. □

Corollary 2.26. *The forcing P_B is proper.* □

Lemma 2.27. *If B is stationary and G is a V-generic subset of P_B, then $\omega_1^{V[G]} = \omega_1$, $\omega_2^{V[G]} = \kappa$, and all larger cardinals are preserved.*

Proof. Corollary 2.26 implies that ω_1 is preserved, and Corollary 2.24 implies that κ is preserved. All larger cardinals are preserved since $|P_B| = \kappa$.

Thus we only need to show that each cardinal λ in the interval $\omega_1 < \lambda < \kappa$ is collapsed. To see this, let $Y := \{ M \cap \lambda : \lambda \in M \ \ \& \ \ C_M \in \bigcup G \}$. If C_M and $C_{M'}$ are compatible and $\lambda \in M \cap M'$, then clause 4(b)i of definition 2.1 implies that either $M \cap \lambda \subset M'$ or $M \cap \lambda \subset M$, so Y is linearly ordered by subset. Since each member of Y is countable, it follows that $|Y| \leq \omega_1$ and hence $|\bigcup Y| = \omega_1$ in $V[G]$. But $\bigcup Y = \lambda$, since for any condition $p \in P_B$ and any ordinal $\xi < \lambda$ we can find a countable set $M \prec H_\kappa$ with $\{p, \xi, \lambda\} \subset M$, so that $p \cup \{C_M\}$ is a condition extending p which forces that $\xi \in M \cap \lambda \in Y$. □

Lemma 2.28. *If $\lambda \in D \cap B$, then every function $\tau \colon \omega_1 \to V$ in $V[G]$ such that $\forall \xi < \omega_1 \ (\tau \restriction \xi \in V[G \cap H_\lambda])$ is in $V[G \cap H_\lambda]$.*

Proof. This is immediate from Lemmas 2.22, 2.23 and 2.25. □

The following observation explains why this forcing is relevant to the ideal $I[\omega_2]$:

Proposition 2.29. *Suppose that $B \subset \kappa$ is a set of inaccessible cardinals in V and that G is a generic subset of P_B. Then in $V[G]$ the restriction of the ideal $I[\omega_2]$ to ordinals of cofinality ω_1 is generated by the nonstationary ideal on ω_2 together with the single set $S = \{ \lambda \in \kappa : B \cap \lambda$ is nonstationary in $\lambda \}$.*

Furthermore, any stationary subset of $B \setminus S$ in V remains stationary in $V[G]$.

Proof. To see that $S \in I[\omega_2]$, pick for each $\lambda \in S$ a closed unbounded set $E_\lambda \subset \lambda$ in V which is disjoint from B. If $\lambda \in \lim(D) \cap S$ and $\mathrm{cf}(\lambda) > \omega$, then the set $c_\lambda := E_\lambda \cap D$ is cofinal in λ, but has order type ω_1 since any member of D of uncountable cofinality is in B.

Let $A_\nu = \{ a_\nu \cap D : \nu < \kappa \}$ where $\{ a_\nu : \nu < \kappa \}$ enumerates in V the bounded subsets of κ, and let F be the closed unbounded set of $\lambda < \kappa$ such that every bounded subset of λ in V is a member of $\{ a_\nu : \nu < \lambda \}$. Then $S \cap (\lim(D) \cap F) \subset B(A)$.

To see that no stationary subset of $B \setminus S$ is in $I[\omega_2]$, let $A = \langle a_\nu : \nu < \kappa \rangle$ be an arbitrary sequence in $V[G]$, and let $\dot{A}$ be a name for A. Fix a continuous increasing elementary chain $\langle X_\nu : \nu < \kappa \rangle$ of elementary substructures of H_{κ^+} with $\dot{A} \in X_0$, and let F be the closed unbounded set of cardinals $\lambda < \kappa$ such that $X_\lambda \cap H_\kappa = H_\lambda$.

We will show that $F \cap D \cap (B \setminus S)$ is disjoint from $B(A)$. Suppose to the contrary that $\lambda \in F \cap D \cap (B \setminus S)$ and let $c \subset \lambda$ witness that $\lambda \in B(A)$. Since the strongly X_λ-generic condition $\{I_\lambda\}$ is in G, the set $a_\nu = \dot{a}_\nu^G \in V[G \cap H_\lambda]$ for all $\nu < \lambda$. Hence $c \cap \nu \in V[G \cap H_\lambda]$ for each $\nu < \lambda$, and it follows by Lemma 2.28 that $c \in V[G \cap H_\lambda]$. However this is impossible: $G \cap H_\lambda$ is a generic subset of $P_B \cap H_\lambda = P_{B \cap \lambda}$ and $B \cap \lambda$ is a stationary subset of λ, so Lemma 2.27 implies that λ is not collapsed in $V[G \cap H_\lambda]$.

To see that any stationary subset of $B \setminus S$ remains nonstationary in $V[G]$, let $T \subseteq B \setminus S$ be stationary and let $\dot{E}$ be a name for a closed unbounded subset E of κ. Now pick a continuously increasing sequence of elementary substructures X_ν of H_{κ^+} with $\dot{E} \in X_0$ and let F be the closed unbounded set of cardinals $\lambda \in \kappa$ such that $X_\lambda \cap H_\kappa = H_\lambda$. Then $T \cap F$ is unbounded in κ, and since $T \subset B$ any condition $p \in P_B$ is compatible with $\{I_\lambda\}$ for some $\lambda \in T \cap F$. Since $\{I_\lambda\}$ is strongly H_λ-generic, it forces that $\dot{E}$ is unbounded in λ, and hence that $\lambda \in \dot{E}$. Thus $p \cup \{I_\lambda\} \Vdash \lambda \in T \cap \dot{E}$. $\square$

One other application of this forcing is of interest: like the forcing described in [Mit73] it gives a model with no special ω_2-Aronszajn trees if κ is Mahlo in V, and no ω_2-Aronszajn trees if κ is weakly compact in V. The proof is the same as in [Mit73], with Lemmas 2.22, 2.23 and 2.25 taking the place of the main lemma in that paper. It would perhaps be hard to argue that this construction is simpler than that of [Mit73], especially in view of the fact that (as is pointed out in [Mit06]) the proof of the main lemma of [Mit73] can be substantially simplified by using the idea of the proof of Lemma 2.22. However it is shown in [Mit05] that if the current forcing is simplified by eliminating requirements of the forms I_λ and $O_{(\eta',\eta]}$, and using clause 4(b)i of Definition 2.1 as the only compatibility condition, then the generic extension is still a model with no special ω_2-Aronszajn trees, or no ω_2-Aronszajn trees. This is certainly the simplest construction known of such a model, and is likely the simplest possible.

3. Adding κ^+ closed, unbounded subsets of κ

We will now extend the forcing from section 2 in order to construct a sequence $\langle D_\alpha : \alpha < \kappa^+ \rangle$ of closed, unbounded subsets of κ. This sequence will be continuously diagonally decreasing, which means that $D_{\alpha+1} \subset D_\alpha$ for all α, and that if α is a limit ordinal, then D_α is equal to the diagonal intersection $\triangle_{\alpha'<\alpha} D_{\alpha'} = \{ \nu : (\forall \alpha' \in \pi_\alpha``\nu)\ \nu \in D_{\alpha'} \}$. The definition of this diagonal intersection will depend on a choice of maps $\pi_\alpha \colon \kappa \cong \alpha$. In addition, the sets D_α will be subsets of B^*_α, where $B_\alpha = \{ \nu < \kappa : \nu \text{ is } f_\alpha(\nu)\text{-Mahlo} \}$, and the definition of the set B_α depends on the choice of the function f_α representing α in the nonstationary ideal. The first subsection describes how to use $\square_\kappa$ to define the functions π_α and f_α.

We assume throughout this section that $\square_\kappa$ holds. We also assume throughout the section that κ is inaccessible and that $2^\kappa = \kappa^+$, but only in the final subsection 3.7 will we make use of the assumption that κ is κ^+-Mahlo.

3.1. Using $\square_\kappa$.

Let $\langle C_\alpha : \alpha < \kappa^+ \rangle$ be a $\square_\kappa$ sequence. This means that if $\alpha < \kappa$, then C_α is a closed unbounded subset of α with ordertype at most κ, and if β is a limit point of C_α, then $C_\beta = C_\alpha \cap \beta$. It will be convenient to assume that $C_{\alpha+1} = \{\alpha\}$ for all α, that $C_\alpha = \alpha$ for limit $\alpha \le \kappa$, and that $\min(C_\alpha) = \kappa$ for all limit $\alpha > \kappa$. We will write $c_{\alpha,\xi}$ for the ξth member of C_α.

The desired functions π_α and f_α will be defined by writing α as a union $\bigcup_{\xi<\kappa} A_{\alpha,\xi}$ of sets $A_{\alpha,\xi}$ of size less than κ:

Definition 3.1. We define $A_{\alpha,\xi}$ for $\alpha < \kappa^+$ and $\xi < \kappa$ by recursion on α:

(1) If $\alpha = \eta + 1$, then $A_{\alpha,0} = \varnothing$ and $A_{\alpha,\xi} = A_{\eta,\xi} \cup \{\eta\}$ for $0 < \xi < \kappa$.
(2) If α is a limit point of $\lim(C_\alpha)$, then $A_{\alpha,\xi} = \bigcup\{ A_{\eta,\xi} : \eta \in \lim(C_\alpha) \}$.

(3) If α is a limit ordinal but $\lim(C_\alpha)$ is bounded in α, then set $\bar{\alpha} = \sup(\{0\} \cup \lim(C_\alpha))$, and let $\{\, \alpha_n : n < \omega \,\}$ enumerate $C_\alpha \setminus \alpha_0$ in increasing order. Thus $\bar{\alpha} = \alpha_0$ if $\mathrm{otp}(C_\alpha) > \omega$, and $\bar{\alpha} = 0$ otherwise. Then

$$A_{\alpha,\xi} = A_{\bar{\alpha},\xi} \cup \bigcup_{n<k} (A_{\alpha_n,\xi} \cup \{\alpha_n\})$$

where $k \leq \omega$ is least such that either (i) $\mathrm{otp}(C_{\alpha_0}) + k \geq \xi$, (ii) $k > 0$ and $\alpha_{k-1} \notin A_{\alpha_k,\xi}$, or (iii) $k = \omega$.

Proposition 3.2. *(1) If $\xi' < \xi < \kappa$, then $A_{\alpha,\xi'} \subseteq A_{\alpha,\xi}$.*
(2) If $\xi < \mathrm{otp}(C_\alpha)$, then $A_{\alpha,\xi} \subseteq c_{\alpha,\xi}$.
(3) $\lim(C_\alpha) \cap c_{\alpha,\xi} \subset A_{\alpha,\xi}$.
(4) $\bigcup_{\xi<\kappa} A_{\alpha,\xi} = \alpha$.

Proof. Each of the four clauses in this proposition is proved by induction on α. In the successor case 3.1(1) all clauses of this lemma follow from the induction hypothesis applied to $A_{\alpha-1,\xi}$, so we only need to consider cases 3.1(2,3).

For clause 1, the induction argument follows easily from an inspection of the terms of the definition.

In the case that α falls into case 3.1(2), clause 2 follows immediately from the induction hypotheses together with the fact that $C_\eta = C_\alpha \cap \eta$ for all $\eta \in \lim(C_\alpha)$. In the case that α falls into case 3.1(3), it follows similarly by applying the induction hypothesis to $A_{\bar{\alpha}}$ when $c_{\alpha,\xi} \leq \bar{\alpha}$, and it follows from clause (i) in the definition of k for larger ξ.

In the case that α falls into case 3.1(2), clause 3 follows from the induction hypothesis in the same way as did clause 2. Also similarly, the induction hypothesis applied to $A_{\bar{\alpha},\xi}$ verifies clause 3 when $c_{\alpha,\xi} \leq \bar{\alpha}$, and the definition of k ensures that $\alpha_0 \in A_{\alpha,\xi}$ when $c_{\alpha,\xi} > \alpha_0$.

To prove clause 4 in the case that α falls into case 3.1(2), we have $\bigcup_{\xi<\kappa} A_{\alpha,\xi} = \bigcup_{\alpha' \in \lim(C_\alpha} \bigcup_{\xi<\kappa} A_{\alpha',\xi}$ and by the induction hypthesis $\alpha' = \bigcup_{\xi<\kappa} A_{\alpha',\xi}$ for all $\alpha' \in \lim(C_\alpha)$. In case 3.1(3) we have $\bar{\alpha} = \bigcup_{\xi<\kappa} A_{\bar{\alpha},\xi} \subseteq \bigcup_{\xi<\kappa} A_{\alpha,\xi}$ and $\alpha_n = \bigcup_{\xi<\kappa} A_{\alpha_n,\xi}$ for each $n < \omega$ by the induction hypothesis. To complete the proof it will be sufficient to show that for each $n < \omega$ there is an ordinal $\xi < \kappa$ such that $k > n$, where k is the integer used in case 3.1(3) to define $A_{\alpha,\xi}$. For $n = 0$ this is true for $\xi = \mathrm{otp}(C_{\bar{\alpha}})+1$. Assume as an induction hypothesis that there is ξ_0 such that $k > n$ for $\xi \geq \xi_0$. By the induction hypothesis on α there is ξ_1 such that $\alpha_n \in A_{\alpha_{n+1},\xi_1}$, and, then $k > n+1$ whenever $\xi \geq \max(\xi_0, \xi_1, \mathrm{otp}(C_{\bar{\alpha}}) + n + 2)$. □

The next lemma states the most important property of the sets $A_{\alpha,\xi}$:

Lemma 3.3. *If $\gamma \in A_{\alpha,\xi} \cup \lim(A_{\alpha,\xi}) \cup \lim(C_\alpha)$, then $A_{\gamma,\xi} = A_{\alpha,\xi} \cap \gamma$.*

Proof. Again we prove this lemma by induction on α, and the successor case 3.1(1) is straightforward.

When α falls into case 3.1(2), we first observe that if $\gamma < \gamma'$ are in $\lim(C_\alpha)$, then $A_{\gamma,\xi} = A_{\gamma',\xi} \cap \gamma$ by the induction hypothesis, and it follows that $A_{\gamma,\xi} = A_{\alpha,\xi} \cap \gamma$ for all $\gamma \in \lim(C_\alpha)$. If $\gamma \in A_{\alpha,\xi} \cup \lim(A_{\alpha,\xi})$, then pick $\gamma' \in \lim(C_\alpha) \setminus \gamma$. Then by the induction hypothesis $A_{\gamma,\xi} = A_{\gamma',\xi} \cap \gamma = A_{\alpha,\xi} \cap \gamma$.

Now suppose α falls into case 3.1(3). Then the lemma holds for $\gamma \leq \bar{\alpha}$ by the same argument. For $\gamma > \bar{\alpha}$, note that if k is as used in the definition of $A_{\alpha,\xi}$, then for any $n < n' < k$ we have $A_{\alpha_n,\xi} = A_{\alpha_{n'},\xi} \cap \alpha_n = A_{\alpha,\xi} \cap \alpha_n$. Now for any

$\gamma \in A_{\alpha,\xi} \cup \lim(A_{\alpha,\xi})$ we must have $\alpha_n \geq \gamma$ for some $n < k$, and the lemma then follows in the same way as in case 3.1(2). □

Corollary 3.4. *If $\omega \leq \xi < \kappa$, then $|A_{\alpha,\xi}| \leq |\xi|$.*

Proof. The proof is by induction on α. The only problematic case is 3.1(3), in which case $A_{\alpha,\xi}$ is defined as a union of $|\lim(C_\alpha)|$ many sets. However Lemma 3.3 and Proposition 3.2(2) imply that in this case $A_{\alpha,\xi} = \bigcup\{ A_{\eta,\nu} : \eta \in \lim(C_\alpha) \cap c_{\alpha,\xi} \}$, a union of $|\xi|$ many sets. Since the induction hypothesis implies that each of these sets $A_{\eta,\xi}$ has size at most $|\xi|$, it follows that $|A_{\alpha,\xi}| \leq |\xi|$. □

Corollary 3.5. *If $\gamma \in \lim(A_{\alpha,\xi}) \cap \alpha$, then $\xi \geq \mathrm{otp}(C_\gamma)$ and $\gamma \in A_{\alpha,\xi+1}$. Furthermore $\gamma \in A_{\alpha,\xi}$ unless $\xi = \mathrm{otp}(C_\gamma)$.*

Proof. The proof is by induction on α. The conclusion follows immediately from the induction hypothesis and Lemma 3.3 unless $\gamma = \sup(A_{\alpha,\xi})$. It also follows easily from the induction hypothesis if α falls into one of the first two cases of Definition 3.1, so we can assume that α falls into case 3.1(3). If $k > 0$, then $\sup(A_{\alpha,\xi}) = \alpha_{k-1} \in A_{\alpha,\xi}$, and if $\gamma < \bar{\alpha}$, then the conclusion follows from the induction hypothesis. This only leaves the case $\gamma = \bar{\alpha} = \alpha_0$. Now $\alpha_0 \in \lim(A_{\alpha_0,\xi})$ implies that $\xi \geq \mathrm{otp}(C_{\alpha_0})$ by Lemma 3.2(2), and if $\xi \geq \mathrm{otp}(C_{\alpha_0}) + 1$, then $k > 0$ and so $\alpha_0 \in A_{\alpha,\xi}$. □

Corollary 3.6. *Suppose $\gamma \in \alpha \setminus C_\alpha$, and let $\bar{\gamma} = \min(C_\alpha \setminus \gamma)$. Then $\gamma \in A_{\alpha,\xi}$ if and only if $\gamma \in A_{\bar{\gamma},\xi}$ and $\bar{\gamma} \in A_{\alpha,\xi}$.* □

The following corollary, giving some other useful properties of the sets $A_{\alpha,\lambda}$, is easily proved using the definition and previous results:

Corollary 3.7. *(1) Suppose that $\nu := \sup(A_{\alpha,\lambda} \cap A_{\alpha',\lambda'}) < \min(\alpha, \alpha')$ and $\lambda > \lambda'$. Then $\nu \in A_{\alpha,\lambda} \cap A_{\alpha',\lambda'+1}$.*
(2) If λ is a limit ordinal, then $A_{\alpha,\lambda} = \bigcup_{\lambda'<\lambda} A_{\alpha,\lambda'}$.
(3) If $\alpha' < \alpha$ and $A_{\alpha',\lambda} \subset A_{\alpha,\lambda}$, then $A_{\alpha',\lambda} = A_{\alpha,\lambda} \cap \sup(A_{\alpha',\lambda})$.

Proof. For clause (1), we have $\nu \in \lim(A_{\alpha',\lambda'}) \cap \lim(A_{\alpha,\lambda})$. Then Corollary 3.5 implies that $\nu \in A_{\alpha',\lambda'+1}$. Furthermore it implies $\mathrm{otp}(C_\nu) \leq \lambda' < \lambda$, so $\nu \in A_{\alpha,\lambda}$ by the second sentence of Corollary 3.5.

The other clauses of Corollary 3.7 are straightforward. □

Definition 3.8. (1) We define $f_\alpha(\lambda) = \mathrm{otp}(A_{\alpha,\lambda})$.
(2) We write B_α for the set of cardinals λ which are $f_\alpha(\lambda)$-Mahlo. Thus κ is $(\alpha+1)$-Mahlo if and only if B_α is stationary.
(3) We write $\pi_\alpha(\eta, \lambda)$ for the ηth member of $A_{\alpha,\lambda}$, if $\mathrm{otp}(A_{\alpha,\lambda}) > \eta$, and otherwise $\pi_{\alpha,}(\eta, \lambda)$ is undefined.

Proposition 3.9. (i) *$[f_\alpha]_{NS} = \alpha$ for all $\alpha < \kappa^+$.* (ii) *If $\alpha' \in A_{\alpha,\lambda}$, then $\pi_\alpha(\eta, \lambda') = \pi_{\alpha'}(\eta, \lambda')$ for all $\lambda' \geq \lambda$ and $\eta < \mathrm{otp}(A_{\alpha',\lambda'})$.* (iii) *If $\alpha' \in \lim(C_\alpha)$, then $\pi_\alpha(\eta, \lambda) = \pi_{\alpha'}(\eta, \lambda)$ for all λ and all $\eta < \mathrm{otp}(A_{\alpha',\lambda})$.* □

We will normally write π_α"X instead of the correct, but cumbersome, expression π_α"$\big(X^2 \cap \mathrm{domain}(\pi_\alpha)\big)$.

Proposition 3.10. *Suppose that $X \prec (H_{\kappa^+}, \vec{C})$ and $\alpha \in X \setminus \lim(X)$. Then, $\alpha' := \sup(X \cap \alpha)$ is a limit point of C_α, and $X \cap \lim(C_\alpha)$ is cofinal in α'.*

Hence $C_{\alpha'} = C_\alpha \cap \alpha'$, $A_{\alpha',\xi} = A_{\alpha,\xi} \cap \alpha'$ *for every* $\xi < \kappa$, *and* $\pi_{\alpha'} = \pi_\alpha \restriction \{ (\eta, \lambda) : \pi_\alpha(\eta, \lambda) < \alpha' \}$.

Proof. By elementarity we have $\alpha' \in \lim(C_\alpha)$, and a second application of elementarity shows that $\lim(C_\alpha) \cap X$ is cofinal in α'. □

Definition 3.11. (1) If $\alpha < \kappa^+$ and $\vec{X} = \langle X_{\alpha'} : \alpha' < \alpha \rangle$ is a sequence of subsets of κ, then the *diagonal intersection* of the sequence $\vec{X}$ is the set $\triangle_{\alpha'<\alpha} X_{\alpha'} = \{ \nu < \kappa : \forall \alpha' \in A_{\alpha,\nu}\ \nu \in X_{\alpha'} \}$.

(2) A sequence $\vec{X} = \langle X_\alpha : \alpha < \kappa^+ \rangle$ is *diagonally decreasing* if $X_\alpha \setminus \lambda \subset X_{\alpha'}$ whenever $\alpha' \in A_{\alpha,\lambda}$.

(3) The sequence $\vec{X}$ is *continuously diagonally decreasing* if, in addition, $X_\alpha = \triangle_{\alpha'<\alpha} X_{\alpha'}$ whenever α is a limit ordinal.

Proposition 3.12. $\langle B_\alpha : \alpha < \kappa^+ \rangle$ *is continuously diagonally decreasing.* □

3.2. **The requirements $I_{\alpha,\lambda}$ and $O_{\alpha,(\lambda',\lambda]}$.** As in the forcing in section 2 for one closed, unbounded set, the conditions in P^* will be finite sets of requirements, inversely ordered by subset (that is, $p \leq q$ if $p \supseteq q$). The counterparts to I_λ and $O_{(\lambda',\lambda]}$ are relatively straightforward and are described in Definition 3.13; the counterparts to C_M are more complex and will be introduced in subsection 3.3. As in section 2, the subscripts of the three types of requirements are distinct and hence we can simply identify the symbols with their subscripts.

Definition 3.13. (1) $I_{\alpha,\lambda}$ is a requirement whenever $\alpha < \kappa^+$ and $\lambda \in B^*_\alpha$.

(2) $O_{\gamma,(\eta',\eta]}$ is a requirement whenever $\eta' < \eta < \kappa$, and either $\gamma = 0$ or γ is a successor ordinal smaller than κ^+.

As in the forcing in section 2, the requirements $I_{\alpha,\lambda}$ will be used to determine the new closed unbounded sets D_α: if G is a generic set, then we will define $\lambda \in D_\alpha$ if and only if there is $p \in G$ with $I_{\alpha,\lambda} \in p$. The definition of compatibility for these requirements will be determined by the analogy to the forcing of section 2, together with the desire that the sequence of sets D_α be diagonally decreasing: The analogy with section 2 suggests that $I_{\gamma,\lambda}$ should be incompatible with $O_{\gamma,(\eta',\eta]}$ whenever $\eta' < \lambda \leq \eta$, and the desire that the sets be diagonally decreasing suggests that if $\gamma \in A_{\alpha,\lambda}$, then $I_{\alpha,\lambda}$ should be incompatible with $O_{\gamma,(\eta',\eta]}$ as well.

The desire that the sequence $(D_\alpha : \alpha < \kappa^+)$ be *continuously* diagonally decreasing motivates the stipulation that the ordinal γ in a requirement $O_{\gamma,(\eta',\eta]}$ cannot be a nonzero limit ordinal: No condition should force that $\lambda \notin D_\gamma$, where γ is a nonzero limit ordinal, without also forcing that $\lambda \notin D_{\gamma'}$ for some $\gamma' \in A_{\gamma,\lambda}$.

3.3. **The requirements $C_{M,a}$.** The next three definitions give the formal definition of the requirements $C_{M,a}$. In addition to the $\square_\kappa$ sequence $\vec{C}$, we fix a well ordering $\lhd$ of H_{κ^+}, which will be used to provide Skolem functions for that set.

Definition 3.14. As used in this section, a model is a structure M such that (i) $M \prec (H_{\kappa^+}, \in, \vec{C}, \lhd)$, (ii) $M \cap \lim(C_{\sup(M)})$ is cofinal in M, and (iii) $\mathrm{otp}(C_{\sup(M)}) \notin M$.

For the remainder of this section we will write $M \prec H_{\kappa^+}$ rather than $M \prec (H_{\kappa^+}, \in, \vec{C}, \lhd)$, leaving the predicates $\in$, $\vec{C}$ and $\lhd$ to be understood. Other predicates, when needed for the construction of particular models, will be specified: thus

if X is a model and $\tau = \sup(X)$, then we may write $M \prec (X, C_\tau)$ to indicate that M is elementary with respect to the extra predicate C_τ as well as the standard predicates $\in$, $\vec{C}$ and $\lhd$.

Proposition 3.15. *If M and M' are models, then $M \cap M'$ is a model.*

Proof. The presence of the well ordering $\lhd$ provides Skolem functions which ensure that an intersection of elementary substructures is an elementary substructure. Hence $M \cap M'$ satisfies clause 3.14(i).

To verify clause 3.14(ii), set $\bar{\alpha} = \sup(M \cap M')$, and note that each of $\lim(C_{\bar{\alpha}}) \cap M$ and $\lim(C_{\bar{\alpha}}) \cap M'$ is cofinal in $\bar{\alpha}$. If $\bar{\alpha} = \sup(M)$ or $\bar{\alpha} = \sup(M')$, then this is Definition 3.14(iii); otherwise it follows from Definition 3.14(i), together, if $\bar{\alpha}$ is not in the model, with Proposition 3.10. Fix any $\gamma < \bar{\alpha}$ and let $\alpha \in M \setminus \gamma + 1$ and $\alpha' \in M' \setminus \gamma + 1$ be limit points of $C_{\bar{\alpha}}$. Then $C_{\bar{\alpha}} \cap \alpha = C_\alpha$ and $C_{\bar{\alpha}} \cap \alpha' = C_{\alpha'}$, so the least limit point of $C_{\bar{\alpha}} \setminus \gamma$ is also the least limit point of both $C_\alpha \setminus \gamma$ and of $C_{\alpha'} \setminus \gamma$, and hence is in $M \cap M'$.

To verify clause 3.14(iii), note that $\bar{\alpha} \notin M \cap M'$, as otherwise we would have $\bar{\alpha} + 1 \in M \cap M'$. If $\bar{\alpha} = \sup(M)$, then $\mathrm{otp}(C_{\bar{\alpha}}) \notin M$ and if $\bar{\alpha} = \sup(M')$, then $\mathrm{otp}(C_{\bar{\alpha}}) \notin M'$, and in either case $\bar{\alpha} \notin M \cap M'$. Otherwise set $\alpha = \min(M \setminus \bar{\alpha})$ and $\alpha' = \min(M' \setminus \bar{\alpha})$ and let $\nu = \mathrm{otp}(C_{\bar{\alpha}})$. Then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha} = C_{\alpha'} \cap \bar{\alpha}$, so $\bar{\alpha} = c_{\alpha,\nu} = c_{\alpha',\nu}$. Thus $\nu \in M \cap M'$ would imply $\bar{\alpha} \in M \cap M'$. $\square$

For most of this subsection, and all of the following two subsections, we will only be considering countable models, but in subsections 3.6 and 3.7 we will discuss models M of two other types: models M with $|M| < \kappa$ and $M \cap \kappa \in \kappa$ (corresponding to the requirement $I_{\sup(M),\sup(M\cap\kappa)}$) and transitive models M of size κ. We say that a model M of any of these three types is *simple* if $\mathrm{otp}(C_{\sup(M)}) = \sup(M \cap \kappa)$. We will show in subsection 3.7 that there are stationarily many simple models of any of these three types..

Definition 3.16. A *proxy* is a finite set of pairs (α, λ) such that $\lambda < \kappa$ and α is a limit ordinal less than κ^+. If a is a proxy, then we write $a(\lambda) = \{\, \alpha : \exists \lambda' \leq \lambda\ (\alpha, \lambda') \in a \,\}$.

Definition 3.17. $C_{M,a}$ is a requirement if M is a countable model and a is a proxy such that (i) If $(\alpha, \lambda) \in a$, then $\lambda < \sup(M \cap \kappa)$ and $\alpha > \sup(M)$, (ii) $\pi_\alpha(\zeta, \lambda) \in M$ whenever $\lambda \in M$, $\alpha \in a(\lambda)$ and $\zeta \in M$, (iii) if $\lambda \in M$ and $\alpha \in a(\lambda)$, then either $\sup(A_{\alpha,\lambda}) \in M$ or $M \cap A_{\alpha,\lambda}$ is cofinal in M, and (iv) if $\lambda \notin B_\alpha$, then $\lambda \notin B_{\sup(M \cap A_{\alpha,\lambda})}$.

We will write C_M for the requirement $C_{M,\varnothing}$ with an empty proxy, and we say that a requirement C_M is *simple* if M is.

Note that if $C_{M,a}$ is a requirement in this forcing, then $C_{M\cap H_\kappa}$ is a requirement in the forcing P_B of section 2. The effect of a requirement C_M in this forcing will be roughly the same as if the requirement $C_{M\cap H_\kappa} \in P_{B_\alpha}$ were used for each set D_α with $\alpha \in M$.

We will complete this subsection with some further useful observations about the behavior of the requirements C_M and $C_{M,a}$; but first we will briefly explain why the proxies are needed. We will want to prove, for any simple countable model M, that the condition $\{C_M\}$ is strongly M-generic. To do so we will need to define a witness function $p \mapsto p|M$. Consider the special case $p = \{C_N, C_M\}$, where N is another simple model with $M \cap N \in M$. The analogy with section 2 suggests trying

$\{C_N, C_M\}|M = \{C_{M\cap N}\}$. The problem with this is that there may be $\xi \in M$ such that $\xi > \sup(M \cap N)$ but $\xi \in A_{\eta,\lambda}$ for some $\eta \in N$ and $\lambda \in M \cap N \cap \kappa$. In that case any requirement $O_{\xi,(\lambda',\lambda]} \in M$ would be compatible with $C_{M\cap N}$; however I claim that it must be incompatible with C_N. The reason for this deals with the need for a function $q \mapsto q|N$ witnessing that $\{C_N\}$ is strongly N-generic. If η is the least ordinal in N such that $\xi \in A_{\eta,\lambda}$, then the requirement $O_{\xi,(\lambda',\lambda]}$ is incompatible with $I_{\eta,\lambda}$, but compatible with $I_{\eta',\lambda}$ for any $\eta' \in N \cap \eta$. The same should be true of $\{O_{\xi,(\lambda',\lambda]}, C_N\}|N$, and the only condition in N which would have this effect would seem to be $\{O_{\eta,(\lambda',\lambda]}\}$. However $O_{\eta,(\lambda',\lambda]}$ is not a requirement since η is a nonzero limit ordinal, and hence $\{O_{\eta,(\lambda',\lambda]}\}$ is not a condition. Since there is no good choice for $\{O_{\xi,(\lambda',\lambda]}, C_N\}|N$, our definition of the forcing will have to specify that $O_{\xi,(\lambda',\lambda]}$ is incompatible with C_N.

Thus the correct choice of $\{C_N, C_M\}|M$ must be a condition which is incompatible with every requirement $O_{\xi,(\lambda',\lambda]}$ as in the last paragraph. This will be accomplished by setting $\{C_N\}|M = \{C_{N\cap M,b}\}$ where b is a proxy chosen so that for any requirement $O_{\xi,(\lambda',\lambda]} \in M$ as in the last paragraph there is some $\eta' \in b(\lambda)$ such that $\xi \in A_{\eta',\lambda}$. The construction of $C_N|M$ will be given in section 3.6, with the construction of the proxy b given in Lemma 3.46.

Proposition 3.18. *If $C_{M,a}$ is a requirement and $\alpha \in M \cup \lim(M)$, then $M \cap \alpha = \pi_\alpha``(M \cap \kappa)$ and $M \cap C_\alpha = \{\, c_{\alpha,\nu} : \nu \in M \cap \mathrm{otp}(C_\alpha)\,\}$.*

Proof. If $\alpha \in M$, then the proposition is immediate since π_α and C_α are in M. If $\alpha \in \lim(M) \setminus M$ and $\alpha < \sup(M)$, then set $\alpha' := \min(M \setminus \alpha)$. Then $M \cap \alpha' = \pi_{\alpha'}``(M \cap \kappa)$ by the previous sentence, and Proposition 3.10 implies that this is equal to $\pi_\alpha``(M \cap \kappa)$. The second clause follows from Proposition 3.10 and the observation that $M \cap C_\alpha = M \cap C_{\alpha'}$.

Finally, if $\alpha = \sup(M)$, then $M \cap \alpha = \bigcup\{\, M \cap \nu : \nu \in \lim(C_\alpha\,\} = \bigcup\{\, \pi_\nu``(M \cap \kappa) : \nu \in \lim(C_{\alpha)}\,\} = \pi_\alpha``(M \cap \kappa)$. □

Notice that Proposition 3.18 implies in particular that the set of ordinals of any requirement C_M is determined by $M \cap \kappa$ together with $\sup(M)$. It follows, by using the well ordering $\lhd$ specified at the beginning of section 3.3, that these determine C_M itself.

Corollary 3.19. *If $C_{M,a}$ and $C_{N,b}$ are requirements, then $M \cap N = \pi_{\bar\alpha}``(M \cap N \cap \kappa) \subset A_{\bar\alpha,\sup(M\cap N\cap\kappa)}$ where $\bar\alpha = \sup(M \cap N)$.* □

Corollary 3.20. *Suppose that $C_{M,a}$ and $C_{N,b}$ are requirements with $M \cap N \cap \kappa \in M$. Then $M \cap N \in M$.*

Proof. First note that $\bar\alpha := \sup(M \cap N) < \sup(M)$, since $\mathrm{otp}(C_{\bar\alpha}) \in \lim(M \cap N \cap \kappa) \subset M$ while $\mathrm{otp}(C_{\sup(M)}) \notin M$. Then $M \cap N = \pi_{\bar\alpha}``(M \cap N \cap \kappa) = \pi_{\min(M\setminus\bar\alpha)}``(M \cap N \cap \kappa) \in M$. □

Definition 3.21. If $C_{M,a}$ is a requirement, then we write $A_{M,a,\lambda}$ for $\bigcup\{\, A_{\alpha,\lambda} : \alpha \in M \cup a(\lambda)\,\}$.

The following observation will be used frequently.

Lemma 3.22. *Suppose that $C_{M,a}$ and $C_{N,b}$ are requirements such that $N \cap \sup(M \cap N \cap \kappa) \subset M$. Then $A_{M,a,\lambda} \cap N \subset M$ for all $\lambda < \sup(M \cap N \cap \kappa)$.*

Proof. Suppose that $\eta' \in N \cap A_{\eta,\lambda}$, where $\lambda < \sup(M \cap N \cap \kappa)$ and $\eta \in M \cup a(\lambda)$. By increasing λ if necessary, we can assume that $\lambda \in M \cap N$. Then $\gamma := \mathrm{otp}(A_{\eta',\lambda}) \in N \cap \lambda^+ \subset M$, since $|A_{\eta',\lambda}| \leq \lambda$ and both $N \cap \kappa$ and $M \cap \kappa$ are closed under cardinal successor. However $A_{\eta',\lambda} = A_{\eta,\lambda} \cap \eta'$ so $\eta' = \pi_\eta(\gamma, \lambda) \in M$. □

Lemma 3.23. *Suppose that $M \cap N \cap \kappa \subseteq M$. Furthermore, suppose that $\lambda \in M \cap N \cap \kappa$ and $\alpha \in N \cup b(\lambda)$, and let $\bar{\alpha} := \sup(M \cap N)$.*

(1) If $A_{\alpha,\lambda} \cap \bar{\alpha}$ is bounded in $\bar{\alpha}$, then $\sup(A_{\alpha,\lambda} \cap M) \in M \cap N$.
(2) If $A_{\alpha,\lambda} \cap \bar{\alpha}$ is unbounded in $\bar{\alpha}$ and $\bar{\alpha} < \sup(N)$, then $\alpha' := \min(N \setminus \bar{\alpha}) \in \lim(A_{\alpha,\lambda})$ and $A_{\alpha,\lambda} \cap M \subset \alpha'$.

Proof. For clause 1, suppose that $A_{\alpha,\lambda}$ is bounded in $\bar{\alpha}$ and set $\gamma := \sup(A_{\alpha,\lambda} \cap A_{\bar{\alpha},\delta})$ where $\delta := \sup(M \cap N \cap \kappa)$. Since $\lambda < \delta$, Corollary 3.7(1) implies that $\gamma \in A_{\alpha,\lambda+1} \cap A_{\bar{\alpha},\delta}$. Also, since $\bar{\alpha}$ and δ are limit ordinals there are $\bar{\alpha}' \in (\lim(C_{\bar{\alpha}}) \cap M \cap N) \setminus \gamma$ and $\delta' \in M \cap N \cap \kappa$ so that $\gamma \in A_{\bar{\alpha}',\delta'}$. Then $\gamma = \sup(A_{\alpha,\lambda} \cap A_{\bar{\alpha}',\delta'}) \in N$, and it follows by Lemma 3.22 that $\gamma \in M$ as well. Thus $\gamma \in M \cap N$, and it remains to show that $M \cap A_{\alpha,\lambda} \subset \gamma + 1$. Suppose to the contrary that there is $\eta > \gamma$ in $A_{\alpha,\lambda} \cap M$, and set $\gamma' := \min(A_{\alpha,\lambda} \setminus \gamma + 1)$. Then $\gamma' \in N$, and $\gamma' = \min\left(A_{\eta,\lambda} \cup \{\eta\} \setminus \gamma + 1\right) \in M$. Thus $\gamma' > \gamma$ is in $M \cap N$, contradicting the choice of γ.

Now suppose that the hypothesis to clause 2 holds. First we show that we can assume that $\alpha \in N$: Otherwise $\alpha \in b(\lambda)$, but in that case clause 3.17(iii) implies that $\alpha'' := \sup(N \cap A_{\alpha,\lambda})$ is either a member of N or else is equal to $\sup(N)$. If $\alpha'' \in N$, then it will be sufficient to show that clause 2 holds with α'' in place of α, and if $\alpha'' = \sup(N)$ it will be sufficient to show that clause 2 holds for any member of $(N \setminus \alpha') \cap A_{\alpha,\lambda}$ in place of α.

Now $\sup(\alpha' \cap A_{\alpha,\lambda}) \in N$ because α', α and λ are in N. Since $\bar{\alpha} \leq \sup(\alpha' \cap A_{\alpha,\lambda}) \leq \alpha' = \min(N \setminus \bar{\alpha})$ it follows that $\alpha' \in \lim(A_{\alpha,\lambda})$.

It remains to show that $A_{\alpha,\lambda} \cap M \subseteq \alpha'$. Suppose to the contrary that there is some ordinal $\eta \in A_{\alpha,\lambda} \cap M \setminus \alpha'$. Then $\alpha' \in M$, either because $\eta = \alpha'$ or because $\eta > \alpha'$, in which case $\alpha' \in \lim(A_{\alpha,\lambda}) \cap \eta = \lim(A_{\eta,\lambda})$, so $\alpha' \in A_{\eta,\lambda+1}$ by Corollary 3.5 and thus $\alpha' \in M$ by Lemma 3.22. However $\alpha' \in M \cap N$ would imply $\alpha' + 1 \in M \cap N$, contradicting the fact that $\alpha' \geq \bar{\alpha} = \sup(M \cap N)$. □

3.4. **Definition of the forcing P^*.** The definition of the forcing P^*, given in Definitions 3.25 and 3.26 below, is very nearly a word for word copy—with the mechanical addition of the extra subscripts—of Definitions 2.1 and 2.2 of the forcing P_B in section 2. The most significant changes appear in clauses 1 and 2. The change in clause 1, which was alluded to at the end of subsection 3.2, is needed to account for the added subscripts α in $I_{\alpha,\lambda}$ and γ in $O_{\gamma,(\eta',\eta]}$. The change in clause 2, using $M[a]$ in place of M, was alluded to in subsection 3.3 and is needed to take account of proxies.

A more subtle change comes in the definition of an M-fence: if $I_{\alpha,\lambda}$ is an M-fence, then λ is required to be a member of M, but α is not. We will see in subsection 3.6 that if M is simple, then α can also be taken to be a member of M. For this reason we will have strongly generic conditions for simple models, but only for simple models.

Except for these changes, the definition is essentially a word for word copy of Definitions 2.1 and 2.2 with the additional subscripts mechanically added to the requirements.

Definition 3.24. We write $M[a]$ for $\{\, I_{\alpha,\lambda} : \lambda \in M \cap B^*_\alpha \ \ \&\ \ \alpha \in M \cup a(\lambda) \,\}$.

Definition 3.25. (1) Two requirements $O_{\gamma,(\eta',\eta]}$ and $I_{\alpha,\lambda}$ are *incompatible* if $\eta' < \lambda \leq \eta$ and $\gamma \in A_{\alpha+1,\lambda}$; otherwise they are *compatible*.

(2) Two requirements $O_{\gamma,(\eta',\eta]}$ and $C_{M,a}$ are *compatible* if either $O_{\gamma,(\eta',\eta]} \in M$ or every requirement $I_{\alpha,\lambda} \in M[a]$ is compatible with $O_{\gamma,(\eta',\eta]}$.

(3) (a) An *M-fence for a requirement* $I_{\alpha,\lambda}$ is a requirement $I_{\alpha',\lambda'}$ with $\lambda' \in M$ such that any requirement $O_{\gamma,(\eta',\eta]}$ in M incompatible with $I_{\alpha,\lambda}$ is also incompatible with $I_{\alpha',\lambda'}$.

(b) Two requirements $C_{M,a}$ and $I_{\alpha,\lambda}$ are *compatible* if either $\lambda \geq \sup(M \cap \kappa)$ or there exists an M-fence for $I_{\alpha,\lambda}$.

(4) (a) An *M-fence* for a requirement $C_{N,b}$ is a finite set x of requirements $I_{\alpha,\lambda}$, with $\lambda \in M \cap B_\alpha$, with the following property: Suppose that $O_{\gamma,(\eta',\eta]} \in M$ is a requirement such that $\eta \geq \sup(M \cap N \cap \kappa)$, $\eta' \geq \sup(M \cap N \cap \kappa)$ if $M \cap N \in M$, and $O_{\gamma,(\eta',\eta]}$ is incompatible with $C_{N,b}$. Then there is some requirement $I_{\alpha,\lambda} \in x$ which is incompatible with $O_{\gamma,(\eta',\eta]}$.

(b) A model M *is fenced from* a requirement $C_{N,b}$ if (i) either $M \cap N \cap H_\kappa \in M$ or $M \cap N \cap H_\kappa = M \cap H_{\sup(M \cap N \cap \kappa)}$, and (ii) there is an M-fence for $C_{N,b}$.

(c) Two requirements $C_{M,a}$ and $C_{N,b}$ are *compatible* if M is fenced from $C_{N,b}$ and N is fenced from $C_{M,a}$.

Definition 3.26. A condition p in the forcing P^* is a finite set of requirements such that each pair of requirements in p is compatible. The conditions are ordered by the reverse inclusion: $p' \leq p$ if $p' \supseteq p$.

Although this forcing is somewhat more complicated than the forcing P_B, our exposition will parallel the exposition in section 2. Like P_B, the forcing P^* is not separative and we will write $p' \leq^* p$ if $p' \Vdash p \in \dot{G}$ and $p =^* p'$ if $p \leq^* p'$ and $p' \leq^* p$. In addition we introduce the following notation for a special case of the failure of separation:

Definition 3.27. We say that $I_{\alpha,\lambda} \in^* p$ if there is $I_{\alpha',\lambda} \in p$ such that either $\alpha \in A_{\alpha'+1,\lambda}$ or α is a limit ordinal and $A_{\alpha,\lambda}$ is a subset (and hence an initial segment) of $A_{\alpha',\lambda}$.

Proposition 3.28. *If $p \in P^*$ and $I_{\alpha,\lambda} \in^* p$, then $p \cup \{I_{\alpha,\lambda}\} \in P^*$. Hence $p \cup \{I_{\alpha,\lambda}\} =^* p$.*

Proof. Let $I_{\alpha',\lambda} \in p$ witness that $I_{\alpha,\lambda} \in^* p$. Then any requirement $O_{\gamma,(\eta',\eta]}$ which is incompatible with $I_{\alpha,\lambda}$ is also incompatible with $I_{\alpha',\lambda}$, and it follows that $I_{\alpha,\lambda}$ is compatible with any requirement $O_{\gamma,(\eta',\eta]} \in p$. In addition, if M is a model, then any M-fence for $I_{\alpha',\lambda}$ is also an M-fence for $I_{\alpha,\lambda}$, and it follows that any requirement $C_{M,a} \in p$ is compatible with $I_{\alpha,\lambda}$. Hence $p \cup \{I_{\alpha,\lambda}\} \in P^*$.

To see that $p \Vdash I_{\alpha,\lambda} \in \bigcup \dot{G}$, note that the first paragraph implies that $q \cup \{I_{\alpha,\lambda}\} \in P^*$ for any condition $q \leq p$. $\square$

Proposition 3.29. *If $C_{M,a}$ is a requirement, then any requirement $R \in M$ is compatible with $C_{M,a}$. Thus $p \cup \{C_{M,a}\} \in P^*$ for any $p \in M \cap P^*$.*

Proof. Any requirement $I_{\alpha,\lambda} \in M$ is its own fence for compatibility with $C_{M,a}$, and any requirement $O_{\alpha,(\eta',\eta]} \in M$ is compatible with $C_{M,a}$. If $C_{N,b} \in M$, then, since

$M \cap N = N$, the empty set $\varnothing$ is both an M-fence for $C_{N,b}$ and an N-fence for $C_{M,a}$. $\square$

Unlike the case in the forcing P_B, the fences specified in Definition 3.25 are not unique. In the next two lemmas we will give an alternate characterization of compatibility, and show that if any M-fence exists, then there is a unique minimal M-fence:

Proposition 3.30. *Suppose that M is a model and $I_{\alpha,\lambda}$ is a requirement with $\lambda < \sup(M \cap \kappa)$. Set*

$$\lambda' = \min(M \setminus \lambda) \qquad \text{and} \qquad \alpha' = \sup(\{\, \gamma + 1 : \gamma + 1 \in M \cap A_{\alpha+1,\lambda} \,\}).$$

Then $I_{\alpha,\lambda}$ is compatible with $C_{M,a}$ if and only if $\lambda' \in B^*_{\alpha'}$.

Furthermore, in this case $I_{\alpha',\lambda'}$ is an M-fence for $I_{\alpha,\lambda}$ which is minimal in the sense that if $I_{\alpha'',\lambda''}$ is any other M-fence for $I_{\alpha,\lambda}$, then (i) $\lambda'' = \lambda'$, (ii) $\alpha' \leq \alpha''$, *and* (iii) $I_{\alpha',\lambda'} \in^* \{I_{\alpha'',\lambda''}\}$.

Notice that $\alpha' = \alpha$ if $\alpha = 0$ or α is a successor ordinal in M. We will call the fence $I_{\alpha',\lambda'}$ of Proposition 3.30 the *minimal M-fence* for $I_{\alpha,\lambda}$.

Proof. First, suppose that $\lambda' \in B^*_{\alpha'}$, so that $I_{\alpha',\lambda'}$ is a requirement. If $O_{\gamma,(\eta',\eta]}$ is a requirement in M which is incompatible with $I_{\alpha,\lambda}$, then, because $\gamma \in A_{\alpha+1,\lambda}$ and γ cannot be a limit ordinal, the choice of α' ensures that $\gamma \in A_{\alpha'+1,\lambda}$. Also $\eta' < \sup(M \cap \lambda) \leq \lambda \leq \lambda' \leq \eta$, so $O_{\gamma,(\eta',\eta]}$ is incompatible with $I_{\alpha',\lambda'}$. It follows that $I_{\alpha',\lambda'}$ is an M-fence for $I_{\alpha,\lambda}$, and hence $I_{\alpha,\lambda}$ is compatible with $C_{M,a}$.

For the other direction, suppose that $I_{\alpha,\lambda}$ is compatible with $C_{M,a}$ and let $I_{\alpha'',\lambda''}$ be an arbitrary M-fence for $I_{\alpha,\lambda}$. First we observe that $\lambda'' = \lambda'$; otherwise pick $\eta \in M \cap \lambda$ such that $\eta > \lambda''$ if $\lambda'' < \lambda$. Then the requirement $O_{0,(\eta,\lambda']}$ is incompatible with $I_{\alpha,\lambda}$ but is compatible with $I_{\alpha'',\lambda''}$.

If α' is a successor ordinal, then it must be a member of M. In that case $O_{\alpha',(\eta,\lambda']}$ is a requirement in M which is incompatible with $I_{\alpha,\lambda}$ and hence must be incompatible with $I_{\alpha'',\lambda'}$, and it follows that $\alpha' \in A_{\alpha''+1,\lambda}$.

If α' is a limit ordinal, then let S be the set of ordinals $\gamma+1$ such that $O_{\gamma+1,(\eta,\lambda']}$ is in M and incompatible with $I_{\alpha,\lambda}$. Then each ordinal $\gamma+1 \in S$ must be a member of $A_{\alpha''+1,\lambda}$. Since S is cofinal in α', it follows that α' is a limit point of $A_{\alpha'',\lambda}$, and hence $\alpha' \leq \alpha''$ and $I_{\alpha',\lambda'} \in^* \{I_{\alpha'',\lambda'}\}$. $\square$

Proposition 3.31. *Suppose that M is a model, $\alpha, \lambda \in M$, and $\gamma \in A_{\alpha,\lambda} \setminus M$. Then $\gamma \in A_{\min(M\setminus\gamma),\lambda}$.*

Proof. Set $\alpha' = \min(M \setminus \gamma)$. Then $\sup(A_{\alpha,\lambda} \cap \alpha') \in M$, since α, λ and α' are. But $\gamma \leq \sup(A_{\alpha,\lambda} \cap \alpha') \leq \alpha'$, and since $\alpha' = \min(M \setminus \gamma)$ it follows that $\alpha' = \sup(A_{\alpha,\lambda} \cap \alpha') \in \lim(A_{\alpha,\lambda})$. Thus Lemma 3.3 implies that $A_{\alpha',\lambda} = A_{\alpha,\lambda} \cap \alpha'$, so $\gamma \in A_{\alpha',\lambda}$. $\square$

Lemma 3.32. *Let M and N be any two models, and set $\bar{\alpha} = \sup(M \cap N)$ and $\delta = \sup(M \cap N \cap \kappa)$. Then for any $\{\alpha, \lambda\} \subset N$ with $\delta \leq \lambda < \kappa$ we have $A_{\alpha,\lambda} \cap M \cap \bar{\alpha} \subseteq A_{\bar{\alpha},\lambda}$.*

Proof. Fix $\gamma \in A_{\alpha,\lambda} \cap M \cap \bar{\alpha}$. If $\gamma \in N$, then Corollary 3.19 implies that $\gamma \in A_{\bar{\alpha},\delta}$, so we can assume that $\gamma \notin N$. By Proposition 3.31 we can assume that $\alpha = \min(N \setminus \gamma) < \bar{\alpha}$. We will show, by induction on ν, that $\gamma \in A_{\nu,\lambda}$ for all $\nu \geq \alpha$ in $(M \cap N \cap \bar{\alpha})$. Since $\lim(C_{\bar{\alpha}}) \cap (M \cap N)$ is cofinal in $\bar{\alpha}$ it will follow that $\eta \in A_{\bar{\alpha},\lambda}$.

Fix such an ordinal ν. If $\alpha \in \lim(C_\nu)$, then $\gamma \in A_{\alpha,\lambda} = A_{\nu,\lambda} \cap \alpha$, so we can assume that $\alpha \notin \lim(C_\nu)$.

Set $\nu' = \min(C_\nu \setminus \alpha)$, so $\nu' \in N$. Since $\nu' \notin \lim(C_\nu)$ there is $\nu'' \in C_\nu \cup \{0\}$ such that $\nu' = \min(C_\nu \setminus \nu'' + 1)$. Then $\nu'' \in N$ and it follows by the minimality of ν' that $\nu'' < \alpha \leq \nu'$. Since $\alpha = \min(N \setminus \gamma)$ it follows that $\nu'' < \gamma \leq \alpha$. This implies that $\nu' = \min(C_\nu \setminus \gamma) \in M$, so $\nu' \in M \cap N$ and the induction hypothesis implies that $\gamma \in A_{\nu',\lambda}$. Furthermore, since $\nu' \in M \cap N$ the least ordinal λ' such that $\nu' \in A_{\nu,\lambda'}$ is also in $M \cap N$, so $\lambda' < \delta$ and hence $\nu' \in A_{\nu,\delta} \subseteq A_{\nu,\lambda}$. Thus $\gamma \in A_{\nu',\lambda} = A_{\nu,\lambda} \cap \nu'$. □

We now consider the compatibility of requirements C_M and C_N. It is easy to see that if C_M and C_N are compatible, then $C_{M\cap H_\kappa}$ and $C_{N\cap H_\kappa}$ are compatible in the forcing P_{B_0} of section 2. In particular $M \cap \kappa$ and $N \cap \kappa$ fall into the pattern of Figure 1: a common initial segment which is followed by a finite alternating sequence of disjoint intervals. For pairs M and N which satisfy Definition 3.25(4(b)i), so that $M \cap N \cap \kappa$ is an initial segment of at least one of M and N, this can be concisely expressed by the statement $\lim(M \cap \kappa) \cap \lim(N \cap \kappa) = \lim(M \cap N \cap \kappa)$. The following proposition shows that this equality also holds above κ:

Proposition 3.33. *If M and N are countable models such that* $\lim(M \cap \kappa) \cap \lim(N \cap \kappa) = \lim(M \cap N \cap \kappa)$*, then* $\lim(M) \cap \lim(N) = \lim(M \cap N)$.

Proof. Suppose $\alpha \in \lim(M) \cap \lim(N)$. Then Proposition 3.18 implies that $M \cap C_\alpha = \{\, c_{\alpha,\nu} : \nu \in M \cap \mathrm{otp}(C_\alpha) \,\}$ and $N \cap C_\alpha = \{\, c_{\alpha,\nu} : \nu \in N \cap \mathrm{otp}(C_\alpha) \,\}$. Since $C_\alpha \cap M$ is a cofinal subset of M and $C_\alpha \cap N$ is a cofinal subset of N, it follows that $\mathrm{otp}(C_\alpha) \in \lim(M) \cap \lim(N) \cap \kappa = \lim(M \cap N \cap \kappa)$. Thus $\{\, c_{\alpha,\nu} : \nu \in M \cap N \cap \mathrm{otp}(C_\nu) \,\} \subset M \cap N$ is cofinal in α. □

Lemma 3.34. *Suppose $C_{N,b}$ is a requirement, M is a countable model, and the models M and N satisfy Definition* 3.25*(*4(b)i*). Let y be the set of requirements $I_{\alpha,\lambda}$ such that*

(1) $\sup(M \cap N \cap \kappa) \leq \lambda = \sup(N \cap \lambda') < \lambda'$ *for some* $\lambda' \in M \cap \kappa$*, and*
(2) either (i) $\alpha \in b(\lambda)$*,* (ii) $\alpha = \sup(N \cap M)$*, or* (iii) $\alpha = \min(N \setminus \alpha') > \alpha'$ *for some* $\alpha' \in M \setminus \sup(M \cap N)$.

Then there is an M-fence for $C_{N,b}$ if and only if $\lim(M \cap N) = \lim(M) \cap \lim(N)$ *and each of the requirements in y is compatible with C_M.*

Furthermore, in this case let x be the set of minimal M-fences for requirements in y. Then x is an M-fence for $C_{N,b}$, and x is minimal in the sense that if x' is any other M-fence for $C_{N,b}$, then $I_{\alpha,\lambda} \in^* x'$ for any $I_{\alpha,\lambda} \in x$.

Proof. Note that every member $I_{\alpha,\lambda} \in y$ is a requirement since $\mathrm{cf}(\lambda) = \omega$ and hence $\lambda \in B^*_\alpha$. Let us say that a requirement $O_{\gamma,(\eta',\eta]} \in M$ *clashes* with $C_{N,b}$ if it is incompatible with $C_{N,b}$, $\eta \geq \sup(M \cap N \cap \kappa)$, and $\eta' \geq \sup(M \cap N \cap \kappa)$ if $M \cap N \in M$. Thus an M-fence for $C_{N,b}$ is a finite set x of requirements $I_{\alpha,\lambda}$ such that $\lambda \in M$ and any requirement $O_{\gamma,(\eta',\eta]} \in M$ which clashes with $C_{N,b}$ is incompatible with some member of x.

We begin by showing that every requirement $O_{\gamma,(\eta',\eta]} \in M$ which clashes with C_N is incompatible with some member of y. To this end let $O_{\gamma,(\eta',\eta]}$ be a requirement

in M which clashes with $C_{N,b}$, and let I_{α_0,λ_0} be a requirement in $N[b]$ which is incompatible with $O_{\gamma,(\eta',\eta]}$.

Set $\lambda = \sup(N \cap \eta)$. Then $I_{\alpha_0,\lambda}$ is a requirement since $\mathrm{cf}(\lambda) = \omega$, and $\gamma \in A_{\lambda_0,\alpha_0} \subseteq A_{\lambda,\alpha_0}$, so $O_{\gamma,(\eta',\eta]}$ is incompatible with $I_{\alpha_0,\lambda}$.

If $\alpha_0 \in b(\lambda)$, then $I_{\alpha_0,\lambda} \in y$. If $\sup(M \cap N) < \alpha_0 \in N$, then set $\alpha = \min(N \setminus \gamma)$. Then $\gamma \in A_{\alpha,\lambda_0}$ by Proposition 3.31, and since $A_{\alpha,\lambda_0} \subseteq A_{\alpha,\lambda}$ it follows that $O_{\gamma,(\eta',\eta]}$ is incompatible with $I_{\alpha,\lambda} \in y$.

Thus we can assume that $\gamma < \sup(M \cap N)$. If $\lambda_0 \geq \delta := \sup(M \cap N \cap \kappa)$, then Lemma 3.32 implies that $\gamma \in A_{\bar{\alpha},\lambda_0} \subseteq A_{\bar{\alpha},\lambda}$, where $\bar{\alpha} = \sup(M \cap N)$, and hence $O_{\gamma,(\eta',\eta]}$ is incompatible with $I_{\bar{\alpha},\lambda} \in y$.

The only remaining case has $\gamma < \sup(M \cap N)$ and $\lambda_0 < \delta$. Since $\lambda_0 \in (\eta', \eta]$, this implies that $\eta' < \delta$ and by Definition 3.25 we must have $M \cap N \notin M$, so $M \cap \sup(M \cap N \cap \kappa) \subset N$. By Lemma 3.22 (with M and N switched) it follows that $\gamma \in A_{\alpha_0,\lambda_0} \cap M \subseteq A_{N,b,\lambda_0} \cap M \subset N$. Thus $\gamma \in M \cap N \subset A_{\bar{\alpha},\delta} \subseteq A_{\bar{\alpha},\lambda}$, so again $O_{\gamma,(\eta',\eta]}$ is incompatible with $I_{\bar{\alpha},\lambda} \in y$.

This completes the proof that any requirement $O_{\gamma,(\eta',\eta]} \in M$ which clashes with $C_{N,b}$ is incompatible with some requirement $I_{\alpha,\lambda} \in y$. Now suppose that each requirement $I_{\alpha,\lambda}$ in y is compatible with C_M, and let x be the set of minimal M-fences for members of y. Then any requirement $O_{\gamma,(\eta',\eta]} \in M$ which clashes with $C_{N,b}$ is incompatible with some member $I_{\alpha,\lambda}$ of y and hence with its minimal M-fence $I_{\alpha',\lambda'} \in x$. If, in addition, $\lim(M) \cap \lim(N) = \lim(M \cap N)$, then y, and hence x, is finite: If y were infinite, then there would be an ordinal in $\lim(M) \cap \lim(N) \setminus \lim(M \cap N)$ either as the limit of infinitely many cardinals λ from clause 1, or else as the limit of infinitely many ordinals α from clause 2(iii).

This completes the proof that if each member of y is compatible with C_M and $\lim(M) \cap \lim(N) = \lim(M \cap N)$, then there is an M-fence for $C_{N,b}$.

Now we verify the final paragraph of the lemma. Let $I_{\alpha,\lambda}$ be any member of y, let $I_{\alpha',\lambda'}$ be the minimal M-fence for $I_{\alpha,\lambda}$, and suppose that x' is an M-fence for $C_{N,b}$. Suppose that $I_{\alpha',\lambda'} \notin^* x'$, and pick $\eta \in M \cap \alpha$ such that $\eta > \lambda''$ for all $I_{\alpha'',\lambda''} \in x'$ with $\lambda'' < \lambda$. Then the proof of Proposition 3.30 shows that $O_{\alpha',(\eta,\lambda']} \in M$ is compatible with every requirement $I_{\alpha'',\lambda''} \in x'$, but $O_{\alpha',(\eta,\lambda']}$ is incompatible with $I_{\alpha,\lambda}$. Now pick $\bar{\lambda} \in N \cap \lambda$ such that $\bar{\lambda} > \eta$ and $\mathrm{cf}(\bar{\lambda}) = \omega$. If α was given by clause 2(i) or 2(iii), then set $\bar{\alpha} = \alpha$; otherwise pick $\bar{\alpha} \in \lim(C_\alpha)$ such that $\gamma < \bar{\alpha}$. Then $I_{\bar{\alpha},\bar{\lambda}}$ is a member of N and is incompatible with $O_{\gamma,(\eta,\lambda']}$, contradicting the assumption that x' is an M-fence for $C_{N,b}$. This completes the proof that the fence x is minimal among all M-fences for $C_{N,b}$.

The last paragraph shows something more: it did not assume that members of y are compatible with C_M or that $\lim(M) \cap \lim(N) = \lim(M \cap N)$, and hence it implies that if there exists an M-fence x' for $C_{N,b}$, then x' must include a minimal M-fence for each member of y. This implies that each member of y has an M-fence, and hence is compatible with C_M. Also, since x' is finite it follows that y is finite, but it is easy to see that this implies that $\lim(M) \cap \lim(N) = \lim(M \cap N)$. This completes the proof of the right to left direction of the equivalence, and hence of Lemma 3.34. $\square$

3.5. **Completeness.** At the end of this subsection we will give a complete characterization, for any condition $p \in P^*$, of the set of requirements $I_{\alpha,\lambda}$ such that $p \Vdash I_{\alpha,\lambda} \in \bigcup \dot{G}$. For the proof of Theorem 1.2, however, we will not use this characterization but rather two intermediate results. The first of these will be needed in

order to define the witness $p \mapsto p|M$ for the strong genericity of a countable simple model M:

Definition 3.35. If $p \in P^*$ and $C_{M,a} \in p$, then $\mathrm{cp}^M(p)$ is the set of all requirements $I_{\alpha,\lambda}$ such that $\alpha = \min(M \setminus \alpha')$ for some requirement $I_{\alpha',\lambda}$ which is a minimal M-fence for some requirement in p.

Notice that every member of $\mathrm{cp}^M(p)$ is a member of M, and is an M-fence for the minimal M-fence from which it was defined and hence for the requirement which demanded that minimal M-fence. In general $\mathrm{cp}^M(p)$ need not include a complete set of M-fences for members of p, since a minimal M-fence $I_{\alpha',\lambda}$ may have $\alpha' = \sup(M)$. We will see later that if M is simple, then this cannot happen.

Lemma 3.36. *Suppose that $p \in P^*$ and $C_{M,a} \in p$. Then $p \cup \mathrm{cp}^M(p) \in P^*$, $p \cup \mathrm{cp}^M(p) =^* p$, and $\mathrm{cp}^M(p \cup \mathrm{cp}^M(p)) = \mathrm{cp}^M(p)$.*

The second asserts that the forcing P^* does in fact add new closed unbounded sets D_α:

Definition 3.37. If $G \subset P^*$ is generic and $\alpha < \kappa^+$, then we write $D_\alpha = \{\, \lambda < \kappa : I_{\alpha,\lambda} \in \bigcup G \,\}$.

Lemma 3.38. *The sequence $\vec{D} = \langle\, D_\alpha : \alpha < \kappa^+ \,\rangle$ is a continuously diagonally decreasing sequence of closed unbounded subsets of κ.*

The difficulty here is in showing that the sets D_α are closed; the rest of Lemma 3.38 can easily be proved with the machinery already developed.

The proof of Lemma 3.36 will be given after the next two lemmas, which contain the substance of the proof.

Lemma 3.39. *Suppose $p \in P^*$ and $C_{M,a} \in p$, and let x be the set of minimal M-fences for requirements in p. Then $p \cup x \in P^*$, $p \cup x =^* p$, and $p \cup x$ includes an M-fence for every requirement in $p \cup x$.*

Proof. Let $I_{\alpha,\lambda}$ be a minimal M-fence for one of the requirements $I_{\chi,\tau}$ or $C_{N,b}$ in p. We will show that $I_{\alpha,\lambda}$ is compatible with all requirements $O_{\gamma,(\eta',\eta]}$ and $C_{M',a'}$ in p. Since $I_{\alpha,\lambda}$ is also a fence for any $q \le p$ in P^*, this will imply that $q \cup \{I_{\alpha,\lambda}\} \in P^*$. It follows that $p \Vdash I_{\alpha,\lambda} \in \bigcup \dot{G}$, that is, that $p =^* p \cup \{I_{\alpha,\lambda}\}$. This will be sufficient to prove the lemma, since it follows by an easy induction that $p \cup x =^* p$.

First we show that $I_{\alpha,\lambda}$ is compatible with every requirement $O_{\gamma,(\eta',\eta]} \in p$. In the case that $O_{\gamma,(\eta',\eta]} \notin M$ the compatibility of $O_{\gamma,(\eta',\eta]}$ and $C_{M,a}$ implies that $O_{\gamma,(\eta',\eta]}$ is compatible with every requirement $I_{\alpha',\lambda'} \in M[a]$, and since $\lambda \in M$ and $\alpha \in M \cup \lim(M)$ this implies that $O_{\gamma,(\eta',\eta]}$ is compatible with $I_{\alpha,\lambda}$. Thus we can assume that $O_{\gamma,(\eta',\eta]} \in M$. If $I_{\alpha,\lambda}$ is the minimal fence for $I_{\chi,\tau} \in p$, then $O_{\gamma,(\eta',\eta]}$ is compatible with $I_{\chi,\tau}$, since both are in p, and by the minimality of $I_{\alpha,\lambda}$ it follows that $O_{\gamma,(\eta',\eta]}$ is compatible with $I_{\alpha,\lambda}$. On the other hand, if $I_{\alpha,\lambda}$ is a minimal fence for $C_{N,b} \in p$, then it follows from $\eta \ge \lambda \ge \sup(M \cap N \cap \kappa)$ that $O_{\gamma,(\eta',\eta]} \notin N$, and hence the compatibility of $O_{\gamma,(\eta',\eta]}$ with $C_{N,b}$ implies that $O_{\gamma,(\eta',\eta]}$ is compatible with every requirement $I_{\alpha',\lambda'} \in N[b]$. Again, the minimality of $I_{\alpha,\lambda}$, then implies that $O_{\gamma,(\eta',\eta]}$ is compatible with $I_{\alpha,\lambda}$.

Now we show that $I_{\alpha,\lambda}$ is compatible with any requirement $C_{M',a'} \in p$. The proof proceeds by verifying the final statement of the lemma, by showing if $I_{\alpha,\lambda}$

is not its own M'-fence, then the minimal M'-fence for $I_{\alpha,\lambda}$ is the same as the minimal M'-fence for some requirement in p.

If $\lambda \geq \sup(M \cap M' \cap \kappa)$, then any M'-fence for $C_{M,a}$ includes an M'-fence for $I_{\alpha,\lambda}$, so we can assume that $\lambda < \sup(M \cap M' \cap \kappa)$. If $\lambda \in M'$, then $I_{\alpha,\lambda}$ is its own M'-fence, so we can assume that $\lambda \notin M'$. Hence $M \cap M' \cap \kappa \in M$. We will show that any M'-fence for $I_{\chi,\tau}$ or for $C_{N,b}$ is or includes an M'-fence for $I_{\alpha,\lambda}$.

To this end, suppose $O_{\gamma,(\eta',\eta]}$ is some requirement in M' which is incompatible with $I_{\alpha,\lambda}$. Since $\lambda < \sup(M \cap M' \cap \kappa)$ we can assume that $\eta < \sup(M \cap M' \cap \kappa)$, so that $M \cap M' \in M$ implies that $\eta', \eta \in M$. Since $O_{\gamma,(\eta',\eta]}$ is incompatible with $I_{\alpha,\lambda}$, we have $\gamma \in A_{\alpha,\lambda} \cap M'$ and it follows by Lemma 3.22 that $\gamma \in M$. Hence $O_{\gamma,(\eta',\eta]}$ is in M.

If $I_{\alpha,\lambda}$ is the minimal M-fence for $I_{\chi,\tau}$, then the minimality of $I_{\alpha,\lambda}$ implies that $O_{\gamma,(\eta',\eta]}$ is incompatible with $I_{\chi,\tau}$, and hence is incompatible with the M'-fence for $I_{\chi,\tau}$. This shows that any M'-fence for $I_{\chi,\tau}$ is an M'-fence for $I_{\alpha,\lambda}$, and thus implies that $I_{\alpha,\lambda}$ is compatible with $C_{M',a'}$.

If $I_{\alpha,\lambda}$ is a minimal M-fence for $C_{N,b}$, then it follows similarly that $O_{\gamma,(\eta',\eta]}$ is incompatible with some $I_{\alpha',\lambda'} \in N[b]$. We will show that $\eta \geq \sup(M' \cap N \cap \kappa)$. It then follows that $O_{\gamma,(\eta',\eta]}$ is incompatible with some member of any M'-fence for $C_{N,b}$, and this implies that any M'-fence for $C_{N,b}$ includes an M'-fence for $I_{\alpha,\lambda}$ and hence completes the proof of Lemma 3.36.

Suppose to the contrary that $\eta < \sup(M' \cap N \cap \kappa)$. If $M' \cap N \cap \kappa$ is an initial segment of M', that is, $M' \cap \sup(M' \cap N \cap \kappa) \subseteq N$, then $\eta \in M \cap M' \cap \sup(M' \cap N \cap \kappa) \subset M \cap N$; however this is impossible since the fact that $I_{\alpha,\lambda}$ is the minimal M-fence for $C_{N,a}$ implies that $\sup(M \cap N \cap \kappa) \leq \lambda < \eta$. Hence we must have $M' \cap N \in M'$, and it follows that $\sup(N \cap \lambda) \in M' \cap \lambda \subset M$. However by Lemma 3.34, the fact that $I_{\alpha,\lambda}$ is in the minimal M-fence for $C_{N,b}$ implies that $\sup(N \cap \lambda) < \lambda = \min(M \setminus \sup(N \cap \lambda))$, so that $\sup(N \cap \lambda) \notin M$. $\square$

Lemma 3.40. *Suppose that $p \in P^*$, $\alpha < \kappa^+$ is a nonzero limit ordinal and $p \Vdash \lambda \notin \dot{D}_\alpha$. Then there is a successor ordinal $\gamma \in A_{\alpha,\lambda}$ such that $p \Vdash \lambda \notin \dot{D}_\gamma$.*

Furthermore if $C_{M,a} \in p$, $\lambda \in M \cup \lim(M)$, and $\alpha \in M \cup a(\lambda)$, then the least such ordinal γ is a member of M.

Proof. The hypothesis that $p \Vdash \lambda \notin \dot{D}_\alpha$ could hold either because $\lambda \notin B^*_\alpha$, so that $I_{\alpha,\lambda}$ is not a requirement, or because $I_{\alpha,\lambda}$ is incompatible with some requirement in p.

If $\lambda \notin B^*_\alpha$, then there is a successor $\gamma \in A_{\alpha,\lambda}$ such that $\lambda \notin B^*_\gamma$, and hence $\varnothing \Vdash \lambda \notin \dot{D}_\gamma$. Furthermore, if $C_{M,\alpha}$ is as in the second paragraph, then $\lambda \in M$, since $\lambda \notin B^*_\alpha$ implies that $\mathrm{cf}(\lambda) > \omega$, and hence $\lambda \notin \lim(M)$. Then it is easy to see that there is some such γ in M, using elementarity if $\alpha \in M$ and Definition 3.17(iv) if $\alpha \in a(\lambda)$.

Thus we can assume that $I_{\alpha,\lambda}$ is a requirement and that $I_{\alpha,\lambda}$ is incompatible with some requirement in p. Now if a requirement $O_{\gamma,(\eta',\eta]} \in p$ is incompatible with $I_{\alpha,\lambda}$, then $p \leq \{O_{\gamma,(\eta',\eta]}\} \Vdash \lambda \notin \dot{D}_\gamma$. If the hypothesis of the second paragraph holds, then the compatibility of $O_{\gamma,(\eta',\eta]}$ with $C_{M,a}$ implies that $O_{\gamma,(\eta',\eta]} \in M$ and thus $\gamma \in M$.

The last possibility is that $I_{\alpha,\lambda}$ is incompatible with $C_{N,b} \in p$. Then $\lambda' := \min(N \setminus \lambda) \notin B^*_\beta$ where $\beta = \sup\{\, \xi + 1 : \xi + 1 \in A_{\alpha+1,\lambda} \cap N \,\}$. It follows that there is a successor ordinal $\gamma \in N \cap A_{\alpha,\lambda}$ such that $\lambda' \notin B^*_\gamma$. Thus $p \leq \{C_{N,b}\} \Vdash \lambda \notin \dot{D}_\gamma$.

Now suppose that the hypothesis of the second paragraph holds. If $\lambda \geq \sup(M \cap N \cap \kappa)$, then any N-fence for $C_{M,a}$ includes an N-fence for $I_{\alpha,\lambda}$, contradicting the assumption that $I_{\alpha,\lambda}$ is incompatible with $C_{N,b}$. Thus we must have $\lambda < \sup(M \cap N \cap \kappa)$. Also $\lambda \notin N$, or else $I_{\alpha,\lambda}$ would be its own N-fence, and hence we must have $N \cap M \in M$. Then Lemma 3.22 implies that $\gamma \in A_{\alpha,\lambda} \cap N \subset M$.

This completes the proof of Lemma 3.40, except that under the hypothesis of the second paragraph we have only shown that there exists a successor ordinal $\gamma \in A_{\alpha,\lambda} \cap M$ such that $p \Vdash \lambda \notin \dot{D}_\gamma$, not that the least such γ is a member of M. Now let γ' be the least ordinal in $A_{\alpha,\lambda} \cap M$ such that $p \Vdash \lambda \notin \dot{D}_{\gamma'}$. Then γ' cannot be a limit ordinal, since in that case we could apply the lemma with γ' in place of α. Thus γ' must be a successor ordinal, say $\gamma' = \gamma'' + 1$, but then $\gamma'' \in M$ and hence $p \nVdash \lambda \notin \dot{D}_{\gamma''}$. Thus γ' is the least member γ of $A_{\alpha,\lambda}$ such that $p \Vdash \lambda \notin \dot{D}_\gamma$. □

Proof of Lemma 3.36. We need to show that every requirement $I_{\alpha,\lambda} \in \mathrm{cp}^M(p)$ is compatible with p. If $I_{\alpha,\lambda}$ is not compatible with p, then $p \Vdash \lambda \notin \dot{D}_\alpha$, and by Lemma 3.40 it follows that $p \Vdash \lambda \notin \dot{D}_\gamma$ for some $\gamma \in M \cap A_{\alpha,\lambda}$. However by the definition of $\mathrm{cp}^M(p)$ we have $\alpha = \min(M \setminus \alpha')$ where $I_{\alpha',\lambda}$ is a minimal M-fence for some requirement in p. Now $A_{\alpha,\lambda} \cap M = A_{\alpha',\lambda} \cap M$, so it follows that $\gamma \in A_{\alpha',\lambda}$ and hence $I_{\alpha',\lambda}$ is incompatible with p; however this contradicts Lemma 3.39. □

One more lemma is needed for the proof of Lemma 3.38.

Lemma 3.41. *Suppose that $C_{M,a} \in p$, and either* (i) $I_{\alpha,\lambda} = I_{\sup(M),\sup(M\cap\kappa)}$, *or else* (ii) $\lambda = \sup(M \cap \lambda') < \lambda'$ *for some $I_{\alpha,\lambda'} \in^* p$ with $\lambda' \in M$ and $\alpha \in M \cup \lim(M) \cup a(\lambda)$. Then $I_{\alpha,\lambda}$ is compatible with p, and indeed $p \cup \{I_{\alpha,\lambda}\} =^* p$.*

Proof. As in previous lemmas, it will be sufficient to show that $I_{\alpha,\lambda}$ is compatible with p, since this implies that $I_{\alpha,\lambda}$ is compatible with any $p' \leq p$ and hence $p \leq^* p \cup \{I_{\alpha,\lambda}\}$.

Since $\mathrm{cf}(\lambda) = \omega$, $\lambda \in B^*_\alpha$ and hence $I_{\alpha,\lambda}$ is a requirement. We first show that $I_{\alpha,\lambda}$ is compatible with any requirement $O_{\gamma,(\eta',\eta]} \in p$. In case (i), where $\lambda = \sup(M)$, any requirement $O_{\gamma,(\eta',\eta]}$ which is incompatible with $I_{\alpha,\lambda}$ is incompatible with $C_{M,a}$, and hence is not in p.

In case (ii), with $\lambda = \sup(M \cap \lambda')$ where $\lambda' \in M$ and $\alpha \in M \cup \lim(M) \cup a(\lambda)$, any requirement $O_{\gamma,(\eta',\eta]}$ incompatible with $I_{\alpha,\lambda}$ is incompatible with some requirement $I_{\alpha'',\lambda''} \in M[a]$ and hence must be a member of M, but this implies that $\eta \geq \lambda'$ so that $O_{\gamma,(\eta',\eta]}$ is incompatible with $I_{\alpha,\lambda'} \in^* p$.

Thus $I_{\alpha,\lambda}$ is compatible with every requirement $O_{\gamma,(\eta',\eta]} \in p$. Now we show that $I_{\alpha,\lambda}$ is compatible with any requirement $C_{N,b}$ in p. If $\lambda \geq \sup(M \cap N \cap \kappa)$, then any requirement $O_{\gamma,(\eta',\eta]} \in N$ which is incompatible with $I_{\alpha,\lambda}$ is incompatible with the requirements $I_{\alpha'',\lambda''} \in M[a]$ and hence must be incompatible with some member of any N-fence for $C_{M,a}$. Hence the N-fence for $C_{M,a}$ includes an N-fence for $I_{\alpha,\lambda}$.

Thus we can assume that $\lambda < \sup(M \cap N \cap \kappa)$. In particular, $\lambda \neq \sup(M \cap \kappa)$, so $I_{\alpha,\lambda}$ comes from clause (ii) for some requirement $I_{\alpha,\lambda'} \in^* p$. If $N \cap \sup(M \cap N \cap \kappa) \subset M$, then $\min(N \setminus \lambda) = \min(N \setminus \lambda')$, so any N-fence for $I_{\alpha,\lambda'}$ is an N-fence for $I_{\alpha,\lambda}$. Otherwise $M \cap N \in N$, so $\lambda \in N$ and hence $I_{\alpha,\lambda} \in p'$ is its own N-fence. □

Proof of Lemma 3.38. To see that D_α is unbounded in κ, let p be any condition in P^* and suppose $\zeta < \kappa$. Pick $\tau > \zeta$ of cofinality ω so that $\tau > \eta$ for all requirements $O_{\alpha,(\eta',\eta]} \in p$ and $\tau > \sup(M \cap \kappa)$ for all $C_{M,a} \in p$. Then $I_{\alpha,\tau}$ is a requirement, $p' := p \cup \{I_{\alpha,\tau}\} \leq p$, and $p' \Vdash \tau \in \dot{D}_\alpha \setminus \eta$.

Proposition 3.28 implies that if $\lambda \in D_\alpha$ and $\alpha' \in A_{\alpha,\lambda}$, then $\lambda \in D_{\alpha'}$, so $\vec{D}$ is diagonally decreasing. Lemma 3.40 implies that if α is a limit ordinal and $p \Vdash \lambda \in \dot{D}_{\alpha'}$ for all $\alpha' \in A_{\alpha,\lambda}$, then $p \Vdash \lambda \in \dot{D}_\alpha$, so $\vec{D}$ is continuously diagonally decreasing.

Thus it only remains to show that D_α is closed for each $\alpha < \kappa^+$. We will show that for any condition p and ordinals α and λ such that $p \nVdash \lambda \in \dot{D}_\alpha$, there is a requirement $O_{\alpha,(\eta',\eta]}$, compatible with p, such that $\eta' < \lambda \leq \eta$. Then $p \cup \{O_{\alpha,(\eta',\eta]}\}$ is a condition extending p which forces that $\dot{D}_\alpha \cap \lambda \subseteq \eta'$, so that λ is not a limit point of D_α.

By extending p if necessary, and taking α to be minimal, we may assume that p forces that $p \Vdash \lambda \in \left(\bigcap_{\alpha' \in A_{\alpha,\lambda}} \dot{D}_{\alpha'}\right) \setminus \dot{D}_\alpha$. It follows by Lemma 3.40 that α is either 0 or a successor ordinal, say $\alpha = \alpha_0 + 1$. The case $\alpha = 0$ is identical to Lemma 2.11, so we will assume $\alpha > 0$. By further extending p if necessary, we may assume that there is an ordinal $\tau > \lambda$ such that $p \Vdash \tau = \min(\dot{D}_\alpha \setminus \lambda)$.

Let $Y = \{\, C_{M,a} \in p : \alpha \in M \ \ \& \ \ \tau \in \lim(M) \,\}$. If $Y = \varnothing$, then set $\eta = \lambda$. Otherwise note that for any two members $C_{M,a}$ and $C_{M',a'}$ of Y, the fact that $\tau \in \lim(M) \cap \lim(M') = \lim(M \cap M')$ implies that $\sup(M \cap M' \cap \kappa) \geq \tau$ and hence one of $M \cap \tau$ and $M' \cap \tau$ is contained in the other. Thus $\{\, M \cap \tau : C_{M,a} \in Y \,\}$ is linearly ordered by $\subseteq$. Pick $C_{M,a} \in Y$ with $M \cap \tau$ minimal, and let $\eta = \min(M \setminus \lambda)$. The desired requirement will be $O_{\alpha,(\eta',\eta]}$ for some suitably chosen $\eta' < \lambda$.

If $Y \neq \varnothing$, then the choice of η ensures that $O_{\alpha,(\eta',\eta]}$ is compatible with any requirement $C_{M',a'} \in Y$ so long as $\eta' \in M$. It remains to show that η' can be chosen so that $O_{\alpha,(\eta',\eta]}$ is also compatible with the requirements $I_{\gamma,\xi} \in p$ and $C_{M',a'} \in p \setminus Y$.

Let $I_{\gamma,\xi}$ be a requirement in p. If $\alpha \notin A_{\gamma+1,\xi}$, then $I_{\gamma,\xi}$ is compatible with $O_{\alpha,(\eta',\eta]}$ for any $\eta' < \lambda$, so we can assume that $\alpha \in A_{\gamma+1,\xi}$. It follows that $p \Vdash \xi \in \dot{D}_\alpha$, so the choice of τ ensures that $\xi < \lambda$ or $\xi \geq \tau$. If $\xi \geq \tau$, then $O_{\alpha,(\eta',\eta]}$ is compatible with $I_{\alpha,\lambda}$ for any $\eta' < \lambda$, so we can assume that $\xi < \lambda$. Then $O_{\alpha,(\eta',\eta]}$ is compatible with $I_{\gamma,\xi}$ for any $\eta' \in \lambda \setminus \xi$. If $Y = \varnothing$, then we are done; otherwise we need to show that $\sup(M \cap \lambda) > \xi$. To see this, note that if $\xi' = \min(M \setminus \xi)$, then $p \Vdash \xi' \in \dot{D}_\alpha$ because of the M-fence for $I_{\gamma,\xi}$. Since $\xi' \leq \eta < \tau$ it follows that $\xi' < \lambda$.

It remains to consider the requirements $C_{M',a'} \in p \setminus Y$. We first show that if $C_{M',a'} \in p$ and $\alpha \notin M'$, then $O_{\alpha,(\eta',\eta]}$ is compatible with $C_{M',a'}$ for any $\eta' < \lambda$. Suppose to the contrary that $O_{\alpha,(\eta',\eta]}$ is incompatible with $C_{M',a'}$. Then there is some ordinal $\alpha' \in M' \cup a'(\eta)$ such that $\alpha \in A_{\alpha',\lambda}$. The least such ordinal α' is a limit ordinal, and $p \Vdash \eta \notin \dot{D}_{\alpha'}$, so Lemma 3.40 implies that there is a successor ordinal $\gamma \in M' \cap A_{\alpha',\lambda}$ such that $p \Vdash \eta \notin \dot{D}_\gamma$. By the choice of α', we must have $\gamma < \alpha$ and hence $\gamma \in A_{\alpha,\lambda} = A_{\alpha_0+1,\lambda}$, and it follows that $p \Vdash \eta \notin \dot{D}_{\alpha_0}$. If $\eta = \lambda$ this contradicts the choice of α. If $\eta > \lambda$, then $I_{\alpha_0,\eta} \in^* p$ as the M-fence for $I_{\alpha_0,\lambda} \in^* p$, and so again $p \Vdash \eta \in \dot{D}_{\alpha_0}$. This contradiction completes the proof that $O_{\alpha,(\eta',\eta]}$ is compatible with $C_{M',\alpha'}$.

The only remaining requirements to consider are $C_{M',a'} \in p \setminus Y$ with $\alpha \in M'$. Now note that Lemma 3.36 implies that $p \Vdash \sup(M' \cap \tau) \in \dot{D}_\alpha$, so $\sup(M' \cap \tau) < \lambda$. If $Y = \varnothing$, then it follows that $O_{\alpha,(\eta',\eta]}$ is compatible with $C_{M',a'}$ so long as $\eta' > \sup(M' \cap \lambda)$. If $Y \neq \varnothing$, then we must show that $\sup(M \cap \lambda) > \sup(M' \cap \lambda)$, so that η' can be chosen to be a member of M. Suppose first that $\delta :=$

$\sup(M \cap M' \cap \kappa) \geq \tau$. Since $\eta \in M \setminus M'$ this implies that $M \cap M' \in M$, and in particular $\sup(M \cap M' \cap \lambda) \in M$, so $\sup(M \cap \lambda) > \sup(M' \cap \lambda)$. Now suppose that $\delta < \tau$, and hence $\delta < \lambda$. If $M \cap M' \in M$ and $\delta = \sup(M' \cap \lambda)$, then again $M' \cap \lambda \subset M$ and hence $\sup(M \cap \lambda) > \sup(M' \cap \lambda)$. Otherwise, $p \Vdash \min(M \setminus \sup(M' \cap \lambda)) \in \dot{D}_\alpha$ because of the M-fence for $C_{M',\alpha}$, so as in the case of $I_{\gamma,\xi}$ it follows that $\min(M \setminus \sup(M' \cap \lambda)) < \lambda$ and hence $\sup(M \cap \lambda) > \sup(M' \cap \lambda)$.

It follows that if $\eta' < \lambda$ is chosen so that $\eta' > \xi$ for all $I_{\gamma,\xi} \in p$ with $\xi < \lambda$ and $I_{\alpha,\xi} \in^* \{I_{\gamma,\xi}\}$, and $\eta' > \sup(M' \cap \lambda)$ for all $C_{M',a'} \in p \setminus Y$ with $\alpha \in M'$, then $O_{\alpha,(\eta',\eta]}$ is compatible with all requirements in $p \setminus Y$; furthermore, we have shown that if $Y \neq \varnothing$, then such ordinals η' can be found in $M \cap \lambda$. Such a choice of η' gives a condition $O_{\alpha,(\eta',\eta]}$ compatible with p which forces that $\dot{D}_\alpha$ is bounded in λ, and it follows that D_α is closed. $\square$

This completes the proof of Lemmas 3.36 and 3.38. In the remainder of this subsection, which is not needed for the proof of Theorem 1.2, we briefly explain how the proofs of these lemmas can be used to give a characterization, for an arbitrary condition p, of the pairs (α, λ) such that $p \Vdash \lambda \in \dot{D}_\alpha$.

This characterization generates the set of such pairs through four steps. We write $A^*_{\alpha,\lambda}$ for the intersection of $A_{\alpha,\lambda}$ with $\kappa^+ \setminus \lim(\kappa^+)$; that is, $A^*_{\alpha,\lambda}$ contains only 0 and the successor ordinals from $A_{\alpha,\lambda}$.

Step 1. By Lemma 3.36, we can assume without loss of generality that p includes, for each requirement $C_{M,a} \in p$, the minimal M-fence for each requirement in p.

Step 2. Suppose $C_{M,a} \in p$, $\lambda \in M \cup \lim(M)$, and $\alpha \in M \cup a(\lambda)$. If $I_{\alpha',\lambda} \in^* p$, where $\alpha' = \sup(M \cap A^*_{\alpha,\lambda})$, then $p \Vdash \lambda \in \dot{D}_\alpha$.

This follows from Lemma 3.40. With some care it can be shown that any condition p can be extended to a condition $p' \leq p$ with $p' =^* p$ so that $I_{\alpha,\lambda} \in^* p$ for each pair (α, λ) as in this step. A key point in the argument is that the requirements $C_{M,a} \in p$ should be considered in the order of their size: if $\lambda \in M \cap M' \in M'$, then the pairs (α, λ) from $C_{M,a}$ should be dealt with before those from $C_{M',a'}$.

Step 3. If $C_{M,a} \in p$, $\lambda \in M$, and $\lambda' = \sup(M \cap \lambda) < \lambda$, then $p \Vdash \lambda' \in \dot{D}_\alpha$ whenever $\alpha \in M \cup a(\lambda)$ and $p \Vdash \lambda \in \dot{D}_\alpha$.

This follows from Lemma 3.41, and it is straightforward to verify that any condition p can be extended to a condition $p' \leq p$ with $p' =^* p$ such that $I_{\alpha,\lambda'} \in^* p'$ for all (α, λ') as in this step.

Step 4. If $\alpha' = \sup(A^*_{\alpha,\lambda})$ and $p \Vdash \lambda \in \dot{D}_{\alpha'}$, then $p \Vdash \lambda \in \dot{D}_\alpha$.

This follows immediately from the fact that the sequence $\vec{D}$ is continuously decreasing. This situation is actually an artifact of our definition of the sets $A_{\alpha,\lambda}$: if these sets had been defined to be closed under successor, then it would never happen that $\sup(A_{\alpha,\lambda}) > \sup(A^*_{\alpha,\lambda})$. Such a change would make this characterization more natural.

The proof of Lemma 3.38 shows that for any condition p which has been extended as described in steps 1-3, the only pairs (α, λ) for which $p \Vdash \lambda \in \dot{D}_\alpha$ are those such that $I_{\alpha,\lambda} \in^* p$ and those coming from step 4 in which $I_{\alpha',\lambda} \in^* p$.

3.6. **Strongly generic conditions.** Earlier we described three types of simple models: in addition to the countable models we have uncountable models $X \prec H_{\kappa^+}$ of size less less than κ with $X \cap H_\kappa$ transitive, and transitive models $X \prec H_{\kappa^+}$ with $\kappa \subset X$. The main result of this section asserts that each of these has a strongly generic condition.

Lemma 3.42. *Suppose that X is a simple model, and that* $\sup(X \cap \kappa) \in B_{\sup(X)}$ *if* $\omega < |X| < \kappa$. *Set*

$$p^X = \begin{cases} \{C_X\} & \text{if } X \text{ is countable,} \\ \{I_{\sup(X),\sup(X\cap\kappa)}\} & \text{if } X \cap \kappa \in \kappa, \\ \varnothing & \text{if } X \text{ is transitive.} \end{cases}$$

Then p^X is a tidy strongly X-generic condition.

The reason for requiring that X be a simple model is given by the following observation, which will be used to define the function $p \mapsto p|X$ witnessing strong genericity.

Proposition 3.43. *Suppose that X is a simple model, $\alpha < \kappa^+$, and $\lambda < \sup(X \cap \kappa)$. Then $A_{\alpha,\lambda} \cap X$ is bounded in X.*

Proof. Suppose to the contrary that $A_{\alpha,\lambda}$ is unbounded in X. Set $\bar{\alpha} = \sup(X) \leq \sup(A_{\alpha,\lambda}) \leq \alpha$ and let α' be the least member of $A_{\alpha+1,\lambda} \setminus \bar{\alpha}$. Then $A_{\alpha,\lambda} \cap \alpha' = A_{\alpha',\lambda} \subset c_{\alpha',\lambda}$. Now $\bar{\alpha} \in \lim(C_{\alpha'})$ since $A_{\alpha,\lambda}$ is unbounded in X, so $C_{\bar{\alpha}} = C_{\alpha'} \cap \bar{\alpha}$. However, since X is simple we have $\operatorname{otp}(C_{\bar{\alpha}}) = \sup(X \cap \kappa) > \lambda$ and hence $\sup(X \cap A_{\alpha',\lambda}) \leq c_{\alpha',\lambda} = c_{\bar{\alpha},\lambda} < \bar{\alpha}$, contrary to assumption. □

In order to make use of this fact we extend to arbitrary models some of the notation previously associated to countable models M. Recall that $A^*_{\alpha,\lambda} = A_{\alpha,\lambda} \cap (\kappa^+ \setminus \lim(\kappa^+))$.

Definition 3.44. If X is an uncountable model, then $I_{\alpha',\lambda}$ is an X-fence for $I_{\alpha,\lambda}$ if $\lambda \in X$ and every requirement $O_{\gamma,(\eta',\eta]} \in X$ which is incompatible with $I_{\alpha,\lambda}$ is also incompatible with $I_{\alpha',\lambda}$.

We say that $I_{\alpha',\lambda}$ is the *minimal X-fence* for $I_{\alpha,\lambda}$ if $\alpha' = \sup(A^*_{\alpha,\lambda} \cap X)$.

We write $\operatorname{cp}^X(p)$ for the set of requirements $I_{\alpha'',\lambda}$ such that $\alpha'' = \min(X \setminus \alpha')$ where $I_{\alpha',\lambda}$ is the minimal X-fence for some requirement $I_{\alpha,\lambda} \in p$.

Note that the definitions are identical to those given previously for countable models (except that if M is countable, then $\operatorname{cp}^M(p)$ also includes M-fences for requirements $C_{N,b} \in p$).

The X-fences from Definition 3.44 have the same properties as M-fences.

Proposition 3.45. *If X is a model of any type, and $p \leq p^X$ if X is countable, then $p \cup \operatorname{cp}^X(p) \in P^*$, and $p \cup \operatorname{cp}^X(p) =^* p$. Furthermore if X is simple, then $\operatorname{cp}^X(p)$ includes an X-fence for every requirement $I_{\alpha,\lambda} \in p \cup \operatorname{cp}^X(P)$ with $\lambda < \sup(X \cap \lambda)$, and if X is countable as well as simple, then $\operatorname{cp}^X(p)$ also includes an X-fence for every requirement $C_{N,b} \in p$.*

Proof. The first statement was proved for countable models as Lemma 3.36, so we can assume that X is uncountable and hence $X \cap \kappa$ is transitive, Suppose that $I_{\alpha,\lambda} \in \operatorname{cp}^X(p)$, say that $\alpha = \min(X \setminus \alpha'')$ where $\alpha'' = \sup(A^*_{\alpha',\lambda} \cap X)$ for some $I_{\alpha',\lambda} \in p$. Then $I_{\alpha'',\lambda} \in^* p$, so $I_{\alpha'',\lambda}$ is compatible with p. But since $X \cap \lambda$ is transitive and λ and α are in X, the set $A_{\alpha,\lambda}$ is a subset of X and hence is contained in $A_{\alpha'',\lambda}$. Thus any fence for $I_{\alpha'',\lambda}$ is also a fence for $I_{\alpha,\lambda}$.

For the second statement, if X is uncountable, then $\lambda < \sup(X \cap \lambda)$ implies that $\lambda \in X$, and if X is countable, then the compatibility of p, together with the assumption that $p \leq p^X$, imply that every stated requirement has a fence which

is a requirement $I_{\alpha,\lambda}$ (or a finite set of such requirements) with $\lambda \in X$. Thus it is enough to show that if $I_{\alpha',\lambda}$ is the minimal X-fence for any requirement $I_{\alpha,\lambda}$ with $\lambda \in X$, then $\alpha' < \sup(X)$. Since $\alpha' = \sup(A^*_{\alpha,\lambda} \cap X)$, this follows from Proposition 3.43. □

We are now ready to start the proof of Lemma 3.42. The function $p \mapsto p|X$ witnessing the strong X-genericity of p^X is defined by the equation

$$p|X = (p \cap X) \cup \mathrm{cp}^X(p) \cup \{\, C_{M,a}|^*X : C_{M,a} \in p \ \ \&\ \ M \cap X \in X \,\},$$

where $C_{M,a}|^*X$ is given by the following lemma:

Lemma 3.46. *Suppose that X is a simple model and $C_{M,a}$ is a requirement compatible with p^X such that $M \cap X \in X$. Then there is a requirement $C_{M,a}|^*X$ in X such that* (i) *every requirement R in X which is compatible with $\{C_{M,a}|^*X\} \cup \mathrm{cp}^X(p)$ is also compatible with $C_{M,a}$, and* (ii) *every requirement R which is compatible with $\{C_{M,a}\} \cup p^X$ is also compatible with $C_{M,a}|^*X$.*

The proof of Lemma 3.46 will take up most of this subsection. We first show that Lemma 3.42 follows from Lemma 3.46.

Proof of Lemma 3.42 *from Lemma* 3.46. First we verify that $p|X \in P^*$, that is, that any two requirements in $p|X$ are compatible. Any two requirements in $p \cup \mathrm{cp}^X(p)$ are compatible by Proposition 3.45, and if $C_{M,a} \in p$ and $M \cap X \in X$, then $C_{M,a}|^*X$ is compatible with every requirement in $p \cup \mathrm{cp}^X(p)$ by clause 3.46(ii). Finally, if $C_{N,b}$ is another member of p such that $X \cap N \in X$, then the compatibility of $C_{M,a}|^*X$ with $C_{N,b}|^*X$ follows from clause 3.46(ii) together with the fact that $C_{M,a}|^*X$ is compatible with $p \leq \{C_{N,b}\} \cup p^X$.

Next we verify that the function $p \mapsto p|X$ is tidy. Suppose that $p, p' \leq p^X$ are compatible conditions. Then $p \wedge p' = p \cup p'$, and $(p \wedge p')|X = p|X \cup p'|X = p|X \wedge p'|X$ since each member of $(p \wedge p')|X$ is determined by the model X together with a single requirement from $p \cup p'$.

It remains to show that $p \mapsto p|X$ witnesses that p^X is strongly generic. We need to show that any condition $q \leq p|X$ in X is compatible with p, and for this it is enough to show that q is compatible with every requirement $R \in p$.

In the case $R = I_{\alpha,\lambda} \in p$ and $\lambda < \sup(X \cap \kappa)$ there is an X-fence for $I_{\alpha,\lambda}$ in $\mathrm{cp}^X(p) \subseteq p|X$, and any requirement in X which is compatible with this X-fence is compatible with $I_{\alpha,\lambda}$.

Now consider $R = O_{\gamma,(\eta',\eta]} \in p$. If $O_{\gamma,(\eta',\eta]} \in X$, then $O_{\gamma,(\eta',\eta]} \in p \cap X \subseteq p|X \subseteq q$, so it will be enough to show that if $O_{\gamma,(\eta',\eta]} \notin X$, then $O_{\gamma,(\eta',\eta]}$ is compatible with every requirement which is a member of X. Now if $O_{\gamma,(\eta',\eta]}$ is incompatible with any requirement in X, then $O_{\gamma,(\eta',\eta]}$ is incompatible with a requirement of the form $I_{\alpha,\lambda} \in X$. In the case that X is countable, it then follows from the definition of compatibility of $O_{\gamma,(\eta',\eta]}$ with C_X that $O_{\gamma,(\eta',\eta]} \in X$, so we can assume that X is uncountable. It follows that $X \cap \kappa$ is transitive, and since $I_{\alpha,\lambda} \in X$ it follows that $A_{\alpha,\lambda} \subset X$ and therefore $\gamma \in X$. If $|X| = \kappa$, then $\eta', \eta \in \kappa \subset X$, so $O_{\gamma,(\eta',\eta]} \in X$. Otherwise we have $\eta' < \lambda < \sup(X \cap \kappa)$, and therefore $\eta < \sup(X \cap \kappa)$ since $O_{\gamma,(\eta',\eta]}$ is compatible with $p^X = \{I_{\sup(X),\sup(X\cap\kappa)}\}$, so again $O_{\gamma,(\eta',\eta]} \in X$.

In the case $R = C_{M,a}$ with $M \cap X \in X$, clause 3.46(i) asserts that $C_{M,a}$ is compatible with any requirement in X which is compatible with $p|X$.

It only remains to consider the case $R = C_{M,a}$ when $M \cap X \notin X$. In this case X must be countable and $M \cap X \cap H_\kappa = X \cap H_\delta$, where $\delta := \sup(M \cap X \cap \kappa)$.

If $I_{\alpha,\lambda} \in q$ and $\lambda < \delta$, then $\lambda \in M$, and in this case $I_{\alpha,\lambda}$ is its own M-fence. If $\sup(M \cap \kappa) > \lambda \geq \delta$, then the M-fence for C_X, required for the compatibility of $C_{M,a}$ with C_X, is an M-fence for $I_{\alpha,\lambda}$. Hence any requirement $I_{\alpha,\lambda} \in q$ is compatible with $C_{M,a}$.

Now we show that $C_{M,a}$ is compatible with any requirement $O_{\gamma,(\eta',\eta]} \in q$ by showing that if $O_{\gamma,(\eta',\eta]}$ is a requirement in X which is compatible with $p|X$ but incompatible with some requirement $I_{\alpha,\lambda} \in M[a]$, then $O_{\gamma,(\eta',\eta]} \in M$. First, we must have $\eta < \delta$, as otherwise $O_{\gamma,(\eta',\eta]}$ would be incompatible with the X-fence for $C_{M,a}$, which is a member of $\mathrm{cp}^X(p) \subseteq p|X$. Thus $\{\eta', \eta\} \subset M$. Next, we have $\gamma \in A_{\alpha+1,\lambda}$, where $\lambda \in M$ and $\alpha \in M \cup a(\lambda)$. If $\gamma < \alpha$, then, since $M \cap X \notin X$, Lemma 3.22 implies that $A_{\alpha,\lambda} \cap X \subset M$, so $\gamma \in M$. If $\gamma = \alpha$, on the other hand, then $\gamma = \alpha \in M[a] = M \cup a(\lambda)$, and $a(\lambda)$ is a set of nonzero limit ordinals while γ is either zero or a successor ordinal. Thus it again follows that $\gamma \in M$.

Finally, suppose that $C_{N,b} \in q$. Since $N \in X$ we have $N \cap M \cap \kappa = N \cap (M \cap X) \cap \kappa = N \cap \delta \in X$, and since ${}^{\omega}\delta \cap X \subset X \cap H_\delta = X \cap M \cap H_\kappa$ it follows that $N \cap M \cap \kappa \in M$. Hence $C_{M,a}$ and $C_{N,b}$ satisfy clause 4(b)i of Definition 3.25. We can obtain an N-fence for M by taking the minimal N-fences for the members of the minimal X-fence for M, which is contained in $p|X$. The M-fence for X is also an M-fence for $C_{N,b}$. Hence $C_{M,a}$ is compatible with $C_{N,b}$. □

As a preliminary to the proof of Lemma 3.46, we give a structural characterization of the desired requirement $C_{M',a'} = C_{M,a}|^*X$. Recall that $A^*_{M,a,\lambda}$ is the set containing 0 together with the successor ordinals from $A_{M,a,\lambda} = \bigcup\{\, A_{\alpha,\lambda} : \alpha \in M \cup a(\lambda)\,\}$.

Lemma 3.47. *Suppose that X and $C_{M,a}$ are as in Lemma* 3.46*, and that $C_{M',a'}$ is a requirement such that $M' = M \cap X$, $a' \in X$, and $A^*_{M',a',\lambda} \cap X = A^*_{M,a,\lambda} \cap X$ for all $\lambda \in M' \cap \kappa$. Then $C_{M',a'}$ satisfies the conclusion of Lemma* 3.46.

Following the proof of Lemma 3.47 we will construct such a requirement $C_{M',a'}$.

Proof. The proof breaks into 3 cases, numbered from 1 to 3, depending on whether the requirement R has the form $I_{\alpha,\lambda}$, $O_{\gamma,(\eta',\eta]}$ or $C_{N,b}$. Furthermore, each of these three cases has two subcases, which are labeled (a) and (b) to correspond to the two clauses in the conclusion of Lemma 3.46.

Note that the hypothesis of Lemma 3.46 implies that $M' \cap \kappa$ is an initial segment of $M \cap \kappa$.

(*Case 1a*) First suppose that $R = I_{\alpha,\lambda} \in X$ and $I_{\alpha,\lambda}$ is compatible with $C_{M',a'}$. We must show that $I_{\alpha,\lambda}$ is compatible with $C_{M,a}$. If $\sup(M \cap \kappa) > \lambda \geq \sup(M \cap X \cap \kappa)$, then any M-fence for X includes an M-fence for $I_{\alpha,\lambda}$, so we can assume that $\lambda < \sup(M \cap X \cap \kappa) = \sup(M' \cap \kappa)$. Set $\lambda' := \min(M \setminus \lambda) = \min(M' \setminus \lambda)$. Now $A_{\alpha,\lambda} \cap M' = A_{\alpha,\lambda} \cap (M \cap X) = A_{\alpha,\lambda} \cap M$ since by Lemma 3.22, $M \cap X \in X$ implies that $A_{\alpha,\lambda} \cap M \subset X$. Thus any M' fence for $I_{\alpha,\lambda}$ is also an M-fence for $I_{\alpha,\lambda}$.

(*Case 1b*) Now suppose that $R = I_{\alpha,\lambda}$ is compatible with $\{C_{M,a}\} \cup p^X$. We will show that R is compatible with $C_{M',a'}$. This is immediate if $\lambda \geq \sup(M' \cap \kappa)$, so we assume that $\lambda < \sup(M' \cap \kappa)$. Then $\lambda' := \min(M' \setminus \lambda) = \min(M \setminus \lambda)$ and $A_{\alpha,\lambda} \cap M' \subset M$ since $M' \subset M$. Hence any M-fence for $I_{\alpha,\lambda}$ is also an M'-fence for $I_{\alpha,\lambda}$.

(*Case 2a*) Here we assume that $R = O_{\gamma,(\eta',\eta]} \in X$ and R is compatible with $\{C_{M',a'}\} \cup \mathrm{cp}^X(p)$, and we will show that R is compatible with $C_{M,a}$. This is

immediate unless there is some $I_{\alpha,\lambda} \in M[a]$ which is incompatible with $O_{\gamma,(\eta',\eta]}$, and in this case we must have $\lambda < \sup(M \cap X \cap \kappa)$, or else $O_{\gamma,(\eta',\eta]}$ would be incompatible with the X-fence for $C_{M,a}$, which is contained in $\mathrm{cp}^X(p)$. Thus $\lambda \in X \cap M = M'$. Furthermore $\gamma \in A^*_{M,a,\lambda} \cap X = A^*_{M',a',\lambda} \cap X$, so $O_{\gamma,(\eta',\eta]}$ is incompatible with a requirement in $M'[a']$ and hence is a member of $M' \subset M$. Thus $O_{\gamma,(\eta',\eta]}$ is compatible with $C_{M,a}$.

(*Case 2b*) Now suppose that $R = O_{\gamma,(\eta',\eta]}$ is compatible with $\{C_{M,a}\} \cup p^X$. Then $O_{\gamma,(\eta',\eta]}$ is compatible with $C_{M',a'}$ unless there is some $I_{\alpha,\lambda} \in M'[a']$ which is incompatible with $O_{\gamma,(\eta',\eta]}$. Since M' and a' are in X this implies that $I_{\alpha,\lambda} \in X$, and since $O_{\gamma,(\eta',\eta]}$ is compatible with p^X it follows that $O_{\gamma,(\eta',\eta]} \in X$. Since $A^*_{M,a,\lambda} \cap X = A^*_{M',a',\lambda} \cap X$ it follows that $O_{\gamma,(\eta',\eta]}$ is incompatible with a requirement in $M[a]$, and hence $O_{\gamma,(\eta',\eta]} \in M$ since $O_{\gamma,(\eta',\eta]}$ is compatible with $C_{M,a}$. Hence $O_{\gamma,(\eta',\eta]} \in M \cap X = M'$.

(*Case 3a*) Suppose that $R = C_{N,b} \in X$ and R is compatible with $C_{M',a'} \cup \mathrm{cp}^X(p)$. We need to show that R is compatible with $C_{M,a}$. Since $C_{N,b}$ is compatible with $C_{M',a'}$, $M \cap N \cap \kappa = M' \cap N \cap \kappa$ is either a member of or an initial segment of N. Also, since $M' \cap \kappa$ is an initial segment of M the set $M \cap N \cap \kappa$ is also either a member of or an initial segment of M according as it is a member or initial segment of M'.

Let x' be an M'-fence for $C_{N,b}$ and let x be an M-fence for C_X. Then $x' \cup x$ is an M-fence for $C_{N,b}$: If $O_{\gamma,(\eta',\eta]} \in M$ is incompatible with some $I_{\alpha,\lambda} \in N[b]$, then $O_{\gamma,(\eta',\eta]}$ is incompatible with some member of x if $\lambda > \sup(M \cap X \cap \kappa)$, and otherwise $O_{\gamma,(\eta',\eta]}$ is incompatible with some member of x'.

In the other direction, let x' be an N-fence for $C_{M',a'}$, let y be an X-fence for $C_{M,a}$ which is contained in $\mathrm{cp}^X(p)$, and let x be the set of minimal N-fences for members of y. Then $x \cup x'$ is an N-fence for $C_{M,a}$: Let $O_{\gamma,(\eta',\eta]} \in N$ be a requirement which is incompatible with some requirement $I_{\alpha,\lambda} \in M[a]$. If $\lambda \geq \sup(M \cap N \cap \kappa)$, then $O_{\gamma,(\eta',\eta]}$ must be incompatible with some member of x. If $\lambda < \sup(M \cap N \cap \kappa)$, then $\lambda \in M'$ since $N \in X$ implies that $\sup(M \cap N \cap X) \leq \sup(M \cap X \cap \kappa)$, and $M \cap X \cap \kappa = M \cap \sup(M \cap X \cap \kappa)$. Thus the fact that $A^*_{M,a,\lambda} \cap X = A^*_{M',a',\lambda} \cap X$ implies that $O_{\gamma,(\eta',\eta]}$ is incompatible with some member of $M'[a']$ and hence with some member of x'.

(*Case 3b*) Finally, suppose $R = C_{N,b}$ is compatible with $\{C_{M,a}\} \cup p^X$. We need to show that R is also compatible with $C_{M',a'}$. First, $N \cap M' \cap \kappa = N \cap (M \cap X) \cap \kappa$. This is an initial segment of $N \cap M$; thus it is either a member or initial segment of M' depending on whether $N \cap M \cap \kappa$ is a member or initial segment of M, and it is a member or initial segment of N depending on whether $N \cap M \cap \kappa$ is a member or initial segment of N.

If x is any M-fence for $C_{N,b}$, then $\{\, I_{\alpha,\lambda} \in x : \lambda \in M' \,\}$ is an M'-fence for $C_{N,b}$. An N-fence for $C_{M',a'}$ can be obtained by taking the union of an N-fence for C_X and an N-fence for $C_{M,a}$: If $I_{\gamma,\xi} \in M'[a']$, then $\lambda \in M' = M \cap X$ and either $\gamma \in M' \subseteq M$ or $\gamma \in a'(\lambda) \subset X$. $\square$

Proof of Lemma 3.46. It remains to construct a pair M', a' satisfying the hypothesis of Lemma 3.47. We already have $M' = M \cap X$. In order to construct a' we will define a sequence of proxies $a(i)$ and $b(i)$ by recursion on i, each of which satisfies the following recursion hypotheses.

(1) (a) $C_{M',a(i)}$ is a requirement, (b) $C_{M,b(i)}$ satisfies Definition 3.17(ii,iv), and (c) for any $\nu \in M' \cap \kappa$ and any $\alpha \in b(i)(\nu)$, either $A_{\alpha,\nu} \cap M'$ is unbounded in M' or $\sup(A_{\alpha,\nu} \cap X) \in M'$.
(2) $a(i) \in X$.
(3) $A^*_{M',a(i)\cup b(i),\nu} \cap X = A^*_{M,a,\nu} \cap X$ for all $\nu < \sup(M' \cap \kappa)$.
(4) Set $d(i) = \{\, \alpha : \exists \lambda\, (\alpha, \lambda) \in b(i) \,\}$. Then $d(i+1) \lessdot d(i)$ where $\lessdot$ is the ordering of $[\kappa^+]^{<\omega}$ defined by $d' \lessdot d$ if $\max(d' \triangle d) \in d$.

The ordering $\lessdot$ is a well order, so clause 3 implies that there is some $k < \omega$ such that $b(k) = \varnothing$. We will set $a' = a(k)$. Then $C_{M',a'}$ is a requirement by clause (1) of the recursion hypothesis, it is in X by clause 2, and it satisfies $A^*_{M',a',\lambda} \cap X = A_{M,a,\lambda} \cap X$ for $\lambda \in M' \cap \kappa$ by clause 3. Hence $C_{M',a'}$ satisfies the conclusion of Lemma 3.46.

Note that clause 1b is a modification of clause iii of Definition 3.17 of a requirement of the type $C_{M,a}$.

(*Case* $i = 0$) The recursion starts with $a(0) = \varnothing$ and

$$b(0) = b \cup \{\, (\alpha, 0) : \alpha \in M \setminus \sup(M') \ \ \&\ \ X \cap \alpha \not\subset \sup(M \cap \alpha) \,\}.$$

The set $b(0)$ is finite, since by Proposition 3.33 there can be only finitely many $\alpha > \sup(M \cap X)$ in M such that $X \cap \alpha \not\subset \sup(M \cap \alpha)$.

Clause 2 of the recursion hypotheses is immediate and clause 4 does not apply, so we only need to verify clauses 1 and 3.

Clause 1a is immediate since $a(0) = \varnothing$. Since the requirement $C_{M,a}$ satisfies 3.17(ii,iv) and $b(0)(\lambda) \subseteq M \cup a(\lambda)$ for $\lambda \in M \cap \kappa$, clause 1b holds for $b(0)$. Finally, clause 1b follows from Lemma 3.23(1).

Now we verify clause 4 of the recursion hypothesis:

Claim. $A_{M',a(0)\cup b(0),\nu} = A_{M,a,\nu}$ for all $\nu \in M' \cap \kappa$.

Proof. We have $a(0) = \varnothing$, and it is clear that $A_{M',b(0),\nu} \subseteq A_{M,a,\nu}$. Since $a \subseteq b(0)$ it only remains to show that $A_{M,\varnothing,\nu} \cap X \subseteq A_{M',b(0),\nu}$. Suppose $\gamma \in X \cap A_{\alpha,\nu}$ where $\alpha \in M$ and $\nu \in M' \cap \kappa$. If $\gamma < \sup(M \cap X)$ and $A_{\alpha,\nu} \cap X$ is bounded in $\sup(M \cap X)$, then clause 1 of Lemma 3.23 implies that $\alpha' := \sup(X \cap A_{\alpha,\nu}) \in M \cap X = M'$, and then $\gamma \in A_{\alpha'+1,\nu} \subseteq A_{M',\varnothing,\nu}$. If $\gamma < \sup(M \cap X)$ and $A_{\alpha,\nu} \cap X$ is unbounded in X, then $\gamma \in A_{\alpha',\nu} \subseteq A_{\alpha,\nu}$ for any $\alpha' \in A_{\alpha,\nu} \cap X \setminus \gamma$. Thus we can assume that $\gamma \geq \sup(M \cap X)$. Then $(\alpha', 0) \in b(0)$, where $\alpha' := \min(M \setminus \gamma)$, so $A_{\alpha',\nu} \subseteq A_{M',b(0),\nu}$, and it follows by Proposition 3.31 that $\gamma \in A_{\alpha,\nu} \cap \alpha' = A_{\alpha',\nu} \subseteq A_{M',b(0),\nu}$. $\square$

(*Case* $i+1$) Now assume that $a(i)$ and $b(i)$ have been defined, and $b(i) \neq \varnothing$. To define $a(i+1)$ and $b(i+1)$, let (α, λ) be the lexicographically least member of $b(i)$, and set $b'(i) = \{\, (\alpha', \lambda') \in b(i) : \alpha' > \alpha \,\}$. Note that, while there may be more than one ordinal λ such that $(\alpha, \lambda) \in b(i)$, all but the least of these are redundant and may be discarded.

We begin with several special cases: If $\lambda \geq \sup(M' \cap \kappa)$, then (α, λ) can be discarded since $\nu < \sup(M' \cap \kappa)$; in this case we set $a(i+1) = a(i)$ and $b(i+1) = b'(i)$. If $\alpha \in X$, then we set $a(i+1) = a(i) \cup \{(\alpha, \min(X \setminus \lambda)\}$ and $b(i+1) = b'(i)$. If $\alpha \leq \alpha' + \omega$ for some limit ordinal α', then we set $a(i+1) = a(i)$ and $b(i+1) = \{(\alpha', \lambda)\} \cup b'(i)$.

The recursion hypotheses are clear in each of these three cases. For the remainder we can assume that $\alpha \notin X$ and that all members of C_α are limit ordinals. If C_α is bounded in X, then set $\eta := \max(\lim(C_\alpha) \cap \bar{X})$ where $\bar{X}$ is the closure $\bar{X} = X \cup \lim(X)$ of X. Otherwise, if C_α is cofinal in X, set $\eta = c_{\alpha,\sup(M' \cap \kappa)}$. Then

$A_{\alpha,\nu} = A_{\eta,\nu}$ for all $\nu < \sup(M' \cap \kappa)$, and $\eta \in X$ since $M' \in X$ and $\eta = c_{\alpha',\sup(M'\cap\kappa)}$ for any $\alpha' \in \lim(C_\alpha \cap X) \setminus \eta$.

Claim. If $\eta \leq \sup(M')$, then $A_{\alpha,\nu} \cap \eta \subseteq A_{M',\varnothing,\nu}$ for all $\nu \leq \sup(M' \cap \kappa)$.

Proof. For any ν such that $A_{\alpha,\nu}$ is cofinal in M' we have $A_{\eta,\nu} = A_{\alpha,\nu} \cap \eta \subseteq A_{\alpha,\nu} \cap \sup(M') = A_{\sup(M'),\nu} \subseteq A_{M',\varnothing,\nu}$ since $\lim(C_{\sup(M')}) \cap M'$ is cofinal in M'. If $\nu \in M' \cap \kappa$ and $A_{\alpha,\nu}$ is bounded in $\sup(M')$, then $\xi := \sup(A_{\alpha,\nu} \cap X) \in M'$ by clause 1c of the recursion hypothesis. Then $A_{\alpha,\nu} \cap \eta = A_{\xi,\nu} \cap \eta \subseteq A_{M',\varnothing,\nu}$. $\square$

Thus if $\eta \leq \sup(M')$ we can set $a(i+1) = a(i)$. Otherwise set $\eta' = \min(X \setminus \eta)$ and $\lambda' = \min(X \setminus \lambda)$, and set $a(i+1) = a(i) \cup \{(\eta', \lambda')\}$.

If $A_{\alpha,\nu} \cap X \subseteq A_{\eta,\nu}$ for all $\nu < \sup(M' \cap \kappa)$, then set $b(i+1) = b'(i)$. Otherwise let $(\gamma_j : j < m)$ enumerate the set $\{\min(C_\alpha \setminus \xi) : \eta < \xi \in X \cap A^*_{\alpha,\sup(M'\cap\kappa)}\}$. Note that m is finite since otherwise $\sup_{j<\omega} \gamma_j$ would be in $\lim(C_\alpha) \cap \bar{X}$. For each $j < m$ let χ_j be the least ordinal $\chi \in X \setminus \lambda$ such that $\gamma_j \in A_{\alpha,\chi}$, and set $b(i+1) = b'(i) \cup \{(\gamma_j, \chi_j) : j < m\}$.

This completes the definition of $a(i+1)$ and $b(i+1)$. Again, Clauses 2 and 4 of the recursion hypotheses are clear. Clause 1a is also immediate unless $a(i+1) \neq a(i)$, in which case we need to show that each clause of Definition 3.17 holds with $C_{M',a(i+1)}$ for $C_{M,a}$ and (η', λ') for (α, λ). Clause 3.17(i) is clear. For clause 3.17(ii), note that if $\nu \in M' \cap (\kappa \setminus \lambda')$, $\gamma \in M' \cap \kappa$ and $\pi_{\eta'}(\gamma, \nu)$ is defined, then $\pi_{\eta'}(\gamma, \nu) \in X$ since $\{\nu, \gamma, \eta'\} \subset X$. In addition $\pi_{\eta'}(\gamma, \nu) = \pi_\eta(\gamma, \nu) = \pi_\alpha(\gamma, \nu) \in M$, and hence $\pi_{\eta'}(\gamma, \nu) \in M \cap X = M'$. For clause 3.17(iii), $A_{\eta,\nu}$ is an initial segment of $A_{\alpha,\nu}$, and clause 2 asserts that either $A_{\alpha,\nu} \cap M'$ is unbounded in M' or else $\sup(A_{\alpha,\nu} \cap X) \in M'$. Since $\eta > \sup(M')$, the first alternative implies that $A_{\eta',\nu}$ is cofinal in M'. Since η' and ν are in X, the second alternative implies that $\sup(A_{\eta',\nu} \cap X) = \sup(A_{\eta,\nu} \cap X) = \sup(A_{\alpha,\nu} \cap X) \in M'$. For clause 3.17(iv), if $\nu \notin B_{\eta'}$ and γ is least such that $\gamma \in A_{\eta',\nu}$ and $\nu \notin B_\gamma$, then $\gamma \in X$, and hence $\gamma \in A_{\eta',\nu} = A_{\alpha,\nu} \cap \eta'$. Thus $\gamma \in M$ by clause 1b of the recursion hypothesis, so $\gamma \in M \cap X = M'$.

To verify clauses 1b and 1c, use the recursion hypothesis and the fact that $A_{\gamma_j,\nu} = A_{\alpha,\nu} \cap \gamma_j$ for $\nu \geq \chi_{i+1}$.

It only remains to verify clause 3 in the final case of the definition:

Claim. In the final case of the definition of $a(i+1)$ and $b(i+1)$ we have $A^*_{M',a(i)\cup b(i),\nu} \cap X = A^*_{M,a,\nu} \cap X$ for all $\nu \in M' \cap \kappa$.

Proof. The change from $a(i) \cup b(i)$ to $a(i+1) \cup b(i+1)$ consists of replacing the single pair $(\alpha, \lambda) \in b(i)$ with the finite set $\{(\gamma_j, \chi_j) : j < m\} \subseteq b(i+1)$, together with $(\eta', \min(X \setminus \lambda)) \in a(i+1)$ if $\eta > \sup(M')$. If $\nu < \lambda$, then none of these contributes any members to either of the sets $A^*_{M',a(i)\cup b(i),\nu}$ or $A^*_{M,a,\nu}$, so it will be sufficient to verify the conclusion of the claim for $\nu \in M' \cap (\kappa \setminus \lambda)$. Since $M' \subseteq X$, this implies that $\nu \geq \min(X \setminus \lambda)$. Thus it will be sufficient to verify that

$$A^*_{\alpha,\nu} \cap X = \Big(A^*_{\eta',\nu} \cup \bigcup\{A^*_{\gamma_j,\nu} : j < m \ \ \& \ \ \chi_j \leq \nu\}\Big) \cap X \tag{3.1}$$

for any $\nu > \lambda$ in M'.

Since $\eta \in \lim(C_\alpha) \cap \lim(C_{\eta'})$, we have $A_{\alpha,\nu} \cap \eta = A_{\eta,\nu} = A_{\eta',\nu} \cap \eta$ for every $\nu < \kappa$. Furthermore $A_{\gamma_j,\nu} \subseteq A_{\alpha,\nu}$ for all $\nu \geq \chi_j$. Thus it it will be sufficient to show that $A^*_{\alpha,\nu} \cap (X \setminus \eta) \subset \bigcup\{\, A^*_{\gamma_j,\nu} : j < m \ \ \&\ \ \chi_j \leq \gamma_j \,\}$.

Suppose that $\gamma \in A^*_{\alpha,\nu} \cap (X \setminus \eta)$. Then there is some $j < m$ such that $\gamma'_j < \gamma \leq \gamma_j$, where $\gamma'_j = \sup(C_\alpha \cap \gamma_j)$. However $\gamma \notin C_\alpha$ since it is a successor ordinal, so $\gamma < \gamma_j$. Furthermore $\gamma \in A_{\alpha,\nu}$ implies that $\gamma_j \in A_{\alpha,\nu}$ and hence $\nu \geq \chi_j$, so $\gamma_j \in b(i+1)(\nu)$ and $\gamma \in A_{\alpha,\nu} \cap \gamma_j = A_{\gamma_j,\nu} \subseteq A_{b(i),\nu}$. □

This completes the proof of Lemma 3.46 and hence of the strong genericity Lemma 3.42. □

3.7. **Completion of the proof of Theorem 1.2.** We first verify that there are stationarily many models satisfying the hypothesis of Lemma 3.42:

Lemma 3.48. (i) *The set of transitive simple models $X \prec H_{\kappa^+}$ is stationary.* (ii) *The set of countable simple models $M \prec H_{\kappa^+}$ is stationary.* (iii) *If κ is κ^+-Mahlo, then the set of simple models $Y \prec H_{\kappa^+}$ with $Y \cap \kappa \in B_{\sup(Y)}$ is stationary.*

Proof. For clause (i), any transitive set $X \prec H_{\kappa^+}$ with $\mathrm{cf}(\sup(X)) = \kappa$ is a simple model.

For the remaining clauses, let X be any model as in the last paragraph and set $\tau = \sup(X)$.

For clause (ii), let M be any countable elementary substructure of the structure $(X, C_{\sup(X)})$. Because $C_{\sup(X)}$ was included as a predicate, $M \cap C_{\sup(X)}$ is unbounded in $\delta := \sup(M)$ and hence $C_\delta = C_{\sup(X)} \cap \delta$. Finally, $c_{\delta,\xi} = c_{\sup(X),\xi} \in M$ for all $\xi \in M \cap \kappa$, so $\mathrm{otp}(C_\delta) = \sup(M \cap \kappa)$ and $\lim(C_\delta)$ is cofinal in M. Thus M is a simple model.

For clause (iii), let E be the closed and unbounded set of cardinals $\lambda < \kappa$ such that there is a set $X_\lambda \prec (X, C_{\sup(X)})$ with $X_\lambda \cap H_\kappa = H_\lambda$. As in the last paragraph the models X_λ are simple. Set $\tau = \sup(X)$. Since κ is $(\tau+1)$-Mahlo there is a stationary set of $\lambda \in E \cap B_\tau$. Pick $\lambda \in E \cap B_\tau$, and set $\tau' = \sup(X_\lambda)$. Then $A_{\tau,\lambda} = A_{\tau',\lambda}$, so $f_\tau(\lambda) = f_{\tau'}(\lambda)$, and since $\lambda \in B_\tau$ it follows that $\lambda \in B_{\tau'}$ as well. Thus the set $Y = X_\lambda$ satisfies clause (iii). □

Corollary 3.49. *The forcing P^* has the κ^+-chain condition and is ω_1-presaturated. If κ is κ^+-Mahlo, then P^* is κ-presaturated.*

Proof. The proof is immediate from Lemma 2.18, Corollary 2.19, Lemma 3.42 and Lemma 3.48. □

Corollary 3.50. *If κ is κ^+-Mahlo and G is a generic subset of P^*, then $\omega_1^{V[G]} = \omega_1^V$, $\omega_2^{V[G]} = \kappa$, and all cardinals larger than κ are preserved.*

Proof. By Corollary 3.49, P^* is ω_1-presaturated, κ-presaturated and has the κ^+-chain condition. Hence these three cardinals, and all cardinals greater than κ^+, are preserved, and it only remains to show that all cardinals between ω_1 and κ are collapsed. This follows by the proof of the corresponding Lemma 2.27 from section 2, using B_0 in place of B, D_0 in place of D, and $I_{0,\alpha}$ instead of I_α. □

Corollary 3.51. *If κ is κ^+-Mahlo, then every subset of* $\mathrm{Cof}(\omega_1)$ *in* $V[G]$ *in* $I[\omega_2]$ *is nonstationary.*

Proof. Assume to the contrary that $A := \langle a_\xi : \xi < \kappa \rangle$ is a sequence of countable subsets of κ in $V[G]$ such that the set $B(A) \cap \mathrm{Cof}(\omega_1)$ is stationary, where $B(A)$ is the set defined in Definition 1.1. Let $\dot{A}$ be a name for A. Fix a transitive simple model $X \prec (H_{\kappa^+}, \dot{A})$, so that $\Vdash \dot{A} \in V[\dot{G} \cap X]$, and as in the proof of Lemma 3.48, let E be the set of $\lambda < \kappa$ such that there is a model $X_\lambda \prec (X, \dot{A}, C_{\sup(X)})$ with $X_\lambda \cap H_\kappa = H_\lambda$. Then E contains a closed and unbounded subset of κ. Since $B(A) \cap \mathrm{Cof}(\omega_1)$ is stationary there is an ordinal $\lambda \in E \cap D_{\tau+1} \cap B(A) \cap \mathrm{Cof}(\omega_1)$, where $\tau = \sup(X)$. Then X_λ is simple, $I_{\sup(X_\lambda),\lambda} \in \bigcup G$, and $\{I_{\sup(X_\lambda),\lambda}\} \Vdash \forall \nu < \lambda\ \dot{a}_\nu \in V[\dot{G} \cap X_\lambda]$. Thus $a_\nu \in V[G \cap X_\lambda]$ for all $\nu < \lambda$.

Now let $c \subset \lambda$ witness that $\lambda \in B(A)$. Thus $\mathrm{otp}(c) = \omega_1$, $\bigcup c = \lambda$, and $c \cap \beta \in \{ a_\nu : \nu < \lambda \} \subset V[G \cap X_\lambda]$ for all $\beta < \lambda$. It follows by Lemma 2.22 that $c \in V[G \cap X_\lambda]$.

We complete the proof by showing that this is impossible. Let $\dot{c}$ be an $(P^* \cap X_\lambda)$-name for c. For a closed unbounded set E' of cardinals $\lambda' < \lambda$ there is a model $X' \prec (X_\lambda, \dot{c})$ with $X' \cap H_\kappa = H_{\lambda'}$. Since $\lambda \in B_{\tau+1} \cap D_{\tau+1}$ there is a cardinal $\lambda' \in E' \cap D_\tau$. As in the previous argument, X' is a simple model and $\{I_{\tau,\lambda'}\} \in G$ is a strongly X'-generic condition. Since $\mathrm{otp}(c) = \omega_1 \subset X'$, it follows that $\{I_{\tau,\lambda'}\} \Vdash \dot{c} \subset X'$, contradicting the fact that c is cofinal in λ. $\square$

This completes the proof of Theorem 1.2.

4. Discussion and questions

Several related questions and ideas are discussed in the paper [Mit05], and we will only summarize some of them here.

The first problem is whether these techniques can be applied at larger cardinals. One easy answer to this problem is given for any regular cardinal κ by substituting "of size less than κ" for "finite" and using models of size κ instead of countable models. The resulting forcing adds closed unbounded subsets of κ^{++} and demonstrates the consistency of the statement that every subset of $\mathrm{Cof}(\kappa^+)$ in $I[\kappa^{++}]$ is nonstationary.

No such generalization is known for cardinals κ^+ where κ is a limit cardinal. This problem is of particular interest in the case when κ is a singular cardinal. Shelah has shown that if κ is singular, then $I[\kappa^+]$ includes a stationary subset of $\mathrm{Cof}(\lambda)$ for every regular $\lambda < \kappa$, but it is open whether $\mathrm{Cof}(\lambda) \in I[\kappa^+]$ for any regular λ in the interval $\omega_1 < \lambda < \kappa$.

Another natural question is whether the techniques of this paper can be applied at multiple cardinals, giving a model in which, for example, neither $I[\omega_2] \cap \mathrm{Cof}(\omega_1)$ nor $I[\omega_3] \cap \mathrm{Cof}(\omega_2)$ contains a nonstationary set. This problem seems to be quite difficult, and a useful test problem comes from considering the much simpler argument, alluded to at the send of section 2 and given in [Mit05], which uses the techniques of this paper to give a model with no ω_2-Aronszajn trees. Can this construction by used to duplicate the results of [Abr83] by obtaining, from a supercompact cardinal κ and a weakly compact cardinal $\lambda > \kappa$, a model with no ω_2- or ω_3-Aronszajn trees? Two approaches to this problem have been attempted. The first, an iteration of the basic method analogous to Abraham's construction in [Abr83], initially seemed quite promising; however the author has withdrawn

previous claims to have such a proof. The second approach would operate simultaneously on both cardinals by using forcing with finite conditions as in the present technique, but containing as requirements models of size less than κ (that is, less than ω_2 in the generic extension) as well as countable models. This would give a structure analogous to a gap-2 morass. The combinatorics of this approach are quite complicated.

It seems plausible that a solution for the problem concerning $I[\omega_2]$ and $I[\omega_3]$ will require solutions to *both* approaches to the Aronszajn tree problem, with the second of the two approaches being used to provide a structure at λ like the $\square_\kappa$ sequence needed in this paper.

A third question is whether it is possible for $I[\omega_2]$ to be ω_3-generated, that is, that $I[\omega_2]$ cannot be normally generated by any of its subsets of size less than ω_2. Note that the continuum hypothesis implies that $I[\omega_2]$ is trivial, that is, $\omega_2 \in I[\omega_2]$, and this paper presents a model in which $I[\omega_2]$ is generated by $\mathrm{Cof}(\omega)$. Either the model of section 2 or the original model [Mit73] with no Aronszajn trees on ω_2 gives an example in which the restriction of $I[\omega_2]$ to $\mathrm{Cof}(\omega_1)$ is generated by the single set $\{\, \nu < \omega_2 : \mathrm{cf}^V(\nu) = \omega_1 \,\}$. If $I[\omega_2]$ is generated by fewer than ω_3 many sets, then it is generated by the diagonal intersection of these sets, so if $2^{\omega_2} = \omega_3$, then the only remaining possibility is that $I[\omega_2]$ requires ω_3 generators.

It is likely that it is possible to obtain such a model by using the techniques of this paper to add closed, unbounded subsets $D_{\alpha,\lambda} \subset \lambda \cap B^*_\alpha$ for $\alpha < \kappa^+$ and $\lambda \in B_\alpha$, with the sets $\{\, D_{\alpha,\lambda} : \lambda \in B_\alpha \,\}$ forming a $\square_\kappa$-like tree. A witness that $A := B_{\alpha+1} \setminus B_\alpha \in I[\omega_2]$ would then be given by $\{\, D_{\alpha,\lambda} \cap C_\lambda : \lambda \in A \,\}$, where C_λ is a closed, unbounded subset of λ such that $C_\lambda \cap B_\alpha = \varnothing$.

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