STRONG $L^p$-SOLUTIONS TO THE NAVIER-STOKES FLOW PAST MOVING OBSTACLES: THE CASE OF SEVERAL OBSTACLES AND TIME DEPENDENT VELOCITY

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Abstract. Consider the Navier-Stokes flow past several moving or rotating obstacles with possible time-dependent velocity. It is shown that under suitable assumptions on the data, there exists a unique, local strong solution in the $L^p - L^q$-setting for suitable $p, q \in (1, \infty)$. Moreover, it is proved that this strong solution coincides with the known mild solution in the very weak sense.

1. Introduction

The mathematical description of the Navier-Stokes flow past rotating or moving obstacles gained quite a bit of attention in the last years. The motion is hereby described by the equations of Navier-Stokes in an exterior domain depending on the time variable $t$. More precisely, consider the equation

\begin{equation}
\begin{cases}
\partial_t v - \Delta v + (v \cdot \nabla)v + \nabla q = f & \text{in } \Omega(t), \quad t \in (0, T), \\
\text{div } v = 0 & \text{in } \Omega(t), \quad t \in (0, T), \\
v(x, t) = M_i(t)x & \text{on } \Gamma_i(t), \ i = 1, \ldots, m, \ t \in (0, T), \\
v(x, 0) = v_0(x) & \text{in } \Omega(0).
\end{cases}
\end{equation}

Here $v = v(x, t)$ and $q = q(x, t)$ denote the velocity and the pressure of the fluid, respectively. In this paper we consider time-dependent domains $\Omega(t)$ of the following form: let $\mathcal{O}_1, \ldots, \mathcal{O}_m \subset \mathbb{R}^n$, $n \geq 2$, be compact sets with boundaries $\Gamma_1, \ldots, \Gamma_m$ of class $C^{1,1}$. We denote by $\Omega := \mathbb{R}^n \setminus \bigcup_{i=1}^m \mathcal{O}_i$ the exterior domain. For time-dependent matrices

\[ M_i \in C^\infty([0, T]; \mathbb{R}^{n \times n}), \]

with $\text{tr } M_i(t) = 0$ for all $t \in [0, T], \ i \in \{1, \ldots, m\}$, we define the time dependent exterior domain

\[ \Omega(t) := \mathbb{R}^n \setminus \bigcup_{i=1}^m \mathcal{O}_i(t) \]

with $\mathcal{O}_i(t) := \{ y = G_i(t)x, x \in \mathcal{O}_i \}, \Gamma_i(t) := \{ y = G_i(t)x, x \in \Gamma_i \}$ for $t \in [0, T]$ and a suitably defined isomorphism $G_i(t) : \overline{\mathcal{O}_i} \to \overline{\mathcal{O}_i(t)}$; for details see [2,3]. As the obstacles shall not collide, we require

\[ \text{dist } (\overline{\mathcal{O}_i(t)}, \overline{\mathcal{O}_j(t)}) > 0, \quad i \neq j, \ t \in [0, T]. \]

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The boundary condition on $\Gamma_i(t)$ is the usual no-slip condition. It is the aim of this paper to construct a strong $L^p$-solution to (1.1).

It is interesting to compare our solution to problem (1.1) with the results which have recently been obtained by several different approaches. The situation of one obstacle rotating with constant angular velocity (i.e. $M$ equals the rotation matrix) was first considered by Hishida [HS99]. He proved the existence of a unique local mild solution to (1.1) in the context of $L^2$.

Strong solutions, again in the $L^2$-context and for one obstacle, were obtained by Galdi and Silvestre in [GS05] using Galerkin methods as well as by Cumsille and Tucsnak [CT06]. In the case of two dimensions, these strong solutions are even global in time under appropriate assumptions on the data; see [CT06].

The situation where the data belong to $L^p$ for $1 < p < \infty$ was first considered in [GHH06a], where the existence of a unique, local, mild $L^p$-solution to (1.1) was established. It was shown by Hishida and Shibata in [HS06] that this solution is even a global one, provided the data are small enough.

In this paper, we consider the situation of strong $L^p$-solutions. We prove the existence of a local, strong solution to (1.1) in $L^p$ even for several non-colliding obstacles which may rotate or move with a time-dependent angular velocity. One of the main tools in the proof of our results will be the maximal $L^p$-regularity of the Stokes operator in exterior domains.

It is a natural question to ask whether the strong solution to (1.1) obtained in Theorem 3.1 below coincides with the mild solution constructed in [GHH06a]. We give an affirmative answer to this question in Theorem 3.3 below. Of course, we need to explain first the meaning of coincides, since mild and strong solutions are defined on different spaces. In this section we make use of the concept of very weak solutions which was introduced in [FJR72] for $\mathbb{R}^n$ and in [Ama00] for domains.

For more information about the Navier-Stokes equation in the rotating frame and for one obstacle, were obtained by Inoue and Wakimoto [IW77].

We then consider a ball $B_r(0)$ of radius $r > 0$ such that $B_r(0) \supset \mathcal{O}(t)$ for all $t \in [0, T]$ for some $T > 0$. Choose a cut-off function $\eta \in C_c^\infty(\mathbb{R}^n)$ satisfying $0 \leq \eta \leq 1$, $\eta = 1$ on $B_r(0)$ and $\eta = 0$ on $B_{2r}(0)^c$. Define $b : \mathbb{R}^n \to \mathbb{R}^n$ by

\begin{equation}
(2.1) \quad b(y) := \eta(y)M y - B_\kappa ((\nabla \eta)M \cdot ) (y),
\end{equation}

where $\kappa$ is a smooth positive constant.

2. PRELIMINARIES

We start by transforming equation (1.1) defined in the time-dependent domain $\Omega(t)$ to an equation on a fixed cylindrical domain. More precisely, following the approach introduced by Inoue and Wakimoto [IW77], we introduce a change of coordinates which coincides in the special case of pure rotation, i.e. $M$ equals the rotation matrix, with the rotation in a neighborhood of the rotating obstacle, but equals the identity far away from the rotating body; see also [CT06].

For the time being, assume there is only one moving obstacle. We then make the following assumption:

(A1) Let $\mathcal{O} \subset \mathbb{R}^n$, $n \geq 2$, be a compact set (the obstacle) with boundary $\Gamma := \partial \mathcal{O}$ of class $C^1$. Denote by $\Omega := \mathbb{R}^n \setminus \mathcal{O}$ the exterior domain corresponding to $\mathcal{O}$. For $M \in \mathbb{R}^{n \times n}$ with $\operatorname{tr} M = 0$ define $\Omega(t), \Gamma(t), \mathcal{O}(t)$ as $\Omega(t) := \{y = e^{tM}x, x \in \Omega\}$, $\Gamma(t) := \{y = e^{tM}x, x \in \Gamma\}$, $\mathcal{O}(t) := \{y = e^{tM}x, x \in \mathcal{O}\}$. We then consider a ball $B_r(0)$ of radius $r > 0$ such that $B_r(0) \supset \mathcal{O}(t)$ for all $t \in [0, T]$ for some $T > 0$. Choose a cut-off function $\eta \in C_c^\infty(\mathbb{R}^n)$ satisfying $0 \leq \eta \leq 1$, $\eta = 1$ on $B_r(0)$ and $\eta = 0$ on $B_{2r}(0)^c$. Define $b : \mathbb{R}^n \to \mathbb{R}^n$ by
where $K := \text{supp} \left( \nabla \eta \right)$. Here $B_K$ denotes the Bogovskii operator. For the definition and properties of this operator, we refer e.g. to [Bog79], [Gal94], and [GHH06b]. Note that $b(y) = M y$ for $y \in \overline{O}(t)$. Moreover, since $\int_K (\nabla \eta)(y) M y \, dy = 0$, thanks to $\text{tr} \, M = 0$, it follows by construction that $\text{div} \, b = 0$ in $\mathbb{R}^n$ and $b \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n)$.

Then consider the initial value problem

\begin{equation}
\begin{aligned}
\partial_t X(y, t) &= b(X(y, t)), & t > 0, & y \in \mathbb{R}^n, \\
X(y, 0) &= y, & y \in \mathbb{R}^n.
\end{aligned}
\end{equation}

(2.2)

Then, by standard theory of ODE’s, there exists a unique vector field $X \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+)$ satisfying (2.2). Moreover, $X(\cdot, t)$ is a $C^\infty$-diffeomorphism from $\Omega$ onto $\Omega(t)$. Its inverse $Y(\cdot, t)$ is of the same class of regularity and satisfies the initial value problem

\begin{equation}
\begin{aligned}
\partial_t Y(x, t) &= -b(Y(x, t)), & t > 0, & x \in \mathbb{R}^n, \\
Y(x, 0) &= x, & x \in \mathbb{R}^n.
\end{aligned}
\end{equation}

(2.3)

In fact, we only need the restrictions of $X$ and $Y$ on $[0, T]$, nevertheless even (2.2) and (2.3) can be solved on the whole of $\mathbb{R}^n \times \mathbb{R}_+$.

Denote by $J_X(\cdot, t)$ and $J_Y(\cdot, t)$ the Jacobian of $X(\cdot, t)$ and $Y(\cdot, t)$, respectively. Since $\text{div} \, b = 0$, Liouville’s theorem, see e.g. [Arn92], implies that

\begin{equation}
\begin{aligned}
J_X(y, t)J_Y(X(y, t), t) &= \text{id} \quad \text{and} \quad \det J_X(y, t) = \det J_Y(x, t) = 1
\end{aligned}
\end{equation}

(2.4)

for all $t \geq 0$ and $x, y \in \mathbb{R}^n$.

In the situation of several obstacles moving with time dependent velocity we make the following assumption:

(A2) Let $O_1, \ldots, O_m \subset \mathbb{R}^n$, $n \geq 2$, be compact sets with boundaries $\Gamma_1, \ldots, \Gamma_m$ of class $C^{1,1}$. We denote by $\Omega := \mathbb{R}^n \setminus \bigcup_{i=1}^m O_i$ the exterior domain. For time-dependent matrices

\[ M_i(t) \in C^\infty([0, T]; \mathbb{R}^{n \times n}), \]

with $\text{tr} \, M_i(t) = 0$ for all $t \in [0, T]$, $i \in \{1, \ldots, m\}$, we define the time-dependent sets $\Omega(t), \Gamma_i(t)$ and $O_i(t)$ for $t \in [0, T]$ by aid of the unique solution $G_{(i)}(t) : \overline{O_i(t)} \to \overline{O_i(t)}$ of the following ODE:

\begin{equation}
\begin{aligned}
G_{(i)}(t)x &= M_i(t)G_{(i)}(t)x, & x \in \overline{O_i(t)}, & t \in (0, T), \\
G_{(i)}(0)x &= x, & x \in \overline{O_i(t)}.
\end{aligned}
\end{equation}

(2.5)

Note that $G_{(i)}(t) : \overline{O_i(t)} \to \overline{O_i(t)}$ is an isomorphism, as

\begin{equation}
\begin{aligned}
\left[ G_{(i)}(t)x \right]_k &= M_i(t)[G_{(i)}(t)x]_k, & x \in \overline{O_i(t)}, & t \in (0, T), \\
\left[ G_{(i)}(0)x \right]_k &= x_k, & x \in \overline{O_i(t)},
\end{aligned}
\end{equation}

with $1 \leq k \leq n$, $i \in \{1, \ldots, m\}$, has a unique fundamental system of linear independent solutions $G_{(i)}(t) = \left( G_{(i)}^1(t), \ldots, G_{(i)}^n(t) \right)$, due to the non-vanishing Wronski-determinant at zero. In particular, for the case of constant matrices $M_i \in \mathbb{R}^{n \times n}$, we are left with $G_{(i)}(t) = e^{tM_i}$ and inverse $G_{(i)}^{-1}(t) = e^{-tM_i}$.

As the obstacles shall not collide, we require

\[ \text{dist} \left( \overline{O_i(t)}, \overline{O_j(t)} \right) > 0, \quad i \neq j, \quad t \in [0, T]. \]
We now choose open sets $B_{1_k}, B_{2_k} \subset \mathbb{R}^n$, such that $\overline{O_i} \subset B_{1_k} \subset \overline{B_{1_k}} \subset B_{2_k}$ and set
\[
B_{k_i}(t) := \{ y = G_{(i)}(t)x, x \in B_{k_i} \}, \quad t \in [0, T], \quad k = 1, 2, \quad i \in \{1, \ldots, m\}.
\]
Then
\[
\overline{O_i}(t) \subset B_{1_k}(t) \subset \overline{B_{1_k}}(t) \subset B_{2_k}(t)
\]
for $t \in [0, T]$. Moreover, we demand the sets $B_{2_k}$ to be so small that the sets $B_{2_k}(t)$ are pairwise disjoint for $t \in [0, T]$. Next, we introduce a time-dependent cut-off function $\eta \in C^\infty(\mathbb{R}^n \times [0, T]), \ 0 \leq \eta \leq 1$, such that
\[
\eta(y, t) := \begin{cases} 1 & \text{on } \bigcup_{i=1}^m B_{1_k}(t), \quad t \in [0, T], \\ 0 & \text{on } \mathbb{R}^n \setminus \bigcup_{i=1}^m B_{2_k}(t), \quad t \in [0, T]. 
\end{cases}
\]
We define the sets
\[
K_i(t) := (\text{supp} \nabla y \eta(t, \cdot)) \cap \overline{B_{2_k}(t)}, \quad t \in [0, T], \quad i \in \{1, \ldots, m\},
\]
and the time dependent vector fields $b, b^* : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ by
\[
b(y, t) := \eta(y, t) \sum_{i=1}^m M_i(t)y - \sum_{i=1}^m B_{K_i}(t) ((\nabla y \eta)(\cdot, t)M_i(t)\cdot)(y),
\]
\[
b^*(y, t) := \eta(y, t) \sum_{i=1}^m G_i^{-1}(t)M_i(t)G_i(t)y
\]
\[= - \sum_{i=1}^m B_{K_i}(t) ((\nabla y \eta)(\cdot, t)G_i^{-1}(t)M_i(t)G_i(t)\cdot)(y).
\]
Note, that due to $\text{tr} \ M_i(t) = 0, \ t \in [0, T], \ i = 1, \ldots, m$, and properties of the Bogovskii operator, the vector field $b$ is solenoidal for $t \in [0, T]$ on all of $\mathbb{R}^n$. Further, $b(y, t) = M_i(t)y$ for $y \in \overline{O_i}(t), t \in [0, T], i \in \{1, \ldots, m\}$. Note that for fixed $t^* \in [0, T]$
\[
b(y, t^*) = \eta(y, t^*) \sum_{i=1}^m M_i(t^*)y - \sum_{i=1}^m B_{K_i^*} ((\nabla y \eta)(\cdot, t^*)M_i(t^*)\cdot)(y).
\]
Since $\eta$ is smooth in the first variable, the mapping properties of the Bogovskii operator imply that $b(\cdot, t^*) \in C^\infty_c(\mathbb{R}^n)$. Now, freeze $y^* \in \mathbb{R}^n$ and consider
\[
b(y^*, t) = \eta(y^*, t) \sum_{i=1}^m M_i(t)y^* - \sum_{i=1}^m B_{K_i^*} ((\nabla y \eta)(\cdot, t)M_i(t)\cdot)(y^*).
\]
The smoothness of the cut-off function $\eta$ and the matrices $M_i(\cdot)$ is inherited by $b$, and we thus see that $b \in C^\infty_{c, c}(\mathbb{R}^n \times [0, T]; \mathbb{R}^n)$.

In particular, $b$ is uniformly Lipschitz continuous with respect to the first variable and is bounded on $[0, T] \times \mathbb{R}^n$.

Thus, the ordinary differential equation
\[
(2.6) \quad \begin{cases}
\partial_t X(y, t) = b(X(y, t), t), \quad t > 0, \quad y \in \mathbb{R}^n, \\
X(y, 0) = y, \quad y \in \mathbb{R}^n,
\end{cases}
\]
admits a unique solution by the Picard-Lindelöf theorem. Moreover,
\[
X \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+).
\]
As above, let \( Y(\cdot, t) \) be the inverse of \( X(\cdot, t) \). Then \( Y \) satisfies
\[
\begin{aligned}
(2.7) \quad \begin{cases}
\partial_t Y(x, t) &= -(J_X)^{-1}(Y(x, t), t)b(x, t), \quad t > 0, \quad x \in \mathbb{R}^n, \\
Y(x, 0) &= x, \quad x \in \mathbb{R}^n,
\end{cases}
\end{aligned}
\]
and (2.4). Again, only the restrictions of \( X, Y \) to \([0, T]\) will be relevant in the sequel.

We set
\[
\begin{aligned}
U(y, t) &:= J_Y(X(y, t), t) v(X(y, t), t), \quad y \in \Omega, \quad t \in [0, T], \\
\pi(y, t) &:= q(X(y, t), t), \quad y \in \Omega, \quad t \in [0, T].
\end{aligned}
\]

Then (similar to \([\text{IW77}], [\text{CT06, Prop.3.5}]\)), a function
\[
v \in L^p(0, T; W^{2,q}(\Omega(\cdot))) \cap W^{1,p}(0, T; L^q(\Omega(\cdot))), \quad q \in L^p(0, T; \hat{W}^{1,q}(\Omega(\cdot)))
\]
is a solution to (1.1) if and only if
\[
U \in L^p(0, T; W^{2,q}(\Omega(\cdot))) \cap W^{1,p}(0, T; L^q(\Omega)), \quad \pi \in L^p(0, T; \hat{W}^{1,q}(\Omega))
\]
and \((U, \pi)\) satisfies the following set of equations
\[
(2.8) \quad \begin{cases}
\partial_t U + (\mathcal{M} - \mathcal{L}) U &= f - NU - G\pi, \quad \text{in } \Omega \times (0, T), \\
div U &= 0, \quad \text{in } \Omega \times (0, T), \\
U &= G_i^{-1}(t)M_i(t)G_i(t)y, \quad \text{on } \Gamma_i \times (0, T), \quad i = 1, \ldots, m, \\
U(0) &= \nu_0, \quad \text{in } \Omega.
\end{cases}
\]
Here
\[
(\mathcal{L}U)_i := \sum_{j,k=1}^n \partial_j (g^{jk} \partial_k U_i) + 2 \sum_{j,k,l=1}^n g^{kl} \Gamma^i_{jk} \partial_l U_j
\]
\[
+ \sum_{j,k,l=1}^n \left( \partial_k (g^{kl} \Gamma^i_{jl}) + \sum_{m=1}^n g^{kl} \Gamma^m_{jl} \Gamma^i_{km} \right) U_j,
\]
\[
(\mathcal{N}U)_i := \sum_{j=1}^n U_j \partial_j U_i + \sum_{j,k=1}^n \Gamma^i_{jk} U_j U_k,
\]
\[
(\mathcal{M}U)_i := \sum_{j=1}^n Y_j \partial_j U_i + \sum_{j,k=1}^n \left( \Gamma^i_{jk} Y_k + (\partial_k Y_i)(\partial_j X_k) \right) U_j,
\]
\[
(\mathcal{G}\pi)_i := \sum_{j=1}^n g^{ij} \partial_j \pi,
\]
with the metric contravariant tensor
\[
g^{ij} = \sum_{k=1}^n (\partial_k Y_i)(\partial_k Y_j),
\]
the metric covariant tensor
\[
g_{ij} = \sum_{k=1}^n (\partial_i X_k)(\partial_j X_k)
\]
and Christoffel’s symbol
\[
\Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^n g^{kl} (\partial_j g_{il} + \partial_i g_{jl} - \partial_i g_{ij}).
\]
Note that $\mathcal{L}$ is the transformed Stokes operator, while $\mathcal{M}$ arises from transforming the time derivative. The non-linearity $\mathcal{N}$ and modified gradient $\mathcal{G}$ correspond to $(v \cdot \nabla)v$ and $\nabla$, respectively.

Setting $u(y, t) := \hat{U}(y, t) - b(y, t)$, we see that

$$U \in L^p(0, T; W^{2,q}(\Omega)) \cap W^{1,p}(0, T; L^q(\Omega)), \quad \pi \in L^p(0, T; \dot{W}^{1,q}(\Omega)),$$

is a solution of (2.23) if and only if

$$u \in L^p(0, T; W^{2,q}(\Omega)) \cap W^{1,p}(0, T; L^q(\Omega)), \quad \pi \in L^p(0, T; \dot{W}^{1,q}(\Omega)),$$

and $(u, \pi)$ solves

$$\begin{cases}
\partial_t u - \Delta u + \nabla \pi &= F - \mathcal{N}u - \mathcal{B}u + (\mathcal{L} - \Delta)u \\
\text{div} u &= 0 \\
u &= 0 \\
u(0) &= v_0 - b(0)
\end{cases}$$

in $\Omega \times (0, T)$, $\Omega \times (0, T)$, $\Gamma_i \times (0, T)$, $i = 1, \ldots, m$, respectively.

In the sequel, maximal $L^p$-regularity of the Stokes operator in $L^2_\sigma(\Omega)$ plays an important role. More precisely, for $1 < q < \infty$, we define the Stokes operator $A_q$ in $L^2_\sigma(\Omega)$ by

$$\begin{cases}
A_q u := P_q \Delta u, \\
D(A_q) := W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^2_\sigma(\Omega).
\end{cases}$$

As usual, $P_q$ denotes the Helmholtz projection on $L^q(\Omega)$.

Let $T_0 > 0$, $1 < p < \infty$, $T \in (0, T_0)$, $f \in L^p(0, T; L^q_2(\Omega))$ and $u_0 \in (L^2_\sigma(\Omega), D(A_q))_{\frac{1}{2}, p}$. Then it follows from a classical result of Solonnikov [Sol77] (also see Gig81, Fr02 or GHHSS06) that there exists a unique solution

$$u \in X^{T}_{p,q} := W^{1,p}(0, T; L^2_\sigma(\Omega)) \cap L^p(0, T; W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^2_\sigma(\Omega))$$

of the inhomogeneous Stokes problem

$$\begin{cases}
u'(t) - A_q u(t) &= f(t), \\
u(0) &= u_0.
\end{cases}$$

Moreover, there exists $C > 0$, independent of $T$, $f$ and $u_0$, such that

$$\|u\|_{X^{T}_{p,q}} \leq C(\|f\|_{L^p(0, T; L^q(\Omega))} + \|u_0\|_{(L^2(\Omega), D(A_q))_{\frac{1}{2}, p}}).$$

Setting $\nabla \pi := (\text{Id} - P_q)\Delta u$, we see that $(u, \pi)$ is a solution to

$$\begin{cases}
u'(t) - \Delta u(t) - \nabla \pi(t) &= f(t), \\
u(0) &= u_0,
\end{cases}$$

satisfying

$$\|u\|_{X^{T}_{p,q}} + \|\pi\|_{Y^{T}_{p,q}} \leq C(\|f\|_{L^p(0, T; L^q(\Omega))} + \|u_0\|_{(L^2(\Omega), D(A_q))_{\frac{1}{2}, p}}),$$

where $Y^{T}_{p,q} := L^p(0, T; \dot{W}^{1,q}(\Omega))$ and $C > 0$ is a constant independent of $T \in (0, T_0)$, $f$ and $u_0$. 
For the rest of this section, assume that $M = -M'$, (A1) holds and consider
\begin{equation}
(2.11)
\begin{cases}
\partial_t v - \Delta v + (v \cdot \nabla)v + \nabla q = 0 & \text{in } \Omega(t), \ t \in (0, T), \\
\operatorname{div} v = 0 & \text{in } \Omega(t), \ t \in (0, T), \\
v(x, t) = Mx & \text{on } \Gamma(t), \ t \in (0, T), \\
v(x, 0) = v_0(x) & \text{in } \Omega(0).
\end{cases}
\end{equation}

It was shown in [GHH06a] that (2.11) admits a unique, local, mild solution $u$. In order to compare the solutions obtained in [GHH06a] and in Corollary 3.2, we introduce the notion of very weak solutions to equation (2.11). To this end, let $D \in (\Omega)$. Moreover, this solution $\hat{u}$ to (2.11) if
\begin{align*}
\langle v_0, \varphi \rangle_\Omega - \int_0^{T_0} \langle u(t), \varphi'(t) + \Delta \varphi(t) + (u(t) \cdot \nabla)\varphi(t) \rangle_{\Omega(t)} \, dt \\
+ \int_0^{T_0} \langle Mx, (\nu(t) \cdot \nabla)\varphi(t) \rangle_{\Gamma(t)} \, dt = 0,
\end{align*}
for $\varphi \in D$, where
\begin{equation*}
D := \left\{ \varphi \in C^1 \left([0, T_0]; C_c^\infty(\overline{\Omega(\cdot)}) \right) : \varphi(T_0) = 0, \operatorname{div} \varphi(t) = 0, \varphi(t)|_{\Gamma(t)} = 0 \text{ for all } t \in [0, T_0] \right\}
\end{equation*}
and $\nu(t)$ is the outer normal. We say that two very weak solutions
\begin{equation*}
u, w \in L^p(0, T_0; L^q(\Omega))
\end{equation*}
to (2.11) with initial value $v_0 \in L^q(\Omega)$ coincide in the very weak sense if there exists $T \in (0, T_0)$ such that $u(t) = v(t)$ for a.a. $t \in (0, T)$.

In [GHH06a], problem (2.11) is transformed into the equation
\begin{equation}
(2.12)
\begin{cases}
\dot{\hat{u}}' - A_{\Omega, b}\hat{u} + (\hat{u} \cdot \nabla \hat{u}) = F_2, & t \in (0, T), \\
\hat{u}(0) = v_0 - b,
\end{cases}
\end{equation}
where $\hat{u}(x, t) := e^{-t M} v(e^{t M} x, t) - b(x)$. Here,
\begin{equation*}
A_{\Omega, b} \hat{u} := P_q(\Delta \hat{u} + Mx \cdot \nabla \hat{u} - M \hat{u} - b \cdot \nabla \hat{u} - \hat{u} \cdot \nabla b)
\end{equation*}
with $D(A_{\Omega, b}) := \left\{ \hat{u} \in W^{2, q}(\Omega) \cap W^{1, q}_0(\Omega) \cap L^\infty_b(\Omega) : Mx \cdot \nabla \hat{u} \in L^q(\Omega) \right\}$ and $F_2 := \Delta b + Mx \cdot \nabla b - Mb - b \cdot \nabla b$. It is proved that, for $v_0 - b \in L^r_\sigma(\Omega)$ with $r \geq n$, there exists a mild solution $\hat{u}$ to (2.12), i.e. $\hat{u} \in C([0, T]; L^r_\sigma(\Omega))$ satisfies the integral equation
\begin{equation}
\dot{\hat{u}}(t) = T_{\Omega, b}(v_0 - b) - \int_0^t T_{\Omega, b}(t - s) P_q(\hat{u} \cdot \nabla \hat{u})(s) \, ds + \int_0^t T_{\Omega, b} P_q F_2(s) \, ds, \ t \in (0, T).
\end{equation}
Here, $(T_{\Omega, b}(t))_{t \geq 0} := (e^{t A_{\Omega, b}})_{t \geq 0}$ is the semigroup generated by $(A_{\Omega, b}, D(A_{\Omega, b}))$ in $L^r_\sigma(\Omega)$. Moreover, this solution $\hat{u}$ satisfies:
\begin{equation}
t \mapsto t \tilde{\Phi}(\frac{t}{T} - \frac{1}{2}) \hat{u}(\cdot) \in C([0, T]; L^r_\sigma(\Omega)),
\end{equation}
\begin{equation}
t \mapsto t \tilde{\Phi}(\frac{t}{T} - \frac{1}{2}) + \frac{1}{2} \nabla \hat{u}(\cdot) \in C([0, T]; L^q(\Omega)).
\end{equation}
We show in Section 4 that \(w\) given by
\[
(2.16) \quad w(x,t) := e^{tM} \hat{u}(e^{-tM}x,t) + e^{tM}b(e^{-tM}x),
\]
with \(\hat{u}\) given by the variation-of-constants formula, \eqref{2.13} is a very weak solution to \eqref{2.11} for suitable choices of \(p, q\) and \(n\). We are in the position to state our main results.

3. Main results

Our existence and uniqueness result for equation \eqref{1.1} reads as follows.

**Theorem 3.1.** Assume \((A2)\) and let \(p, q \in (1, \infty)\) such that \(\frac{n}{2q} + \frac{1}{p} \leq \frac{3}{2}\). Assume that

\begin{align*}
\text{a) } & f \in L^p(0, T; L^q_0(\Omega(\cdot))), \\
\text{b) } & v_0 - b(\cdot, 0) \in (L^q_0(\Omega), W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega) \cap L^q_0(\Omega))_{1-1/p,p}.
\end{align*}

Then there exists a \(T > 0\) such that the problem \eqref{1.1} admits a unique strong solution
\[
v \in L^p(0, T; W^{2,q}(\Omega(\cdot))) \cap W^{1,p}(0, T; L^q_0(\Omega(\cdot))), \quad q \in L^p(0, T; \hat{W}^{1,q}(\Omega(\cdot))).
\]

Moreover, one may choose \(T > 0\) such that either \(T = +\infty\) or the function
\[
t \mapsto \|v(t)\|_{(L^q_0(\Omega(t)), W^{1,q}_0(\Omega(t)) \cap W^{2,q}(\Omega(t)) \cap L^q_0(\Omega(t)))_{1-1/p,p}}
\]
is unbounded on its maximal interval of existence \([0, T]\).

**Corollary 3.2.** Assume \((A1)\) and let \(p, q \in (1, \infty)\) such that \(\frac{n}{2q} + \frac{1}{p} \leq \frac{3}{2}\). Assume that

\begin{align*}
\text{a) } & f \in L^p(0, T; L^q_0(\Omega(\cdot))), \\
\text{b) } & v_0 - b \in (L^q_0(\Omega), W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega) \cap L^q_0(\Omega))_{1-1/p,p}.
\end{align*}

Then there exists a \(T > 0\) such that the problem \eqref{2.11} admits a unique strong solution
\[
v \in L^p(0, T; W^{2,q}(\Omega(\cdot))) \cap W^{1,p}(0, T; L^q_0(\Omega(\cdot))), \quad q \in L^p(0, T; \hat{W}^{1,q}(\Omega(\cdot))).
\]

Moreover, either \(T = +\infty\) or
\[
t \mapsto \|v(t)\|_{(L^q_0(\Omega(t)), W^{1,q}_0(\Omega(t)) \cap W^{2,q}(\Omega(t)) \cap L^q_0(\Omega(t)))_{1-1/p,p}}
\]
is unbounded on its maximal interval of existence \([0, T]\).

The following theorem says that the two solutions \(v\) and \(w\) coincide in the very weak sense. More precisely, the following holds true.

**Theorem 3.3.** Let \(1 < p, q < \infty\) such that \(\frac{n}{2q} + \frac{1}{p} \leq \frac{3}{2}\). Assume that

\begin{align*}
\text{a) } & v_0 - b \in (L^q_0(\Omega), D(A_q))_{1-1/p,p}, \\
\text{b) } & v_0 - b \in L^q_0(\Omega) \text{ for some } r > n \text{ provided } \frac{n}{2q} + \frac{1}{p} = \frac{3}{2}.
\end{align*}

Then \(w\) given by \eqref{2.16} and \(v\) given in Corollary 3.2 coincide in the very weak sense.
4. PROOF OF THE FIRST MAIN RESULT

Note that in order to prove our main result it suffices to construct a unique solution \((u, \pi)\) to (2.9). The strategy of the proof is as follows: first, we derive estimates on the coefficients of the operators \(\mathcal{N}, \mathcal{B}, \mathcal{L}, \mathcal{M}\) and \(G\) in problem (2.9). Then, a fixed point argument in a suitable closed subspace of \(X_{p,q}^T \times Y_{p,q}^T\) yields a unique solution \((u, \pi) \in X_{p,q}^T \times Y_{p,q}^T\) via maximal \(L^p\)-regularity of the Stokes operator.

Observe that for a multi-index \(\alpha\) and \(k \in \mathbb{N}\) there is some constant \(K_{|\alpha|,k,T} > 0\) such that

\[
\|\partial^\alpha_y \partial^k_t b\|_{L^\infty(\mathbb{R}^n \times [0,T])} \leq K_{|\alpha|,k,T}, \quad |\alpha| + k > 0.
\]

The following lemma yields estimates of the transformation mappings \(X\) and \(Y\), respectively, that are defined by (2.6) and (2.7). Clearly, the assertions remain true in the case of \(b\) being independent from the time variable and \(X, Y\) defined as in (2.2) and (2.3).

Lemma 4.1. Let \(T_0 > 0\), \(k \in \mathbb{N}\) and \(\alpha\) be a multi-index satisfying \(|\alpha| + k > 0\). Then there exists \(C_{|\alpha|,k,T_0} > 0\) such that

\[
\|\partial^\alpha_y \partial^k_t X\|_{L^\infty(\mathbb{R}^n \times [0,T])} \leq C_{|\alpha|,k,T_0}.
\]

The above estimates remain valid when \(X(\cdot, t)\) is replaced by its inverse \(Y(\cdot, t)\).

Proof. Let \(T_0 > 0\), \(k \in \mathbb{N}\) and \(\alpha\) be a multi-index satisfying \(|\alpha| + k > 0\). By a direct calculation, we see that \(X(t, y) = y\) for \(y \notin \text{supp} \ b\). Hence,

\[
\|\partial^\alpha_y \partial^k_t X\|_{L^\infty((\text{supp} \ b)^c \times [0,T_0])} \leq 1.
\]

Since \(\text{supp} \ b\) is compact and \(X \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+\), there exists \(C_{|\alpha|,k,T_0}\) such that

\[
\|\partial^\alpha_y \partial^k_t X\|_{L^\infty((\text{supp} \ b) \times [0,T_0])} \leq C_{|\alpha|,k,T_0}.
\]

It follows from the definition of \(g^{ij}, g_{ij}\) and \(\Gamma^k_{ij}\) and the previous lemma that all coefficients of \(\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{B}\) and \(\mathcal{G}\) are smooth and bounded on finite time intervals \([0,T_0]\) for any \(T_0 > 0\). Moreover, by the mean value theorem, for \(x \in \mathbb{R}^n\) we have

\[
g^{ij}(x,t) - \delta_{ij} = t\partial_t g^{ij}(x,\tau) \quad \text{for some } \tau \in (0,t).
\]

Hence, it follows from Lemma 4.1 that

\[
g^{ij} - \delta_{ij} \leq t\|\partial_t g^{ij}\|_{L^\infty(\mathbb{R}^n \times [0,T_0])} \leq Ct, \quad t \in (0,T_0).
\]

The following embedding of \(X_{p,q}^T\) in \(L^k(0,T; W^{s,m}(\Omega))\) is needed to cope with the gradient terms. It mainly relies on the mixed derivative theorem. Precisely, the following lemma holds:

Lemma 4.2. Let \(1 < p,q < \infty\), \(T_0 > 0\) and \(s = 0\) or \(s = 1\). Assume that \(k, m \in (1, \infty)\) obey \(\frac{2s}{p} + \frac{n}{m} - \frac{s}{2} \geq \frac{p}{2} - \frac{k}{2}\). Then \(X_{p,q}^T\) is continuously embedded in \(L^k(0,T; W^{s,m}(\Omega))\) for any \(T \in (0,T_0)\). Moreover, there exists \(C_{T_0} > 0\) such that

\[
\|u\|_{L^k(0,T; W^{s,m}(\Omega))} \leq C_{T_0}\|u\|_{X_{p,q}^T}, \quad T \in (0,T_0), \quad u \in X_{p,q}^T, \quad u(0) = 0.
\]

Proof. By the mixed derivative theorem (see [Sob75] or [DHP07]), for \(\theta \in (0,1)\) there exists \(C > 0\) such that

\[
\|u\|_{H^2,p(0,T_0; H^2-2\theta, q(\Omega))} \leq C\|u\|_{X_{p,q}^T}, \quad u \in X_{p,q}^T\].
It then follows from Sobolev embeddings that
\[ \|u\|_{L^k(0,T;W^{s,m}(\Omega))} \leq C_{T_0} \|u\|_{X^{T_0}_{p,q}}, \quad u \in X^{T_0}_{p,q}. \]
Let \( S : \{ u \in X^{T_0}_{p,q} : u(0) = 0 \} \to X^{T_0}_{p,q} \) be defined by
\[ S_u(t) := \begin{cases} u(t - (T_0 - T)), & t \geq T_0 - T, \\ 0, & 0 \leq t < T_0 - T. \end{cases} \]
Then \( \|S_u\|_{X^{T_0}_{p,q}} = \|u\|_{X^{T_0}_{p,q}} \) and \( \|S_u\|_{L^k(0,T;W^{s,m}(\Omega))} = \|u\|_{L^k(0,T;W^{s,m}(\Omega))}. \) We thus obtain
\[ \|u\|_{L^k(0,T;W^{s,m}(\Omega))} = \|S_u\|_{L^k(0,T;W^{s,m}(\Omega))} \leq C_{T_0} \|S_u\|_{X^{T_0}_{p,q}} = C_{T_0} \|u\|_{X^{T_0}_{p,q}} \]
for all \( u \in X^{T_0}_{p,q} \) satisfying \( u(0) = 0. \) \( \square \)

Next, we prove estimates for the terms on the right-hand side of (2.9).

**Lemma 4.3.** Fix \( T_0 > 0 \) and \( u \in X^{T_0}_{p,q}. \) Then, for \( T \in (0,T_0) \) there exists \( C > 0, \) independent of \( T \) and \( K(T) > 0 \) with \( \lim_{T \to 0} K(T) = 0, \) such that for \( (v_1,q_1), (v_2,q_2) \in X^{T}_{p,q} \times Y^{T}_{p,q} \) satisfying \( v_1(0) = v_2(0) = 0, \)

(a) \( \|N u\|_{L^p(0,T;L^q(\Omega))} \leq (K(T))^2, \)

(b) \( \|N(v_1 + u) - N(v_2 + u)\|_{L^p(0,T;L^q(\Omega))} \leq C(K(T) + \sum_{i=1}^{2} \|v_i\|_{X^{T}_{p,q}} \|v_1 - v_2\|_{X^{T}_{p,q}}), \)

(c) \( \|B(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} \leq CT^{\frac{1}{p'}} \|v_1 - v_2\|_{X^{T}_{p,q}}, \)

(d) \( \|M(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} \leq CT^{\frac{1}{p'}} \|v_1 - v_2\|_{X^{T}_{p,q}}, \)

(e) \( \|(\mathcal{L} - \Delta)(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} \leq C(T + T^{\frac{1}{p'}}) \|v_1 - v_2\|_{X^{T}_{p,q}}, \)

(f) \( \|(\nabla - \mathcal{G})(q_1 - q_2)\|_{L^p(0,T;L^q(\Omega))} \leq CT \|q_1 - q_2\|_{Y^{T}_{p,q}}. \)

The estimates above are valid even if \( v_1(0) \neq 0 \) and \( v_2(0) \neq 0. \) However, in this case, \( C \) depends on \( T \) as well.

**Proof.** Set \( k = 3p, \ k' = 3p/2, \ m = 3q, \ m' = 3q/2. \) By Hölder’s inequality, we obtain for the first term of \( \mathcal{N} u \)
\[ \|u \cdot \nabla u\|_{L^p(0,T;L^q(\Omega))} \leq \|u\|_{L^k(0,T;L^m(\Omega))} \|u\|_{L^{k'}(0,T;W^{1,m'}(\Omega))}. \]

We now define
\[ K(T) := \|u\|_{L^k(0,T;L^m(\Omega))} + \|u\|_{L^{k'}(0,T;W^{1,m'}(\Omega))}. \]
Then,
\[ \|u \cdot \nabla u\|_{L^p(0,T;L^q(\Omega))} \leq (K(T))^2. \]
Since \( \lim_{T_0 \to 0} K(T) = 0, \) assertion [(a)] follows.
Next, set \( w_1 = v_1 + u \) and \( w_2 = v_2 + u \). By Hölder’s inequality, we similarly obtain for the first term of \( \mathcal{N}w_1 - \mathcal{N}w_2 \)

\[
\| w_1 \cdot \nabla w_1 - w_2 \cdot \nabla w_2 \|_{L^p(0,T;L^q(\Omega))} \\
\leq \| (v_1 - v_2) \cdot \nabla v_1 + v_2 \cdot (\nabla v_1 - \nabla v_2) \|_{L^p(0,T;L^q(\Omega))} \\
+ \| (v_1 - v_2) \cdot \nabla u + u \cdot (\nabla v_1 - \nabla v_2) \|_{L^p(0,T;L^q(\Omega))} \\
\leq \left( \| v_1 \|_{L^p(0,T;W^{1,m'}(\Omega))} + \| v_2 \|_{L^p(0,T;W^{1,m'}(\Omega))} \right) \| v_1 - v_2 \|_{L^{p}(0,T;L^m(\Omega))} \\
+ \left( \| v_2 \|_{L^p(0,T;L^m(\Omega))} + \| u \|_{L^p(0,T;L^m(\Omega))} \right) \| v_1 - v_2 \|_{L^{p}(0,T;W^{1,m'}(\Omega))}.
\]

Hence, by Lemma 4.2 there exists \( C > 0 \), independent of \( T \in (0,T_0) \) and \( v_1, v_2 \), such that

\[
\| w_1 \cdot \nabla w_1 - w_2 \cdot \nabla w_2 \|_{L^p(0,T;L^q(\Omega))} \leq C \left( \| v_1 \|_{X^{T,q}_{p,q}} + \| v_2 \|_{X^{T,q}_{p,q}} + K(T) \right) \| v_1 - v_2 \|_{X^{T,q}_{p,q}}.
\]

Since the second term of \( \mathcal{N}w_1 - \mathcal{N}w_2 \) can be estimated similarly, assertion (b) follows.

Similarly, (c) and (d) follow from the estimates

\[
\| B(v_1 - v_2) \|_{L^p(0,T;L^q(\Omega))} \leq CT^{\frac{1}{q}} \| v_1 - v_2 \|_{L^p(0,T;L^m(\Omega))} \leq CT^{\frac{1}{q}} \| v_1 - v_2 \|_{X^{T,q}_{p,q}}
\]

and

\[
\| M(v_1 - v_2) \|_{L^p(0,T;L^q(\Omega))} \leq CT^{\frac{1}{q}} \| v_1 - v_2 \|_{L^p(0,T;W^{1,m}(\Omega))} \leq CT^{\frac{1}{q}} \| v_1 - v_2 \|_{X^{T,q}_{p,q}},
\]

respectively, where \( C > 0 \) is independent of \( T \in (0,T_0) \) and \( v_1, v_2 \).

As all coefficients of \( L \) are smooth we may rewrite it in non-divergence form, and, by (4.1), we obtain

\[
\| (L - \Delta)(v_1 - v_2) \|_{L^p(0,T;L^q(\Omega))} \\
\leq C \| \delta_{jk} - \delta_{jk} \|_{L^\infty(\Omega)} \| D^2(v_1 - v_2) \|_{L^p(0,T;L^q(\Omega))} \\
+ CT^{\frac{1}{q}} \| v_1 - v_2 \|_{L^p(0,T;W^{1,m}(\Omega))} \\
\leq C(T + T^{\frac{1}{q}}) \| v_1 - v_2 \|_{X^{T,q}_{p,q}}, \quad T \in (0,T_0), \; v_1, v_2 \in X^{T}_{p,q}, \; v_1(0) = 0, \; v_2(0) = 0.
\]

Hence (6) follows. Assertion (4) similarly follows from (4.1). \( \square \)

Let \( T_0 > 0, \; T \in (0,T_0) \) and \( p, q \in (1, \infty) \) satisfy \( \frac{1}{T_0} + \frac{1}{p} \leq \frac{3}{2} \). Next, we introduce the space for the fixed point argument. In order to do this, consider

\[
\begin{align*}
\partial_t u^* - \Delta u^* + \nabla \pi^* & = \tilde{F}(0,0) \quad \text{in} \; \Omega \times (0,T), \\
\text{div} \; u^* & = 0 \quad \text{in} \; \Omega \times (0,T), \\
u^* & = 0 \quad \text{on} \; \Gamma \times (0,T), \\
u^*(0) & = u_0 \quad \text{in} \; \Omega,
\end{align*}
\]

where \( \tilde{F} : X^{T_0}_{p,q} \times Y^{T_0}_{p,q} \to L^p(0,T_0;L^q(\Omega)) \) is defined by

\[
\tilde{F}(v,q) := F - \mathcal{N}v - Bv + (L - \Delta)v - Mv + (\nabla - G)q.
\]

Note that \( \tilde{F} \) is well-defined by Lemma 4.3 thanks to \( \frac{1}{T_0} + \frac{1}{p} \leq \frac{3}{2} \). Hence, by (2.10), there exists a unique solution \((u^*, \pi^*) \in X^{T}_{p,q} \times Y^{T}_{p,q}\). We equivalently rewrite (2.9)
in terms of a fixed point problem by

$$\begin{aligned}
\partial_t \tilde{u} - \Delta \tilde{u} + \nabla \tilde{\pi} &= F + \tilde{F}(v + u^*, q + \pi^*) \quad &\text{in } \Omega \times (0, T), \\
\text{div } \tilde{u} &= 0 \quad &\text{in } \Omega \times (0, T), \\
\tilde{u} &= 0 \quad &\text{on } \Gamma \times (0, T), \\
\tilde{u}(0) &= 0 \quad &\text{in } \Omega,
\end{aligned}$$

where we used the notation

$$u := \tilde{u} - u^*, \quad \pi := \tilde{\pi} - \pi^*.$$  

In view of the Banach fixed point theorem we define for a given radius $R > 0$ and $T \in (0, T_0)$ the closed set

$$\mathcal{K}_{R,T} := \left\{ (v, q) \in X^T_{p,q} \times Y^T_{p,q} : v(0) = 0 \text{ and } \|v\|_{X^T_{p,q}} + \|q\|_{Y^T_{p,q}} \leq R \right\}$$

and a mapping

$$\Phi_{R,T} : \left\{ \begin{array}{l}
\mathcal{K}_{R,T} \\
(v, q)
\end{array} \rightarrow \begin{array}{l}
X^T_{p,q} \times Y^T_{p,q} \\
(\tilde{u}, \tilde{\pi})
\end{array} \text{ such that (4.2) holds.} \right.$$  

In order to apply the Banach fixed point theorem to $\Phi_{R,T}$ we have to show that the mapping is well-defined, maps $\mathcal{K}_{R,T}$ into itself and is a contraction.

Note that $\phi_{R,T}$ is well-defined due to Lemma 4.3 and (2.10). The next lemma shows that for suitable choices of $R > 0$ and $T > 0$ the closed set $\mathcal{K}_{R,T}$ is mapped by $\Phi_{R,T}$ into itself.

**Lemma 4.4.** There exist $R > 0$ and $T_1 > 0$, such that $\Phi_{R,T} : \mathcal{K}_{R,T} \rightarrow \mathcal{K}_{R,T}$ for all $T \in (0, T_1)$.

**Proof.** By (2.10) and Lemma 4.3 we obtain

$$\|\Phi_{R,T}(v, q)\|_{X^T_{p,q} \times Y^T_{p,q}} \leq C \|F + \tilde{F}(v + u^*, q + \pi^*)\|_{L^p(0,T;L^q(\Omega))} \leq C \left( \|F\|_{L^p(0,T;L^q(\Omega))} + \|\mathcal{N}(v + u^*) - \mathcal{N}u^*\|_{L^p(0,T;L^q(\Omega))} + \|\mathcal{N}u^*\|_{L^p(0,T;L^q(\Omega))} \\
+ \|\mathcal{B}(v + u^*)\|_{L^p(0,T;L^q(\Omega))} \right) \leq C \left( \|F\|_{L^p(0,T;L^q(\Omega))} + K(T)R + R^2 + (K(T))^2 \right) \leq C \left( \|F\|_{L^p(0,T;L^q(\Omega))} + K(T)R + R^2 + (K(T))^2 \right)$$

Since

$$\lim_{T \rightarrow 0} \|F\|_{L^p(0,T;L^q(\Omega))} + K(T) = 0,$$

we obtain $\|\Phi_{R,T}(v, q)\|_{X^T_{p,q} \times Y^T_{p,q}} \leq R$ provided $R$ and $T$ are small enough.  

**Lemma 4.5.** There exists $T_0 > 0$ such that $\Phi_{R,T} : \mathcal{K}_{R,T} \rightarrow \mathcal{K}_{R,T}$ is a contraction for all $T \in (0, T_0)$.  

Proof. Again, by (2.10) and Lemma 4.3, we obtain
\[ \| \Phi_{R,T}(v_1, q_1) - \Phi_{R,T}(v_2, q_2) \|_{X_{p,q}^T \times Y_{p,q}^T} \]
\[ \leq C \left( \| N(v_1 + u^*) - N(v_2 + u^*) \|_{L^p(0,T;L^q(\Omega))} + \| B(v_1 - v_2) \|_{L^p(0,T;L^q(\Omega))} \right. \]
\[ + \| (L - \Delta)(v_1 - v_2) \|_{L^p(0,T;L^q(\Omega))} + \| M(v_1 - v_2) \|_{L^p(0,T;L^q(\Omega))} \]
\[ \left. + \| (\nabla - \mathcal{G})(q_1 - q_2) \|_{L^p(0,T;L^q(\Omega))} \right) \]
\[ \leq C(K(T) + R + T + T^\frac{1}{p}) \| (v_1, q_1) - (v_2, q_2) \|_{X_{p,q}^T \times Y_{p,q}^T}, \]
\[ (v_1, q_1), (v_2, q_2) \in K_{R,T}. \]
Choosing \( T \) and \( R \) small enough, we obtain \( C(K(T) + R + T + T^\frac{1}{p}) < 1 \). \( \square \)

Proof of Theorem 3.1. The existence of a unique strong solution now follows from Lemma 4.4, Lemma 4.5, and the Banach fixed point theorem. Now, the theorem follows in a standard way from the fact that \( T > 0 \) is uniform with respect to \( v_0 \), provided
\[ \| v_0 \|_{L^q_1(\Omega),W^{2,q}_0(\Omega) \cap W^{2,q}(\Omega \cap L^q_2(\Omega))_{1-1/p, p}} < C_0, \]
and the continuous embedding
\[ X^T_{p,q} \hookrightarrow C \left( 0, T; \left( W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega \cap L^q_2(\Omega), L^q_2(\Omega))_{1-1/p, p} \right) \right). \] \( \square \)

5. Comparison of strong and mild solutions

In this section we prove Theorem 3.3. For the notion of very weak solutions we refer back to Section 2. In a first step, we show that a mild solution is a very weak solution.

Lemma 5.1. Let \((v_0 - b) \in L^q_2(\Omega)\) for some \( q \geq n \) and denote the mild solution to (2.12) on \([0, T]\) for some \( T > 0 \) by \( \hat{u} \) (for a representation by the variation-of-constants formula see (2.13)). Then \( w \), defined by (2.10), is a very weak solution to (2.11).

Proof. By (2.14) and (2.15), we obtain
\[ \hat{u} \in C([0, T]; L^q_2(\Omega)) \quad \text{and} \quad \nabla \hat{u} \in L^1(0, T; L^q(\Omega)). \]
Choose \( \hat{u}_n^1 \in C^1([0, T]; C^\infty_{c, 0}(\Omega)) \) and \( \hat{u}_n^2 \in C^1([0, T]; C^\infty(\Omega)) \) such that
\[ \lim_{n \to \infty} \| \hat{u}_n^1 - \hat{u} \|_{C([0, T]; L^q(\Omega))} = 0 \quad \text{and} \quad \lim_{n \to \infty} \| \nabla \hat{u}_n^2 - \nabla \hat{u} \|_{L^1(0, T; L^q(\Omega))} = 0. \]
Then, by [Paz83] Chapter 4, Theorem 2.9, there exists a solution
\[
\hat{v}_n \in L^1(0, T; D(A_{\Omega, b})) \cap W^{1,1}(0, T; L^q_2(\Omega))
\]
satisfying
\[
\begin{aligned}
\hat{v}_n'(t) - A_{\Omega, b} \hat{v}_n(t) + P_q \left( \left( \hat{u}_n^1 \cdot \nabla \hat{u}_n^2 \right)(t) \right) &= P_q F_2(t), \\
\hat{v}_n(0) &= \hat{u}_n(0),
\end{aligned}
\]
for \( t \in (0, T) \).
By the representation of \( \hat{v}_n \) via the variation-of-constants formula, we obtain
\[
(5.2) \quad \lim_{n \to \infty} \| \hat{v}_n - \hat{u} \|_{L^1(0,T;L^2(K))} = 0
\]
for any compact \( K \subset \Omega \). Setting
\[
\nabla \hat{\pi}_n = (\mathrm{Id} - P_q)[F_2(t) + \hat{u}_n^1 \cdot \nabla \hat{u}_n^2(t) - \Delta \hat{v}_n(t) - Mx \cdot \nabla \hat{v}_n(t) + M\hat{v}_n(t) + b \cdot \nabla \hat{v}_n(t) + \hat{v}_n(t) \cdot \nabla b]
\]
for \( t \in (0,T) \), we see that \((\hat{v}_n, \hat{\pi}_n)\) solves
\[
\begin{cases}
\hat{v}'_n - \Delta \hat{v}_n - Mx \cdot \nabla \hat{v}_n + M\hat{v}_n \\
+ b \cdot \nabla \hat{v}_n + \hat{v}_n \cdot \nabla b + \hat{u}_n^1 \cdot \nabla \hat{u}_n^2 + \nabla \hat{\pi}_n = F_2 & \text{in } \Omega, \ t \in (0,T), \\
\div \hat{v}_n = 0 & \text{in } \Omega, \ t \in (0,T), \\
\hat{v}_n = 0 & \text{on } \Gamma, \ t \in (0,T), \\
\hat{v}_n(0) = \hat{u}_n^1(0) & \text{in } \Omega.
\end{cases}
\]
Hence, \((v_n, \pi_n) := (e^{tM} \hat{v}_n(e^{-tM}x,t) + e^{tM}b(e^{-tM}x), \hat{\pi}(e^{-tM}x,t))\) is a solution to
\[
\begin{cases}
\partial_t v_n - \Delta v_n + u_n^1 \cdot \nabla u_n^2 + (b \cdot \nabla)(v_n - u_n^2) \\
+ ((v_n - u_n^1) \cdot \nabla) b + \nabla \pi_n = 0 & \text{in } \Omega(t), \ t \in (0,T), \\
\div v_n = 0 & \text{in } \Omega(t), \ t \in (0,T), \\
v_n = Mx & \text{on } \Gamma(t), \ t \in (0,T), \\
v_n(0) = u_n^1(0) & \text{in } \Omega,
\end{cases}
\]
where
\[
u_1 := e^{tM} \hat{u}_n^1(e^{-tM}x,t) + e^{tM}b(e^{-tM}x)
\]
and
\[
u_2 := e^{tM} \hat{u}_n^2(e^{-tM}x,t) + e^{tM}b(e^{-tM}x).
\]
Moreover,
\[
(v_n, \pi_n) \in (L^1(0,T;W^{2,q}(\Omega))) \cap W^{1,1}(0,T;L^2(\Omega)) \times L^1(0,T;W^{1,q}(\Omega))
\]
thanks to \( \hat{v}_n \in L^1(0,T;D(A_{\Omega,b})) \cap W^{1,1}(0,T;L^q(\Omega)) \). Therefore, integration by parts yields
\[
\langle v_n(0), \varphi \rangle_{\Omega} - \int_0^T \langle v_n(t), \varphi'(t) + \Delta \varphi(t) \rangle + \langle u_n^2(t), (u_n^1(t) \cdot \nabla) \varphi(t) \rangle_{\Omega(t)} \, dt
\]
\[
+ \int_0^T \langle (b \cdot \nabla)(v_n - u_n^2) + ((v_n - u_n^1) \cdot \nabla) b, \varphi \rangle_{\Omega(t)} \, dt
\]
\[
+ \int_0^T \langle Mx, \nu(t) \cdot \nabla \varphi(t) \rangle_{\Gamma(t)} \, dt = 0, \quad \varphi \in D.
\]
Letting \( n \to \infty \), it follows from (5.1) and (5.2) that \( u \) is a very weak solution. \( \square \)

In a second step, the following lemma shows uniqueness of the very weak solution for suitable values of \( p \) and \( q \). Especially, we will conclude coincidence of the mild and strong notion of solutions, as stated in Theorem 3.3.
Lemma 5.2. Let $1 < p, q < \infty$ be such that $\frac{2}{2q} + \frac{1}{p} \leq \frac{1}{2}$ and let $v_1, v_2 \in L^p(0, T_0; L^q(\Omega(\cdot)))$ be two very weak solutions to (2.11) for some $T_0 > 0$ with initial value $v_0 \in L^q(\Omega)$. Assume that $v_1 - v_2 \in L^p(0, T_0; L^q(\Omega(\cdot)))$. Then there exists $T \in (0, T_0)$ such that $v_1(t) = v_2(t)$ for a.e. $t \in (0, T)$.

Proof. For the time being, let us assume that for some $T \in (0, T_0)$ and all $f \in L^p(0, T; L^q(\Omega(\cdot)))$, $\frac{1}{p} + \frac{1}{q} = 1$, there exists a solution

$$(5.4)$$

$$(\varphi, \pi) \in D_{\text{ext}} := \left( W^{1,p}(0, T; L^q(\Omega(\cdot))) \cap L^p(0, T; W^{2,q}(\Omega(\cdot))) \right) \times L^p(0, T; W^{1,q}(\Omega(\cdot)))$$

to the dual backward problem

$$\begin{cases}
-\partial_t \varphi - \Delta \varphi + \nabla \pi - (v_1 + v_2) \cdot \nabla \varphi = f & \text{in } \Omega(t), \ t \in (0, T), \\
\text{div } \varphi = 0 & \text{in } \Omega(t), \ t \in (0, T), \\
\varphi = 0 & \text{on } \Gamma(t), \ t \in (0, T), \\
\varphi(T) = 0 & \text{in } \Omega.
\end{cases}$$

Then, we obtain for all $f \in L^p(0, T; L^q(\Omega(\cdot)))$

$$\begin{align*}
\int_0^T \langle v_1(t) - v_2(t), f(t) \rangle_{\Omega(t)} \, dt &= \int_0^T \langle v_1(t) - v_2(t), (-\partial_t \varphi - \Delta \varphi + \nabla \pi - (v_1 + v_2) \cdot \nabla \varphi)(t) \rangle_{\Omega(t)} \, dt \\
&= \int_0^T \langle v_1(t) - v_2(t), (-\partial_t \varphi - \Delta \varphi - (v_1 + v_2) \cdot \nabla \varphi)(t) \rangle_{\Omega(t)} \, dt = 0.
\end{align*}$$

Here, we have used that $\varphi$ can be approximated by functions in $D$. This implies $v_1 = v_2$ in $(0, T)$.

It thus remains to show that for $f \in L^p(0, T; L^q(\Omega(\cdot)))$, there exists a solution $(\varphi, \pi) \in D_{\text{ext}}$ to (5.3). In order to do so, we first consider the forward problem

$$\begin{cases}
\partial_t \varphi - \Delta \varphi + \nabla \pi - (v_1T + v_2T) \cdot \nabla \varphi = f^T & \text{in } \Omega(t), \ t \in (0, T), \\
\text{div } \varphi = 0 & \text{in } \Omega(t), \ t \in (0, T), \\
\varphi = 0 & \text{on } \Gamma(t), \ t \in (0, T), \\
\varphi(0) = 0 & \text{in } \Omega,
\end{cases}$$

where $v_1T(t) = v_1(T - t)$, $v_2T(t) = v_2(T - t)$ and $f^T(t) = f(T - t)$ for $t \in (0, T)$. Note first that by a scaling argument, we may assume that $\|f^T\|_{L^p(0, T; L^q(\Omega(\cdot)))}$ is arbitrarily small. Then, similar to the proof of Theorem [3.1], it follows that there exists $T > 0$, independent of $f^T$, and a solution $(\varphi, \pi) \in D_{\text{ext}}$ to (5.3). Indeed, we have an additional term coming from $f^T$, which is no problem since $f^T$ is arbitrarily small. Moreover, the non-linear term has to be replaced by a term coming from $(v_1T + v_2T) \cdot \nabla \varphi$. For the convenience of the reader, we will give the estimates for $(v_1T + v_2T) \cdot \nabla \varphi$. We choose $r, s \in (1, \infty)$ such that $\frac{1}{p} = \frac{1}{p} + \frac{1}{r}$ and $\frac{1}{q} = \frac{1}{q} + \frac{1}{s}$. 

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Then, it follows from Hölder’s inequality and Lemma 4.2 that
\[
\| (\tilde{v}_1^T - \tilde{v}_2^T) \cdot \nabla \varphi \|_{L^p(0,T;L^q(\Omega))} \leq \| \tilde{v}_1^T - \tilde{v}_2^T \|_{L^p(0,T;L^q(\Omega))} \| \nabla \varphi \|_{L^r(0,T;L^s(\Omega))} \\
\leq C \| \tilde{v}_1^T - \tilde{v}_2^T \|_{L^p(0,T;L^q(\Omega))} \| \nabla \varphi \|_{X_{p\sigma}}.
\]
Since \(\| \tilde{v}_1^T - \tilde{v}_2^T \|_{L^p(0,T;L^q(\Omega))} \to 0\) for \(T \to 0\), we get a solution \((\varphi^T, \pi^T)\) to (5.4) for some \(T > 0\). Finally, \((\varphi, \pi)\), where \(\varphi(t) := \varphi^T(T - t)\) and \(\pi(t) := \pi^T(T - t)\) for \(t \in (0, T)\), is a solution to (5.3).

We finally prove Theorem 3.3.

Proof of Theorem 3.3. Let us first assume that \(\frac{n}{2q} + \frac{1}{p} < \frac{3}{2}\). Then, by Sobolev embeddings,
\[
(L^2_q(\Omega), D(A_q))_{1-\frac{1}{p}, p} \hookrightarrow L^p_\sigma(\Omega)
\]
for some \(\sigma > n\). Hence, \(v, w \in C([0, T_0]; L^\sigma(\Omega))\) for some \(T_0 > 0\). In particular, \(v, w\) satisfy the assumption of Lemma 5.2. Hence, the assertion follows from iteration in this case.

Let us now assume that \(\frac{n}{2q} + \frac{1}{p} = \frac{3}{2}\) and \(v_0 - b \in L^s(\Omega)\) for some \(r > n\). By Sobolev embeddings, we have
\[
v \in L^s(0, T; L^r(\Omega))
\]
for some \(s \in (n, r)\) and \(1 < s < \infty\) satisfying \(\frac{n}{2q} + \frac{1}{s} \leq \frac{3}{2}\). Moreover, since \(v_0 - b \in L^s(\Omega)\), we have \(v \in C([0, T]; L^s(\Omega))\). Hence, in this case the assertion follows similar to the above.

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