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# APPROXIMATION PROPERTIES AND APPROXIMATE IDENTITIES OF $A_p(G)$

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Abstract. For a locally compact group $G$ and $1 < p < \infty$, let $A_p(G)$ be the Figà-Talamanca-Herz algebra. Then the multiplier algebra $MA_p(G)$ of $A_p(G)$ is a dual space. We say that $A_p(G)$ has the approximation property (or simply, AP) in $MA_p(G)$ if there is a net $\{u_\alpha\}$ in $A_p(G)$ such that $u_\alpha \to 1$ in the associated *weak*$^*$ topology. We prove that $A_p(G)$ has the AP in $MA_p(G)$ if and only if there exists a net $\{a_\alpha\}$ in $A_p(G)$ such that $\|a_\alpha a - a\|_{A_p(G)} \to 0$ uniformly for $a$ in any compact subset of $A_p(G)$. Consequently, we have that if $A_p(G)$ has the AP in $MA_p(G)$, then $A_p(G)$ has the approximation property as a Banach space in the sense of Grothendieck for a discrete group $G$. We also study the relationship between the AP of $A_p(G)$ in $MA_p(G)$ and the weak amenability of $G$.

## 1. Introduction

The investigation of the approximation properties for Banach spaces was initiated by Grothendieck [10]. Other versions of Grothendieck's approximation properties were introduced and studied recently, such as Effros and Ruan [6] for operator spaces and Haagerup and Kraus [11] for locally compact groups. These new versions of approximation property are closely related in some cases. For example, it was shown in [11] that a discrete group $G$ has the approximation property of Haagerup and Kraus [11] if and only if its reduced group $C^*$-algebra $C^*_\rho(G)$ has the operator space approximation property of Effros and Ruan [6]. It is also known that a discrete group $G$ has the approximation property if and only if its Fourier algebra $A(G)$ has the operator space approximation property (see Junge and Ruan [13]). A weaker version of the approximation property for locally compact groups was introduced by Haagerup and Kraus [11] (see page 683). By the main result of this paper, we will show that all of these versions of the approximation property are stronger than the approximation property for Banach spaces in the sense of Grothendieck [10] for a discrete group.

In this paper, we study the approximation properties of the Figà-Talamanca-Herz algebra $A_p(G)$ for locally compact groups. With the multiplier norm, the multiplier algebra $MA_p(G)$ of $A_p(G)$ is a dual space (see Miao [16]). We say that $A_p(G)$ has the approximation property (or simply, AP) in the multiplier algebra $MA_p(G)$ if there is a net $\{a_\alpha\}$ in $A_p(G)$ such that $a_\alpha \to 1$ in the associated *weak*$^*$

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topology. When $p = 2$, saying that $A_p(G)$ has the AP in the multiplier algebra $MA_p(G)$ is the same as saying that the locally compact group $G$ has the $AP'$ as defined in [11].

It is obvious that $A_p(G)$ has the AP in the multiplier algebra $MA_p(G)$ if and only if $A_p(G)$ is $weak^*$ dense in $MA_p(G)$. When $p = 2$ and $G$ is amenable, then $MA_p(G) = B(G)$, the Fourier-Stieltjets algebra of $G$, $A_p(G) = A(G)$, the Fourier algebra and the predual of $MA_p(G)$ is the group $C^*$-algebra $C^*(G)$. It is well known that $A(G)$ is $weak^*$ dense in $B(G)$ if and only if $A(G)$ has a bounded approximate identity which is equivalent to the fact that $G$ is amenable (the weak containment property). For $p = 2$, it is a result of Haagerup and Kraus ([11], Proposition 1.19) that if $A_p(G)$ has the AP in the multiplier algebra $MA_p(G)$, then there is a net $\{u_\alpha\}$ in $A_p(G)$ (maybe unbounded) such that $\|u_\alpha v - v\|_{A_p(G)} \to 0$ for any $v \in A_p(G)$. Motivated by an argument given in Haagerup and Kraus [11, Proposition 1.3] and by a result of Miao [16] about the predual of $MA_p(G)$, we characterize the AP in terms of the approximate identity of $A_p(G)$ as follows. $A_p(G)$ has the AP in the multiplier algebra $MA_p(G)$ if and only if there exists a net $\{a_\alpha\}$ of functions in $A_p(G)$ such that $\|a_\alpha a - a\|_{A_p(G)} \to 0$ uniformly for $a$ in any compact subset of $A_p(G)$. Also, this net of functions from $A_p(G)$ may be chosen to have compact supports. Consequently, we have that if $A_p(G)$ has the AP in the multiplier algebra $MA_p(G)$, then $A_p(G)$ has the approximation property as a Banach space in the sense of Grothendieck for a discrete group $G$. $G$ is said to have the approximation property in the completed bounded multiplier algebra $M_0A(G)$ if there is a net $\{u_\alpha\}$ in $A(G)$ such that $u_\alpha \to 1$ in the associated $weak^*$ topology of $M_0A(G)$ (it is said that $G$ has the AP in [11]). It is known that if $G$ has the AP in the completed bounded multiplier algebra $M_0A(G)$, then $A(G)$ has the AP in the multiplier algebra $MA_2(G)$ (see Haagerup and Kraus [11], page 683). Hence, we have that both the operator space version of Grothendieck's approximation property introduced in Effros and Ruan [6] and the approximation property of Haagerup and Kraus in [11] are stronger than Grothendieck's approximation property for Banach spaces if $G$ is a discrete group.

If $G$ is weakly amenable with respect to the multiplier algebra $MA_p(G)$, i.e. $A_p(G)$ has an approximate identity that is bounded in the multiplier norm, then $A_p(G)$ has the AP in the multiplier algebra $MA_p(G)$. But the converse is not true since there is an example of the locally compact group $G = SL(R, 2) \rtimes R^2$ such that $A_p(G)$ does not have an approximate identity that is bounded in the multiplier norm, while $A_p(G)$ has the AP in the multiplier algebra $MA_p(G)$. We show that if $G$ is $\sigma$-compact and $A_p(G)$ has the sequential AP in the multiplier algebra $MA_p(G)$ ($\Leftrightarrow$ there is a sequence $\{u_n\}$ of functions in $A_p(G)$ such that $u_n \to 1$ in the associated $weak^*$ topology of $MA_p(G)$), then $G$ is weakly amenable with respect to the multiplier algebra $MA_p(G)$. We do not know of an example of a locally compact group $G$ for which $A_p(G)$ does not have the AP in the multiplier algebra $MA_p(G)$ for some $p$.

We organize the paper as follows. In Section 2, we recall some necessary notations and present some preliminary results. In Section 3, we show the main result of this paper (Theorem 3.2), characterizing the AP of $A_p(G)$ in the multiplier algebra $MA_p(G)$ in terms of the approximate identity of $A_p(G)$. In Section 4, we investigate the relationship between the AP of $A_p(G)$ in $MA_p(G)$ and approximation property of $A_p(G)$ as a Banach space, and the weak amenability of $G$.

## 2. Preliminaries and some notation

Let $G$ be a locally compact group equipped with a fixed left Haar measure $\lambda$. If $G$ is compact, we assume $\lambda(G) = 1$. Let $L^p(G)$, $1 \leq p \leq \infty$, be the usual Lebesgue spaces on $G$ with norm $\|\cdot\|_p$.

Suppose that $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The Figà-Talamanca-Herz algebra $A_p(G)$ is the space of continuous functions $u$ which can be represented as

$$u = \sum_{n=1}^{\infty} f_i * \check{g}_i \quad \text{with} \quad f_i \in L^q(G), g_i \in L^p(G), \quad \text{and} \quad \sum_{n=1}^{\infty} \|f_i\|_q \|g_i\|_p < \infty,$$

where $\check{g} \in L^p(G)$ is defined by $\check{g}(x) = g(x^{-1})$, $x \in G$. The norm of $u$ is defined by

$$\|u\|_{A_p(G)} = \inf \sum_{n=1}^{\infty} \|f_i\|_q \|g_i\|_p,$$

where the infimum is taken over all the representations of $u$ above. It is known that $A_p(G)$ is a subspace of $C_0(G)$ and, equipped with the norm $\|\cdot\|_{A_p(G)}$ above and the pointwise multiplication is a regular tauberian algebra whose Gelfand spectrum is $G$. Furthermore, the algebra $A_p(G)$ has a bounded approximate identity if and only if the group $G$ is amenable (see Herz [12], Theorem 6). We emphasize that our $A_p(G)$ coincides with $A_q(G)$, $\frac{1}{p} + \frac{1}{q} = 1$, in [18]. It follows that the dual $A_p(G)^*$ is the space of convolution operators on $L^p(G)$, denoted by $PM_p(G)$ as in Herz [12]. Let $PF_p(G)$ be the norm closure of $L^1(G)$ in $A_p(G)^*$. Then $PF_p(G)^* = W_p(G)$ is a Banach algebra such that $A_p(G)$ is dense in the associated $w^*$-topology. For $p = 2$, $A_p(G) = A(G)$, the Fourier algebra, $PM_p(G) = VN(G)$, the group Von Neumann algebra and $PF_p(G) = C^*_\rho(G)$, the reduced group $C^*$ algebra of $G$ (see Eymard [7]). For $u \in A_p(G)$ and $T \in PM_p(G)$, $uT \in PM_p(G)$ is defined by $\langle uT, v\rangle = \langle T, uv\rangle$ for $v \in A_p(G)$. For the definitions and properties of $PM_p(G)$ and $PF_p(G)$, see Pier [18].

For a Banach space $X$, let $X^*$ be the conjugate Banach space of $X$. For $x \in X$ and $f \in X^*$, the value of $f$ at $x, f(x)$, is sometimes denoted by $\langle f, x\rangle$ or $\langle x, f\rangle$ in duality. The norm of $x$ (respectively, $f$) is sometimes written as $\|x\|_X$ (respectively, $\|f\|_{X^*}$) or $\|x\|_{X^*}$ (respectively, $\|f\|_X$).

**Definition 2.1.** A complex-valued function $m$ on $G$ is called a multiplier of $A_p(G)$ if $mu \in A_p(G)$ for all $u \in A_p(G)$.

Note that a multiplier of $A_p(G)$ is necessarily a continuous function on $G$. It follows from the closed graph theorem that for every multiplier $m$ of $A_p(G)$ the map

$$m : u \to mu, \quad u \in A_p(G),$$

is a bounded linear operator on $A_p(G)$. The space of multipliers of $A_p(G)$ is denoted by $MA_p(G)$. Clearly, $MA_p(G)$ is a Banach algebra under the pointwise multiplication. Its norm is called the multiplier norm and is denoted by $\|m\|_{MA_p(G)}$.

It is obvious that $A_p(G) \subseteq MA_p(G)$ and $\|u\|_{MA_p(G)} \leq \|u\|_{A_p(G)}$ if $u \in A_p(G)$. It is known that $MA_p(G)$ is a dual Banach space and its predual, denoted by $Q_p(G)$ or $Q_p$ for short, can be characterized by the following:

(a) $Q_p(G)$ is equal to the norm closure of $L^1(G)$ in the dual space of $MA_p(G)$;

(b) an element $f$ is in $Q_p(G)$ if and only if there are $u_i$ in $A_p(G)$ and $f_i$ in $PF_p(G)$ for $i = 1, 2, \ldots$ with $\sum_{i=1}^{\infty} \|u_i\|_{A_p(G)} \|f_i\|_{PF_p(G)} < \infty$ such that $f = \sum_{i=1}^{\infty} u_i f_i$ on $MA_p(G)$ (see Miao [16]).

**Proposition 2.2.** *Let $m$ be a continuous function on a locally compact group $G$. Then $m$ is a multiplier of $A_p(G)$ if and only if $m\omega \in W_p(G)$ for all $\omega \in W_p(G)$. Furthermore, $\|m\omega\|_{W_p(G)} \leq \|m\|_{MA_p(G)} \|\omega\|_{W_p(G)}$ if $m \in MA_p(G)$ and $\omega \in W_p(G)$.*

*Proof.* Suppose $m\omega \in W_p(G)$ for all $\omega \in W_p(G)$. It follows from the closed graph theorem that the map

$$m : u \to mu, \quad u \in W_p(G),$$

is a bounded linear operator on $W_p(G)$. Hence $ma \in A_p(G)$ if $a \in A_p(G)$ has a compact support. Since $m$ is a bounded operator on $W_p(G)$, it follows from the closedness of $A_p(G)$ in $W_p(G)$ that $ma \in A_p(G)$ for any $a \in A_p(G)$. Hence, $m \in MA_p(G)$.

Conversely, suppose $m$ is a multiplier of $A_p(G)$ and $\omega \in W_p(G)$. Then there is a net $a_\alpha$ in $A_p(G)$ such that $a_\alpha \to \omega$ in the $\sigma(W_p(G), PF_p(G))$-topology and $\|a_\alpha\|_{A_p(G)} \to \|\omega\|_{W_p(G)}$ (see Cowling [1], Theorem 4). It follows easily from the fact $mPF_p(G) \subseteq PF_p(G)$ that $ma_\alpha \to m\omega$ in the $\sigma(W_p(G), PF_p(G))$-topology, where $mf \in PF_p(G)$ is defined by $\langle mf, a\rangle = \langle f, ma\rangle$ for $f \in PF_p(G)$ and $a \in A_p(G)$. Hence, $m\omega \in W_p(G)$. Furthermore, we have

$$\begin{aligned}\|m\omega\|_{W_p(G)} &\leq \lim_\alpha \|ma_\alpha\|_{A_p(G)} \\ &\leq \lim_\alpha \|m\|_{MA_p(G)} \|a_\alpha\|_{A_p(G)} = \|m\|_{MA_p(G)} \|\omega\|_{W_p(G)}.\end{aligned}$$

□

**Proposition 2.3.** *Let $H$ be a closed subgroup of a locally compact group $G$. For any function $\varphi$ on $G$, let $\varphi_{|H}$ denote the restriction of $\varphi$ to $H$. If $\varphi \in MA_p(G)$, then $\varphi_{|H} \in MA_p(H)$ and*

$$\|\varphi_{|H}\|_{MA_p(H)} \leq \|\varphi\|_{MA_p(G)}.$$

*Proof.* Let $a \in A_p(H)$ and $\epsilon > 0$. Then there is a function $\tilde{a}$ in $A_p(G)$ such that $\tilde{a}_{|H} = a$ and $\|\tilde{a}\|_{A_p(G)} < \|a\|_{A_p(H)} + \epsilon$ (see Theorem 1b in Herz [12]). Hence, $\varphi_{|H} a = (\varphi\tilde{a})_{|H} \in A_p(H)$ and

$$\|(\varphi_{|H})a\|_{A_p(H)} = \|(\varphi\tilde{a})_{|H}\|_{A_p(H)} \leq \|\varphi\tilde{a}\|_{A_p(G)} \leq \|\varphi\|_{MA_p(G)}(\|a\|_{A_p(H)} + \epsilon),$$

where we use the fact that the restriction map is from $A_p(G)$ to $A_p(H)$ and it is also a contraction (see Herz [12], Theorem 1a). It follows that $\varphi_{|H} \in MA_p(H)$ and $\|\varphi_{|H}\|_{MA_p(H)} \leq \|\varphi\|_{MA_p(G)}$. □

Let $M_0A(G)$ be the completely bounded multiplier algebra of $A(G)$. It is known that $M_0A(G)$ is a dual space and its predual, denoted by $Q(G)$, is equal to the norm closure of $L^1(G)$ in $M_0A(G)^*$ (see De Cannière and Haagerup [3]). $G$ is said to have the approximation property (AP for short) if there is a net $\{a_\alpha\}$ in $A(G)$ such that $a_\alpha \to 1$ in the $\sigma(M_0A(G), Q(G))$-topology. Since $M_0A(G) \subseteq MA_2(G)$ and $\|\cdot\|_{MA_2(G)} \leq \|\cdot\|_{M_0A(G)}$ on $M_0A(G)$, it is easy to see that the restriction $f_{|M_0A(G)}$ of any $f \in Q_2(G)$ to $M_0A(G)$ is in $Q(G)$. Hence, if $G$ has the AP, i.e. there is a net $\{a_\alpha\}$ in $A(G)$ such that $a_\alpha \to 1$ in the $\sigma(M_0A(G), Q(G))$-topology, then $a_\alpha \to 1$ in the $\sigma(MA_2(G), Q_2(G))$-topology as well. Therefore, we have the following:

**Proposition 2.4.** *If $G$ has the AP, then $A(G)$ has the AP in $MA_2(G)$.*

*Remark* 2.5. It is unknown whether the converse is true (see [11], page 683).

## 3. Characterizations of the approximation property of $A_p(G)$

In this section, we will characterize the AP of $A_p(G)$ in $MA_p(G)$ by the approximate identities of $A_p(G)$. Theorem 3.2 is our main result in this paper. We denote the set of functions in $A_p(G)$ with compact support by $A_p(G)_c$. For $u \in A_p(G)$, the left and the right translations of $u$ by $t \in G$ is defined by ${}_tu(x) = u(tx)$ and $u_t(x) = u(xt)$ for $x \in G$. For a subset $S$ of $G$, the characteristic function of $S$ is denoted by $1_S$.

Let $v \in A_p(G)_c$ and $u \in MA_p(G)$. Then the convolution $v * u$ of $v$ and $u$ is an element of $MA_p(G)$ and $\|v * u\|_{MA_p(G)} \leq \|u\|_{MA_p(G)} \int_G |v(t)|\, dt$. In fact, for $f \in L^1(G)$ with $\|f\|_{Q_p(G)} \leq 1$, we have

$$\begin{aligned} |\langle v * u, f\rangle| &= |\int_G v(t)u(t^{-1}x)f(x)\, dtdx| = |\int_G v(t)\langle {}_{t^{-1}}u, f\rangle\, dt| \\ &\leq \int_G |v(t)|\|\, {}_{t^{-1}}u\|_{MA_p(G)}\, dt = \|u\|_{MA_p(G)} \int_G |v(t)|\, dt. \end{aligned}$$

Therefore, it is obvious that, for any $T \in PM_p(G), a \in A_p(G)$ and $v \in A_p(G)_c$,

$$\omega_{T,a,v}(u) = \langle T, (v * u)a\rangle, \qquad u \in MA_p(G),$$

defines a bounded linear functional on $MA_p(G)$, and its norm satisfies

$$\|\omega_{T,a,v}\| \leq \|T\|\, \|a\| \int_G |v(t)|\, dt.$$

To prove our main result, we use the characterization of $Q_p(G)$ in Miao [16] and the following lemma. The proof of Lemma 3.1 is a modification of the proof of Proposition 1.3 (a) in [11]. For a function $f$ on $G$, let $supp\,(f)$ denote the support of $f$.

**Lemma 3.1.** *Let $G$ be a locally compact group. For any $T \in PM_p(G), a \in A_p(G)$ and $v \in A_p(G)_c$ with $v(t) \geq 0$ for all $t \in G$ and $\int_G v(t)\, dt = 1$, we have that $\omega_{T,a,v}$ is in $Q_p(G)$ and $\|\omega_{T,a,v}\| \leq \|T\|\, \|a\|$.*

*Proof.* Since $Q_p(G)$ is norm closed in $MA_p(G)^*$ and $\|\omega_{T,a,v}\| \leq \|T\|\|a\| \int_G |v(t)|\, dt$, we assume that $a \in A_p(G)_c$ without loss of generality. We will complete the proof by showing that there is a $g \in L^1(G)$ such that $\omega_{T,a,v}(u) = \int_G u(x)g(x)\, dx$ for all $u \in MA_p(G)$. Let $S$ denote the compact set $(supp(v))^{-1}supp(a)$ and let $u \in MA_p(G)$. Then it can be shown by calculation (see [2], page 510) that

$$[(v * u)a](x) = [(v * (1_S u))a](x) \qquad \forall x \in G.$$

Moreover, we have

$$[(v * (1_S u))a](x) = \int_G v(xt)(1_S u)(t^{-1})a(x)\, dt = \int_G v_t(x)a(x)(1_S u)(t^{-1})\, dt.$$

Define a map $\Phi : G \to A_p(G)$ by $\Phi(t)(x) = v_t(x)a(x)$ for $t, x \in G$. It follows from the fact that the right translation of $v$ is a norm continuous map from $G$ to $A_p(G)$ that $\Phi$ is norm continuous. It is obvious that if $t \notin supp(v)supp(a)^{-1}$, then $\Phi(t) = 0$. Hence $\Phi$ is bounded by the compactness of $supp(v)supp(a)^{-1}$. Let $\mu$ be the measure on $G$ defined by $d\mu(t) = (1_S u)(t^{-1})dt$. Since $u$ is bounded and $S$ is

compact, $d\mu$ is a bounded Radon measure. Hence there is an element $\eta$ in $A_p(G)$ such that $\eta = \int_G \Phi(t)\, d\mu(t)$ and

$$\langle F, \eta \rangle = \int_G \langle F, \Phi(t) \rangle \, d\mu(t) \quad \forall F \in PM_p(G).$$

Let $F = \delta_x$ for $x \in G$; then we have

$$\eta(x) = \int_G \Phi(t)(x)\, d\mu(t) = \int_G v_t(x) a(x) (1_S u)(t^{-1})\, dt = [(v * (1_S u))a](x).$$

Thus, $\eta = (v * (1_S u))a = (v * u)a$, and so we have

$$\begin{aligned}
\omega_{T,a,v}(u) &= \langle T, (v * u)a \rangle = \langle T, \eta \rangle \\
&= \int_G \langle T, \Phi(t) \rangle \, d\mu \\
&= \int_G u(t^{-1}) \langle T, \Phi(t) \rangle 1_S(t^{-1})\, dt \\
&= \int_G u(t) \langle T, \Phi(t^{-1}) \rangle 1_S(t) \Delta(t^{-1})\, dt \\
&= \int_G u(t) g(t)\, dt,
\end{aligned}$$

where $g(t) = \langle T, \Phi(t^{-1}) \rangle 1_S(t) \Delta(t^{-1})$. Since

$$\begin{aligned}
\int_G |g(t)|\, dt &= \int_G |\langle T, \Phi(t^{-1}) \rangle| 1_S(t) \Delta(t^{-1}) dt \\
&\leq \|T\| \int_G \|\Phi(t^{-1})\| 1_S(t) \Delta(t^{-1}) dt \\
&\leq M \|T\| \ \lambda(S^{-1}) < \infty,
\end{aligned}$$

where $M$ is a upper bound of $\Phi$, $g \in L^1(G)$, as required. □

Now we are ready to prove our main theorem. This result is an analogue of the weak containment property (see Paterson [17]).

**Theorem 3.2.** *Let $G$ be a locally compact group. Then the following statements are equivalent:*

(*i*) *$A_p(G)$ has the AP in the multiplier algebra $MA_p(G)$;*

(*ii*) *for any sequence $\{u_n\}$ in $A_p(G)$ and any sequence $\{f_n\}$ in $PF_p(G)$ with $\sum_{n=1}^{\infty} \|u_n\|_{A_p(G)} \|f_n\|_{PF_p(G)} < \infty$,*

$$\langle a, \sum_{n=1}^{\infty} u_n f_n \rangle = 0 \quad \forall a \in A_p(G) \quad \textit{implies} \quad \sum_{n=1}^{\infty} \langle f_n, u_n \rangle = 0;$$

(*iii*) *for any sequence $\{u_n\}$ in $A_p(G)$ and any sequence $\{T_n\}$ in $PM_p(G)$ with $\sum_{n=1}^{\infty} \|u_n\|_{A_p(G)} \|T_n\|_{PM_p(G)} < \infty$,*

$$\langle a, \sum_{n=1}^{\infty} u_n T_n \rangle = 0 \quad \forall a \in A_p(G) \quad \textit{implies} \quad \sum_{n=1}^{\infty} \langle T_n, u_n \rangle = 0;$$

(*iv*) *there is a net $\{a_\alpha\}$ in $A_p(G)$ such that for any sequence $\{u_n\}$ in $A_p(G)$ with $\|u_n\|_{A_p(G)} \to 0$, we have $\lim_\alpha \|a_\alpha a - a\|_{A_p(G)} \to 0$ uniformly for $a$ in $\{u_n\}$;*

(v) *there is a net* $\{a_\alpha\}$ *in* $A_p(G)$ *such that* $\lim_\alpha \|a_\alpha a - a\|_{A_p(G)} \to 0$ *uniformly for* $a$ *in any compact subset of* $A_p(G)$*;*

(vi) *there is a net* $\{a_\alpha\}$ *in* $A_p(G)$ *such that* $\langle a_\alpha, f\rangle \to \langle 1, f\rangle$ *uniformly for* $f$ *in any compact subset of* $Q_p(G)$*.*

*Proof.* We will show $(i) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i), (iv) \Leftrightarrow (v)$ and $(i) \Leftrightarrow (vi)$.

$(i) \Rightarrow (iv)$ Let $X = (A_p(G) \oplus A_p(G) \oplus \dots)_0$, i.e. all sequences $(a_n)$ in $A_p(G)$ with $\|a_n\|_{A_p(G)} \to 0$ and with the sup-norm $\|(a_n)\| = \sup_n \|a_n\|_{A_p(G)}$. If $(u_n) \in X$, then $(au_n - u_n)$ is also in $X$ for any $a \in A_p(G)$. We define

$$\Phi : A_p(G) \to X \quad \text{by} \quad \Phi(a) = (au_n - u_n) \qquad \forall a \in A_p(G).$$

We will show that 0 is in the closure of $\Phi(A_p(G))$ in the weak topology of $X$. It follows from $(i)$ that there is a net $\{a_\alpha\}$ in $A_p(G)$ such that $a_\alpha \to 1$ in the $\sigma(MA_p(G), Q_p(G))$-topology. Choose $v \in A_p(G)_c$ with $v(t) \geq 0$ for all $t \in G$ and $\int_G v(t)\,dt = 1$. Since $v$ has a compact support, $v * a_\alpha$ is in $A_p(G)$ for all $\alpha$. Claim $\Phi(v * a_\alpha) \to 0$ weakly in $X$. In fact, let $T \in X^*$. Since $X^* = (A_p(G)^* \oplus A_p(G)^* \oplus \dots)_{\ell_1}$, there is a sequence $\{T_n\}$ in $PM_p(G)$ such that $\|T\| = \sum_{n=1}^\infty \|T_n\| < \infty$ and $\langle T, (a_n)\rangle = \sum_{n=1}^\infty \langle T_n, a_n\rangle$ for $(a_n) \in X$. Hence,

$$\begin{aligned}
\langle T, \Phi(v * a_\alpha)\rangle &= \sum_{n=1}^\infty \langle T_n, (v * a_\alpha)u_n - u_n\rangle \\
&= \sum_{n=1}^\infty \langle T_n, (v * a_\alpha)u_n\rangle - \sum_{n=1}^\infty \langle T_n, u_n\rangle \\
&= \sum_{n=1}^\infty \langle \omega_{T_n,u_n,v}, a_\alpha\rangle - \sum_{n=1}^\infty \langle T_n, u_n\rangle \\
&= \langle a_\alpha, \sum_{n=1}^\infty \omega_{T_n,u_n,v}\rangle - \sum_{n=1}^\infty \langle T_n, u_n\rangle.
\end{aligned}$$

By Lemma 3.1, each $\omega_{T_n,u_n,v} \in Q_p(G)$ and also we have

$$\sum_{n=1}^\infty \|\omega_{T_n,u_n,v}\|_{Q_p(G)} \leq \sum_{n=1}^\infty \|T_n\|_{PM_p(G)} \|u_n\|_{A_p(G)} < \infty.$$

Thus, $\sum_{n=1}^\infty \omega_{T_n,u_n,v}$ is in $Q_p(G)$. Therefore,

$$\langle a_\alpha, \sum_{n=1}^\infty \omega_{T_n,u_n,v}\rangle \to \langle 1, \sum_{n=1}^\infty \omega_{T_n,u_n,v}\rangle = \sum_{n=1}^\infty \langle T_n, (v * 1)u_n\rangle = \sum_{n=1}^\infty \langle T_n, u_n\rangle.$$

Hence we have $\langle T, \Phi(v * a_\alpha)\rangle \to 0$.

Since $\Phi(A_p(G))$ is convex, 0 is also in the norm closure of $\Phi(A_p(G))$. For each positive integer $i$, there is an $a_{(i,E)}$ in $A_p(G)$ such that$\|\Phi(a_{(i,E)})\| < \frac{1}{i}$, where $E = \{u_n\}$ is considered as a subset of $A_p(G)$. Thus, for any $n$,

$$\|a_{(i,E)}u_n - u_n\|_{A_p(G)} \leq \|\Phi(a_{(i,E)})\|_X < \frac{1}{i}.$$

The net $(a_{(i,E)})$ satisfies the requirements, where $(i, E)$ is directed by $(i_1, E_1) \leq (i_2, E_2)$ if $i_1 \leq i_2$ and $E_1 \subseteq E_2$, where $i_1$ and $i_2$ are positive integers and $E_1$ and $E_2$ are countable subsets of $A_p(G)$ that can be listed as sequences converging to zero in the norm.

$(iv) \Rightarrow (iii)$ Let $\{a_\alpha\}$ be a net in $A_p(G)$ satisfying the condition $(iv)$. Suppose that the sequences $\{T_n\}$ in $PM_p(G)$ and $\{u_n\}$ in $A_p(G)$ have the property $\sum_{n=1}^{\infty} \|u_n\|_{A_p(G)} \|T_n\|_{PM_p(G)} < \infty$. We assume that $\|u_n\| = 1$ for all $n$ without loss of generality. Then $\sum_{n=1}^{\infty} \|T_n\|_{PM_p(G)} < \infty$. By the convergence of the series $\sum_{n=1}^{\infty} \|T_n\|$, there are positive integers $i_1 < i_2 < i_3 < \dots$ such that for each $k$, $\sum_{n=i_k}^{i_{k+1}-1} \|T_n\| < \frac{1}{2^k}$. We denote

$$E = \{u_1, u_2, \dots, u_{i_1-1}\} \cup \{\frac{u_n}{k} : k = 1, 2, 3 \dots,\ i_k \le n < i_{k+1}\}.$$

Then there is an obvious order to make this a sequence that converges to zero in the norm. It follows from $(iv)$ that for any $\epsilon > 0$ there is $\alpha_0$ such that $\|a_\alpha u - u\| < \epsilon$ and $\sum_{n=1}^{i_1-1} \|a_\alpha u_n - u_n\| < \epsilon$ for all $u \in E$ and $\alpha \ge \alpha_0$. Therefore, for $\alpha \ge \alpha_0$, we have

$$\begin{aligned}
&|\langle a_\alpha, \sum_{n=1}^{\infty} u_n T_n\rangle - \sum_{n=1}^{\infty} \langle T_n, u_n\rangle| \\
&= |\sum_{n=1}^{\infty} \langle T_n, a_\alpha u_n - u_n\rangle| \\
&\le \sum_{n=1}^{\infty} \|T_n\| \, \|a_\alpha u_n - u_n\| \\
&= \sum_{n=1}^{i_1-1} \|T_n\| \, \|a_\alpha u_n - u_n\| + \sum_{k=1}^{\infty} \sum_{n=i_k}^{i_{k+1}-1} \|T_n\| \, \|a_\alpha u_n - u_n\| \\
&\le (\sum_{n=1}^{\infty} \|T_n\|) \sum_{n=1}^{i_1-1} \|a_\alpha u_n - u_n\| + \sum_{k=1}^{\infty} \sum_{n=i_k}^{i_{k+1}-1} \|T_n\| \, \|a_\alpha \frac{u_n}{k} - \frac{u_n}{k}\| \, k \\
&< \epsilon \sum_{n=1}^{\infty} \|T_n\| + \sum_{k=1}^{\infty} \sum_{n=i_k}^{i_{k+1}-1} \|T_n\| \, \epsilon \, k \\
&\le \epsilon \sum_{n=1}^{\infty} \|T_n\| + \epsilon \sum_{k=1}^{\infty} \frac{k}{2^k}.
\end{aligned}$$

Let $\langle a_\alpha, \sum_{n=1}^{\infty} u_n T_n\rangle = 0$ for all $\alpha$ by the assumption of $(iii)$. Since $\epsilon$ is arbitrary, $\sum_{n=1}^{\infty} \langle T_n, u_n\rangle = 0$, as required.

$(iii) \Rightarrow (ii)$ is trivial. Now we prove $(ii) \Rightarrow (i)$ If the function 1 is not in the closure of $A_p(G)$ in the $\sigma(MA_p(G), Q_p(G))$-topology, then by the Hahn-Banach theorem there is $f \in Q_p(G)$ such that $\langle 1, f\rangle = 1$ and $\langle a, f\rangle = 0$ for all $a \in A_p(G)$. By the characterization of $Q_p(G)$ (see Miao [16]), there are sequences $\{u_n\}$ in $A_p(G)$ and $\{f_n\}$ in $PF_p(G)$ with $\sum_{n=1}^{\infty} \|u_n\|_{A_p(G)} \|f_n\|_{PF_p(G)} < \infty$ such that $f = \sum_{n=1}^{\infty} u_n f_n$. Thus, we have $\langle 1, \sum_{n=1}^{\infty} u_n f_n\rangle = \sum_{n=1}^{\infty} \langle u_n, f_n\rangle = 1$, while $\langle a, \sum_{n=1}^{\infty} u_n f_n\rangle = 0$ for all $a \in A_p(G)$. Therefore $(ii)$ fails.

$(v) \Rightarrow (iv)$ is trivial. We show $(iv) \Rightarrow (v)$. Let $\{a_\alpha\}$ be a net in $A_p(G)$ satisfying the condition in $(iv)$. Then $\{a_\alpha\}$ satisfies the condition in $(v)$. In fact, if $K$ is a compact subset of $A_p(G)$, then there is a sequence $\{u_n\}$ in $A_p(G)$ with $\|u_n\| \to 0$ such that $K \subseteq \overline{conv}\,\{u_n\}$ (see Proposition 1.e.2 in Lindenstrauss and Tzafriri [15]). For any $\epsilon > 0$, there is $\alpha_0$ such that $\|a_\alpha u_n - u_n\| < \epsilon$ for any $n$ and $\alpha \ge \alpha_0$ by $(iv)$. It follows from a simple convex argument that $\|a_\alpha u - u\| < \epsilon$ for any $\alpha \ge \alpha_0$

and for any $u \in conv\,\{u_n\}$. For any $v \in \overline{conv}\,\{u_n\}$, there is a sequence $\{v_i\}$ in $conv\,\{u_n\}$ such that $v_i \to v$ in the norm. Hence, $\|a_\alpha v - v\| = \lim_i \|a_\alpha v_i - v_i\| \le \epsilon$ for any $\alpha \ge \alpha_0$, as required.

$(vi) \Rightarrow (i)$ is trivial. Now we prove $(i) \Rightarrow (vi)$. Let $a_\alpha \to 1$ in the associated *weak*$^*$ topology of $MA_p(G)$. For any compact subset $E$ of $Q_p(G)$, there is a sequence $\{f_n\}$ in $Q_p(G)$ such that $E \subseteq \overline{conv}\,\{f_n\}$ (see Proposition 1.e.2 in Lindenstrauss and Tzafriri [15]). It suffices to show that $\langle a_\alpha, f\rangle \to \langle 1, f\rangle$ uniformly for $f$ in $\{f_n\}$ (see the proof of $(iv) \Rightarrow (v)$). Let $X = (Q_p(G) \oplus Q_p(G) \oplus \dots)_0$. Then the net $\Phi_\alpha = ((a_\alpha - 1)f_n)_{n=1}^\infty$ is in $X$ as $\|f_n\| \to 0$. Claim $\Phi_\alpha \to 0$ weakly in $X$. In fact, let $F \in X^*$; then there is a sequence $\{m_i\}$ in $Q_p(G)^* = MA_p(G)$ such that $\|F\| = \sum_{i=1}^\infty \|m_i\|$ and $\langle F, (g_i)\rangle \sum_{i=1}^\infty \langle m_i, g_i\rangle$ for any $(g_i) \in X$. For each $i$, $m_i f_i \in MA_p(G)^*$ is defined by $\langle m_i f_i, m\rangle = \langle f_i, m_i m\rangle$ for $m \in MA_p(G)$. Since $Q_p(G)$ is the norm closure of $L^1(G)$ in $MA_p(G)^*$, it is obvious that $m_i f_i \in Q_p(G)$. Hence, $\sum_{i=1}^\infty m_i f_i$ is in $Q_p(G)$. Thus we have

$$\begin{aligned}\langle F, \Phi_\alpha\rangle &= \sum_{i=1}^\infty \langle m_i, (a_\alpha - 1)f_i\rangle \\ &= \sum_{i=1}^\infty \langle (a_\alpha - 1), m_i f_i\rangle \\ &= \langle a_\alpha - 1, \sum_{i=1}^\infty m_i f_i\rangle \to 0,\end{aligned}$$

since $a_\alpha \to 1$ in the *weak*$^*$-topology. For any $k$, there is a convex combination of $\{\Phi_\alpha\}$, say $\sum_{i=1}^{m_k} \beta_i \Phi_{\alpha_i}$, such that $\|\sum_{i=1}^{m_k} \beta_i \Phi_{\alpha_i}\| < \frac{1}{k}$. Let $v_{(E,k)} = \sum_{i=1}^{m_k} \beta_i a_{\alpha_i}$. Then for every $n$,

$$\begin{aligned}&|\langle v_{(E,k)}, f_n\rangle - \langle 1, f_n\rangle| \\ &= |\langle v_{(E,k)} - 1, f_n\rangle| = |\langle 1, (v_{(E,k)} - 1)f_n\rangle| \\ &\le \|(v_{(E,k)} - 1)f_n\| = \|(\sum_{i=1}^{m_k} \beta_i a_{\alpha_i} - 1)f_n\| \\ &\le \|\sum_{i=1}^{m_k} \beta_i \Phi_{\alpha_i}\| < \frac{1}{k}.\end{aligned}$$

Hence the net $\{v_{(E,k)}\}$ satisfies the reqirements, where the set $\{(E,k)\}$ is directed as usual (see the proof of $(i) \Rightarrow (iv)$). □

*Remarks* 3.3. a. Since $A_p(G) \subseteq MA_p(G)$ and $\|\cdot\|_{MA_p(G)} \le \|\cdot\|_{A_p(G)}$ on $A_p(G)$, for any subset $E \subseteq A_p(G)$, the $\sigma(MA_p(G), Q_p(G))$-closure of $E$ contains the $\|\cdot\|_{A_p(G)}$-closure of $E$. Hence $A_p(G)$ has the AP in the multiplier algebra $MA_p(G)$ if and only if the function 1 is in the $\sigma(MA_p(G), Q_p(G))$-closure of $A_p(G)_c$. Also, it is clear that the net $\{a_\alpha\}$ of functions in Theorem 3.2 can be taken from $A_p(G)_c$.

b. This result improves Proposition 1.19 in [11].

**Corollary 3.4.** *Let $H$ be a closed subgroup of a locally compact group $G$. If $A_p(G)$ has the AP in $MA_p(G)$, then $A_p(H)$ also has the AP in $MA_p(H)$.*

*Proof.* If $A_p(G)$ has the AP in $MA_p(G)$, then there is a net $\{a_\alpha\}$ in $A_p(G)$ such that $\|a_\alpha a - a\|_{A_p(G)} \to 0$ uniformly for $a$ in any compact subset of $A_p(G)$ by Theorem 3.2. Let $\{u_n\}$ be a sequence in $A_p(H)$ with $\|u_n\|_{A_p(H)} \to 0$. Then there

exists a sequence $\{\tilde{u}_n\}$ in $A_p(G)$ with $\|\tilde{u}_n\|_{A_p(G)} \to 0$ and $\tilde{u}_n(h) = u_n(h)$ for $h \in H$ and for any $n$ (see Herz [12], Theorem 1b). Hence $\|a_\alpha \tilde{u}_n - \tilde{u}_n\|_{A_p(G)} \to 0$ uniformly for all $n$. Since the restriction map from $A_p(G)$ to $A_p(H)$ is a contraction (see Herz [12], Theorem 1a), $\|(a_\alpha)_{|H} u_n - u_n\|_{A_p(H)} \to 0$ uniformly for all $n$. Therefore, $\{(a_\alpha)_{|H}\}$ is the net in $A_p(H)$ satisfied by $(iv)$ in Theorem 3.2. Hence $A_p(H)$ has the AP in $MA_p(H)$. □

## 4. Approximation properties and weak amenability

In this section, we study the relationship between the AP of $A_p(G)$ in the multiplier algebra and the approximation property of $A_p(G)$ as a Banach space, and the weak amenability of $G$.

**Definition 4.1.** A locally compact group $G$ is said to be weakly amenable with respect to the multiplier algebra $MA_p(G)$ if there exists an approximate identity $\{a_\alpha\}$ in $A_p(G)$ which is bounded in the multiplier norm $\|\cdot\|_{MA_p(G)}$.

Since a $\|\cdot\|_{MA_p(G)}$-bounded approximate identity $\{a_\alpha\}$ of $A_p(G)$ is contained in a closed bounded ball of $MA_p(G)$, there must be a subnet of $\{a_\alpha\}$ which converges in the $\sigma(MA_p(G), Q_p(G))$-topology. Obviously, the limit of this subnet must be the function 1. Hence, $A_p(G)$ has the AP in the multiplier algebra $MA_p(G)$. It is well known that a locally compact group $G$ is amenable if and only if $A_p(G)$ has a bounded approximate identity. Since $\|\cdot\|_{MA_p(G)} \leq \|\cdot\|_{A_p(G)}$, if $G$ is amenable, then $G$ must be weakly amenable with respect to the multiplier algebra $MA_p(G)$. But the converse is not true since the free group on two generators of $F_2$ is weakly amenable while $F_2$ is not amenable (see Furuta [8]). We have the following:

**Proposition 4.2.** *If $G$ is weakly amenable with respect to the multiplier algebra $MA_p(G)$, then $A_p(G)$ has the AP in $MA_p(G)$.*

**Corollary 4.3.** *Let $G$ be a locally compact group. If $G$ is weakly amenable with respect to the multiplier algebra $MA_p(G)$, there is a net $\{a_\alpha\}$ of functions in $A_p(G)$ such that $\|a_\alpha a - a\|_{A_p(G)} \to 0$ uniformly for $a$ in any compact subset of $A_p(G)$.*

*Remark* 4.4. There is a direct proof for this result (see the proof of Theorem 4.8 $(i) \Rightarrow (iii)$).

The following example shows that the AP of $A_p(G)$ in $MA_p(G)$ does not imply the weak amenability of $G$.

**Counterexample 4.5.** Let $G = SL(2, R) \rtimes R^2$ be the semidirect product of $SL(2, R)$ and $R^2$ with the standard action of $SL(2, R)$ on $R^2$. Then both $R^2$ and $SL(2, R)$ have $\|\cdot\|_{M_0A(G)}$-bounded approximate identities (see Theorem 3.7 and Remark 3.8 in De Cannière and Haagerup [3]). Hence they have the AP in the sense of the completely bounded multiplier algebra $M_0A(G)$ (see Proposition 4.2). $G$ has the AP as well (see Haagerup and Kraus [11], Corollary 1.17). Thus $A(G)$ has the AP in $MA_2(G)$ (see Proposition 2.4). It is proved by Dorofaeff [4] (Theorem 12.1) that there is no approximate identity in $A(G)$ that is bounded in the multiplier norm (see also [5]). Hence $G$ is not weakly amenable with respect to the multiplier algebra $MA_2(G)$.

*Remark* 4.6. Presently, we do not know of an example of a locally compact group $G$ such that $A_p(G)$ does not have the AP in $MA_p(G)$ for some $p$. However, if we replace the net which converges to 1 in the *weak*$^*$ topology by a sequence in the

definition of the approximation property, then we will have the equivalence between the weak amenability and the AP as follows.

**Definition 4.7.** For a locally compact group $G$, $A_p(G)$ is said to have a sequential AP in the multiplier algebra $MA_p(G)$ if there is a sequence $\{u_n\}$ in $A_p(G)$ such that $u_n \to 1$ in the associated *weak*$^*$ topology of $MA_p(G)$.

**Theorem 4.8.** *Let $G$ be a $\sigma$-compact locally compact group. Then the following statements are equivalent:*

- *(i) $G$ is weakly amenable with respect to the multiplier algebra $MA_p(G)$;*
- *(ii) $A_p(G)$ has the sequential AP in $MA_p(G)$;*
- *(iii) there is a sequence $\{u_n\}$ of functions in $A_p(G)$ such that $\|u_n a - a\|_{A_p(G)} \to 0$ uniformly for $a$ in any compact subset of $A_p(G)$.*

*Proof.* $(i) \Rightarrow (iii)$ Since $G$ is $\sigma$-compact and weakly amenable with respect to the multiplier algebra $MA_p(G)$, there is a sequence $\{a_n\}$ of functions in $A_p(G)$ and a positive number $C$ such that $\|a_n\|_{MA_p(G)} \leq C$ for all $n$ and $\|a_n a - a\|_{A_p(G)} \to 0$ for each $a \in A_p(G)$ (see Granirer [9], Remark 8 (iii)). Then $\|a_n a - a\|_{A_p(G)} \to 0$ uniformly for $a$ in any compact subset of $A_p(G)$. In fact, for any compact subset $E$ of $A_p(G)$, there is a sequence $\{u_k\}$ of functions in $A_p(G)$ with $\|u_k\|_{A_p(G)} \to 0$ such that $E \subseteq \overline{conv}\,\{u_k\}$ (see Lindenstrauss and Tzafriri [15], Proposition 1.e.2). For any $\epsilon > 0$, there is $K$ such that $\|u_k\|_{A_p(G)} < \epsilon$ for $k > K$. Since $\|a_n u_k - u_k\|_{A_p(G)} \to 0$ for $k = 1, 2, \ldots, K$, there is $N$ such that $\|a_n u_k - u_k\|_{A_p(G)} < \epsilon$ for any $n \geq N$ and $k = 1, 2, \ldots, K$. For $k = K+1, K+2, \ldots,$ we have $\|a_n u_k - u_k\|_{A_p(G)} \leq C\epsilon + \epsilon$. Hence we have

$$\|a_n u_k - u_k\|_{A_p(G)} \leq C\epsilon + \epsilon \text{ for any } \ n \geq N \ \text{ and for all } k.$$

It is easy to see that for any $n \geq N$, we have $\|a_n u - u\|_{A_p(G)} \leq C\epsilon + \epsilon$ for any $u \in conv\,\{u_k\}$, and so it is also true for any $u \in \overline{conv}\,\{u_k\}$, as required.

$(iii) \Rightarrow (i)$ Let $\{u_n\}$ be a sequence of functions in $A_p(G)$ such that $\|u_n a - a\|_{A_p(G)} \to 0$ uniformly for $a$ in any compact subset of $A_p(G)$. If $\{u_n\}$ is not bounded in the multiplier norm, then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\|u_{n_k}\|_{MA_p(G)} > k$ for all $k$. For each $k$, choose a $v_k$ in $A_p(G)$ such that $\|v_k\|_{A_p(G)} \leq 1$ and $\|u_{n_k} v_k\|_{A_p(G)} > k$. So

$$\|u_{n_k}\frac{v_k}{k} - \frac{v_k}{k}\| \geq \|u_{n_k}\frac{v_k}{k}\| - \|\frac{v_k}{k}\| \geq 1 - \frac{1}{k}$$

for $k = 1, 2, 3, \ldots$. Hence $\|u_n a - a\| \not\to 0$ uniformly for $a$ in $E = \{\frac{v_k}{k} : k = 1, 2, \ldots\}$. This is a contradiction. Therefore, $\{u_n\}$ is bounded in the multiplier norm.

$(ii) \Rightarrow (i)$ Let $\{u_n\}$ be a sequence of functions in $A_p(G)$ such that $u_n \to 1$ in the $\sigma(MA_p(G), Q_p(G))$-topology. By the uniform boundedness principle, there is a positive number $C$ such that $\|u_n\|_{MA_p(G)} \leq C$ for all $n$. Choose a nonnegative function $v$ from $A_p(G)$ such that $\int_G v(t)\,dt = 1$. For any finite subset $S = \{a_1, a_2, \ldots, a_m\}$ of $A_p(G)$, let

$$X = A_p(G) \oplus A_p(G) \oplus \cdots \oplus A_p(G)\,(m \text{ copies}).$$

We denote $\phi_n = ((v * u_n)a_i - a_i)_{i=1}^m$. Claim that the sequence $\phi_n$ converges to 0 weakly in $X$. In fact, let $T \in X^*$. Then there are $T_i \in PM_p(G)\,(i = 1, 2, \ldots, m)$

such that $\|T\| = \sum_{i=1}^m \|T_i\|$ and

$$\begin{aligned}\langle T, \phi_n\rangle &= \sum_{i=1}^m \langle T_i, (v * u_n)a_i - a_i\rangle \\ &= \sum_{i=1}^m \langle \omega_{T_i,a_i,v}, u_n\rangle - \sum_{i=1}^m \langle T_i, a_i\rangle \\ &\to \sum_{i=1}^m \langle \omega_{T_i,a_i,v}, 1\rangle - \sum_{i=1}^m \langle T_i, a_i\rangle = 0\end{aligned}$$

as $n \to \infty$ because each $\omega_{T_i,a_i,v}$ is in $Q_p(G)$ by Lemma 3.1 and $\langle \omega_{T_i,a_i,v}, 1\rangle = \langle T_i, a_i\rangle$ for each $i$. For each positive integer $j$, there is a convex combination $\sum_{i=1}^k \beta_i\phi_{n_i}$ from $\{\phi_n\}$ such that $\|\sum_{i=1}^k \beta_i\phi_{n_i}\| < \frac{1}{j}$. Let $v_{(j,S)} = \sum_{i=1}^k \beta_i\, v * u_{n_i}$. Then $\|v_{(j,S)}\|_{MA_p(G)} \le C$ since $\|v * u_{n_i}\|_{MA_p(G)} \le \|u_{n_i}\|_{MA_p(G)} \le C$ for each $i$. Thus, for each $a \in S$, we have

$$\|v_{(j,S)}a - a\| = \|\sum_{i=1}^k \beta_i \big((v * u_{n_i})a - a\big)\| \le \|\sum_{i=1}^k \beta_i\phi_{n_i}\| < \frac{1}{j}.$$

Hence the net $\{v_{(j,S)}\}$ is an approximate identity of $A_p(G)$ which is bounded in the multiplier norm by $C$, where the set $\{(j,S)\}$ is directed by $(j_1, S_1) \le (j_2, S_2)$ if $j_1 \le j_2$ and $S_1 \subseteq S_2$ for positive integers $j_1, j_2$ and finite subset $S_1, S_2$ of $A_p(G)$.

*(iii)* $\Rightarrow$ *(ii)* Let $\{a_n\}$ be a sequence of functions in $A_p(G)$ such that $\|a_n a - a\|_{A_p(G)} \to 0$ uniformly for $a$ in any compact subset of $A_p(G)$. For any $f \in Q_p(G)$, by the characterization of $Q_p(G)$ (see Miao [16]), there are sequences $\{u_i\}$ in $A_p(G)$ and $\{f_i\}$ in $PF_p(G)$ with $\sum_{i=1}^\infty \|u_i\|_{A_p(G)}\|f_i\|_{PF_p(G)} < \infty$ such that $f = \sum_{i=1}^\infty u_i f_i$. As in the proof of Theorem 3.2 (*(iv)* $\Rightarrow$ *(iii)*), we have $\langle a_n, \sum_{i=1}^\infty u_i f_i\rangle \to \langle 1, \sum_{i=1}^\infty u_i f_i\rangle$, as required. □

Next, we investigate the relationship between the AP in the multiplier algebra and the approximation property as a Banach space.

**Definition 4.9.** A Banach space $X$ is said to have the approximation property as a Banach space if, for every compact subset $K$ in $X$ and every $\epsilon > 0$, there is an operator $T : X \to X$ of finite rank so that $\|Tx - x\| \le \epsilon$ for every $x \in K$.

**Theorem 4.10.** *Let $G$ be a discrete group. Consider the following conditions:*
*(i) $G$ has the AP in the completely bounded multiplier algebra $M_0A(G)$;*
*(ii) $A_p(G)$ has the AP in the multiplier algebra $MA_p(G)$;*
*(iii) $A_p(G)$ has the approximation property as a Banach space.*
*Then (i) $\Rightarrow$ (ii) if $p = 2$, and (ii) $\Rightarrow$ (iii) for any $p$.*

*Proof.* We have seen *(i)* $\Rightarrow$ *(ii)* in Proposition 2.4 if $p = 2$. Now we show *(ii)* $\Rightarrow$ *(iii)*. Let *(ii)* be true. Then there exists an approximate identity $\{a_\alpha\}$ in $A_p(G)$ such that $\|a_\alpha a - a\|_{A_p(G)} \to 0$ uniformly for $a$ in any compact subset of $A_p(G)$ by Theorem 3.2. Since each $a_\alpha$ can be chosen to have a compact support (see Remarks 3.3) and $G$ is discrete, we can assume that each $a_\alpha$ has a finite support. Hence the operator $\eta_\alpha : A_p(G) \to A_p(G)$ defined by $\eta_\alpha(a) = a_\alpha a$ has a finite rank and the net $\{a_\alpha\}$ satisfies the requirement. □

**Proposition 4.11.** *Let $G$ be a locally compact group. Then the following conditions are equivalent:*

*(i) For any positive integer $m$, if $u_i \in A_p(G)$ and $T_i \in PM_p(G)$ $(i = 1, 2, \ldots, m)$,*

$$\sum_{i=1}^{m} u_i T_i = 0 \text{ implies } \sum_{i=1}^{m} \langle T_i, u_i \rangle = 0;$$

*(ii) $A_p(G)$ has an approximate identity.*

*Proof.* $(i) \Rightarrow (ii)$ For any finite subset $S = \{a_1, a_2, \ldots, a_m\}$ of $A_p(G)$, let

$$X = A_p(G) \oplus A_p(G) \oplus \cdots \oplus A_p(G) \quad (m \text{ copies})$$

have the sup-norm. We denote the subset of $X$ by

$$A_p(G)(a_1, a_2, \ldots, a_m) = \{(aa_1, aa_2, \ldots, aa_m) : a \in A_p(G)\}.$$

Claim that $(a_1, a_2, \ldots, a_m)$ is in the norm closure of $A_p(G)(a_1, a_2, \ldots, a_m)$ in $X$. If this is not true, by the Hahn Banach theorem there is an $T \in X^*$ such that $\langle T, (a_1, a_2, \ldots, a_m) \rangle = 1$ and $T = 0$ on $A_p(G)(a_1, a_2, \ldots, a_m)$. Then there are $T_i \in PM_p(G)$ $(i = 1, 2, \ldots, m)$ such that, for any $(u_1, u_2, \ldots, u_m) \in X$,

$$\langle T, (u_1, u_2, \ldots, u_m) \rangle = \sum_{i=1}^{m} \langle T_i, u_i \rangle \text{ and } \|T\| = \sum_{i=1}^{m} \|T_i\|.$$

Hence, $\langle T, (a_1, a_2, \ldots, a_m) \rangle = \sum_{i=1}^{m} \langle T_i, a_i \rangle = 1$ and

$$\langle T, (aa_1, aa_2, \ldots, aa_m) \rangle = \sum_{i=1}^{m} \langle T_i, aa_i \rangle = \sum_{i=1}^{m} \langle a_i T_i, a \rangle = 0$$

for all $a \in A_p(G)$. Hence $\sum_{i=1}^{m} a_i T_i = 0$. This contradicts $(i)$. Thus, for any positive integer $j$, there is a function, denoted by $a_{(j,S)}$, in $A_p(G)$ such that

$$\|(a_1, a_2, \ldots, a_m) - (a_{(j,S)} a_1, a_{(j,S)} a_2, \ldots, a_{(j,S)} a_m)\|_X \leq \frac{1}{j}.$$

Hence $\|a_{(j,S)} a_i - a_i\|_{A_p(G)} \leq \frac{1}{j}$ for $i = 1, 2, \ldots, m$. Therefore, the net $\{a_{(j,S)}\}$ is an approximate identity of $A_p(G)$, where the set $\{(j, S)\}$ is directed as usual.

$(ii) \Rightarrow (i)$ Let $\{a_\alpha\}$ be an approximate identity of $A_p(G)$. If $\sum_{i=1}^{m} u_i T_i = 0$ for $u_i \in A_p(G)$ and $T_i \in PM_p(G)$ $(i = 1, 2, \ldots, m)$, then

$$\langle a_\alpha, \sum_{i=1}^{m} u_i T_i \rangle = \sum_{i=1}^{m} \langle a_\alpha u_i, T_i \rangle \to \sum_{i=1}^{m} \langle u_i, T_i \rangle.$$

Since $\langle a_\alpha, \sum_{i=1}^{m} u_i T_i \rangle = 0$ for all $\alpha$, we have $\sum_{i=1}^{m} \langle u_i, T_i \rangle = 0$. This shows $(i)$. $\square$

*Remark* 4.12. Finally, we summarize the relationship between the various approximation properties and some of the other properties related to a locally compact group. For a locally compact group $G$ and $1 < p < \infty$, we consider the following properties:

(1) $G$ is amenable;
(2) $G$ is weakly amenable with respect to the multiplier algebra $MA_p(G)$;
(3) $A_p(G)$ has the sequential AP in $MA_p(G)$;
(4) $A_p(G)$ has the AP in $MA_p(G)$;
(4′) $A_p(G)$ has the approximation property as a Banach space;

(5) for any positive integer $m$, if $u_i \in A_p(G)$ and $T_i \in PM_p(G)$ $(i = 1, 2, \ldots, m)$,

$$\sum_{i=1}^{m} u_i T_i = 0 \text{ implies } \sum_{i=1}^{m} \langle T_i, u_i \rangle = 0;$$

(6) $A_p(G)$ has an approximate identity;

(7) for any $u \in A_p(G)$ and $T \in PM_p(G)$,

$$\text{if } uT = 0, \text{ then} \langle T, u \rangle = 0$$

($T$ is said to have the (H) property by Eymard [7]);

(8) for any $a \in A_p(G)$

$$a \in \overline{aA_p(G)}$$

(see Kaniuth and Lau [14], page 191, question 8).

Then (1) $\Rightarrow$ (2) is well known and (2) $\not\Rightarrow$ (1) by the counterexample $F_2$, a free group on two generators. (2) $\Leftrightarrow$ (3) for a $\sigma$-compact $G$ by Theorem 4.8. (2) $\Rightarrow$ (4) by Proposition 4.2. (4) $\not\Rightarrow$ (2) by Counterexample 4.5. For a $\sigma$-compact $G$, (3) $\Rightarrow$ (4) is trivial, but (4) $\not\Rightarrow$ (3) by Counterexample 4.5. For a discrete group $G$, (4) $\Rightarrow$ (4′) by Theorem 4.10. (4) $\Rightarrow$ (5) by Theorem 3.2. (5) $\Leftrightarrow$ (6) by Proposition 4.11. (6) $\Rightarrow$ (7) is trivial. (7) $\Leftrightarrow$ (8) follows immediately from the Hahn Banach Theorem.

If $p = 2$, then we consider the following properties:

(a) $G$ is weakly amenable with respect to the completely bounded multiplier algebra $M_0A(G)$;

(b) $G$ has the AP in $M_0A(G)$.

Then $(a) \Rightarrow (b)$ is trivial (see Proposition 4.2). $(b) \Rightarrow$ (4) by Proposition 2.4. $(a) \Rightarrow$ (2) follows from the fact that the multiplier norm is less than or equal to the completely bounded multiplier norm.

It is an open question whether there exists a locally compact group $G$ such that any of the properties from (4) to (8) fail.

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