HOMOTOPY ON SPATIAL GRAPHS
AND THE SATO-LEVINE INVARIANT

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Abstract. Edge-homotopy and vertex-homotopy are equivalence relations on
spatial graphs which are generalizations of Milnor’s link-homotopy. We intro-
duce some edge (resp. vertex)-homotopy invariants of spatial graphs by ap-
plying the Sato-Levine invariant for the 2-component constituent algebraically
split links and show examples of non-splittable spatial graphs up to edge (resp.
vertex)-homotopy, all of whose constituent links are link-homotopically trivial.

1. Introduction

Throughout this paper we work in the piecewise linear category. Let $G$ be a
finite graph which does not have isolated vertices and free vertices. An embedding
$f$ of $G$ into the 3-sphere $S^3$ is called a spatial embedding of $G$ or simply a spatial
graph. For a spatial embedding $f$ and a subgraph $H$ of $G$ which is homeomorphic
to the 1-sphere $S^1$ or a disjoint union of 1-spheres, we call $f(H)$ a constituent knot
or a constituent link of $f$, respectively. A graph $G$ is said to be planar if there
exists an embedding of $G$ into the 2-sphere $S^2$, and a spatial embedding of a planar
graph is said to be trivial if it is ambient isotopic to an embedding of the graph into
a 2-sphere in $S^3$. A spatial embedding $f$ of a graph $G$ is said to be split if there
exists a 2-sphere $S$ in $S^3$ such that $S \cap f(G) = \emptyset$ and each component of $S^3 - S$
has intersection with $f(G)$, and otherwise $f$ is said to be non-splittable.

Two spatial embeddings of a graph $G$ are said to be edge-homotopic if they
are transformed into each other by self crossing changes and ambient isotopies,
where a self crossing change is a crossing change on the same spatial edge, and
vertex-homotopic if they are transformed into each other by crossing changes on
two adjacent spatial edges and ambient isotopies[1]. These equivalence relations were
introduced by Taniyama [19] as generalizations of Milnor’s link-homotopy on links
[8]; namely if $G$ is homeomorphic to a disjoint union of 1-spheres, then these are
none other than link-homotopy. There are many studies about link-homotopy. In
particular, the link-homotopy classification was given for 2- and 3-component links

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In [19], edge-homotopy and vertex-homotopy were called homotopy and weak homotopy,
respectively.

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by Milnor [8], for 4-component links by Levine [7] and for all links by Habegger and Lin [2]. On the other hand, there are very few studies about edge (resp. vertex)-homotopy on spatial graphs [18], [9], [13], [11].

In [18], Taniyama defined an edge (resp. vertex)-homotopy invariant of spatial graphs called the \( \alpha \)-invariant by applying the Casson invariant (or equivalently the second coefficient of the Conway polynomial) of the constituent knots and showed that there exists a non-trivial spatial embedding \( f \) of a planar graph up to edge (resp. vertex)-homotopy, even in the case where \( f \) does not contain any constituent link. But the \( \alpha \)-invariant cannot detect a non-splittable spatial embedding of a disconnected graph up to edge (resp. vertex)-homotopy. As far as the authors know, an example of a non-splittable spatial embedding of a disconnected graph up to edge (resp. vertex)-homotopy, all of whose constituent links are link-homotopically trivial, has not yet been demonstrated.

Our purpose in this paper is to study spatial embeddings of disconnected graphs up to edge (resp. vertex)-homotopy by applying the Sato-Levine invariant [14] (or equivalently the third coefficient of the Conway polynomial) for the constituent 2-component algebraically split links and show that there exist infinitely many non-splittable spatial embeddings of a certain disconnected graph up to edge (resp. vertex)-homotopy, all of whose constituent links are link-homotopically trivial. These examples show that edge (resp. vertex)-homotopy on spatial graphs behaves quite differently than link-homotopy on links. In the next section we give the definitions of our invariants and state their invariance up to edge (resp. vertex)-homotopy.

2. Definitions of invariants

We call a subgraph of a graph \( G \) a cycle if it is homeomorphic to the 1-sphere, and a cycle is called a \( k \)-cycle if it contains exactly \( k \) edges. For a subgraph \( H \) of \( G \), we denote the set of all cycles of \( H \) by \( \Gamma(H) \). We set \( \mathbb{Z}_m = \{0, 1, \ldots, m-1\} \) for a positive integer \( m \) and \( \mathbb{Z}_0 = \mathbb{Z} \). We regard \( \mathbb{Z}_m \) as an abelian group in the obvious way. We call a map \( \omega : \Gamma(H) \to \mathbb{Z}_m \) a weight on \( \Gamma(H) \) over \( \mathbb{Z}_m \).

Let \( G = G_1 \cup G_2 \) be a disjoint union of two connected graphs and \( \omega_i : \Gamma(G_i) \to \mathbb{Z}_m \) a weight on \( \Gamma(G_i) \) over \( \mathbb{Z}_m \) \((i = 1, 2)\). Let \( f \) be a spatial embedding of \( G \) such that

\[
\omega_1(\gamma)\omega_2(\gamma')\text{lk}(f(\gamma), f(\gamma')) = 0
\]

\(^2\) A weight \( \omega \) on \( \Gamma(H) \) over \( \mathbb{Z}_m \) is said to be balanced on an edge \( e \) of \( H \) if \( \sum_{\gamma \in \Gamma_e(H)} \omega(\gamma)[\gamma] = 0 \) in \( H_1(H; \mathbb{Z}_m) \), where the orientation of \( \gamma \) is induced by the one of \( e \) [18].
in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$, where $\text{lk}(L) = \text{lk}(K_1, K_2)$ denotes the linking number of a 2-component oriented link $L = K_1 \cup K_2$. Then we define $\beta_{\omega_1, \omega_2}(f) \in \mathbb{Z}_m$ by

$$\beta_{\omega_1, \omega_2}(f) \equiv \sum_{\gamma \in \Gamma(G_1), \gamma' \in \Gamma(G_2)} \omega_1(\gamma)\omega_2(\gamma')a_3(f(\gamma), f(\gamma')) \pmod{m},$$

where $a_3(L) = a_3(K_1, K_2)$ denotes the third coefficient of the Conway polynomial of a 2-component oriented link $L = K_1 \cup K_2$. We remark here that $a_3(L)$ coincides with the Sato-Levine invariant $\beta(L)$ of $L$ if $L$ is algebraically split, namely $\text{lk}(K_1, K_2) = 0$ \cite{[1], [7]}. Thus our $\beta_{\omega_1, \omega_2}(f)$ is also the modulo $m$ reduction of the summation of Sato-Levine invariants for the constituent 2-component algebraically split links of $f$.

**Remark 2.1.** For a 2-component algebraically split link $L = K_1 \cup K_2$,

1. The value of $a_3(L)$ does not depend on the orientations of $K_1$ and $K_2$. Actually we can check it easily by the original definition of the Sato-Levine invariant.

2. The value of $a_3(L)$ is not a link-homotopy invariant of $L$ (see also Lemma \ref{lem:3.1}). For example, the Whitehead link $L$ is link-homotopically trivial but $a_3(L) = 1$.

Now we state the invariance of $\beta_{\omega_1, \omega_2}$ up to edge (resp. vertex)-homotopy under some conditions on the graphs.

**Theorem 2.2.** Let $G = G_1 \cup G_2$ be a disjoint union of two connected graphs and $\omega_i$ a weight on $\Gamma(G_i)$ over $\mathbb{Z}_m$ $(i = 1, 2)$. Let $f$ be a spatial embedding of $G$ such that

$$\omega_1(\gamma)\omega_2(\gamma')\text{lk}(f(\gamma), f(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$. Then we have the following:

1. If $\omega_i$ is weakly balanced on any edge of $G_i$ $(i = 1, 2)$, then $\beta_{\omega_1, \omega_2}(f)$ is an edge-homotopy invariant of $f$.

2. If $\omega_i$ is weakly balanced on any pair of adjacent edges of $G_i$ $(i = 1, 2)$, then $\beta_{\omega_1, \omega_2}(f)$ is a vertex-homotopy invariant of $f$.

We prove Theorem 2.2 in the next section. In addition, by using an integer-valued invariant (Theorem \ref{thm:1.2}), we show that there exist infinitely many non-splittable spatial embeddings of a certain disconnected graph up to edge-homotopy all of whose constituent links are link-homotopically trivial (Example \ref{ex:3.3}). We also exhibit an infinite family of non-splittable spatial embeddings of a certain disconnected graph up to vertex-homotopy which can be distinguished by our integer-valued invariant (Example \ref{ex:3.4}).

We note that if a graph $G$ contains a connected component which is homeomorphic to the 1-sphere, then our invariants in Theorem 2.2 are useless. For such cases, we can define edge (vertex)-homotopy invariants that take values in $\mathbb{Z}_2$ on a weaker condition for weights than the one stated in Theorem 2.2. For a subgraph $H$ of a graph $G$, we say that a weight $\omega$ on $\Gamma(H)$ over $\mathbb{Z}_2$ is totally balanced if

$$\sum_{\gamma \in \Gamma(H)} \omega(\gamma)[\gamma] = 0$$
in $H_1(H;\mathbb{Z}_2)$. We note that if a weight $\omega$ on $\Gamma(H)$ over $\mathbb{Z}_2$ is totally balanced, then it is weakly balanced on any edge $e$ of $H$ (Lemma 3.2), but not always weakly balanced on any pair of adjacent edges of $H$ (Remark 3.3). Then we have the following:

**Theorem 2.3.** Let $G = G_1 \cup G_2$ be a disjoint union of two connected graphs and $\omega_i$ a weight on $\Gamma(G_i)$ over $\mathbb{Z}_2$ ($i = 1, 2$). Let $f$ be a spatial embedding of $G$ such that

$$\omega_1(\gamma)\omega_2(\gamma')\text{lk}(f(\gamma), f(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$. Then we have the following:

1. If either $\omega_1$ is totally balanced on $\Gamma(G_1)$ or $\omega_2$ is totally balanced on $\Gamma(G_2)$, then $\beta_{\omega_1, \omega_2}(f)$ is an edge-homotopy invariant of $f$.
2. If either $\omega_1$ is totally balanced on $\Gamma(G_1)$ and weakly balanced on any pair of adjacent edges of $G_1$, or $\omega_2$ is totally balanced on $\Gamma(G_2)$ and weakly balanced on any pair of adjacent edges of $G_2$, then $\beta_{\omega_1, \omega_2}(f)$ is a vertex-homotopy invariant of $f$.

We also prove Theorem 2.3 in the next section and give some examples in Section 5. In particular, we show that there exist infinitely many non-splittable spatial embeddings of a certain disconnected graph up to vertex-homotopy, all of whose constituent links are link-homotopically trivial (Example 5.4). We remark here that the $\mathbb{Z}_2$-valued invariant in Theorem 2.3 cannot always be extended to an integer-valued one (Remark 5.5).

Theorems 2.2 and 2.3 do not work for spatial graphs as illustrated in Figure 2.1, for instance. In Section 6, we state a method to detect such non-splittable spatial graphs up to edge-homotopy by using a planar surface having a graph as a spine (Theorem 6.1). Actually we show that each of the spatial graphs as illustrated in Figure 2.1 is non-splittable up to edge-homotopy (Example 6.2).

![Figure 2.1](image-url)

**Figure 2.1**

### 3. Proofs of Theorems 2.2 and 2.3

We first calculate the change in the third coefficient of the Conway polynomial of 2-component algebraically split links which differ by a single self crossing change.
Lemma 3.1. Let \( L_+ \) and \( L_- \) be two 2-component oriented links and \( L_0 = J_1 \cup J_2 \cup K \) a 3-component oriented link which are identical except inside the depicted regions as illustrated in Figure 3.1. Suppose that \( \text{lk}(L_+) = \text{lk}(L_-) = 0 \). Then it holds that
\[
a_3(L_+) - a_3(L_-) = -\text{lk}(J_1, K)^2 = -\text{lk}(J_2, K)^2.
\]

![Figure 3.1](image)

Proof. By the skein relation of the Conway polynomial and a well-known formula for the second coefficient of the Conway polynomial of a 3-component oriented link (cf. [1], [3], [5]), we have that
\[
a_3(L_+) - a_3(L_-) = \text{lk}(J_1, J_2)\text{lk}(J_2, K)\text{lk}(J_1, K) + \text{lk}(J_1, K)\text{lk}(J_1, J_2)
\]
\[
= \text{lk}(J_1, K)\text{lk}(J_1, J_2) + \text{lk}(J_1, K)\text{lk}(J_2, K).
\]

We note that
\[
\text{lk}(J_1, K) + \text{lk}(J_2, K) = 0
\]
by the condition \( \text{lk}(L_+) = \text{lk}(L_-) = 0 \). Thus by (3.1) and (3.2), we have that
\[
a_3(L_+) - a_3(L_-) = \text{lk}(J_1, J_2)\{-\text{lk}(J_1, K)\} + \text{lk}(J_1, K)\text{lk}(J_1, J_2)
\]
\[
= \text{lk}(J_2, K)\text{lk}(J_1, K).
\]

Therefore by (3.2) we have the result. \( \square \)

Proof of Theorem 2.2. (1) Let \( f \) and \( g \) be two spatial embeddings of \( G \) such that
\[
\omega_1(\gamma)\omega_2(\gamma')\text{lk}(f(\gamma), f(\gamma')) = 0
\]
in \( \mathbb{Z} \) for any \( \gamma \in \Gamma(G_1) \) and \( \gamma' \in \Gamma(G_2) \) and \( g \) is edge-homotopic to \( f \). Then it also holds that
\[
\omega_1(\gamma)\omega_2(\gamma')\text{lk}(g(\gamma), g(\gamma')) = 0
\]
in \( \mathbb{Z} \) for any \( \gamma \in \Gamma(G_1) \) and \( \gamma' \in \Gamma(G_2) \) because the linking number of a 2-component constituent link of a spatial graph is an edge-homotopy invariant. First we show that if \( f \) is transformed into \( g \) by self crossing changes on \( f(G_1) \) and ambient isotopies, then \( \beta_1\omega_1(f) = \beta_1\omega_1(g) \). It is clear that any link invariant of a constituent link of a spatial graph is also an ambient isotopy invariant of the spatial graph. Thus we may assume that \( g \) is obtained from \( f \) by a single crossing change on \( f(e) \) for
an edge $e$ of $G_1$ as illustrated in Figure 3.2. Moreover, by smoothing this crossing point we can obtain the spatial embedding $h$ of $G$ and the knot $J_h$ as illustrated in Figure 3.2. Then by (3.3), (3.4), Lemma 3.1 and the assumption for $\omega_1$ we have that

$$
\beta_{\omega_1, \omega_2}(f) - \beta_{\omega_1, \omega_2}(g) = \sum_{\gamma \in \Gamma(G_1)} \omega_1(\gamma) \omega_2(\gamma') \{a_3(f(\gamma), f(\gamma')) - a_3(g(\gamma), g(\gamma'))\}
$$

$$
= \sum_{\gamma \in \Gamma(G_1)} \sum_{\gamma' \in \Gamma(G_2)} \omega_1(\gamma) \omega_2(\gamma') \{a_3(f(\gamma), f(\gamma')) - a_3(g(\gamma), g(\gamma'))\}
$$

$$
= - \sum_{\gamma \in \Gamma(G_1)} \omega_1(\gamma) \omega_2(\gamma') \text{lk}(h(\gamma'), J_h)^2
$$

$$
= - \left( \sum_{\gamma \in \Gamma(G_1)} \omega_1(\gamma) \right) \sum_{\gamma' \in \Gamma(G_2)} \omega_2(\gamma') \text{lk}(h(\gamma'), J_h)^2
$$

$$
= 0.
$$

Therefore we have that $\beta_{\omega_1, \omega_2}(f) = \beta_{\omega_1, \omega_2}(g)$. In the same way we can show that if $f$ is transformed into $g$ by self crossing changes on $f(G_2)$ and ambient isotopies, then $\beta_{\omega_1, \omega_2}(f) = \beta_{\omega_1, \omega_2}(g)$. Thus we have that $\beta_{\omega_1, \omega_2}$ is an edge-homotopy invariant.

(2) By considering the triple of spatial embeddings as illustrated in Figure 3.3, we can prove (2) in a similar way as the proof of (1). We omit the details. □

Next we prove Theorem 2.3. For a subgraph $H$ of a graph $G$, we have the following.

**Lemma 3.2.** A totally balanced weight $\omega$ on $\Gamma(H)$ over $\mathbb{Z}_2$ is weakly balanced on any edge $e$ of $H$. 
Proof. For an edge $e$ of $H$, we can represent any $\gamma \in \Gamma_e(H)$ as $e + c_\gamma \in Z_1(H \setminus e; \mathbb{Z}_2)$, where $c_\gamma$ is a 1-chain in $C_1(H \setminus e; \mathbb{Z}_2)$. Then we have that

$$0 = \sum_{\gamma \in \Gamma(H)} \omega(\gamma)[\gamma] = \sum_{\gamma \in \Gamma_e(H)} \omega(\gamma)[e + c_\gamma] + \sum_{\gamma' \in \Gamma(H) \setminus \Gamma_e(H)} \omega(\gamma')[\gamma']$$

in $H_1(H; \mathbb{Z}_2)$. This implies that if $\omega$ is not weakly balanced on $e$, then $\omega$ is not totally balanced on $\Gamma(H)$ over $\mathbb{Z}_2$. \hfill $\square$

Remark 3.3. A totally balanced weight $\omega$ on $\Gamma(H)$ over $\mathbb{Z}_2$ is not always weakly balanced on any pair of adjacent edges of $H$. For example, let $\omega$ be a weight on $\Theta_3$ (see Example 4.3) over $\mathbb{Z}_2$ defined by $\omega(\gamma) = 1$ for any cycle $\gamma \in \Gamma(\Theta_3)$. It is easy to see that $\omega$ is totally balanced, but not weakly balanced, on each pair of adjacent edges of $\Theta_3$.

Proof of Theorem 2.3 (1) Let $f$ and $g$ be two spatial embeddings of $G$ which are edge-homotopic such that

$$\omega_1(\gamma)\omega_2(\gamma')\text{lk}(f(\gamma), f(\gamma')) = \omega_1(\gamma)\omega_2(\gamma')\text{lk}(g(\gamma), g(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$. First we show that if $f$ is transformed into $g$ by self crossing changes on $f(G_1)$ and ambient isotopies, then $\beta_{\omega_1, \omega_2}(f) = \beta_{\omega_1, \omega_2}(g)$. In the same way as the proof of Theorem 2.2 we may consider three spatial embeddings $f, g$ and $h$ of $G$ and the knot $J_h$ as illustrated in Figure 3.2.
Then, by the same calculation in the proof of Theorem 2.2 we have that
\[
\beta_{\omega_1, \omega_2}(f) - \beta_{\omega_1, \omega_2}(g) = -\left( \sum_{\gamma \in \Gamma_1(G_1)} \omega_1(\gamma) \right) \sum_{\gamma' \in \Gamma_2(G_2)} \omega_2(\gamma') \text{lk}(h(\gamma'), J_h)^2
\]
\[
= \left( \sum_{\gamma \in \Gamma_1(G_1)} \omega_1(\gamma) \right) \sum_{\gamma' \in \Gamma_2(G_2)} \omega_2(\gamma') \text{lk}(h(\gamma'), J_h)
\]
\[
= \left( \sum_{\gamma \in \Gamma_1(G_1)} \omega_1(\gamma) \right) \text{lk} \left( \sum_{\gamma' \in \Gamma_2(G_2)} \omega_2(\gamma') h(\gamma'), J_h \right).
\]
If \(\omega_1\) is totally balanced on \(\Gamma(G_1)\), then by Lemma 3.2 it is weakly balanced on any edge \(e\) of \(G_1\). This implies that \(\beta_{\omega_1, \omega_2}(f) = \beta_{\omega_1, \omega_2}(g)\). If \(\omega_2\) is totally balanced on \(\Gamma(G_1)\), then we have that
\[
\text{lk} \left( \sum_{\gamma' \in \Gamma_2(G_2)} \omega_2(\gamma') h(\gamma'), J_h \right) \equiv \text{lk}(0, J_h) = 0.
\]
Therefore this also implies that \(\beta_{\omega_1, \omega_2}(f) = \beta_{\omega_1, \omega_2}(g)\). In the same way we can show that if \(f\) is transformed into \(g\) by self crossing changes on \(f(G_2)\) and ambient isotopies, then \(\beta_{\omega_1, \omega_2}(f) = \beta_{\omega_1, \omega_2}(g)\). Thus we have that \(\beta_{\omega_1, \omega_2}\) is an edge-homotopy invariant.

(2) By considering the triple of spatial embeddings as illustrated in Figure 3.3 we can prove (2) in a similar way as the proof of (1). We also omit the details. \(\square\)

Since the Conway polynomial of a split link is zero, our invariants take the value zero for any split (2-component) spatial graph. Therefore if the value of our invariant of a spatial graph is not zero, then it is non-splittable up to edge (resp. vertex)-homotopy.

4. Integer-valued invariants

Let \(G\) be a planar graph. An embedding \(p : G \to S^2\) is said to be cellular if the closure of each of the connected components of \(S^2 - p(G)\) is homeomorphic to the disk. Then we regard the set of the boundaries of all of the connected components of \(S^2 - p(G)\) as a subset of \(\Gamma(G)\) and denote it by \(\Gamma_p(G)\). We say that \(G\) admits a checkerboard coloring on \(S^2\) if there exists a cellular embedding \(p : G \to S^2\) such that we can color all of the connected components of \(S^2 - p(G)\) by two colors (black and white) so that any of the two components which are adjacent by an edge have distinct colors; see Figure 4.1. We denote the subset of \(\Gamma_p(G)\) which corresponds to the black (resp. white) colored components by \(\Gamma_p^b(G)\) (resp. \(\Gamma_p^w(G)\)).

**Proposition 4.1.** Let \(G\) be a planar graph which is not homeomorphic to \(S^1\) and admits a checkerboard coloring on \(S^2\) with respect to a cellular embedding \(p : G \to S^2\). Let \(\omega_p\) be a weight on \(\Gamma(G)\) over \(\mathbb{Z}\) defined by
\[
\omega_p(\gamma) = \begin{cases} 
1 & (\gamma \in \Gamma_p^b(G)), \\
-1 & (\gamma \in \Gamma_p^w(G)), \\
0 & (\gamma \in \Gamma(G) \setminus \Gamma_p(G)).
\end{cases}
\]
Then \(\omega_p\) is weakly balanced on any edge of \(G\).
that of Figure 4.2. We can see that any of the 2-component constituent links of $\Theta_n$ in $\mathbb{Z}$.

Theorem 4.2. Let $G = G_1 \cup G_2$ be a disjoint union of two connected planar graphs such that $G_i$ is not homeomorphic to $S^1$ and admits a checkerboard coloring on $S^2$ with respect to a cellular embedding $p_i : G_i \to S^2$ ($i = 1, 2$). Let $\omega_p$ be a checkerboard weight on $\Gamma(G_i)$ over $\mathbb{Z}$ ($i = 1, 2$) and $f$ a spatial embedding of $G$ such that

$$\omega_p(\gamma)lk(f(\gamma), f(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$. Then $\beta_{\omega_p, \omega_p}(f)$ is an integer-valued edge-homotopy invariant of $f$.

Example 4.3. Let $\Theta_n$ be a graph with two vertices $u$ and $v$ and $n$ edges $e_1, e_2, \ldots, e_n$, each of which joins $u$ and $v$. A spatial embedding of $\Theta_n$ is called a (spatial) theta $n$-curve or simply a theta curve if $n = 3$. For $n \geq 2$, we denote that a cycle of $\Theta_n$ consists of two edges $e_i$ and $e_j$ by $\gamma_{ij}$ ($i < j$). Then it is clear that $\Theta_n$ admits a cellular embedding $p : \Theta_n \to S^2$ so that

$$\Gamma_p(\Theta_n) = \{\gamma_{12}, \gamma_{23}, \ldots, \gamma_{n-1, n}, \gamma_{1n}\}.$$ 

Moreover, for $m \geq 1$, $\Theta_{2m}$ admits a checkerboard coloring on $S^2$ so that

$$\Gamma_p(\Theta_{2m}) = \{\gamma_{12}, \gamma_{34}, \ldots, \gamma_{2m-1, 2m}\},$$

$$\Gamma_p(\Theta_{2m}) = \{\gamma_{23}, \gamma_{45}, \ldots, \gamma_{2m-2, 2m-1}, \gamma_{1, 2m}\}.$$ 

Now let $G$ be a disjoint union of two copies of $\Theta_4$, each of which admits a checkerboard coloring on $S^2$ with respect to the cellular embedding $p$ as above. Let $\omega_p$ be a checkerboard weight on $\Gamma(\Theta_4)$ over $\mathbb{Z}$ and $g_1$ a spatial embedding of $G$ as illustrated in Figure 4.2. We can see that any of the 2-component constituent links of $g_1$ has a zero linking number. More precisely, $g_1$ contains exactly one non-trivial 2-component link $L = g_1(\gamma_{14}) \cup g_1(\gamma'_{14})$ whose linking number is zero. Thus by Theorem 4.2 we have that $\beta_{\omega_p, \omega_p}(g_1)$ is an integer-valued edge-homotopy invariant of $g_1$. Then, by a direct calculation we have that $a_3(L) = 2$, namely $\beta_{\omega_p, \omega_p}(g_1) = 2$. Note that a 2-component link is link-homotopically trivial if and only if its linking number is zero $\mathbb{Z}$. This implies that $g_1$ is non-splitable up to edge-homotopy despite the fact that any of the constituent links of $g_1$ is link-homotopically trivial.
Moreover, for an integer $m$, let $g_m$ be a spatial embedding of $G$ as illustrated in Figure 4.3. If $m \neq 0$, we can see that $g_m$ contains exactly one non-trivial 2-component link $L = g_m(\gamma_{14}) \cup g_m(\gamma'_{14})$ whose linking number is zero. Thus we also have that $\beta_{\omega, \omega}(g_m)$ is an integer-valued edge-homotopy invariant of $g_m$. Then, by a calculation we have that $\alpha_3(L) = 2m$, namely $\beta_{\omega, \omega}(g_m) = 2m$. This implies that there exist infinitely many non-splittable spatial embeddings of $G$ up to edge-homotopy, all of whose constituent links are link-homotopically trivial.

**Example 4.4.** Let $H$ be a graph as illustrated in Figure 4.4. We denote the cycle of $H$ which contains $e_i$ and $e_j$ by $\gamma_{ij}$ ($i < j$). Let $G$ be a disjoint union of two copies of $H$ and $g_1$ a spatial embedding of $G$ as illustrated in Figure 4.5. This spatial embedding $g_1$ contains exactly one 4-component constituent link $L = g_1(\gamma_{12} \cup \gamma_{34} \cup \gamma'_{12} \cup \gamma'_{34})$. Note that if $g_1$ is split up to vertex-homotopy, then $L$ is split up to link-homotopy. Since $|\mu_{1234}(L)| = 1$, where $\mu_{1234}$ denotes Milnor’s $\mu$-invariant of length 4 of 4-component links [8], we have that $L$ is non-splittable up to link-homotopy. Therefore we have that $g_1$ is non-splittable up to vertex-homotopy.

Moreover, let $g_m$ be a spatial embedding of $G$ as illustrated in Figure 4.5, which can be constructed in the same way as in Example 4.3. Then we can see that $\beta_{\omega, \omega}(g_m)$ is an integer-valued vertex-homotopy invariant of $g_m$ and $\beta_{\omega, \omega}(g_m) = 2m$. This implies that $g_m$ is non-splittable up to vertex-homotopy for any integer $m \neq 0$ and $g_i$ and $g_j$ are not vertex-homotopic for any $i \neq j$.

## 5. Modulo two invariants

**Proposition 5.1.** Let $G$ be a planar graph which is not homeomorphic to $S^1$ and $p : G \to S^2$ a cellular embedding. Let $\omega_p : \Gamma(G) \to \mathbb{Z}_2$ be a weight on $\Gamma(G)$ over $\mathbb{Z}_2$. 
defined by

\[ \omega_p(\gamma) = \begin{cases} 
1 & (\gamma \in \Gamma_p(G)), \\
0 & (\gamma \in \Gamma(G) \setminus \Gamma_p(G)). 
\end{cases} \]

Then \( \omega_p \) is totally balanced.
Proof. It holds that
\[
\sum_{\gamma \in \Gamma(G)} \omega_p(\gamma) |\gamma| = \sum_{\gamma \in \Gamma_p(G)} |\gamma| = 2 \left[ \sum_{e \in E(G)} e \right] = 0
\]
in $H_1(G; \mathbb{Z}_2)$, where $E(G)$ denotes the set of all edges of $G$. Thus we have the result. □

Thus by Proposition 5.1 and Theorem 2.3 (1), we can obtain an edge-homotopy invariant as follows.

Theorem 5.2. Let $G = G_1 \cup G_2$ be a disjoint union of two connected graphs such that $G_1$ is planar, not homeomorphic to $S^1$ and admits a cellular embedding $p_1 : G_1 \rightarrow S^2$. Let $\omega_p$, be a weight on $\Gamma(G_1)$ over $\mathbb{Z}_2$ as in Proposition 5.1 $\omega_2$ a weight on $\Gamma(G_2)$ over $\mathbb{Z}_2$ and $f$ a spatial embedding of $G$ such that
\[
\omega_{p_1}(\gamma)\omega_2(\gamma') \text{lk}(f(\gamma), f(\gamma')) = 0
\]
in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$. Then $\beta_{x_\omega_1, \omega_2}(f)$ is an edge-homotopy invariant of $f$. □

Example 5.3. Let $G$ be a disjoint union of $\Theta_3$ and a circle $\gamma$. Let $\omega_p$ be a weight on $\Gamma(\Theta_3)$ over $\mathbb{Z}_2$ as in Proposition 5.1 with respect to a cellular embedding $p : \Theta_3 \rightarrow S^2$ as in Example 4.3 and $\omega$ a weight on $\Gamma(\gamma)$ over $\mathbb{Z}_2$ defined by $\omega(\gamma) = 1$. Let $g$ be a spatial embedding of $G$ as illustrated in Figure 5.1 (1). We can see that $g$ contains exactly one non-trivial 2-component link $L = g(\gamma_1) \cup g(\gamma)$ which is the Whitehead link, so $\text{lk}(L) = 0$ and $a_3(L) = 1$. Thus by Theorem 5.2 we have that
$\beta_{\omega, \omega}(g)$ is an edge-homotopy invariant of $g$ and $\beta_{\omega, \omega}(g) = 1$. Namely $g$ is non-splittable up to edge-homotopy despite the fact that any of the constituent links of $g$ is link-homotopically trivial.

**Example 5.4.** Let $G$ be a disjoint union of the complete bipartite graph on $3 + 3$ vertices $K_{3,3}$ and a circle $\gamma$. Let $\omega_{3,3}$ be a weight on $K_{3,3}$ over $\mathbb{Z}$ defined by $\omega_{3,3}(\gamma') = 1$ if $\gamma'$ is a 4-cycle and 0 if $\gamma'$ is a 6-cycle. Let $\omega$ be a weight on $\Gamma(\gamma)$ over $\mathbb{Z}$ defined by $\omega(\gamma) = 1$. Then it is not hard to see that $\omega_{3,3}$ is totally balanced and weakly balanced on any pair of adjacent edges of $K_{3,3}$. For a positive integer $m$, let $g_m$ be a spatial embedding of $G$ as illustrated in Figure 5.1 (2). Note that $g_i(K_{3,3})$ and $g_j(K_{3,3})$ are not vertex-homotopic for any $i \neq j$; namely $g_i$ and $g_j$ are not vertex-homotopic for any $i \neq j$. Since all of the 2-component constituent links of $g_m$ are algebraically split, by Theorem 2.3 (2) we have that $\beta_{\omega_{3,3}, \omega}(g)$ is a vertex-homotopy invariant of $g_m$. Moreover we can see that there exists exactly one 4-cycle $\gamma'$ of $K_{3,3}$ so that $L = g_m(\gamma \cup \gamma')$ is non-trivial. Since $L$ is the Whitehead link, we have that $\beta_{\omega_{3,3}, \omega}(g_m) = 1$. Therefore $g_m$ is non-splittable up to vertex-homotopy despite the fact that any of the constituent links of $g$ is link-homotopically trivial.

![Figure 5.1](image)

**Remark 5.5.** The $\mathbb{Z}_2$-valued invariant in Theorem 2.3 cannot always be extended to an integer-valued one. For example,

1. Let us consider the graph $G$ and the invariant $\beta_{\omega, \omega}$ as in Example 5.3. Let $f$ be a spatial embedding of $G$ as illustrated in Figure 5.2. We can see that $f$ is edge-homotopic to the trivial spatial embedding $h$ of $G$. But by a calculation we have that $\sum_{1 \leq i < j \leq 3} a_3(f(\gamma_{ij}), f(\gamma)) = -2$.

2. Let $G$ be a disjoint union of $\Theta_4$ and a circle $\gamma$. Let $\omega_p$ be a checkerboard weight on $\Gamma(\Theta_4)$ over $\mathbb{Z}$ as in Example 4.3. Note that the modulo two reduction of a checkerboard weight is totally balanced. So by Theorem 2.3 (1), the modulo two reduction of $\sum_{\gamma_{ij} \in \Gamma(\Theta_4)} \omega_p(\gamma_{ij})a_3(f(\gamma_{ij} \cup \gamma))$ is an edge-homotopy invariant of a spatial embedding $f$ of $G$. Moreover, we can see that the integer-value $\sum_{\gamma_{ij} \in \Gamma(\Theta_4)} \omega_p(\gamma_{ij})a_3(f(\gamma_{ij} \cup \gamma))$ is invariant under the self crossing change on $f(\Theta_4)$ in the same way as in the proof of Theorem 2.2 (1). But this value may change under a self crossing change.
on $f(\gamma)$. For example, let $f$ and $g$ be two spatial embeddings of $G$ as illustrated in Figure 5.3. We can see that $f$ is edge-homotopic to $g$. But by a calculation we have that
\[
\sum_{\gamma_{ij} \in \Gamma(\Theta_4)} \omega_p(\gamma_{ij})a_3(f(\gamma_{ij}), f(\gamma)) = -1,
\]
\[
\sum_{\gamma_{ij} \in \Gamma(\Theta_4)} \omega_p(\gamma_{ij})a_3(g(\gamma_{ij}), g(\gamma)) = 1.
\]

6. APPLYING THE BOUNDARY OF A PLANAR SURFACE

Let $X$ be a disjoint union of a graph $G$ and a planar surface $F$ with boundary. Let $\omega$ be a weight on $\Gamma(G)$ over $\mathbb{Z}_2$ and $\varphi$ an embedding of $X$ into $S^3$ such that
\[
\omega(\gamma)\text{lk}(\varphi(\gamma), \varphi(\gamma')) = 0
\]
in $\mathbb{Z}$ for any $\gamma \in \Gamma(G)$ and $\gamma' \in \Gamma(\partial F)$. Then we define $\beta_{\omega}(\varphi) \in \mathbb{Z}_2$ by
\[
\beta_{\omega}(\varphi) = \sum_{\gamma \in \Gamma(G), \gamma' \in \Gamma(\partial F)} \omega(\gamma)a_3(\varphi(\gamma), \varphi(\gamma')) \pmod 2.
\]
Let $G$ be a disjoint union of a connected graph $G_1$ and a connected planar graph $G_2$. Let $f$ be a spatial embedding of $G$ and $p$ an embedding of $G_2$ into $S^2$. We denote the regular neighborhood of $p(G_2)$ in $S^2$ by $F(G_2; p)$, which is a planar surface having $p(G_2)$ as a spine. Then the spatial embedding $f$ induces an embedding $\tilde{f}_p$ of the disjoint union $G_1 \cup F(G_2; p)$ into $S^3$, so that $\tilde{f}_p(G_1) = f(G_1)$ and $\tilde{f}_p(F(G_2; p))$ has $f(G_2)$ as a spine in the natural way. Note that such an induced embedding $\tilde{f}_p$ is not unique up to ambient isotopy. Let $\omega$ be a weight on $\Gamma(G_1)$ over $\mathbb{Z}_2$ so that

$$\omega(\gamma)\text{lk}(\tilde{f}_p(\gamma), \tilde{f}_p(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(\partial F(G_2; p))$. Then we have the following.

**Theorem 6.1.** If $f$ is split up to edge-homotopy, then $\beta_\omega(\tilde{f}_p) = 0$ for any induced embedding $\tilde{f}_p$ of $G_1 \cup F(G_2; p)$.

**Proof.** By the assumption we have that $f$ is transformed into a split spatial embedding $\tilde{u}$ of $G$ by self crossing changes and ambient isotopies. Then each of the self crossing changes induces a self crossing change on $\tilde{f}_p(G_1)$ or a band-pass move \[0\] (see Figure 6.1) on $\tilde{f}_p(F(G_2; p))$. Namely $\tilde{f}_p$ can be transformed into an induced embedding $\tilde{u}_p$ of $G_1 \cup F(G_2; p)$ by such moves and ambient isotopies. Let $\tilde{g}_p$ be an embedding of $G_1 \cup F(G_2; p)$ into $S^3$ obtained from $\tilde{f}_p$ by a single self crossing change on $\tilde{f}_p(G_1)$ or a single band-pass move on $\tilde{f}_p(F(G_2; p))$. Then it still holds that

$$\omega(\gamma)\text{lk}(\tilde{g}_p(\gamma), \tilde{g}_p(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(\partial F(G_2; p))$.

**Claim.** $\beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{g}_p)$.

Assume that $\tilde{g}_p$ is obtained from $\tilde{f}_p$ by a single self crossing change on $\tilde{f}_p(G_1)$. Since it holds that

$$\sum_{\gamma' \in \Gamma(\partial F(G_2; p))} [\gamma'] = 0$$

in $H_1(F(G_2; p); \mathbb{Z}_2)$, we can see that $\beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{g}_p)$ in a similar way as the proof of Theorem 2.3 (1). Next we assume that $\tilde{g}_p$ is obtained from $\tilde{f}_p$ by a single band-pass move on $\tilde{f}_p(F(G_2; p))$. Then $\tilde{g}_p|_{G_1 \cup \partial F(G_2; p)}$ is obtained from $\tilde{f}_p|_{G_1 \cup \partial F(G_2; p)}$ by a single pass move \[0\] (see Figure 6.1) on $\tilde{f}_p(\partial F(G_2; p))$. We divide our situation into the following two cases.

**Case 1.** Four strings in the pass move belong to $\tilde{f}_p(\gamma_1')$ and $\tilde{f}_p(\gamma_2')$ for exactly two cycles $\gamma_1'$ and $\gamma_2'$ in $\Gamma(\partial F(G_2; p))$. This pass move causes a single self crossing change on $\tilde{f}_p(\gamma_1')$ and a single self crossing change on $\tilde{f}_p(\gamma_2')$. Then the separated components that result from smoothing each of the self crossings are orientation-reversing parallel knots; see Figure 6.2.

So the difference between $\beta_\omega(\tilde{f}_p)$ and $\beta_\omega(\tilde{g}_p)$ is cancelled out in a similar way as in the proof of Theorem 2.2 (1). Thus we have that $\beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{g}_p)$.

**Case 2.** Four strings in the pass move belong to $\tilde{f}_p(\gamma')$ for a cycle $\gamma'$ in $\Gamma(\partial F(G_2; p))$.

It is known that a pass move on the same component of a proper link $L = J_1 \cup J_2 \cup \cdots \cup J_n$ preserves $\overline{\text{Arf}}(L) \equiv \text{Arf}(L) - \sum_{i=1}^{n} \text{Arf}(J_i) \in \mathbb{Z}_2$ (cf. \[15\])\[3\].

\[3\] The value of $\overline{\text{Arf}}(L)$ is called the **reduced Arf invariant** of $L$ \[15\].
Especially, if $n = 2$, then $a_3(L) \equiv \text{Arf}(L) \pmod{2}$ [12, Lemma 3.5 (ii)]. Therefore in this case the pass move preserves $\omega(\gamma) a_3(\tilde{f}_p(\gamma), \tilde{f}_p(\gamma'))$ for any cycle $\gamma \in \Gamma(G_1)$. This implies that $\beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{u}_p)$. 

Now by the argument above, we have that $\beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{u}_p)$. Then, each 2-component link $\tilde{u}_p(\gamma \cup \gamma')$ is split for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(\partial F(G_2; p))$ because $u$ is split. Therefore we have that $\beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{u}_p) = 0$. This completes the proof. □

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{band-pass_move}
\caption{band-pass move}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{pass_move}
\caption{pass move}
\end{figure}

\textbf{Example 6.2.} Let $G$ be a disjoint union of a circle $\gamma$ and the handcuff graph (resp. 2-bouquet) $G_2$. Let $\omega$ be a weight on $\Gamma(\gamma)$ over $\mathbb{Z}_2$ defined by $\omega(\gamma) = 1$. We fix an embedding $p : G_2 \to S^2$ and take a regular neighborhood $F(G_2; p)$ as illustrated in Figure 6.3 (1) (resp. (2)).

Let $f$ be a spatial embedding of $G$ as illustrated in Figure 2.1 (1) (resp. (2)). Let us take an induced embedding $\tilde{f}_p : \gamma \cup F(G_2; p) \to S^3$ as illustrated in Figure 6.4 (1) (resp. (2)). Note that $\text{lk}(\tilde{f}_p(\gamma), \tilde{f}_p(\gamma')) = 0$ for any $\gamma' \in \Gamma(\partial F(G_2; p))$. Then it can be calculated that $\beta_\omega(\tilde{f}_p) = 1$. Thus by Theorem 6.1 we have that $f$ is non-splittable up to edge-homotopy.

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REFERENCES


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