ON ZEROS OF SOME ENTIRE FUNCTIONS

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Abstract. We study the distribution of zeros $z_k$ for entire functions of the form $\sin z + \int_0^1 f(t) e^{iz(1-2t)} \, dt$ with $f$ belonging to a space $X \hookrightarrow L_1(0,1)$. For a large class $\mathcal{F}$ of spaces $X$ (including, e.g., the spaces $L_p(0,1)$ for all $p \in [1, \infty]$) we show that $z_k = \pi k + \zeta_k$, where $(\zeta_k)_{k \in \mathbb{Z}}$ is the sequence of Fourier coefficients for some function $g$ in $X$, and study properties of the induced mapping $g \mapsto f$.

1. Introduction

The aim of this paper is to study the distribution of zeros for a class of entire functions of the form

$$ F(z) = F_f(z) := \sin z + \int_0^1 f(t) e^{iz(1-2t)} \, dt, $$

where $f$ is a function integrable over $(0,1)$. Such a class consists of functions of the so-called sine type [12, Lect. 22]. Moreover, to this form also reduce the functions

$$ \tilde{F}(z) = m_- e^{-iz} + m_+ e^{iz} + \int_{-1}^1 \tilde{f}(t) e^{izt} \, dt, \quad m_- m_+ \neq 0, $$

which are the Fourier–Stieltjes transforms [7, Ch. 12.5] of the measures

$$ m_- \delta_{-1} + m_+ \delta_1 + \tilde{f}(t) \, dt $$

on $[-1,1]$, with $\delta_x$ being the Dirac measure at the point $x$ and $\tilde{f} \in L_1(-1,1)$. Indeed, taking $\alpha := \frac{\pi}{2} + \frac{1}{21} \log(m_-/m_+)$ and $c := -2\sqrt{m_- m_+}$, we get by direct calculations the equality $\tilde{F}(z + \alpha) = cF_f(z)$ for a suitable $f \in L_1(0,1)$.

Asymptotic distribution of zeros for Fourier–Stieltjes transforms is of key significance in many areas of function theory, harmonic analysis, functional analysis, etc. and has been studied for many particular situations; see, e.g., [10,11,13]. Our interest in this topic stems from the spectral theory of Sturm–Liouville and Dirac operators, since their characteristic functions usually are of the above type; e.g., the case $f \in W^s_2(0,1)$ appears in [13], $f \in W^s_2(0,1)$ in [16, 17], $f \in W^s_2(0,1)$ with $s \in [0,1]$ in [12], $f \in L_p(0,1)$ with $p \in [1, \infty]$ in [3,4], and $f \in BV[0,1]$ in [20].

By a refined version of the Riemann–Lebesgue lemma (cf. [16, Lemma 1.3.1]), the integral term in (1.1) is $o(e^{\text{Im} z})$ as $|z| \to \infty$, and Rouché’s theorem shows that the functions $F_f$ and $\sin z$ have the same number of zeros in the discs $|z| \leq \pi (l + \frac{1}{2})$ for all sufficiently large $l \in \mathbb{N}$ and that the zeros of $F_f$ approach those of $\sin z$ for
large $|z|$. Therefore the zeros of $F_f$ can (and always will) be enumerated by $z_n$, $n \in \mathbb{Z}$, in such a way that $z_n = \pi n + \zeta_n$ with $\zeta_n = o(1)$ as $n \to \pm \infty$.

We would like to study dependence between the decay rate of the remainders $\zeta_n$ and the regularity of the function $f$. An example of such an interrelation can be found in the paper by Levin and Ostrovski˘ı [13], namely:

**Theorem A.** Assume that $f \in L_p(0, 1)$ with $p \in (1, 2)$ and let $z_n = \pi n + \zeta_n$ be zeros of the function $F_f$ in (1.1). Then the sequence $(\zeta_n)_{n \in \mathbb{Z}}$ belongs to $\ell_q$ with $q := p/(p-1)$.

It is interesting to know whether the statement of Theorem A can be reverted, i.e., whether for any sequence $(\zeta_n)_{n \in \mathbb{Z}}$ from $\ell_q$ the numbers $\pi n + c_n$ are zeros (of respective multiplicities) of a function $F_f$ in (1.1) with some $f \in L_p(0, 1)$. Further related questions are about the cases when $f \in L_p(0, 1)$ with $p = 1$ or $p > 2$ and, more generally, when the function $f$ belongs to other spaces embedded into $L_1(0, 1)$, as in the above-mentioned examples from the spectral theory.

Our results describe explicitly zero sequences of functions $F_f$ when $f$ runs through $L_p(0, 1)$, $p \in [1, \infty]$, or through some other function spaces and, in particular, show that the statement of Theorem A can be neither reverted nor extended to $p > 2$. In fact, we prove that the remainders $\zeta_n = z_n - \pi n$ of the zeros $z_n$ of the function $F_f$ with $f \in L_p(0, 1)$ form a sequence of Fourier coefficients of some function $g \in L_p(0, 1)$. And conversely, for every $g \in L_p(0, 1)$ the sequence $(\pi n + \hat{g}(n))_{n \in \mathbb{Z}}$, $\hat{g}(n)$ denoting the $n$th Fourier coefficient of $g$, is a zero sequence for a function $F_f$, with a unique $f \in L_p(0, 1)$. Since by the Hausdorff–Young theorem [7, Sect. 13.5] the sequences of Fourier coefficients of functions in $L_p(0, 1)$ form a proper subset of $\ell_q$, $q := p/(p-1)$, if $p \in (1, 2)$ and do not form a subset of $\ell_q$ if $p > 2$, the above claim about Theorem A follows.

The main results of the paper are formulated in terms of the properties of the induced mapping $\phi : g \mapsto f$. We first prove that $\phi$ maps $L_1(0, 1)$ surjectively and analytically onto $L_1(0, 1)$ and is a local analytic homeomorphism when restricted to an open dense subset $\Omega$ of $L_1(0, 1)$ consisting of those $g$ for which the function $F_{\phi(g)}$ has no multiple zeros. The same results are shown to hold for a rich class $\mathcal{X}$ of Banach spaces $X$ that are continuously embedded into $L_1(0, 1)$ (among them $L_p(0, 1)$ for $p \in (1, \infty]$); see Corollary 2.6. Our approach consists of constructing an analytic function

$$\Phi : L_1(0, 1) \times L_1(0, 1) \to L_1(0, 1)$$

such that the relation $\phi(g) = f$ is equivalent to $\Phi(f, g) = 0$ if $g$ belongs to $\Omega$. It turns out that the partial derivatives $\partial_f \Phi$ and $\partial_g \Phi$ are boundedly invertible if and only if all zeros of the function $F_f$ are simple, i.e., if and only if $g \in \Omega$, and the implicit function theorem then gives the properties of $\phi$.

The paper is organized as follows. In the next section some definitions are given and the main results (Theorems 2.2 and 2.4 and Corollary 2.5) are formulated. Some auxiliary statements are proved in Section 3. In Section 4 we introduce the mappings $\phi$ and $\Phi$ and establish some of their properties. The final two sections are devoted to the proofs of Theorems 2.2 and 2.4 and Corollary 2.5.

**Notation.** Throughout the paper we abbreviate $L_1(0, 1)$ with $L_1$. For a Banach space $X$, we denote by $\mathcal{B}(X)$ the algebra of all linear continuous operators in $X$. 

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The space $L_1$ will be regarded as a Banach algebra with respect to the convolution
\[ (f * g)(x) := \int_0^1 f(x - t)g(t) \, dt, \]
where $f$ is continued onto $(-1, 1)$ to a periodic function of period 1. We denote by $e_n$, $n \in \mathbb{Z}$, the functions $e_n(t) = e^{2\pi int}$, $t \in [0, 1]$, and write $\hat{f}$ for the Fourier transform of a function $f \in L_1$, viz.
\[ \hat{f}(n) := \int_0^1 f(t)e^{-2\pi int} \, dt. \]
Occasionally we shall also use functions defined over $(-1, 1)$; if $h \in L_1(-1, 1)$, then $\hat{h}$ will stand for its Fourier transform over $(-1, 1)$, i.e.,
\[ \hat{h}(n) = \int_{-1}^{1} h(t)e^{-2\pi int} \, dt. \]
It will be clear from the context as to which interval the function under consideration is defined, so that no confusion should arise. We note that the classes of Fourier transforms of functions in $L_1$ and $L_1(-1, 1)$ coincide; indeed, if $f \in L_1$ and $h \in L_1(-1, 1)$ are related via
\[ h(t) := \begin{cases} \frac{1}{2}f\left(\frac{t}{2} + 1\right), & \text{if } t < 0, \\ \frac{1}{2}f\left(\frac{t}{2}\right), & \text{if } t \geq 0, \end{cases} \]
then $f$ and $h$ have the same Fourier transform.

For a sequence $\Lambda = (\lambda_k)_{n \in \mathbb{Z}}$ of complex numbers we denote by $\mathcal{S}_\Lambda$ the system of exponentials $(e^{i\lambda_k t})_{n \in \mathbb{Z}}$; as usual, for every $\omega$ that is repeated $l > 1$ times among $\lambda_n$, we replace the repeated occurrences of $e^{i\omega t}$ with the functions $te^{i\omega t}, \ldots, t^{l-1}e^{i\omega t}$.

Finally, V.p. indicates that the corresponding summation or multiplication over the index set $\mathbb{Z}$ is taken in the sense of the principal value.

2. The main results

A sequence $\Lambda := (\lambda_n)_{n \in \mathbb{Z}}$ is called the zero sequence of an entire function $F$ if every $\lambda_n$ is a zero of $F$ and every zero $\lambda$ of $F$ of order $n(\lambda)$ is repeated in $\Lambda$ exactly $n(\lambda)$ times.

**Definition 2.1.** For a function $g \in L_1$, we denote by $\Lambda_g$ the sequence $(\lambda_n)_{n \in \mathbb{Z}}$ with $\lambda_n = \lambda_n(g) := \pi n + \hat{g}(n)$. We also denote by $\Omega$ the subset of $L_1$ consisting of all $g \in L_1$, for which the sequence $\Lambda_g$ has no repeating elements.

For $g \in L_1$, we denote by $G_g$ the entire function given by the product
\[ G_g(z) := (\lambda_0(g) - z) \lim_{n \to \infty} \prod_{0 < k \leq n} \frac{z - \lambda_k(g)}{\pi k}. \]
The above product converges uniformly on compact sets (see Lemma 3.1 below) and thus determines an entire function. It turns out that this entire function has the form (1.1) for some $f \in L_1$, and, moreover, a function $f$ with this property is unique. In this way we get the mapping $\phi$ such that
\[ L_1 \ni g \mapsto \phi(g) := f \in L_1. \]
We observe that φ cannot be univalent; indeed, different enumerations of zeros of $F_f$ lead to different $g$.

One of the main results of the paper is the following theorem.

**Theorem 2.2.** The mapping $\phi : L_1 \to L_1$ is analytic and surjective, and its restriction $\phi|_\Omega$ onto the set $\Omega$ is a local analytic homeomorphism on $\Omega$.

We recall that a mapping $\psi : U \to V$ between subsets $U$ and $V$ of some Banach spaces is a local analytic homeomorphism at a point $u$ if there is a neighbourhood $U'$ of $u$ and a neighbourhood $V'$ of $\psi(u)$ such that $\psi|_{U'} : U' \to V'$ is bijective, analytic, and homeomorphic on $U'$. This mapping $\psi$ is a local analytic homeomorphism on $U$ if it is such at every point $u \in U$. We refer the reader to [8] for the basic properties of analytic mappings between Banach spaces.

Theorem 2.2 shows that, for any $f \in L_1$, the remainders $\zeta_n := z_n - \pi n$ of the zeros $z_n$ of the function $F_f$ are Fourier coefficients of some function $g \in L_1$ and that, conversely, the sequence $(\pi n + \hat{g}(n))_{n \in \mathbb{Z}}$ for any $g \in L_1$ is the zero sequence for a function $F_f$ with some $f \in L_1$. Next we define the class $\mathcal{X}$ of Banach spaces having similar properties.

**Definition 2.3.** We denote by $\mathcal{X}$ the set of all Banach spaces $X$ that are continuously embedded into $L_1$ and satisfy the properties

(i) $\phi(X) = X$, and the mapping $\phi|_X : X \to X$ is analytic;

(ii) the mapping $\phi|_X$ is a local analytic homeomorphism on $\Omega \cap X$.

It turns out that the class $\mathcal{X}$ contains many classical function spaces. They can be constructed from a given $X \in \mathcal{X}$ using the following statement.

**Theorem 2.4.** Assume that $X \in \mathcal{X}$ and that the operator $M$ of multiplication by $i(1 - 2\pi)$ is continuous in $X$. Let $Y$ be a Banach space that is continuously embedded into $X$; then $Y$ belongs to $\mathcal{X}$ if the following conditions (A1) and one of (A2) and (A2') hold:

(A1) the mapping $(f, g) \mapsto f * g$ is continuous from $X \times Y$ into $Y$ and, moreover, for a fixed $f_0 \in X$ and a fixed $g_0 \in Y$ the linear operators

$$Y \ni g \mapsto f_0 * g \in Y, \quad X \ni f \mapsto f * g_0 \in Y$$

are compact;

(A2) for some $m \in \mathbb{N}$, the $m$-fold convolution

$$X^{(m)} := \{f_1 * f_2 * \cdots * f_m \mid f_1, f_2, \ldots, f_m \in X\}$$

of $X$ is contained in $Y$;

(A2') the system $\{e_n\}_{n \in \mathbb{Z}} \cap X$ is complete in $X$, and the system $\{e_n\}_{n \in \mathbb{Z}} \cap Y$ is complete in $Y$.

Some sufficient conditions under which (A1) holds are stated in Lemma [6,1].

**Corollary 2.5.** The following spaces belong to the class $\mathcal{X}$:

(i) $L_p(0, 1)$ for $p \in [1, \infty]$;

(ii) the space $C[0, 1]$ of continuous functions over $[0, 1]$;

(iii) the Hölder spaces $C^\alpha_{\text{per}}[0, 1]$, $\alpha \geq 0$, of periodic functions;

(iv) the Sobolev spaces $W^s_p(0, 1)$, $p \in [1, \infty)$, $s \geq 0$;
(v) the space $BV(0, 1)$ of functions of bounded total variation over $[0, 1]$;

(vi) the closed linear span $C_B$ in $C[0, 1]$ of the set $\{e_n \mid n \in B\}$, for every subset $B \subset \mathbb{Z}$.

The above examples demonstrate to some extent the diversity of the function spaces in $\mathcal{D}$.

3. Preliminaries

For an arbitrary $g \in L_1$, we construct the infinite product $G_g$ of (2.1) and would like to prove that $G_g = F_f$ for some $f \in L_1$. We show in this section that if such a function $f$ exists, then it is unique, so that the above relation determines a mapping $\phi : g \mapsto f$. Put as before $\zeta_n := \hat{g}(n)$ and $z_n = \pi n + \zeta_n$, $n \in \mathbb{Z}$.

**Lemma 3.1.** The product

$$
(z - z_0)V.p. \prod_{n = -\infty}^{\infty} \frac{\pi n - z}{\pi n} = G_g(z),
$$

the prime denoting that the factor corresponding to $n = 0$ is omitted, converges uniformly on compact sets and satisfies the relation

$$
\lim_{y \to \pm \infty} \frac{G_g(iy)}{\sin iy} = 1.
$$

**Proof:** That the product converges uniformly on compact sets follows from the same convergence of the series $V.p. \sum_{n \in \mathbb{Z}} (\zeta_n - z)/(\pi n)$; see [7, 10.1.5]. Using the canonical product for $\sin z$, we find that

$$
\frac{G_g(z)}{\sin z} = \left(1 - \frac{z_0}{z}\right)V.p. \prod_{n = -\infty}^{\infty} \left(1 + \frac{\zeta_n}{\pi n - z}\right),
$$

and it remains to prove that the function

$$
\tilde{G}(iy) := V.p. \prod_{n = -\infty}^{\infty} \left(1 + \frac{\zeta_n}{\pi n - iy}\right)
$$

satisfies the relation $\lim_{y \to \pm \infty} \tilde{G}(iy) = 1$.

To this end it suffices to show that the sum of the series

$$
V.p. \sum_{n = -\infty}^{\infty} \frac{\zeta_n}{\pi n - iy}
$$

vanishes as $y \in \mathbb{R}$ tends to $\pm \infty$. We observe that $1/(\pi n - iy)$ for $y \neq 0$ is the $n$th Fourier coefficient of the function $u_y(t)$ that equals $\exp(1 - 2it)/\sinh y$ on $[0, 1]$ and is periodically continued onto $\mathbb{R}$. Therefore $\zeta_n/(\pi n - iy)$ is the $n$th Fourier coefficient of the convolution $w_y := u_y \ast g$. The function $u_y$ is of bounded variation over $[0, 1]$, whence by [7, 3.1.5] such is $w_y$. It follows [7, 10.1.4] that the Fourier series of $w_y$ converges pointwise to $\frac{1}{2}[w_y(\cdot - 0) + w_y(\cdot + 0)]$; in particular, we have

$$
V.p. \sum_{n = -\infty}^{\infty} \frac{\zeta_n}{\pi n - iy} = \frac{1}{2}[w_y(0) + w_y(1)] = \int_0^1 u_y(1 - t)g(t) dt.
$$

Since $|u_y(1 - t)| \leq \exp y/\sinh y |y| \leq 4$ for all $t \in (0, 1)$ and all $y \in \mathbb{R} \setminus (-1, 1)$ and since $u_y(t)$ go to zero as $|y| \to \infty$ for all $t \in (0, 1)$, the Lebesgue dominated
convergence theorem shows that the integral in (3.1) vanishes as $|y| \to \infty$. The proof is complete. □

We next show that $F_f$ must coincide with $G_g$ if both have the same zeros.

**Lemma 3.2.** Assume that a sequence $(\pi n + \hat{g}(n))_{n \in \mathbb{Z}}$ is the zero sequence for a function $F_f$ with $f \in L_1$. Then $F_f = G_g$.

**Proof.** We assume that none of $z_n := \pi n + \hat{g}(n)$ vanishes; the changes to be made otherwise are straightforward. Being of exponential type 1, the function $F_f$ admits a representation as the Hadamard canonical product, i.e.,

$$F_f(z) = e^{az+b} \text{ V.p. } \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{z_n} \right) e^{z/z_n}$$

with some constants $a$ and $b$. Observe that the series

$$\text{V.p. } \sum_{n=-\infty}^{\infty} \frac{1}{z_n} = \frac{1}{z_0} + \sum_{n=1}^{\infty} \frac{\hat{g}(n) + \hat{g}(-n)}{z_n z_n}$$

converges; hence the factors $e^{z/z_n}$ can be incorporated into $e^{az+b}$ by modifying $a$ accordingly.

Since the integral term in (1.1) is $o(e^{\text{Im} z})$ as $|z| \to \infty$ by [10, Lemma 1.3.1], we have

$$\lim_{y \to \pm \infty} \frac{F_f(iy)}{\sin iy} = 1.$$ 

On the other hand, for $z \in \mathbb{C} \setminus \pi \mathbb{Z}$,

$$\frac{F_f(z)}{\sin z} = e^{az+b} \left( \frac{1}{z} - \frac{1}{z_0} \right) \text{ V.p. } \prod_{n=-\infty}^{\infty} \left(1 + \frac{\hat{g}(n)}{\pi n - z} \right) \text{ V.p. } \prod_{n=-\infty}^{\infty} \left(1 - \frac{\hat{g}(n)}{z_n} \right).$$

Both products above converge due to the convergence of the series V.p.$\sum_{n=-\infty}^{\infty} \hat{g}(n)/n$; see [7, 10.1.5]. Recalling (3.2) and the proof of Lemma 3.1, we conclude that $a = 0$ and that

$$e^b = -z_0 \text{ V.p. } \prod_{n=-\infty}^{\infty} \frac{z_n}{\pi n}.$$ 

Substituting for $e^b$ in the canonical product for $F_f$, we get $F_f = G_g$ as stated. □

We next show that there is at most one $f \in L_1$ such that $F_f = G_g$ for a given $g \in L_1$; see Corollary 3.3. This uniqueness statement follows from Levinson’s theorems on closure $L_p(-1, 1)$ of a system of exponentials [14, Chap. 1] for the case where $f, g \in L_p$ with $p > 1$; however, these theorems are not applicable in the case $p = 1$. We recall that a system $\mathcal{S}$ of functions is called closed $L_p(-1, 1)$ if the only function $h \in L_p(-1, 1)$ such that

$$\int_{-1}^{1} h(t) f(t) \, dt = 0$$

for every $f \in \mathcal{S}$ is the identically zero function.

**Lemma 3.3.** Assume that $r \in L_1(-1, 1)$ and put $z_n := \pi n + \hat{r}(n)$ and $\Lambda := (z_n)_{n \in \mathbb{Z}}$. Then the system of exponentials $\mathcal{E}_\Lambda$ is closed $L_1(-1, 1)$.
Proof. Without loss of generality we may assume that none of $z_k$ vanishes. Indeed, by [14, Theorem VI] finitely many of $z_k$ may be arbitrarily changed without affecting the property of the system $E$ to be closed $L_1(-1,1)$.

Assume that the system $E$ is not closed $L_1(-1,1)$. Then there is $q \in L_1(-1,1)$ that is not identically zero and such that the function

$$Q(z) := \int_{-1}^{1} q(t) e^{izt} \, dt$$

has zeros $z_n, n \in \mathbb{Z}$, counting multiplicities. Observe that the numbers $z_n$ then give all the zeros of the entire function $Q$. Indeed, we clearly have

$$|Q(re^{i\theta})| \leq \max_{-1 \leq t \leq 1} \left| \exp\{ire^{i\theta}t\}\right| \int_{-1}^{1} |q(t)| \, dt = e^{r|\sin \theta|} \int_{-1}^{1} |q(t)| \, dt,$$

and Jensen’s theorem leads to the inequality

$$(3.3) \quad \int_{0}^{r} n(t) \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |Q(re^{i\theta})| \, d\theta - \log |Q(0)|$$

$$\leq \frac{r}{2\pi} \int_{-\pi}^{\pi} |\sin \theta| \, d\theta + C_1 = \frac{2r}{\pi} + C_1,$$

where $n(r)$ is the number of zeros of $Q$ in the disc $\{z \in \mathbb{C} \mid |z| \leq r\}$ and $C_1$ is some constant. If, however, $Q$ had at least one additional zero, then for any $\varepsilon > 0$ and all sufficiently large $k$ (say, for $k \geq m$) we would have the estimate

$$n(\pi k + \pi \varepsilon) \geq 2k + 2.$$

Setting $t_k := \pi(k + \varepsilon)$ and using Stirling’s approximation of the Euler gamma-function [11, Ch. 6], we then find that

$$\int_{t_m}^{t_{m+1}} \frac{n(t)}{t} \, dt \geq \sum_{l=m}^{k} (2l + 2) \log \frac{t_{l+1}}{t_l}$$

$$= (2k + 2) \log t_{k+1} - 2 \sum_{l=m}^{k} \log t_l - 2m \log t_m$$

$$= (2k + 2) \log(t_{k+1}/\pi) - 2 \log \Gamma(t_{k+1}/\pi) + C_2$$

$$\geq \frac{2t_{k+1}}{\pi} + (1 - 2\varepsilon) \log t_{k+1} + C_3$$

for some real $C_2$ and $C_3$ independent of $k$, which contradicts (3.3). (Alternatively, we could then apply an analogue of Theorem III from [14] for exponentials $e^{i\lambda_k t}$ with complex $\lambda_k$ to contradict the assumption that $q$ is not identically zero.)

Repeating the arguments of the proof of Lemma 3.2 we represent $Q$ in the form

$$Q(z) = e^{az+b} \text{V.p.} \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{z_n}\right)$$

with some constants $a$ and $b$ and then show that

$$\lim_{y \to \pm \infty} \frac{Q(iy)e^{-iay-b}}{\sin(iy)} = -\frac{1}{z_0} \text{V.p.} \prod_{n=-\infty}^{\infty} \left(1 - \frac{r(n)}{z_n}\right) \neq 0.$$

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On the other hand, the integral representation of $Q$ implies the estimate $|Q(z)| = o(e^{\text{Im} z})$ as $|z| \to \infty$, which is inconsistent with at least one of the above limits. The contradiction derived shows that $q = 0$ and that the system $\mathcal{E}_\Lambda$ is closed $L_1(-1, 1)$.

\begin{corollary}
Assume that $g \in L_1$; then there exists at most one $f \in L_1$ such that $\Lambda_g$ is the zero sequence of the entire function $F_f$ of (1.1).
\end{corollary}

**Proof.** Assume that there are $f, \tilde{f} \in L_1$ such that the functions $F_f$ and $F_{\tilde{f}}$ vanish at all the points $z_n := \pi n + \hat{g}(n)$, $n \in \mathbb{Z}$, together with their derivatives of respective orders at the multiple points of $\Lambda_g$. We put $h := f - \tilde{f}$ and $H(z) := \int_0^1 h(t)e^{iz(1-2t)} \, dt$ and observe that $\Lambda_g$ is the zero sequence of $H$. Writing $H(z)$ as $\int_0^1 q(t)e^{itz} \, dt$ with $q(t) := \frac{1}{2}h(\frac{1-2}{2})$ and applying Lemma 3.3, we conclude that $q = 0$, i.e., that $f = \tilde{f}$.

\section{4. Properties of the mappings $\phi$ and $\Phi$}

The results of the previous section show that the mapping $\phi : L_1 \to L_1$ given by the relation $\phi(g) = f$ if $F_f = G_g$ on the domain

$$\text{dom} \phi = \{g \in L_1 \mid \exists f \in L_1, F_f = G_g\}$$

is well defined. To derive its properties, we shall show that there exists an analytic map $\Phi : L_1 \times L_1 \to L_1$ such that the following holds:

(a) for every $f$ and $g$ in $L_1$, the equality $\phi(g) = f$ implies that $\Phi(f, g) = 0$;

(b) for every $f \in L_1$ and every $g \in \Omega$, the equality $\Phi(f, g) = 0$ yields $\phi(g) = f$.

Assume that $f, g \in L_1$ and that $\phi(g) = f$, i.e.,

$$F_f(\pi n + \zeta_n) = 0, \quad \forall n \in \mathbb{Z},$$

where $\zeta_n := \hat{g}(n)$. Taking into account (1.1) and the equality

$$e^{i\zeta_n(1-2t)} = \sum_{k=0}^{\infty} \frac{|i\zeta_n(1-2t)|^k}{k!},$$

relation (1.1) can be recast as

$$\sin \zeta_n + \sum_{k=0}^{\infty} \frac{\zeta_n^k}{k!} \int_0^1 [i(1-2t)]^k f(t)e^{-2\pi int} \, dt = 0.$$

We denote by $g^{(k)}$ the $k$-fold convolution of $g$ with itself, with $g^{(0)}$ being equal to the unity $\delta_0$ of the convolution algebra $L_1$ and $g^{(1)} = g$, and put

$$s(g) := \sum_{k=0}^{\infty} \frac{(-1)^k g^{(2k+1)}}{(2k+1)!}.$$

Well-known properties of the convolution yield the relations

$$\sin \zeta_n = \widehat{s(g)}(n), \quad \zeta_n^k = \widehat{g^{(k)}}(n), \quad n \in \mathbb{Z},$$

so that the inverse Fourier transform takes (4.2) into the relation

$$s(g) + f + \sum_{k=1}^{\infty} \frac{(M^k f) \ast g^{(k)}}{k!} = 0,$$

where $(Mf)(x) := i(1-2x)f(x)$ is the operator of multiplication by $i(1-2x)$.  

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Assume now that the above equality holds for some \( f \in L_1 \) and \( g \in \Omega \). Reverting the arguments, we get then that, for every \( n \in \mathbb{Z} \),
\[
F_f(\pi n + \zeta_n) = 0.
\]
Since \( g \in \Omega \), all the numbers \( \lambda_n(g) = \pi n + \zeta_n \) are pairwise distinct. As in the proofs of Lemmas 3.2 and 3.3 it can be shown that \( F_f \) has no other zeros except \( \lambda_n(g) \), i.e., that \( \Lambda_g \) is the zero sequence for the function \( F_f \), so that \( F_f = G_g \).

We have thus shown that the mapping \( \Phi : L_1 \times L_1 \to L_1 \) given by the formula
\[
(4.3) \quad \Phi(f, g) := s(g) + f + \sum_{k=1}^{\infty} \frac{(M^k f) * g^{(k)}}{k!}
\]
satisfies properties (a) and (b). It remains to establish analyticity of \( \Phi \). It is easy to see that the Fréchet derivative \( d\Phi(f, g) \), which is a linear continuous operator from \( L_1 \times L_1 \) into \( L_1 \) \([8, \text{Ch. 2}]\), is equal to
\[
d\Phi(f, g)(h_1, h_2) = \partial_f \Phi(f, g)(h_1) + \partial_g \Phi(f, g)(h_2),
\]
where the partial derivatives \( \partial_f \Phi(f, g) \) and \( \partial_g \Phi(f, g) \) are linear continuous operators in \( L_1 \) and are given by the formulae
\[
(4.4) \quad \partial_f \Phi(f, g)(h_1) = h_1 + \sum_{k=1}^{\infty} \frac{(M^k f) h_1 * g^{(k)}}{k!},
\]
\[
(4.5) \quad \partial_g \Phi(f, g)(h_2) = \left( c(g) + \sum_{k=1}^{\infty} \frac{(M^k f) * g^{(k-1)}}{(k-1)!} \right) * h_2,
\]
with
\[
c(g) := \sum_{k=0}^{\infty} \frac{(-1)^k g^{(2k)}}{(2k)!}.
\]
Continuity of the convolution implies that the mappings
\[
L_1 \times L_1 \ni (f, g) \mapsto \partial_f \Phi(f, g) \in \mathcal{B}(L_1),
\]
\[
L_1 \times L_1 \ni (f, g) \mapsto \partial_g \Phi(f, g) \in \mathcal{B}(L_1)
\]
are continuous, and hence such is the mapping
\[
L_1 \times L_1 \ni (f, g) \mapsto d\Phi(f, g) \in \mathcal{B}(L_1 \times L_1, L_1).
\]
It follows \([8, \text{Ch. 2}]\) that the function \( \Phi \) is analytic.

Having in mind the use of the implicit function theorem, we have to find out when the partial derivatives of \( \Phi \) are invertible. Since, for an arbitrary \( h \in L_1 \), the operator
\[
f \mapsto f * h
\]
is compact in \( L_1 \), we derive from (1.2) and (1.3) that the operators
\[
\partial_f \Phi(f, g) - I, \quad \partial_g \Phi(f, g) - I
\]
are also compact. Henceforth the operators \( \partial_f \Phi(f, g) \) and \( \partial_g \Phi(f, g) \) are not invertible in \( L_1 \) if and only if their null-spaces are nontrivial \([8\text{, Theorem 4.25}]\).

Assume that some \( h_1 \in L_1 \setminus \{0\} \) verifies the relation \( \partial_f \Phi(f, g)(h_1) = 0 \). Taking the \( n \)th Fourier coefficient of (1.1), we find that
\[
(4.6) \quad 0 = \int_0^1 h_1(t) e^{i\lambda_n(g)(1-2t)} dt = \frac{1}{2} \int_{-1}^1 h_1 \left( \frac{1-s}{2} \right) e^{i\lambda_n(g)s} ds.
\]
Such a relation can hold for every $n \in \mathbb{Z}$ if and only if $g \in L_1 \setminus \Omega$. Indeed, for $g \in \Omega$ all $\lambda_n(g)$ are pairwise distinct, and since the system $\mathcal{E}_\lambda$ is closed $L_1(-1,1)$ by Lemma 3.3, we get $h_1 = 0$. Conversely, $g \in L_1 \setminus \Omega$ means that at least two among $\lambda_n(g)$ coincide, say $\lambda_n(g) = \lambda_m(g) = \omega$. Since by [5] Prop. II.4.3 the system $\mathcal{E}_\lambda$ forms a Riesz basis of $L_2(0,1)$, the system $\mathcal{E}_\lambda \setminus \{ t e^{i \omega t} \}$ is not closed $L_1(-1,1)$, and thus there exists a nonzero $h_1$ satisfying (4.6) for all $n \in \mathbb{Z}$. Therefore we have shown that $\Phi(f,g) = 0$ implies the equivalence

$$\partial_f \Phi(f,g) \text{ is invertible } \iff g \in \Omega.$$ 

Assume now that $f$ and $g$ in $L_1$ are such that $\Phi(f,g) = 0$ and $\partial_g \Phi(f,g)$ is noninvertible, i.e., that $\partial_g \Phi(f,g)(h_2) = 0$ for some $h_2 \in L_1 \setminus \{0\}$. It follows that $F_f(\lambda_n(g)) = 0$ for every $n \in \mathbb{Z}$. Using formula (3.4) and equating to zero the $n$th Fourier coefficient of the function $\partial_g \Phi(f,g)(h_2)$, we arrive at the relation

$$F'_f(\lambda_n(g)) \hat{h}_2(n) = 0.$$ 

Since at least one Fourier coefficient of $h_2$ does not vanish, we conclude that at least one of $\lambda_n(g)$ is a multiple zero of $F_f$, i.e., that $g \in L_1 \setminus \Omega$. Conversely, if $\lambda_n(g)$ is a multiple zero of $F_f$, then $\partial_g \Phi(f,g)(h_2) = 0$ for $h_2(t) := e^{2 \pi i n t}$.

Therefore if some $f$ and $g$ in $L_1$ verify the equality $\Phi(f,g) = 0$, then the partial derivatives $\partial_f \Phi(f,g)$ and $\partial_g \Phi(f,g)$ are invertible if and only if $g \in \Omega$. The implicit function theorem now yields the following statement.

**Proposition 4.1.** Assume that $f_0 \in L_1$, $g_0 \in \Omega$, and $\Phi(f_0,g_0) = 0$. Then there exist an open neighbourhood $U$ of the point $f_0$, an open neighbourhood $W$ of the point $g_0$, and a homeomorphism $\phi_W : W \to U$ such that $\Phi(\phi_W(g),g) = 0$ for every $g \in W$, and the mappings $\phi_W : W \to U$ and $\phi_W^{-1} : U \to W$ are analytic.

It is clear that the mapping $\phi$ is well defined in $W$ and coincides therein with $\phi_W$. Since $0 \in \Omega$ and $\Phi(0,0) = 0$, we conclude from Proposition 4.1 that there is $\delta \in (0,1)$ such that $\phi$ is well defined in the ball

$$B_\delta = \{ g \in L_1 \mid \| g \|_{L_1} < \delta \}$$

of radius $\delta$. Next we shall show that $\text{dom } \phi = L_1$.

We fix $l \in \mathbb{N}$ and $a = (a_{-l}, \ldots, a_l) \in \mathbb{C}^{2l+1}$ and show that $\phi$ is well defined and analytic in the $\delta$-neighbourhood of the trigonometric polynomial

$$P_a(t) := \sum_{|k| \leq l} a_k e^{2\pi ik t}.$$ 

**Lemma 4.2.** For any $l \in \mathbb{N}$ and any $a \in \mathbb{C}^{2l+1}$ the set $P_a + B_\delta$ belongs to the domain of $\phi$ and there exist analytic functions

$$\mathbb{C}^{2l+1} \ni \lambda \mapsto h_l(\lambda) \in L_1, \quad \mathbb{C}^{2l+1} \ni \lambda \mapsto A_l(\lambda) \in \mathcal{B}(L_1)$$

such that

$$\phi(g + P_a) = \phi(g) + h_l(\lambda(g)) + A_l(\lambda(g)) \phi(g)$$

for all $g \in B_\delta$, with $\lambda(g) := (\lambda_k(g))_{k=-l}^{l} \in \mathbb{C}^{2l+1}$.

**Proof.** We shall start with the following remark. Fix $b \in \mathbb{C}$ and let $f \in L_1$ and $\zeta \in \mathbb{C}$ be such that $F_f(\zeta) = 0$. As in [11] p. 535 and [14] p. 10 it can be shown that there is $f \in L_1$ such that

$$F_f(z) = \frac{z - \zeta - b}{z - \zeta} F_f(z), \quad z \in \mathbb{C}. $$
The mapping \( f \mapsto \tilde{f} \) is affine and depends only on \( \zeta \); moreover, this dependence is analytic. In fact, a straightforward calculation gives that
\[
\tilde{f} = f - b(h(\zeta) + A(\zeta)f),
\]
where
\[
[h(\zeta)](t) := e^{i\zeta r(t)}, \quad r(t) := \text{sgn} (1 - 2t) - (1 - 2t), \quad t \in [0, 1],
\]
and
\[
[A(\zeta)f](t) = 2i \int_{a(t)}^{t} f(s)e^{2i\zeta(t-s)} \, ds,
\]
with \( a(t) = 0 \) if \( t < \frac{1}{2} \) and \( a(t) = 1 \) otherwise. It is clear that the mappings
\[
\mathbb{C} \ni \zeta \mapsto h(\zeta) \in L_1, \quad \mathbb{C} \ni \zeta \mapsto A(\zeta) \in \mathcal{B}(L_1)
\]
are entire.

Assume now that \( f \in L_1 \) and \( \lambda = (\lambda_k)_{k=-l}^{l} \in \mathbb{C}^{2l+1} \) are such that the numbers \( \lambda_k \) are pairwise distinct and \( F_f(\lambda_k) = 0 \) for every \( k = -l, \ldots, l \). Fixing \( a = (a_{-l}, \ldots, a_l) \in \mathbb{C}^{2l+1} \) and applying the above transformation \( 2l + 1 \) times, we find that there is \( f \) such that
\[
F_f(z) = \left( \prod_{|k| \leq l} \frac{z - \lambda_k - a_k}{z - \lambda_k} \right) F_f(z), \quad z \in \mathbb{C}.
\]
It is easy to see that \( f \mapsto \tilde{f} \), being the composition of \( 2l + 1 \) affine mappings, is affine itself and has the form
\[
\tilde{f} = f + h_l(\lambda) + A_l(\lambda)f,
\]
with analytic functions \( h_l \) and \( A_l \) from \( \mathbb{C}^{2l+1} \) into \( L_1 \) and \( \mathcal{B}(L_1) \), respectively.

Now let \( g \in B_1 \), \( f = \phi(g) \), and \( \lambda(g) = (\lambda_k(g))_{k=-l}^{l} \). As \( \delta \) is less than 1, the numbers \( \lambda_k(g) \) are pairwise distinct. The above arguments show that
\[
\phi(g + P_a) = \phi(g) + h_l(\lambda(g)) + A_l(\lambda(g))\phi(g),
\]
and the lemma is proved. \( \square \)

Since the set of all trigonometric polynomials is dense in \( L_1 \), it follows from Lemma 4.2 that \( \text{dom} \phi = L_1 \) and that \( \phi \) is an analytic mapping on the whole space \( L_1 \).

5. Proof of Theorem 2.2

In view of the results of the previous section it suffices to prove that \( \phi \) is surjective.

We fix an arbitrary \( f \in L_1 \) and denote by \( \Lambda = (z_n)_{n \in \mathbb{Z}} \) the sequence of zeros of the function \( F_f \) numbered so that
\[
z_n = \pi n + o(1), \quad |n| \to \infty.
\]
Then the system of exponentials \( e_{\Lambda} \) forms a Riesz basis of \( L_2(-1, 1) \) [Prop. II.4.3]. Let \( (q_k)_{k \in \mathbb{Z}} \) be a biorthogonal basis. According to Proposition 4.1 there are \( \varepsilon > 0 \) and \( \delta \in (0, 1) \) such that \( B_{\varepsilon} \subset \phi(B_\delta) \). Since \( L_2(-1, 1) \) is dense in \( L_1(-1, 1) \), there exist \( n_0 \in \mathbb{N} \) and \( q \in L_2(-1, 1) \) of the form
\[
q = \sum_{|k| \leq n_0} c_k q_k
\]
It is clear that mapping from $Y$ we have $f$ and thus such is the composition of $\Psi$ and $g$.

$\phi$ (6.1) follows that the sequence $\Lambda_{g_0}$ is the sequence of zeros of the function $F_{f_0}$, i.e., that

$$F_{f_0}(\lambda_k(g_0)) = 0,$$  

$k \in \mathbb{Z}$.

We also notice that

$|\lambda_k(g_0) - \pi k| \leq \|g_0\|_{L_1} \leq \delta < 1$,  

$k \in \mathbb{Z}$.

It is clear that

$$F_{f_0}(z_k) = F_{f_0}(z_k) - F_f(z_k) = -2 \int_0^1 \bar{\eta}(1 - 2t) e^{itz_k(1 - 2t)} dt$$

$$= - \sum_{|l| \leq n_0} c_l \int_{-1}^1 e^{itz_k \eta(s)} ds.$$

Since the sequence $(q_k)_{k \in \mathbb{Z}}$ is biorthogonal to $\phi$, we conclude that $F_{f_0}(z_k) = 0$ for all $k \in \mathbb{Z}$ such that $|k| > n_0$; it follows from (5.2) and (5.3) that $z_k = \lambda_k(g_0)$ for such $k$. Now we set

$$g := g_0 + \sum_{|k| \leq n_0} (z_k - \lambda_k(g_0)) e_k$$

and conclude that $\lambda_k(g) = z_k$ for all $k \in \mathbb{Z}$, so that $F_f = G_g$, and $\phi$ is surjective. The proof of Theorem 2.4 is complete.

6. Proof of Theorem 2.4 and Corollary 2.5

Proof of Theorem 2.4. We shall first prove that $\phi(Y) \subset Y$ and that the restriction $\phi_Y$ of the mapping $\phi$ onto $Y$ is analytic in $Y$. As $X \in \mathcal{X}$, $\phi$ is analytic in $X$, and continuity of the embedding $Y \hookrightarrow X$ implies that $\phi$ is also analytic as a mapping from $Y$ into $X$.

Next, since $\Phi(\phi(g), g) = 0$ for all $g \in Y$, by (4.3) we get

$$\phi(g) = -s(g) - \sum_{k=1}^{\infty} \frac{(M_k \phi(g)) * g^{(k)}}{k!}.$$  

Assumption (A1) implies that $g \mapsto g^{(k)}$ is a continuous $k$-homogeneous operator in $Y$, so that $g \mapsto s(g)$ is an analytic mapping in $Y$. Also analytic is the mapping $\Psi : X \times Y \rightarrow Y$ given by

$$\Psi(f, g) := \sum_{k=1}^{\infty} \frac{(M_k f) * g^{(k)}}{k!},$$

and thus such is the composition of $\Psi$ and $\phi$, viz.

$$Y \ni g \mapsto \Psi(\phi(g), g) \in Y.$$  

In view of (6.1) this establishes analyticity of the mapping $\phi|_Y$. 

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Next we prove that $\phi_{1,\nu}$ is a local analytic homeomorphism on $Y \cap \Omega$. The above reasonings show that the mapping $\Phi$ sends $Y \times Y$ into $Y$, and we denote by $\Phi_{1,\nu}$ its restriction onto $Y \times Y$. The Fréchet derivative and Fréchet partial derivatives of $\Phi_{1,\nu}$ in $Y$ are given by the same formulae as for $L_1$, but considered on elements of $Y$. Assumption (A1) implies that the operators

$$
\partial_f \Phi_{1,\nu}(f,g) - I, \quad \partial_g \Phi_{1,\nu}(f,g) - I
$$

are compact, and the operators $\partial_f \Phi_{1,\nu}(f,g)$ and $\partial_g \Phi_{1,\nu}(f,g)$ have trivial null-spaces as soon as $g \in \Omega \cap Y$ by the reasoning of Section 4. Thus these partial derivatives are boundedly invertible in the algebra $B(\mathcal{Y})$, which by the implicit function theorem yields the claim about $\phi_{1,\nu}$.

It remains to show that $\phi(Y) = Y$ if either (A2) or (A2') holds. Assume first that condition (A2) is satisfied, i.e., that $X^{(m)} \subset Y$. We take an arbitrary $f \in Y$; since $Y \subset X$ and $X \in \mathcal{Y}$, there is $g \in X$ such that $f = \phi(g)$. In other words, such a $g$ satisfies the relation $\Phi(f,g) = 0$, i.e.,

$$
g = -\sum_{k=1}^{\infty} \frac{(-1)^k g^{2k+1}}{(2k+1)!} + f + \sum_{k=1}^{\infty} \frac{(M^k f) \ast g^{(k)}}{k!}.
$$

(6.2)

We set

$$
Z_l := X^{(l)} + X^{(l+1)} + \cdots + X^{(m-1)} + Y
$$

for $l = 1, \ldots, m-1$ and $Z_m := \{y \in X \mid y \in Z_l \text{ for all } l < m \}$. Then the right-hand side of relation (6.2) belongs to $Z_{l+1}$, i.e., $g \in Z_{l+1}$, and, by induction, $g \in Z_m = Y$.

Assume now that (A2') holds, take an arbitrary $f \in Y$, and choose trigonometric polynomials $P_0$ and $P_1$ in $e_n$ such that $P_0 \in Y$, $P_1 \in X$, and

$$
||f - P_0||_Y < \varepsilon_0, \quad ||Mf - P_1||_X < \varepsilon_1,
$$

with $\varepsilon_0$ and $\varepsilon_1$ to be specified later. Consider the mapping $\psi : Y \to Y$ given by

$$
\psi(h) := h - s(h) - (f - P_0) - (Mf - P_1) \ast h - \sum_{k=2}^{\infty} \frac{(M^k f) \ast h^{(k)}}{k!}.
$$

(6.3)

It is clear that $\psi$ is analytic in $Y$ and that $\psi(0) = -(f - P_0)$ and $d \psi(0)(h_1) = -(Mf - P_1) \ast h_1$. By Assumption (A1) there is $\varepsilon_1 > 0$ such that $\|d\psi(0)\|_{\mathcal{B}(Y)} \leq \frac{1}{4}$, whence for some $\varepsilon > 0$ we have $\|d\psi(h)\|_{\mathcal{B}(Y)} \leq \frac{1}{4}$ for all $h$ in the closed ball $B_{\varepsilon,Y} := \{h \in Y \mid \|h\|_Y \leq \varepsilon\}$. We now set $\varepsilon_0 = \varepsilon/2$ and observe that $\psi$ is a contraction of the ball $B_{\varepsilon,Y}$. Thus the equation $h = \psi(h)$ has a unique solution in $B_{\varepsilon,Y}$, which we denote by $g_0$.

Taking into account relations (4.3) and (6.3), we conclude that

$$
\Phi(f,g_0) = g_0 - \psi(g_0) + P_0 + P_1 \ast g_0 = P_0 + P_1 \ast g_0 := P
$$

is a trigonometric polynomial that belongs to the space $Y$. It is straightforward to verify that

$$
F_f(\lambda_k(g_0)) = (-1)^k \tilde{P}(k),
$$

so that all but at most finitely many of the $\lambda_k(g_0)$ are zeros of the entire function $F_f$; moreover, $\tilde{P}(k) \neq 0$ yields $e_k \in Y$. Without loss of generality we can assume $\varepsilon$ to be so small that $\|g_0\|_{L_1} < 1$, and then the numbers $\lambda_k(g_0)$, $k \in \mathbb{Z}$, are pairwise distinct. Denote by $\Lambda = (z_k)_{k \in \mathbb{Z}}$ the zero sequence of the function $F_f$. The numbers
Proof. (a) Since the convolution operation is continuous in \( \| \cdot \| \), therefore continuity and the estimates \( \| z_k - \lambda_k(g_0) \| e_k \) implies that, for every \( f \in Y \) and \( \lambda_n(g) = z_n \) for all \( n \in \mathbb{Z} \), which yields the relation \( F_f = G_g \), i.e., the relation \( \phi(g) = f \). Thus \( \phi_1 \) is surjective under (A2'), and the proof of Theorem 2.4 is complete. \( \square \)

Before proving Corollary 2.5, we give some sufficient conditions for assumption (A1) of Theorem 2.4 to be satisfied.

**Lemma 6.1.** Assume that \( X, Y, \) and \( Z \) are Banach spaces continuously embedded into \( L_1 \) and that \( X \star Y \subset Z \). Then the following holds:

1. the mapping \( (f, g) \mapsto f \star g \) is continuous from \( X \times Y \) to \( Z \);
2. if the set \( \mathcal{P} \cap X \) of all trigonometric polynomials belonging to \( X \) is dense in \( X \), then the operator \( R(f) : g \mapsto f \star g \) is compact from \( Y \) to \( Z \) for every \( f \in X \);
3. if the set \( \mathcal{P} \cap Y \) of all trigonometric polynomials belonging to \( Y \) is dense in \( Y \), then the operator \( L(g) : f \mapsto f \star g \) is compact from \( X \) to \( Z \) for every \( g \in Y \);
4. if \( X \star Y \) is compactly embedded into \( Z \), i.e., if the set \( B_{1,X} \star B_{1,Y} := \{ f \star g \mid f \in B_{1,X}, \ g \in B_{1,Y} \} \) (\( B_{1,X} \) and \( B_{1,Y} \) denoting the unit balls of \( X \) and \( Y \), respectively) is precompact in \( Z \), then the operators \( R(f) \) and \( L(g) \) are compact for every \( f \in X \) and every \( g \in Y \).

**Proof.** (a) Since the convolution operation is continuous in \( L_1 \) [7 Sect. 3.1.6], the operators \( R(f) \) and \( L(g) \) are closed [13 Sect. 13.1] for every \( f \in X \) and every \( g \in Y \) and hence, by the closed graph theorem [13 Thm. 2.15], they are continuous. The inequality
\[
\| f \star g \|_Z \leq \| R(f) \|_{Y \to Z} \| g \|_Y
\]
implies that, for every \( f \in X \), the orbit \( \{ L(g)f \mid g \in B_{1,Y} \} \) is bounded in \( Z \), and hence by the uniform boundedness principle [13 Thm. 2.6] we get that
\[
c := \sup_{\| g \|_Y \leq 1} \| L(g) \|_{X \to Z} < \infty.
\]
Therefore \( \| f \star g \|_Z \leq c \| f \|_X \| g \|_Y \) for all \( f \in X \) and \( g \in Y \), which yields the required continuity and the estimates \( \| R(f) \|_{Y \to Z} \leq c \| f \|_X \) and \( \| L(g) \|_{X \to Z} \leq c \| g \|_X \).

(b) Let \( (p_n) \in \mathcal{P} \cap X \) be a sequence converging to \( f \) in \( X \); then by the above
\[
\| R(f) - R(p_n) \|_{Y \to Z} = \| R(f - p_n) \|_{Y \to Z} \leq c \| f - p_n \|_X \to 0
\]
as \( n \to \infty \). Therefore \( R(f) \) is the limit in the uniform operator topology of the operators \( R(p_n) \); since the latter are of finite rank, the claim follows [13 Thm. 4.18].

The proof of item (c) is completely analogous.

(d) immediately follows from the definition of compact operators [13 Def. 4.16]. \( \square \)

**Proof of Corollary 2.5.** For each of cases (i)–(vi) listed, we establish properties (A1) and one of (A2) and (A2') of Theorem 2.4 for a suitable \( X \in \mathcal{P} \), in which the multiplication operator \( M \) is continuous.
Case (i) with $p < \infty$. The spaces $X := L_1 \in \mathcal{F}$ and $Y := L_p$ satisfy $(A2')$ and $L_1 \ast L_p \subset L_p$ by [7 Sect. 3.1.6]. Thus by Lemma 6.1 condition $(A1)$ is satisfied, and $L_p \in \mathcal{F}$ by Theorem 2.3.

Case (ii) and Case (i) with $p = \infty$. Since by [7 Sect. 3.1.4]

$$L_2 \ast L_2 \subset C[0,1] \subset L_\infty,$$

the operator $T(h) : L_2 \to C[0,1]$ of convolution with an element $h \in L_2$ is compact by Lemma 6.1. Now take $X = L_2 \in \mathcal{F}$ and $Y = C[0,1]$ or $Y = L_\infty$. Then $(A2)$ holds with $m = 2$ by the above formula. The operator $R(f) : g \mapsto f \ast g$ is compact in $Y$ for every $f \in L_2$ as a composition of the continuous embedding of $Y$ into $X$ and the mapping $T(f)$, and $L(g) : f \mapsto f \ast g$ acts compactly from $X$ into $Y$ for every $g \in L_\infty$ because it coincides with $T(g)$. Thus $(A1)$ is verified, and $Y \in \mathcal{F}$.

Case (iii). By definition, the space $C^\alpha_\text{per}[0,1]$, $\alpha = n + s$, with $n \in \mathbb{Z}_+$ and $s \in [0,1)$, is the completion of the set $\mathcal{P}$ by the norm

$$\|p\|_{C^\alpha_\text{per}} := \max_{0 \leq j \leq n} \max_{x \in \mathbb{R}} |p^{(j)}(x)| + \sup_{x,y \in \mathbb{R}, x \neq y} \frac{|p^{(n)}(x) - p^{(n)}(y)|}{|x - y|^s}.$$  

If $s = 0$, the second term in the above definition can be omitted. Set $X := L_1$ and $Y := C^\alpha_\text{per}[0,1]$; then $(A2')$ holds. Moreover, for integer $\alpha$, $X \ast Y \subset Y$ and $(f \ast g)^{(j)} = f \ast g^{(j)}$ for every $f \in X$, every $g \in Y$, and $j = 1, \ldots, \alpha$ by [7 Sect. 3.1.5].

For $\alpha = n + s$ we insert the above formula for $(f \ast g)^{(j)}$ in the definition of the norm to conclude that $f \ast g \in C^\alpha_\text{per}[0,1]$. Now $(A1)$ follows from Lemma 6.1 thus yielding $Y \in \mathcal{F}$.

Case (iv). We refer the reader to [2 Ch. 7] for the definition and basic properties of the Sobolev spaces $W^s_p := W^s_p(0,1)$ with $p \geq 1$ and $s \geq 0$.

Direct verification shows that, for $n \in \mathbb{Z}_+$ and $s \in [0,1)$, we have

$$(6.4) \quad W^{n+s}_p \ast W^{n+t}_p \subset W^{n+s+t}_p;$$

therefore the multilinear interpolation [6 Ch. 4.4] justifies (6.4) for all $s, t \in [0,1]$. By the Arzela-Ascoli theorem the space $W^{n+1}_p$ is compactly embedded into $W^n_p$ for every $n \in \mathbb{Z}_+$, and by the compactness theorem [6 Theorem 3.8] the embedding of $W^n_\ast$ into $W^n_p$ is compact as soon as $s \geq t \geq 0$.

Let $s \in (0,1/p)$ and set $X := L_p = W_0^0$ and $Y := W^s_\ast$. Since the set $\mathcal{P}$ of all trigonometric polynomials is dense in the space $C^\alpha_\text{per}[0,1]$ of continuous 1-periodic functions and the latter is dense in $Y$ by [2 Ch. 7], we conclude that $(A2')$ holds. By (6.1) and Lemma 6.1 condition $(A1)$ is also verified, and thus $Y \in \mathcal{F}$.

Now take $X := W^r_p$ with $r \in (0,1/p)$ and $Y := W^s_p$ with $s \in (r,1]$. By (6.4), $X^{(m)} \subset Y$ if $m \geq 1/r$, thus giving $(A2)$. Since the embedding $X \ast Y \subset Y$ is compact by the above, $(A1)$ holds by Lemma 6.1 and $Y \in \mathcal{F}$.

Assume that $X := W^n_p$, $n \in \mathbb{N}$, is in $\mathcal{F}$ (the case $n = 1$ being established above) and set $Y := W^{n+t}_p$ with $t \in (0,1]$. Observe that $W^n_p \ast W^{n+t}_p \subset W^{n+1+t}_p$ for natural $n$ and $t \in [0,1]$ (the cases $t = 0$ and $t = 1$ can be verified directly, and the case $t \in (0,1)$ follows by interpolation). Then $X \ast X \subset W^{n+1}_p \subset Y$, so that $(A2)$ is satisfied with $m = 2$, and the embedding of $X \ast Y \subset W^{n+1+t}_p$ into $Y$ is compact. Therefore $(A1)$ holds by Lemma 6.1 and $Y \in \mathcal{F}$; in particular, $W^{n+1}_p \in \mathcal{F}$. Using induction on $n \in \mathbb{N}$, we conclude that $W^n_\ast \in \mathcal{F}$ for all $s \geq 0$, thus completing the proof of Case (iv).
Case (v). If \( g \in BV[0, 1] \) and \( V(g) \) is the total variation of \( g \), then \( |n\hat{g}(n)| \leq V(g) \) for all \( n \in \mathbb{Z} \) by [7, Sect. 2.3.6], and thus
\[
\sum_{n \in \mathbb{Z}} (1 + n^2)^{1/3} |\hat{g}(n)|^2 < \infty.
\]
It follows [2, Ch. 7] that \( BV[0, 1] \subset W^{1/3}_2 \). Set \( X := W^{1/3}_2 \subset \mathcal{X} \) and \( Y := BV[0, 1] \); then by [6.3] we have \( X^{(3)} \subset W^{1/3}_2 \subset Y \), and (A2) holds. Take \( f \in X \) and \( g \in Y \); then the Fourier coefficients \( \hat{h}(n) = \hat{f}(n)\hat{g}(n) \) of the convolution \( h := f * g \) satisfy
\[
\sum_{n \in \mathbb{Z}} (1 + n^2)^{1/3} |\hat{h}(n)|^2 \leq 2(|g(0)| + V(g))^2 \sum_{n \in \mathbb{Z}} (1 + n^2)^{1/3} |\hat{f}(n)|^2 < \infty.
\]
It follows [2, Ch. 7] that \( X * Y \) is contained into \( W^{4/3}_2 \) and thus is compactly embedded into \( W^{1/3}_2 \) and \( Y \). Thus (A1) is verified by Lemma 6.1, and \( Y \in \mathcal{X} \).

Case (vi). The spaces \( X := L_1 \) and \( Y := C_b \) verify condition (A2’); moreover, the relation \( f * e_n = e_n \sum_{k=-1}^{1} f e_{-n} \) for every \( f \in X \) yields \( X * Y \subset Y \). Thus (A1) holds by Lemma 6.1 and the proof of (vi) and of the corollary is complete. \( \square \)

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