ON ESTIMATES FOR THE RATIO OF ERRORS IN BEST RATIONAL APPROXIMATION OF ANALYTIC FUNCTIONS

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Abstract. Let $E$ be an arbitrary compact subset of the extended complex plane $\mathbb{C}$ with nonempty interior. For a function $f$ continuous on $E$ and analytic in the interior of $E$ denote by $\rho_n(f; E)$ the least uniform deviation of $f$ on $E$ from the class of all rational functions of order at most $n$. In this paper we show that if $f$ is not a rational function and if $K$ is an arbitrary compact subset of the interior of $E$, then $\prod_{k=0}^{n}(\rho_k(f; K)/\rho_k(f; E))$, the ratio of the errors in best rational approximation, converges to zero geometrically as $n \to \infty$ and the rate of convergence is determined by the capacity of the condenser $(\partial E, K)$. In addition, we obtain results regarding meromorphic approximation and sharp estimates of the Hadamard type determinants.

1. Introduction

Let $E$ be a compact subset of the extended complex plane $\mathbb{C}$ and denote by $C(E)$ the space of continuous functions on $E$ with the supremum norm

$$||f||_E = \sup_{z \in E} |f(z)|.$$ 

By $A(E)$ we mean the algebra of functions in $C(E)$ which are analytic on the interior of $E$. Also, for $f \in A(E)$ and each nonnegative integer $n$, let $\rho_n(f; E)$ denote the error in best rational approximation of $f$ in the supremum norm on $E$ by rational functions of order at most $n$; that is,

$$\rho_n(f; E) = \inf_{r \in \mathcal{R}_n} ||f - r||_E,$$

where $\mathcal{R}_n = \{r : r = p/q, \deg p \leq n, \deg q \leq n, q \neq 0\}$ is the class of all rational functions of order at most $n$.

From now on we will always assume that $E$ has a nonempty interior. In this paper, the main object of study is the ratio of errors in the best rational approximation of $f$ on $E$ and an arbitrary compact subset of its interior. More precisely, we investigate the asymptotic behaviors of the ratio $\rho_n(f; K)/\rho_n(f; E)$ and the product $\prod_{k=0}^{n}(\rho_k(f; K)/\rho_k(f; E))$ as $n \to \infty$, where $K$ denotes a compact subset of the $E$’s interior. We make two trivial observations regarding the ratio of the...
errors. First of all one has to exclude rational functions since in this case the error \( \rho_n(f; E) \) would vanish for all but finitely many \( n \). Secondly, since \( K \subset E \), it follows directly from the definition that \( \rho_n(f; K)/\rho_n(f; E) \leq 1 \) for all \( n \geq 0 \). Our main result is Theorem A. Also note that \( \partial E \) stands for the boundary of the set \( E \) and \( C(F,K) \) is the capacity of the condenser \((F,K)\) for a pair of disjoint compact subsets of \( \mathbb{C} \) (see, for example, [8] and [16] for more details and the exact definition).

**Theorem A.** Let \( E \) be a compact subset of \( \mathbb{C} \) with nonempty interior and suppose that \( K \) is a compact subset of the interior of \( E \). If \( f \in A(E) \) and \( f \) is not a rational function, then

\[
\limsup_{n \to \infty} \left( \prod_{k=0}^{n} \frac{\rho_k(f; K)}{\rho_k(f; E)} \right)^{1/n^2} \leq \exp(-1/C(\partial E, K)).
\]

In [14], the second author proves the above inequality in the case where the complements of \( E \) and \( K \) are both connected. Therefore, Theorem A can be considered as the generalization of the result in [14] with no additional assumptions on the compact sets \( E \) and \( K \). One immediate consequence of Theorem A is the following estimate for the lower limit of \( (\rho_n(f; K)/\rho_n(f; E))^{1/n} \) as \( n \to \infty \).

**Corollary 1.** Under the assumptions of Theorem A, we have

\[
\liminf_{n \to \infty} \left( \frac{\rho_n(f; K)}{\rho_n(f; E)} \right)^{1/n} \leq \exp(-2/C(\partial E, K)).
\]

As another application of Theorem A, we state the following result regarding the degree of rational approximation of analytic functions.

**Corollary 2.** Suppose \( E \) and \( F \) are disjoint compact subsets of \( \mathbb{C} \). If \( f \) is analytic on \( \mathbb{C} \setminus F \), then

\[
\begin{align*}
(1.1) \quad & \limsup_{n \to \infty} \left( \prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n^2} \leq \exp(-1/C(E,F)); \\
(1.2) \quad & \limsup_{n \to \infty} \rho_n(f; E)^{1/n} \leq \exp(-1/C(E,F)); \\
(1.3) \quad & \liminf_{n \to \infty} \rho_n(f; E)^{1/n} \leq \exp(-2/C(E,F)).
\end{align*}
\]

We remark that (1.2) and (1.3) follow directly from (1.1). Inequality (1.2) is the well-known theorem of Walsh (see [19] and [2]). Estimate (1.3) is known as Gonchar’s conjecture [7]. Parfenov [9] gives a proof of (1.1) and (1.3) for the case where \( E \) is a continuum with connected complement. In [12], the second author proves (1.1) and (1.3) for an arbitrary compact set \( E \).

This paper is organized as follows. In Section 2 we present the needed notation and some facts about the theory of Hankel operators which includes the AAK theorem and its generalization. Section 3 contains Theorem [5] related to the estimates of the Hadamard type determinants. The second author (see [13]) has proved the corresponding result for domains bounded by finitely many closed analytic Jordan curves. Finally, in Section 4 we give the proof of Theorem A.
2. Notation and related topics from the theory of Hankel operators

We fix the following notation which will be used throughout this paper. Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. For a compact operator $A : \mathcal{H} \to \mathcal{K}$, denote by $\{s_n(A)\}_{n \geq 0}$ the sequence of singular numbers (counted with multiplicities) of the operator $A$; that is, $\{s_n(A)\}_{n \geq 0}$ is the sequence of eigenvalues of the operator $(A^*A)^{1/2}$, where $A^* : \mathcal{K} \to \mathcal{H}$ denotes the adjoint of $A$. Furthermore, we shall always assume that the sequence $\{s_n(A)\}_{n \geq 0}$ is nonincreasing. Also, one can think of $s_n(A)$ as the minimum distance of $A$, in the operator norm, from the class of operators of rank at most $n$. More precisely,

$$s_n(A) = \inf \|A - L\|,$$

where the infimum is taken over the class of all operators $L : \mathcal{H} \to \mathcal{K}$ of rank at most $n$, and $\| \cdot \|$ is the usual operator norm. In fact, the infimum in (2.1) is always achieved for some finite rank operator; that is, there exists an operator $M : \mathcal{H} \to \mathcal{K}$ of rank at most $n$ for which $s_n(A) = \|A - M\|$ (see [5] for more details and facts about the singular numbers).

Let $\Gamma$ be the union of a finite number of rectifiable Jordan curves. Denote by $L_2(\Gamma)$ the Hilbert space of square integrable functions $\varphi$ with respect to the Lebesgue measure on $\Gamma$, where the usual norm and inner product are given by

$$||\varphi||_2 = \left( \int_{\Gamma} |\varphi(\xi)|^2 \, d\xi \right)^{1/2}$$

and

$$\langle \varphi, \psi \rangle_{L_2(\Gamma)} = \int_{\Gamma} \overline{\varphi(\xi)} \psi(\xi) \, d\xi, \quad \varphi, \psi \in L_2(\Gamma).$$

We also will be concerned with $L_\infty(\Gamma)$, the space of essentially bounded functions $\varphi$ on $\Gamma$ with the norm

$$||\varphi||_\infty = \operatorname{ess sup}_\Gamma |\varphi(\xi)|.$$

Next suppose that $G$ is a bounded domain of the complex plane $\mathbb{C}$ such that $G$’s boundary $\Gamma$ consists of a finite number of closed analytic Jordan curves. Fix $1 \leq p < \infty$. An analytic function $\varphi$ on $G$ belongs to the Smirnov class $E_p(G)$ if there is a sequence of domains $G_1, G_2, \ldots$ with rectifiable boundaries $\partial G_1, \partial G_2, \ldots$ such that $G_k \subset G_{k+1}, \overline{G_k} \subset G, \bigcup_k G_k = G$, and

$$\sup_k \int_{\partial G_k} |\varphi(\xi)|^p \, d\xi < \infty.$$

It should be mentioned that for such domains $G$, the Smirnov class $E_p(G)$ coincides with the Hardy space $H_p(G)$ (see [3], [10], or [15] for more details). The Smirnov class $E_\infty(G)$ is always the same as $H_\infty(G)$ (the class of bounded analytic functions on $G$). Moreover, it follows that each function (or equivalent class functions) in $E_p(G)$, $1 \leq p \leq \infty$, can be identified with its boundary function in the sense of nontangential limit (see [3] and [10]); and, $E_p(G)$ can be considered as a closed subspace of $L_p(\Gamma)$. We will use this fact throughout without further notice.

For a domain $G$ with the boundary $\Gamma$ (described as above) and $f \in C(\Gamma)$, define the Hankel operator $A_{f,G}$ with symbol $f$ by

$$A_{f,G} : E_2(G) \to E_2^+(G) = L_2(\Gamma) \ominus E_2(G)$$
and

\[ A_{f,G}(\varphi) = P_-(\varphi f) \quad \text{for all} \quad \varphi \in E_2(G), \]

where \( P_- \) is the orthogonal projection from \( L_2(\Gamma) \) onto \( E_2^+(G) \). From now on, whenever \( G \) is understood, we shall denote \( A_{f,G} \) simply by \( A_f \). It is not hard to see that \( A_f \) is a compact operator (see, for example, \([\text{II}]\)).

Finally, let \( M_n(G) = \{ h : h = p/q, \ p \in E_\infty(G), \ \deg q \leq n, \ q \neq 0 \} \) be a class of meromorphic functions on \( G \) with at most \( n \) poles (counted with multiplicities), and denote by \( \Delta_n(f;G) \) the least deviation of \( f \) in \( L_\infty(\Gamma) \) from the class \( M_n(G) \); that is,

\[ \Delta_n(f;G) = \inf_{h \in M_n(G)} \| f - h \|_\infty. \]

The AAK theorem (see \([\text{II}]\)) asserts that for the unit disk \( \mathbb{D} \) and \( f \in C(\partial \mathbb{D}) \), \( s_n(A_f) = \Delta_n(f;\mathbb{D}) \) for all \( n \geq 0 \). One of our tools is the following generalization of the AAK theorem obtained by the second author (see \([\text{II}]\)).

If \( G \) is a bounded domain whose boundary \( \Gamma \) consists of \( N \) closed analytic Jordan curves and if \( f \in C(\Gamma) \), then

\[ s_n(A_f) \leq \Delta_n(f;G), \quad n = 0, 1, 2, \ldots, \]

and

\[ \Delta_{n+N-1}(f;G) \leq s_n(A_f), \quad \text{for} \quad n \geq N-1. \]

3. Meromorphic approximation and Hankel operators

Before proving the main results of this section, Theorems 5 and 6 we need some auxiliary results from the theory of Hankel operators. For the sake of simplicity and further references we define the following notation which will be used throughout this paper.

**Definition.** An open subset of the complex plane \( G \) is called an \( m \)-domain \((1 \leq m < \infty)\) if \( G \) is the union of \( m \) bounded domains \( G_1, \ldots, G_m \) with disjoint closures such that the boundary of each \( G_i \), denoted by \( \Gamma_i \), consists of finitely many closed analytic Jordan curves. Furthermore, we let \( \Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m \) denote the boundary of \( G \).

For an \( m \)-domain \( G \), denote by \( E_2(G) \) the direct sum of the Smirnov classes \( E_2(G_i), \ 1 \leq i \leq m \); that is,

\[ E_2(G) = E_2(G_1) \oplus \cdots \oplus E_2(G_m). \]

Let \( f \in C(\Gamma) \). We define the operator \( A_f = A_{f,G} : E_2(G) \to E_2^+(G_1) \oplus \cdots \oplus E_2^+(G_m) \) as the direct sum of the Hankel operators \( A_{f_i} : E_2(G_i) \to E_2^+(G_i), \ 1 \leq i \leq m \) :

\[ A_f = A_{f_1} \oplus \cdots \oplus A_{f_m}, \]

where \( f_i = f|_{\Gamma_i} \) is the restriction of \( f \) to \( \Gamma_i \). Since each \( A_{f_i} \) is compact, it follows that \( A_f \) is a compact operator. We also mention the following facts regarding \( A_f \) :

\[ \| A_f \| = \max(\| A_{f_1} \|, \ldots, \| A_{f_m} \|) \]

and

\[ A_f^* A_f = A_{f_1}^* A_{f_1} \oplus \cdots \oplus A_{f_m}^* A_{f_m}. \]
Equality (3.3) shows that if \( s \) is a singular number of the operator \( A_f \), then \( s \) must be a singular number for at least one of the operators \( A_{f_i} \). Actually more can be said. The sequence \( \{s_n(A_f)\}_{n \geq 0} \) of the singular numbers of \( A_f \) can be put into a one-to-one correspondence with the rearrangement (counting multiplicities) of the sequences \( \{s_n(A_{f_i})\}_{n \geq 0}, \ldots, \{s_n(A_{f_m})\}_{n \geq 0} \) in a nonincreasing order. The next lemma gives the precise statement of this fact.

**Lemma 3.** Suppose \( G = \bigcup_{i=1}^m G_i \) is an \( m \)-domain. If \( f \) is continuous on \( \Gamma = \bigcup_{i=1}^m \Gamma_i \), then the following statements hold.

(a) For each \( n \geq 0 \)

\[
(3.4) \quad s_n(A_f) = \min_{k_1 + \cdots + k_m \leq n} \max \{s_{k_1}(A_{f_1}), \ldots, s_{k_m}(A_{f_m})\}.
\]

(b) There is a one-to-one correspondence between the sequence of singular numbers \( \{s_n(A_f)\}_{n \geq 0} \) and the sequence \( \{\mu_n\}_{n \geq 0} \) obtained by rearranging (counting multiplicities) the sequences \( \{s_n(A_{f_1})\}_{n \geq 0}, \ldots, \{s_n(A_{f_m})\}_{n \geq 0} \) in a nonincreasing order.

**Proof.** We first observe that formula (2.1) together with equality (3.2) imply

\[
(3.5) \quad s_n(A_f) = \inf \max(||A_{f_1} - L_1||, \ldots, ||A_{f_m} - L_m||),
\]

where the infimum is taken over all operators \( L_i : E_2(G_i) \to E_2(G_i), i = 1, \ldots, m, \) of rank at most \( k_i \) such that \( k_1 + \cdots + k_m \leq n \).

Since, by (2.1), \( ||A_{f_i} - L_i|| \geq s_{k_i}(A_{f_i}) \) holds for any operator \( L_i : E_2(G_i) \to E_2(G_i) \) of rank at most \( k_i \), where \( 1 \leq i \leq m \), it is easily seen that

\[
s_n(A_f) \geq \min_{k_1 + \cdots + k_m \leq n} \max(s_{k_1}(A_{f_1}), \ldots, s_{k_m}(A_{f_m})).
\]

To prove the reverse inequality, fix \( k_1 \geq 0, \ldots, k_m \geq 0 \) such that \( k_1 + \cdots + k_m \leq n \). Let \( M_i : E_2(G_i) \to E_2(G_i), i = 1, \ldots, m, \) denote operators of rank at most \( k_i \) for which \( s_{k_i}(A_{f_i}) = ||A_{f_i} - M_i|| \) (see the citation after the formula (2.1)). According to (3.3), we can write

\[
s_n(A_f) \leq \max(||A_{f_1} - M_1||, \ldots, ||A_{f_m} - M_m||) = \max(s_{k_1}(A_{f_1}), \ldots, s_{k_m}(A_{f_m})).
\]

Since the above inequality is valid for all \( k_1 \geq 0, \ldots, k_m \geq 0 \) such that \( k_1 + \cdots + k_m \leq n \), we have

\[
s_n(A_f) \leq \min_{k_1 + \cdots + k_m \leq n} \max(s_{k_1}(A_{f_1}), \ldots, s_{k_m}(A_{f_m})).
\]

This proves part (a).

To prove part (b), we only need to show that if \( s = s_n(A_f) \) is a singular number of \( A_f \) with multiplicity \( \alpha \), then \( s \) is repeated exactly \( \alpha \) times in the sequence \( \{\mu_n\}_{n \geq 0} \). Let \( I \subseteq \{1, \ldots, m\} \) be the set of all indices \( i \) such that \( s \) is the singular number of the operators \( A_{f_i} \). As mentioned earlier, (3.3) implies that \( s \) is a singular number for at least one of the operators \( A_{f_i} \); therefore, \( I \neq \emptyset \). Next, for each \( i \in I \), we let \( \alpha_i \) denote the multiplicity of the singular number \( s \) for the corresponding operator \( A_{f_i} \). Thus we have to show \( \alpha = \sum_{i \in I} \alpha_i \).

For each fixed \( i \in I \), one can find an orthogonal system in \( E_2(G_i) \) of the eigenfunctions \( Q_{1i}, \ldots, Q_{\alpha_i} \) of the operator \( (A_{f_i}^* A_{f_i})^{1/2} \) corresponding to the eigenvalue \( s \). Let

\[
L = \{Q = (0, \ldots, Q_{ik}, \ldots, 0) \in E_2(G) : i \in I \text{ and } 1 \leq k \leq \alpha_i\}.
\]
The set $L$ consists of $\sum_{i \in I} \alpha_i$ orthogonal functions in $E_2(G)$, each of which, by (3.3), is an eigenfunction of the operator $(A_f^* A_f)^{1/2}$ corresponding to the eigenvalue $s$; therefore, $\alpha \geq \sum_{i \in I} \alpha_i$.

Next suppose $\alpha > \sum_{i \in I} \alpha_i$. Consequently, there must exist an eigenfunction $R = (R_1, \ldots, R_m) \in E_2(G)$, $R \neq 0$, of $(A_f^* A_f)^{1/2}$ corresponding to $s$ that is orthogonal to each function $Q$ in $L$. But this would imply that

$$0 = \langle R, Q \rangle_{L_2(\Gamma)} = \langle R_i, Q_i \rangle_{L_2(\Gamma_i)},$$

for all $i \in I$ and $1 \leq k \leq \alpha_i$. Formula (3.8) implies that each nonzero $R_i$, $1 \leq i \leq m$, is an eigenfunction of the operator $(A_f^* A_f)^{1/2}$ corresponding to the eigenvalue $s$. From this and formula (3.6) it follows that $R_i = 0$ for each $i \in I$. Now since $R \neq 0$, we can conclude that $s$ is a singular value for some operator $A_f$ with $i \notin I$. But, this contradicts the definition of $I$. Thus $\alpha = \sum_{i \in I} \alpha_i$ and we are done.

**Remark.** We remark that since for each $1 \leq i \leq m$ the sequence $\{s_n(A_{f_i})\}_{n \geq 0}$ is nonincreasing, (3.3) directly implies

$$s_n(A_f) = \min_{k_1 + \cdots + k_m = n} \max \{s_{k_1}(A_{f_1}), \ldots, s_{k_m}(A_{f_m})\}.$$  \hspace{1cm} (3.7)

In order to state our next lemma, we need the following definition that extends the notion of error in meromorphic approximation (2.2) to an $m$–domain.

Let $G = \bigcup_{i=1}^m G_i$ be an $m$–domain with the boundary $\Gamma = \bigcup_{i=1}^m \Gamma_i$, and suppose $f \in C(\Gamma)$. For $n \geq 0$ define

$$\Delta_n(f; G) = \min_{k_1 + \cdots + k_m \leq n} \max \{\Delta_{k_1}(f_1; G_1), \ldots, \Delta_{k_m}(f_m; G_m)\},$$

where $\Delta_{k_i}(f_i; G_i)$, $1 \leq i \leq m$, are defined as in definition (2.2) and $f_i = f|\Gamma_i$ denotes the restriction of $f$ to $\Gamma_i = \partial G_i$. The following result is a direct consequence of Lemma 3, inequalities (2.3) and (2.4), and definition (3.8). However, for the sake of completeness, we also include a proof.

**Lemma 4.** Let $G = \bigcup_{i=1}^m G_i$ be an $m$–domain and suppose that $f$ is continuous on the boundary $\Gamma = \bigcup_{i=1}^m \Gamma_i$.

(a) For all $n = 0, 1, 2, \ldots$

$$s_n(A_f) \leq \Delta_n(f; G).$$  \hspace{1cm} (3.8)

(b) Suppose each $G_i$, $1 \leq i \leq m$, consists of $N_i$ closed analytic Jordan curves, and put $N = N_1 + \cdots + N_m$. Then there is a positive integer $n^*$ such that

$$\Delta_{n+N-m}(f; G) \leq s_n(A_f) \quad \text{for all} \quad n \geq n^*.$$  \hspace{1cm} (3.9)

**Proof.** Noting that part (a) follows trivially from formula (3.4), inequality (2.2) and definition (3.8), we only give a proof of part (b).

For each fixed $n \geq 0$, by (3.4), there are nonnegative integers $k_1, \ldots, k_m$ such that $k_1 + \cdots + k_m = n$ and

$$s_n(A_f) = \max \{s_{k_1}(A_{f_1}), \ldots, s_{k_m}(A_{f_m})\}.$$  \hspace{1cm} (3.10)

Hence, for some $1 \leq i \leq m$, we have

$$s_n(A_f) = s_{k_i}(A_{f_i}) \geq s_{k_j}(A_{f_j}) \quad \text{for all} \quad j = 1, \ldots, m.$$  \hspace{1cm} (3.11)

In light of Lemma 3 part (b), we can choose $n^* \geq 1$ sufficiently large such that whenever $n \geq n^*$ and $s_n(A_f) = s_k(A_{f_j})$ for some $k$ and $1 \leq i \leq m$, we have $k \geq N_i - 1$. Now if we let $n \geq n^*$, then from (3.10) and Lemma 3 part (b), it
follows that \( k_j \geq N_j - 1 \) for all \( 1 \leq j \leq m \). Since, by inequality (2.4), \( s_k(A_{f_j}) \geq \Delta_{k_j+N_j-1}(f_j;G_j) \), where \( j = 1, \ldots, m \), we can deduce with the help of (3.9) that

\[
s_n(A_f) \geq \max(\Delta_{k_1+N_1-1}(f_1;G_1), \ldots, \Delta_{k_m+N_m-1}(f_m;G_m)) \\
\geq \min_{l_1+\cdots+l_m=n+N-m} \max(\Delta_{l_1}(f_1;G_1), \ldots, \Delta_{l_m}(f_m;G_m)) \\
\geq \Delta_{n+N-m}(f;G).
\]

This completes the proof of part (b).

\[\Box\]

Let \( \varphi, \psi \in E_2(G) \) and \( f \in C(\Gamma) \). Here and in what follows, we use the notation

\[
\int_\Gamma (\varphi f)(\xi)d\xi = \sum_{k=1}^m \int_{\Gamma_k} (\varphi_k f_k)(\xi)d\xi,
\]

where \( \varphi_k = \varphi|_{\Gamma_k} \), \( \psi_k = \psi|_{\Gamma_k} \), and \( f_k = f|_{\Gamma_k} \), \( k = 1, \ldots, m \).

Now we can state the first result of this section. In fact, Theorem 5 is an extension of the second author’s result (see [13]) proved for the case \( m = 1 \).

**Theorem 5.** Suppose \( G \) is an \( m \)-domain with its boundary denoted by \( \Gamma \). If \( f \) is continuous on \( \Gamma \) and \( \varphi_0, \ldots, \varphi_n, \psi_0, \ldots, \psi_n \) belong to \( E_2(G) \), then the following estimate for the absolute value of the Hadamard type determinant of order \( n + 1 \) holds:

\[
\left| \int_\Gamma (\varphi_i \psi_j f)(\xi)d\xi \right|^{\alpha}_{i,j=0} \leq \prod_{k=0}^n s_k(A_f) \left( \langle |\varphi_1\rangle, |\varphi_j\rangle \rangle_{L^2(\Gamma)} \right)_{i,j=0}^{1/2} \left( \langle |\psi_1\rangle, |\psi_j\rangle \rangle_{L^2(\Gamma)} \right)_{i,j=0}^{1/2}
\]

(with Gram determinants of order \( n + 1 \) on the right-hand side).

**Proof.** It should be mentioned that if one follows Weyl’s original proof using antisymmetric tensor products (see e.g. B. Simon [17], pp. 6–7), then one gets the desired inequality (see also [15]). However, the construction developed in our proof (see below) is needed and is referred to in Theorem 6. Therefore, for the sake of completeness and the mentioned fact, we also include a proof.

It may be assumed that \( \Gamma \) is positively oriented with respect to \( G \). For each \( 1 \leq i \leq m \), it is known (see [12] for exact details) that there are orthonormal systems \( \{q_{ik}\}_{k \geq 0} \) and \( \{\alpha_{ik}\}_{k \geq 0} \) of the eigenfunctions of the operator \( (A_{f_i}^*, A_{f_i})^{1/2} \) in \( E_2(G) \), corresponding to the singular numbers \( \{s_k(A_{f_i})\}_{k \geq 0} \), such that

\[
(A_{f_i} q_{ik})(\xi) = s_k(A_{f_i}) \alpha_{ik}(\xi) d\xi \quad \text{a.e. on } \Gamma_i.
\]

Let \( n \geq 0 \). By Lemma 3 part (b), there is a pair \( (i, k) \) such that \( s_n(A_{f_i}) = s_k(A_{f_i}) \).

Define \( q_n = (0, \ldots, q_{ik}, \ldots, 0) \) and \( \alpha_n = (0, \ldots, \alpha_{ik}, \ldots, 0) \); i.e. \( q_n \) and \( \alpha_n \) have only one nonzero entry in their \( i \)-th positions, namely \( q_{ik} \) and \( \alpha_{ik} \), and zero elsewhere. By [13], \( \{q_n\}_{n \geq 0} \) and \( \{\alpha_n\}_{n \geq 0} \) are orthonormal systems in \( E_2(G) \) of eigenfunctions of the operator \( (A_{f_i}^*, A_{f_i})^{1/2} \) corresponding to the sequence of the singular numbers
\{s_n(A_f)\}_{n \geq 0}$. In view of (3.11) and the definitions of \(q_n\) and \(\alpha_n\), we get
\begin{equation}
(3.12) \quad (A_f q_n)(\xi) = s_n(A_f) \alpha_n(\xi) d\xi / d\xi \quad \text{a.e. on } \Gamma.
\end{equation}

We can represent (see, for example, [5]) \(\varphi_i\) and \(\psi_j\) \((i, j = 0, 1, \ldots, n)\) as
\[
\varphi_i = \sum_{k=0}^{\infty} c_{ik} q_k + \eta_i \quad \text{and} \quad \psi_j = \sum_{k=0}^{\infty} b_{jk} \alpha_k + \omega_j,
\]
where \(c_{ik} = \langle \varphi_i, q_k \rangle_{L_2(\Gamma)}\), \(b_{jk} = \langle \psi_j, \alpha_k \rangle_{L_2(\Gamma)}\), \(k = 0, 1, \ldots\) Moreover, we have
\[
\eta_i, \omega_j \in \text{Ker}(A_f), \quad \langle \varphi_i - \eta_i, \eta_j \rangle_{L_2(\Gamma)} = 0, \quad \text{and} \quad \langle \psi_j - \omega_j, \omega_j \rangle_{L_2(\Gamma)} = 0 \quad \text{for } i, j = 0, 1, \ldots.
\]

Now if we let \(\varphi_{ik}, \psi_{jk}\) denote the restrictions of \(\varphi_i, \psi_j\) to \(G_k\), then
\begin{equation}
(3.13) \quad \int_{\Gamma} (\varphi_i \psi_j f)(\xi) d\xi = \sum_{k=1}^{m} \int_{G_k} (A_{f_k} \varphi_{ik})(\xi) \psi_{jk}(\xi) d\xi = \int_{\Gamma} (A_f \varphi_i)(\xi) \psi_j(\xi) d\xi.
\end{equation}

Since \(A_f \varphi_i = \sum_{k=0}^{\infty} c_{ik} A_f q_k\), we can conclude with the help of (3.12) that
\[
\int_{\Gamma} (\varphi_i \psi_j f)(\xi) d\xi = \sum_{k=0}^{\infty} s_k(A_f) c_{ik} b_{jk};
\]
that is,
\[
J = \left| \left| \int_{\Gamma} (\varphi_i \psi_j f)(\xi) d\xi \right| \right|_{i,j=0}^{\infty} = \left| \left| \sum_{k=0}^{\infty} s_k(A_f) c_{ik} b_{jk} \right| \right|_{i,j=0}^{n}.
\]

The last expression in the above equality can be expanded using the Binet-Cauchy formula (see [4]),
\begin{equation}
(3.14) \quad J = \frac{1}{(n+1)!} \sum_{k_0=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} s_{k_0}(A_f) \cdots s_{k_n}(A_f) \left| c_{i_0} \right|_{i,j=0}^{n} \left| b_{i_j} \right|_{i,j=0}^{n}.
\end{equation}

By virtue of the Cauchy-Schwarz inequality, we get
\[
J \leq \left( \frac{1}{(n+1)!} \sum_{k_0=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} s_{k_0}(A_f) \cdots s_{k_n}(A_f) \left| c_{i_0} \right|_{i,j=0}^{n} \left| b_{i_j} \right|_{i,j=0}^{n} \right)^{1/2} \times \left( \frac{1}{(n+1)!} \sum_{k_0=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} s_{k_0}(A_f) \cdots s_{k_n}(A_f) \left| b_{i_0} \right|_{i,j=0}^{n} \right)^{1/2}.
\]

Since the sequence \(\{s_n(A_f)\}_{n \geq 0}\) of the singular numbers of \(A_f\) is decreasing, the last inequality implies
\[
J \leq s_0(A_f) \cdots s_n(A_f) \left( \sum_{k=0}^{\infty} c_{ik} \overline{c_{jk}} \right)_{i,j=0}^{n} \left( \sum_{k=0}^{\infty} b_{ik} \overline{b_{jk}} \right)_{i,j=0}^{n}^{1/2} \left( \sum_{k=0}^{\infty} c_{ik} \overline{c_{jk}} \right)_{i,j=0}^{n} \left( \sum_{k=0}^{\infty} b_{ik} \overline{b_{jk}} \right)_{i,j=0}^{n}^{1/2}.
\]

Finally, since \(\langle \varphi_i - \eta_i, \eta_j \rangle_{L_2(\Gamma)} = 0\) and \(\langle \psi_i - \omega_i, \omega_j \rangle_{L_2(\Gamma)} = 0\), where \(i, j = 0, 1, 2, \ldots\), the properties of the Gram determinants (see, for example, [4]) imply
\[
\left| \langle \varphi_i - \eta_i, \varphi_j - \eta_j \rangle_{L_2(\Gamma)} \right|_{i,j=0}^{n} \leq \left| \langle \varphi_i, \varphi_j \rangle_{L_2(\Gamma)} \right|_{i,j=0}^{n}.
\]
and
\[ |\langle \psi_1 - \omega_1, \psi_j - \omega_j \rangle_{L^2(\Gamma)}|_{i,j=0}^n \leq |\langle \psi_1, \psi_j \rangle_{L^2(\Gamma)}|_{i,j=0}^n. \]

This completes the proof of the theorem. \qed

Our next theorem has an important consequence (Corollary 10) with respect to the estimates of errors in meromorphic approximation.

**Theorem 6.** Suppose \( G \) is an \( m \)-domain with its boundary denoted by \( \Gamma \) and let \( F \) denote a compact subset of \( G \). If \( f \) is a continuous function on \( \Gamma \) which has an analytic extension to \( G \setminus F \) and if \( D \) is an \( m_1 \)-domain such that \( F \subset D \) and \( \overline{D} \subset G \), then
\[
\prod_{k=0}^n s_k(A_{f,G}) \leq \prod_{k=0}^n s_k(A_{f,D}) \prod_{k=0}^n s_k^2(J),
\]
where \( s_k(J) \) denotes the \( k \)-th singular number of the restriction operator \( J : E_2(G) \to L^2(\partial D) \) defined by \( J\varphi = \varphi_{|\partial D} \) for all \( \varphi \in E_2(G) \).

**Proof.** Let \( \{q_n\}, \{\alpha_n\}, n = 0, 1, 2, \ldots \) denote the orthonormal systems of eigenfunctions of the operator \( (A_{f,G}^*A_{f,G})^{1/2} \) corresponding to the sequence of singular numbers \( \{s_n(A_{f,G})\} \) as in the proof of Theorem 5. From (3.13) (with \( \varphi_i = q_i \) and \( \psi_j = \alpha_j \)) and formula (3.12) (with \( n = i \)), together with the fact that \( \{\alpha_n\}_{n \geq 0} \) is an orthonormal system in \( E_2(G) \), it follows that
\[
\int_{\Gamma} (q_i\alpha_j)(\xi)f(\xi)d\xi = s_n(A_{f,G})\delta_{i,j}, \quad i, j = 0, 1, 2, \ldots,
\]
where \( \delta_{i,j} \) is Kronecker’s symbol. Thus the product of singular numbers can be written as a determinant of order \( n + 1 \):
\[
\prod_{k=0}^n s_k(A_{f,G}) = \left| \int_{\Gamma} (q_i\alpha_j)(\xi)f(\xi)d\xi \right|_{i,j=0}^n.
\]

Let \( \gamma \) denote the boundary of \( D \). We may also assume that \( \Gamma \) and \( \gamma \) are positively oriented with regard to \( G \) and \( D \), respectively. Since \( q_i, \alpha_j, i, j = 0, 1, 2, \ldots \), belong to \( E_2(G) \) and \( f \) is analytic on \( G \setminus F \), the Cauchy formula yields
\[
\prod_{k=0}^n s_k(A_{f,G}) = \left| \int_{\gamma} (q_i\alpha_j)(t)f(t)dt \right|_{i,j=0}^n.
\]

As a consequence of Theorem 5 one can estimate the right-hand side of the above equality to obtain
\[
\prod_{k=0}^n s_k(A_{f,G}) \leq \prod_{k=0}^n s_n(A_{f,D}) \left( |\langle q_i, q_j \rangle_{L^2(\gamma)}|_{i,j=0}^n \right)^{1/2} \left( |\langle \alpha_i, \alpha_j \rangle_{L^2(\gamma)}|_{i,j=0}^n \right)^{1/2}.
\]

Noting that \( J \) is a compact operator, the Weyl-Horn Theorem (see, for example, [3]) together with the fact \( \langle \alpha_i, \alpha_j \rangle_{L^2(\gamma)} = \langle q_i, q_j \rangle_{L^2(\Gamma)} = \delta_{i,j} \) imply
\[
|\langle q_i, q_j \rangle_{L^2(\gamma)}|_{i,j=0}^n = |\langle J q_i, J q_j \rangle_{L^2(\gamma)}|_{i,j=0}^n \leq \prod_{k=0}^n s_k^2(J)|\langle q_i, q_j \rangle_{L^2(\Gamma)}|_{i,j=0}^n = \prod_{k=0}^n s_k^2(J).
\]
and

\[
|\langle \alpha_i, \alpha_j \rangle_{L_2(\gamma)}|^n_{i,j=0} = \left| \langle J\alpha_i, J\alpha_j \rangle_{L_2(\gamma)} \right|^n_{i,j=0} \\
\leq \prod_{k=0}^{n} s_k^2(J) |\langle \alpha_i, \alpha_j \rangle_{L_2(\gamma)}|^n_{i,j=0} = \prod_{k=0}^{n} s_k^2(J).
\]

Thus the theorem is proved. \(\square\)

In view of Theorem 6 and Lemma 4, we obtain the following.

**Corollary 7.** Under the assumptions of Theorem 3, if the boundary of each \(G_i\) (1 \(\leq i \leq m\)) consists of \(N_i\) closed analytic Jordan curves and \(N = N_1 + \cdots + N_m\), then there is a positive integer \(n^*\) such that for \(n \geq n^*\)

\[
\prod_{k=0}^{n^*-1} s_k(A_{f,G}) \prod_{k=n^*}^{n} \Delta_{k+N-m}(f;G) \leq \prod_{k=0}^{n} \Delta_k(f;D) \prod_{k=0}^{n} s_k^2(J).
\]

We end this section with a result regarding the rate with which the product of the singular numbers of the restriction operator decreases.

**Lemma 8.** Let \(G\) be an \(m\)-domain and suppose \(D\) is an \(m_1\)-domain such that \(\overline{D} \subset G\). Then

\[
\limsup_{n \to \infty} \left( \frac{\prod_{k=0}^{n} s_k^2(J)}{n^2} \right)^{1/n^2} \leq \exp(-1/C(\partial D, \partial G)),
\]

where \(\{s_n(J)\}, n = 0, 1, 2, \ldots\), denotes the sequence of the singular numbers of the restriction operator \(J: E_2(G) \to L_2(\partial D)\).

**Proof.** If \(G\) is a domain \((m = 1)\), it follows from the result of Zaharjuta and Skiba regarding the \(n\)-widths (see [24] and also [9]) that

\[
\lim_{n \to \infty} s_n^{1/n}(J) = \exp(-1/C(\partial D, \partial G)).
\]

From the above result, (3.15) follows easily. To see how this is done in details (see also [12]), denote by \(\{\varphi_n\}, n = 0, 1, 2, \ldots\), the orthonormal sequence of eigenfunctions of \(J^*J\) corresponding to the sequence \(\{s_n(J)\}, n = 0, 1, 2, \ldots\). Since

\[
\langle J\varphi_i, J\varphi_j \rangle_{L_2(\partial D)} = s_i^2(J) \langle \varphi_i, \varphi_j \rangle_{L_2(\partial D)} = s_i^2(J) \delta_{ij},
\]

we have that

\[
\prod_{k=0}^{n} s_k^2(J) = \left| \int_{\partial D} (\varphi_i \varphi_j)(t)dt \right|^{n}_{i,j=0} = \frac{1}{(n+1)!} \int_{\partial D} \cdots \int_{\partial D} \left| \varphi_i(t_j) \right|^{n}_{i,j=0} |dt_0| \cdots |dt_n|.
\]

Next, let \(U\) be any Jordan domain such that \(\overline{D} \subset U \subset \overline{U} \subset G\), and denote by \(g(z, \zeta)\) the Green function (see, for example, [6]) of the domain \(U\) with singularity at \(\zeta \in U\). Using the fact that \(\|\varphi_i\|_{L_2(\partial U)} = 1\) \((i = 0, 1, \ldots)\), we get \(\|\varphi_i\|_{\partial U} \leq C\) for some positive constant \(C\). Consequently,

\[
\max_{i, \zeta \in \partial U} \left| \langle \varphi_i(t), \varphi_i(t) \rangle^{n}_{i,j=0} \right| \leq (n+1)! C^{n+1}.
\]
Moreover, it is easily seen that
\[
F_n(t_0, \ldots, t_n) = \ln |\varphi_i(t_j)|_{i,j=0}^{n} |^2 + 2 \sum_{0 \leq i < j \leq n} g(t_i, t_j)
\]
defines a subharmonic function in \(U\) for each \(t_i\). Now the maximum principle for subharmonic functions together with (3.16) and (3.17) implies
\[
\prod_{k=0}^{n} s_k^2(J) \leq (n + 1)!C_1^{n+1} \exp(-\tau_n),
\]
where \(C_1\) denotes a positive constant and \(\tau_n = \min_{t_i \in \partial D} \left(2 \sum_{0 \leq i < j \leq n} g(t_i, t_j) \right)\).

Using the fact (see, for example, [6])
\[
\lim_{n \to \infty} \frac{\tau_n}{n^2} = \frac{1}{C(\partial D, \partial U)},
\]
we obtain the desired inequality
\[
\limsup_{n \to \infty} \left( \prod_{k=0}^{n} s_k^2(J) \right)^{1/n^2} \leq \exp\left(-1/C(\partial D, \partial U)\right).
\]
The result now follows from the general properties of capacity together with the fact that \(U\) is an arbitrary Jordan domain satisfying \(\bar{D} \subset U \subset \bar{U} \subset G\).

For the general case \((m > 1)\), let \(G\) be a union of domains \(G_1, \ldots, G_m\) with disjoint closures. Set \(D_k = G_k \cap D\). We may further assume that \(D_k \neq \emptyset\) for all \(k = 1, \ldots, m\). Denote by \(J_k : E_2(G_k) \to L_2(\partial D_k), 1 \leq k \leq m\), the corresponding restriction operator. Since \(J = J_1 \oplus \cdots \oplus J_m\), it follows from a similar argument as in Lemma 3 that
\[
(3.18) \quad s_n(J) = \min_{k_1 + \cdots + k_m \leq n} \max \{ s_{k_1}(J_1), \ldots, s_{k_m}(J_m) \}.
\]

Now for each \(i = 1, \ldots, m\), the simple case \(m = 1\) implies
\[
(3.19) \quad \limsup_{n \to \infty} \left( \prod_{k=0}^{n} s_k^2(J_i) \right)^{1/n^2} \leq \exp\left(-1/C(\partial D_i, \partial G_i)\right).
\]
Furthermore (see [12], Lemma 3)
\[
C(\partial D, \partial G) = \sum_{i=1}^{m} C(\partial D_i, \partial G_i),
\]
which together with (3.18) and (3.19) implies (see [12], Lemma 2)
\[
(3.21) \quad \limsup_{n \to \infty} \left( \prod_{k=0}^{n} s_k^2(J) \right)^{1/n^2} \leq \exp\left(- \sum_{i=1}^{m} \frac{w_i^2}{C(\partial D_i, \partial G_i)}\right),
\]
where \(w_i \geq 0\) for all \(1 \leq i \leq m\) and \(\sum_{i=1}^{m} w_i = 1\). For any \(\theta_i \geq 0\) we have (see [12], Lemma 4)
\[
\frac{1}{\sum_{i=1}^{m} \theta_i} \leq \sum_{i=1}^{m} \frac{w_i^2}{\theta_i}.
\]
Finally, letting \(\theta_i = C(\partial D_i, \partial G_i)\), the result follows from (3.20) and (3.21). \(\square\)
4. Proof of Theorem A

Part I (special case). Here we will use the obtained results from the theory of Hankel operators (Section 3) to prove Theorem A under the assumption that $K$ and $E$ are bounded by finitely many disjoint closed analytic Jordan curves. First of all we remark that in view of the mapping $w = 1/(z - a)$, where $a$ is some fixed point of the interior of $K$, we can confine ourselves to the case where the complement of $K$, denoted by $G$, is bounded.

Denote the interior of $E$ by $\Omega$, and let $w(z)$ be the solution of the Dirichlet problem with respect to the boundary values $1$ on $\partial K$ and $0$ on $\partial \Omega$. Extend $w(z)$ by continuity to $\overline{E}$ such that $w(z) = 1$ for $z \in K$ and $w(z) = 0$ for $z \in \overline{E} \setminus E$. Furthermore, for any $0 < \varepsilon < 1$, let $G(\varepsilon) = \{z : w(z) < \varepsilon\}$ and $\gamma(\varepsilon) = \{z : w(z) = \varepsilon\}$, where it is assumed that $\gamma(\varepsilon)$ is positively oriented with respect to the open set $G(\varepsilon)$.

Next choose $0 < \varepsilon < \varepsilon_1 < 1$, sufficiently close to $0$ and $1$, respectively, so that $\gamma(\varepsilon)$ and $\gamma(\varepsilon_1)$ consist of finitely many closed analytic Jordan curves. It is not hard to see that $G(\varepsilon)$ and $G(\varepsilon_1)$ are $m$-domains satisfying $\overline{G(\varepsilon)} \subset G(\varepsilon_1)$. We also assume that $G(\varepsilon_1)$ and $G(\varepsilon)$ consist of $m$ and $m'$ connected components, respectively. Denote the components of $G(\varepsilon_1)$ by $G_1, \ldots, G_m$, where the boundary of each $G_i$ consists of $N_i$ closed analytic Jordan curves. Put $N = N_1 + \cdots + N_m$. Since $f$ is analytic in $\Omega$, we can assert with the aid of Corollary [7] that there exists a positive integer $n^*$ such that for all $n \geq n^*$

\begin{equation}
(4.1) \quad \prod_{k=n^*}^{n} \Delta_{k+N-m}(f; G(\varepsilon_1)) \leq C_1 \prod_{k=0}^{n} \Delta_k(f; G(\varepsilon)) \prod_{k=0}^{n} s_k^2(J),
\end{equation}

where $C_1$ is a positive constant independent of $n$ and $s_k(J)$ is the $k$-th singular number of the restriction operator $J : E_2(G(\varepsilon_1)) \to L_2(\gamma(\varepsilon))$ (see Corollary [7]).

Next we claim there is a constant $C_2 > 0$ such that

\begin{equation}
(4.2) \quad \rho_k(f; K) \leq C_2 \Delta_k(f; G(\varepsilon_1)), \quad k = 0, 1, \ldots.
\end{equation}

To see this, fix a nonnegative integer $k$. It follows from the definition \([3.8]\) that there are nonnegative integers $k_1, \ldots, k_m$ such that $k_1 + \cdots + k_m \leq k$ and

$$\Delta_k(f; G(\varepsilon_1)) = \max(\Delta_{k_1}(f_1; G_1), \ldots, \Delta_{k_m}(f_m; G_m)).$$

Let $h_i, 1 \leq i \leq m$, denote the function of best approximation of $f_i = f|\partial G_i$ in $L_\infty(\partial G_i)$ by meromorphic functions from the class $\mathcal{M}_{k_i}(G_i)$; that is,

$$||f - h_i||_{\infty} = \Delta_k(f_i; G_i).$$

Now \([4.2]\) follows from the above observation together with the estimation of the Cauchy integral formula

$$\langle r - f \rangle(z) = 2 \prod_{i=1}^{m} \frac{1}{2\pi i} \int_{\partial G_i} \frac{(f - h_i)(\xi)}{\xi - z} d\xi,$$

which holds for some $r \in R_k$ and all $z \in K$. 

In view of (4.11) and (4.12), we obtain
\begin{equation}
\prod_{k=0}^{n} \rho_k(f; K) \leq C^n \prod_{k=0}^{n} \Delta_k(f; G(\varepsilon)) \prod_{k=0}^{n} s_k^2(J),
\end{equation}
where $C > 0$ is a constant independent of $n$.

Moreover, it is easy to verify that
\begin{equation}
\Delta_k(f; G(\varepsilon)) \leq \rho_k(f; \gamma(\varepsilon)).
\end{equation}

First we note that for any rational function $r \in \mathcal{R}_k$ with poles off $\gamma(\varepsilon)$ and for any connected component $D_i$, $i = 1, \ldots, m'$, of $G(\varepsilon)$ we have
\[ ||f - r||_{\gamma(\varepsilon)} \geq ||f - r||\partial D_i \geq \Delta_i(f; \partial D_i), \]
where $l_i$ is the number of poles (counted with multiplicities) of $r$ inside $D_i$ and $f_i = f|_{\partial D_i}$. Therefore it follows from (4.3) that
\[ ||f - r||_{\gamma(\varepsilon)} \geq \max(\Delta_i(f_1; \partial D_1), \ldots, \Delta_{m'}(f_{m'}; \partial D_{m'})). \]

Since $l_1 + \cdots + l_{m'} \leq k$, we get $||f - r||_{\gamma(\varepsilon)} \geq \Delta_k(f; G(\varepsilon))$. Thus (4.4) follows from the fact that $r$ is an arbitrary rational function in the class $\mathcal{R}_k$ with poles off $\gamma(\varepsilon)$.

Using (4.4), together with the fact $\gamma(\varepsilon) \subseteq E$, we have
\[ \rho_k(f; \gamma(\varepsilon)) \leq \rho_k(f; E) \quad \text{and} \quad \Delta_k(f; G(\varepsilon)) \leq \rho_k(f; E). \]

Therefore it follows from (4.3) that
\begin{equation}
\prod_{k=0}^{n} \rho_k(f; K) \leq C^n \prod_{k=0}^{n} \rho_k(f; E) \prod_{k=0}^{n} s_k^2(J).
\end{equation}

Combining (4.5) and (3.15), we get
\begin{equation}
\limsup_{n \to \infty} \left( \prod_{k=0}^{n} \rho_k(f; K)/ \prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n^2} \leq \exp(-1/C(\gamma(\varepsilon), \gamma(\varepsilon_1))).
\end{equation}

Since the left-hand side of (4.6) does not depend on $\varepsilon$ and $\varepsilon_1$, the proof of Theorem A, in the special case, follows from the properties of capacities that (see [8, 14])
\[ \lim_{\varepsilon \to 0, \varepsilon_1 \to 1} C(\gamma(\varepsilon), \gamma(\varepsilon_1)) = C(\partial E, K). \]

**Part II (general case).** Here we consider the general case, where $E$ is an arbitrary compact set with nonempty interior $\Omega$ and $K$ is an arbitrary compact subset of $\Omega$.

We start our proof by observing that $\Omega$ is an open cover of $K$. Therefore, there are finitely many open connected components $\Omega_i$ of $\Omega$ such that $\Omega_i \cap K \neq \emptyset$. Let
\[ \Omega' = \bigcup\{\Omega_i : \Omega_i \cap K \neq \emptyset\}. \]
From the properties of the capacity (see [8] or [16]), we have $C(\partial \Omega', K) = C(\partial \Omega, K)$. Since $\partial \Omega \subset \partial E$, it follows that
\begin{equation}
C(\partial \Omega', K) \leq C(\partial E, K).
\end{equation}

It is well known (see [6]) that one can construct two sequences of compact sets $\{K_m\}_{m \geq 1}$ and open sets $\{\Omega_m\}_{m \geq 1}$ which tend monotonically to $K$ and $\Omega'$, respectively. Furthermore, we may also arrange the sequences such that both $K_m$ and
$\Omega_m$ are bounded by finitely many closed analytic Jordan curves, and $K_m \subset \Omega_m$ for $m = 1, 2, \ldots$. More precisely,

$$K_1 \supset K_2 \supset \cdots \supset K, \quad \bigcap_{m=1}^{\infty} K_m = K, \quad \Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega', \quad \text{and} \quad \bigcup_{m=1}^{\infty} \Omega_m = \Omega'. $$

Set $E_m = \overline{\Omega}_m$. Since $K \subset K_m$ and $E_m \subset E$,

$$\rho_n(f; K) \leq \rho_n(f; K_m) \quad \text{and} \quad \rho_n(f; E_m) \leq \rho_n(f; E), \quad \text{for all} \quad n \geq 0.$$

For each fixed $m \geq 1$, using the fact that $K_m$ and $E_m$ are both bounded by finitely many closed analytic Jordan curves, it follows from Part I that

$$\limsup_{n \to \infty} \left( \prod_{k=0}^{n} \rho_k(f; K_m) / \prod_{k=0}^{n} \rho_k(f; E_m) \right)^{1/n} \leq \exp(-1/C(\partial E_m, K_m)).$$

As a consequence of (4.8), we have

$$\limsup_{n \to \infty} \left( \prod_{k=0}^{n} \rho_k(f; K) / \prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n} \leq \exp(-1/C(\partial E_m, K_m)).$$

Now the definition of $E_m$ and $K_m$ together with the properties of the capacity (see Part I) imply

$$\lim_{m \to \infty} C(\partial E_m, K_m) = C(\partial \Omega', K).$$

Hence, taking the limit on the right-hand side of the inequality (4.9) as $m \to \infty$, we obtain

$$\limsup_{n \to \infty} \left( \prod_{k=0}^{n} \rho_k(f; K) / \prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n} \leq \exp(-1/C(\partial \Omega', K)).$$

Finally use (4.7) to conclude the proof of Theorem A.

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