TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 361, Number 5, May 2009, Pages 2377–2395
S 0002-9947(08)04757-0
Article electronically published on November 25, 2008

NILMANIFOLDS OF DIMENSION ≤ 8 ADMITTING ANOSOV DIFFEOMORPHISMS

JORGE LAURET AND CYNTHIA E. WILL

Abstract. After more than thirty years, the only known examples of Anosov diffeomorphisms are topologically conjugated to hyperbolic automorphisms of infranilmanifolds, and even the existence of an Anosov automorphism is a really strong condition on an infranilmanifold. Any Anosov automorphism determines an automorphism of the rational Lie algebra determined by the lattice, which is hyperbolic and unimodular (and conversely ...). These two conditions together are strong enough to make of such rational nilpotent Lie algebras (called Anosov Lie algebras) very distinguished objects. In this paper, we classify Anosov Lie algebras of dimension less than or equal to 8.

As a corollary, we obtain that if an infranilmanifold of dimension $n \leq 8$ admits an Anosov diffeomorphism f and it is not a torus or a compact flat manifold (i.e. covered by a torus), then $n = 6$ or 8 and the signature of f necessarily equals $\{3, 3\}$ or $\{4, 4\}$, respectively.

1. Introduction

A diffeomorphism f of a compact differentiable manifold M is called *Anosov* if it has a global hyperbolic behavior, i.e. the tangent bundle $\mathrm{T}M$ admits a continuous invariant splitting $\mathrm{T}M = E^+ \oplus E^-$ such that $\mathrm{d}f$ expands E^+ and contracts E^- exponentially. These diffeomorphisms define very special dynamical systems, and it is then a natural problem to understand which are the manifolds supporting them (see [Sm]). After more than thirty years, the only known examples are topologically conjugated to hyperbolic automorphisms of infranilmanifolds (called *Anosov automorphisms*), and it is conjectured that any Anosov diffeomorphism is one of these (see [Mr]). The conjecture is known to be true in many particular cases: J. Franks [Fr] and A. Manning [Mn] proved it for Anosov diffeomorphisms on infranilmanifolds themselves; Y. Benoist and F. Labourie [BL] in the case where the distributions E^+, E^- are differentiable and the Anosov diffeomorphism preserves an affine connection (for instance a symplectic form); and J. Franks [Fr] when $\dim E^+ = 1$ (see also [Gh] for the holomorphic case and [Gr] for expanding maps).

It is also important to note that the existence of an Anosov automorphism is a really strong condition on an infranilmanifold. An infranilmanifold is a quotient N/Γ, where N is a nilpotent Lie group and $\Gamma \subset K \ltimes N$ is a lattice (i.e. a discrete cocompact subgroup) which is torsion-free and K is a compact subgroup of $\mathrm{Aut}(N)$.

Received by the editors March 22, 2007.
2000 *Mathematics Subject Classification.* Primary 37D20; Secondary 22E25, 20F34.
This research was supported by CONICET fellowships and grants from FONCyT and Fundación Antorchas.

Among some other more technical obstructions (see [Ma2] for further information), the first natural obstruction for the infranilmanifold N/Γ to admit an Anosov automorphism is that the nilmanifold $N/(\Gamma \cap N)$, which is a finite cover of N/Γ, has to do so.

In the case of a nilmanifold N/Γ (i.e. when $\Gamma \subset N$), any Anosov automorphism determines an automorphism A of the rational Lie algebra $\mathfrak{n}^{\mathbb{Q}} = \Gamma^{\mathbb{Q}}$, the Lie algebra of the rational Mal'cev completion of Γ, which is *hyperbolic* (i.e. $|\lambda| \neq 1$ for any eigenvalue λ of A) and *unimodular* (i.e. $[A]_\beta \in \mathrm{GL}_n(\mathbb{Z})$ for some basis β of $\mathfrak{n}^{\mathbb{Q}}$). Recall that $\mathfrak{n}^{\mathbb{Q}}$ is a rational form of the Lie algebra $\mathfrak{n}$ of N. These two conditions together are strong enough to make of such rational nilpotent Lie algebras (called *Anosov Lie algebras*) very distinguished objects. It is proved in [I] and [De] that if Γ_1 and Γ_2 are commensurable (i.e. $\Gamma_1^{\mathbb{Q}} \simeq \Gamma_2^{\mathbb{Q}}$), then N/Γ_1 admits an Anosov automorphism if and only if N/Γ_2 does. All this suggests that the class of rational Anosov Lie algebras is the key algebraic structure to study if one attempts to classify infranilmanifolds admitting an Anosov diffeomorphism.

Finally, if one is interested in just those Lie groups which are simply connected and which cover such infranilmanifolds, then the objects to be studied are real nilpotent Lie algebras $\mathfrak{n}$ supporting a hyperbolic automorphism A such that $[A]_\beta \in \mathrm{GL}_n(\mathbb{Z})$ for some $\mathbb{Z}$-basis β of $\mathfrak{n}$ (i.e. with integer structure constants). Such Lie algebras will also be called *Anosov*. We note that a real Lie algebra is Anosov if and only if it has an Anosov rational form.

The following would then be a natural program to classify all the infranilmanifolds up to homeomorphism of a given dimension n which admits an Anosov diffeomorphism:

(i) Find all n-dimensional Anosov Lie algebras over $\mathbb{R}$.
(ii) For each real Lie algebra $\mathfrak{n}$ obtained in (i), determine which rational forms of $\mathfrak{n}$ are Anosov.
(iii) For each rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ from (ii), classify up to isomorphism all the lattices Γ in N, the nilpotent Lie group with Lie algebra $\mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R}$, such that $\Gamma^{\mathbb{Q}} = \mathfrak{n}^{\mathbb{Q}}$. In other words, classify up to isomorphism all the lattices in the commensurability class corresponding to $\mathfrak{n}^{\mathbb{Q}}$.
(iv) Given a nilmanifold N/Γ from (iii), decide which of the finitely many infranilmanifolds N/Λ essentially covered by N/Γ (i.e. $\Lambda \cap N \simeq \Gamma$) admits an Anosov automorphism.

Parts (i) and (ii) have been solved for dimension $n \leq 6$ in [CKS] and [Ma1], yielding only two algebras over $\mathbb{R}$: $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ and $\mathfrak{f}_3$ (see Table 1). There is a construction in [L1] (see [LW] for a generalization of it to an arbitrary number of terms) proving that $\mathfrak{n} \oplus \mathfrak{n}$ is Anosov for any real graded nilpotent Lie algebra $\mathfrak{n}$ which admits at least one rational form $\mathfrak{n}^{\mathbb{Q}}$. We note that the existing Anosov rational form is not necessarily $\mathfrak{n}^{\mathbb{Q}} \oplus \mathfrak{n}^{\mathbb{Q}}$. Since for instance any 2-step nilpotent Lie algebra is graded, this construction shows that part (i) of the program above is already a wild problem for n large (see [DM] for recent explicit examples attached to graphs).

The goal of this paper is to approach an explicit classification in small dimensions. We classify up to isomorphism real and rational Lie algebras of dimension ≤ 8 which are Anosov. In other words, we have solved parts (i) and (ii) of the program for $n = 7$ and $n = 8$. We refer to Tables 1 and 3 for a quick look at the results obtained. Without an abelian factor, there are only three 8-dimensional real Lie algebras which are Anosov and none in dimension 7. This is a really small list, bearing

in mind that there exist several one- and two-parameter families and hundreds of isolated examples of 7- and 8-dimensional nilpotent Lie algebras.

One of the corollaries which might be interesting from a dynamical point of view is that if an infranilmanifold of dimension $n \leq 8$ admits an Anosov diffeomorphism f and it is not a torus or a compact flat manifold (i.e. covered by a torus), then $n = 6$ or 8 and the signature of f, defined by $\{\dim E^+, \dim E^-\}$, necessarily equals $\{3, 3\}$ or $\{4, 4\}$, respectively.

We now give an idea of the structure of the proof. The *type* of a nilpotent Lie algebra $\mathfrak{n}$ is the r-tuple $(n_1, ..., n_r)$, where $n_i = \dim C^{i-1}(\mathfrak{n})/C^i(\mathfrak{n})$ and $C^i(\mathfrak{n})$ is the central descending series. By using the fact that any Anosov Lie algebra admits an Anosov automorphism A which is semisimple and some elementary properties of lattices, one sees that only a few types are allowed in each dimension 7 and 8. We then study these types case by case in Section 3 and strongly use the fact that the eigenvalues of A are algebraic integers (even units). For each of the types we get only one or two real Lie algebras (sometimes not one at all) which are candidates to be Anosov. Some of them are excluded by using a criterion given in terms of a homogeneous polynomial (called the *Pfaffian form*) associated to each 2-step nilpotent Lie algebra, and the remainder are proved to be Anosov by exhibiting an Anosov automorphism.

The results on rational forms obtained in [L2] (see Table 2) help us to classify Anosov Lie algebras over $\mathbb{Q}$ in Section 4, and here we also need a criterion on the Pfaffian form to discard some of them, which has in this case integer coefficients, and hence some topics from number theory as the Pell equation and square free numbers appear. Such criteria and most of the known tools to deal with Anosov automorphisms are given in Section 2.

2. Anosov diffeomorphisms and Lie algebras

Anosov diffeomorphisms play an important and beautiful role in dynamics as the notion represents the most perfect kind of global hyperbolic behavior, giving examples of structurally stable dynamical systems. A diffeomorphism f of a compact differentiable manifold M is called *Anosov* if the tangent bundle $\mathrm{T}M$ admits a continuous invariant splitting $\mathrm{T}M = E^+ \oplus E^-$ such that $\mathrm{d}f$ expands E^+ and contracts E^- exponentially; that is, there exist constants $0 < c$ and $0 < \lambda < 1$ such that

$$||\mathrm{d}f^n(X)|| \leq c\lambda^n ||X||, \quad \forall X \in E^-, \qquad ||\mathrm{d}f^n(Y)|| \geq c\lambda^{-n} ||Y||, \quad \forall Y \in E^+,$$

for all $n \in \mathbb{N}$. The condition is independent of the Riemannian metric. Some of the other very nice properties of these special dynamical systems, all proved mainly by D. Anosov, are: the distributions E^+ and E^- are completely integrable with C^∞ leaves and determine two (unique) f-invariant foliations (unstable and stable, respectively) with remarkable dynamical properties; the set of periodic points (i.e. $f^m(p) = p$ for some $m \in \mathbb{N}$) is dense in the set of those points of M such that for any neighborhood U of p there exist $k \neq m \in \mathbb{N}$ with $f^k(U) \cap f^m(U) \neq \emptyset$; the set of all Anosov diffeomorphisms forms an open subset of $\mathrm{Diff}(M)$ (see [V]).

Example 2.1. Let N be a real simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Let φ be a hyperbolic automorphism of N; that is, all the eigenvalues of its derivative $A = (\mathrm{d}\varphi)_e : \mathfrak{n} \longrightarrow \mathfrak{n}$ have absolute value different from 1. If $\varphi(\Gamma) = \Gamma$ for some lattice Γ of N (i.e. a uniform discrete subgroup), then φ defines

an Anosov diffeomorphism on the nilmanifold $M = N/\Gamma$, which is called an *Anosov automorphism*. The subspaces E^+ and E^- are obtained by left translation of the eigenspaces of eigenvalues of A of absolute value greater than 1 and less than 1, respectively, and so the splitting is differentiable. If more in general, Γ is a uniform discrete subgroup of $K \ltimes N$, where K is any compact subgroup of $\mathrm{Aut}(N)$ for which $\varphi(\Gamma) = \Gamma$ (recall that φ acts on $\mathrm{Aut}(N)$ by conjugation), then φ also determines an Anosov diffeomorphism on $M = N/\Gamma$ which is also called an Anosov automorphism. In this case M is called an infranilmanifold and is finitely covered by the nilmanifold $N/(N \cap \Gamma)$ (for the known examples of infranilmanifolds which are not nilmanifolds and admit Anosov automorphisms we refer to [Sh, P, Ma2]).

In [Sm], S. Smale raised the problem of classifying all compact manifolds (up to homeomorphism) which admit an Anosov diffeomorphism. At this moment, the only known examples are of algebraic nature, namely Anosov automorphisms of nilmanifolds and infranilmanifolds described in the example above. It is conjectured that any Anosov diffeomorphism is topologically conjugate to an Anosov automorphism of an infranilmanifold (see [Mr]).

All this certainly highlights the problem of classifying all nilmanifolds which admit Anosov automorphisms, which are easily seen to be in correspondence with a very special class of nilpotent Lie algebras over $\mathbb{Q}$. Nevertheless, not too much is known on the question, since it is not so easy for an automorphism of a (real) nilpotent Lie algebra to be hyperbolic and unimodular at the same time.

Definition 2.2. A rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ (i.e. with structure constants in $\mathbb{Q}$) of dimension n is said to be *Anosov* if it admits a *hyperbolic* automorphism A (i.e. all their eigenvalues have absolute value different from 1) which is *unimodular*; that is, $[A]_\beta \in \mathrm{GL}_n(\mathbb{Z})$ for some basis β of $\mathfrak{n}^{\mathbb{Q}}$, where $[A]_\beta$ denotes the matrix of A with respect to β. A hyperbolic and unimodular automorphism is called an *Anosov automorphism*. We also say that a real Lie algebra is *Anosov* when it admits a rational form which is Anosov. An automorphism of a real Lie algebra $\mathfrak{n}$ is called *Anosov* if it is hyperbolic and $[A]_\beta \in \mathrm{GL}_n(\mathbb{Z})$ for some $\mathbb{Z}$-basis β of $\mathfrak{n}$ (i.e. with integer structure constants).

The unimodularity condition on A in the above definition is equivalent to the fact that the characteristic polynomial of A has integer coefficients and constant term equal to ± 1 (see [De]). It is well known that any Anosov Lie algebra is necessarily nilpotent, and it is easy to see that the classification of nilmanifolds which admit an Anosov automorphism is essentially equivalent to that of Anosov Lie algebras (see [L1, D, I, De]). If $\mathfrak{n}$ is a rational Lie algebra, we call the real Lie algebra $\mathfrak{n} \otimes \mathbb{R}$ the *real completion* of $\mathfrak{n}$.

Let $\mathfrak{n}$ be a nilpotent Lie algebra over K, where K is $\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{C}$.

Definition 2.3. Consider the central descendent series of $\mathfrak{n}$ defined by $C^0(\mathfrak{n}) = \mathfrak{n}$, $C^i(\mathfrak{n}) = [\mathfrak{n}, C^{i-1}(\mathfrak{n})]$. When $C^r(\mathfrak{n}) = 0$ and $C^{r-1}(\mathfrak{n}) \neq 0$, $\mathfrak{n}$ is said to be r-step nilpotent, and we denote by $(n_1, ..., n_r)$ the *type* of $\mathfrak{n}$, where

$$n_i = \dim C^{i-1}(\mathfrak{n})/C^i(\mathfrak{n}).$$

We also take a decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus ... \oplus \mathfrak{n}_r$, a direct sum of vector spaces, such that $C^i(\mathfrak{n}) = \mathfrak{n}_{i+1} \oplus ... \oplus \mathfrak{n}_r$ for all i.

We now give some necessary conditions a real Lie algebra has to satisfy in order to be Anosov (see [Ma1] and [LW]).

Proposition 2.4. *Let $\mathfrak{n}$ be a real nilpotent Lie algebra which is Anosov. Then there exist a decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus ... \oplus \mathfrak{n}_r$ satisfying $C^i(\mathfrak{n}) = \mathfrak{n}_{i+1} \oplus ... \oplus \mathfrak{n}_r$, $i = 0, ..., r$, and a hyperbolic $A \in \mathrm{Aut}(\mathfrak{n})$ such that*

(i) *$A\mathfrak{n}_i = \mathfrak{n}_i$ for all $i = 1, ..., r$.*
(ii) *A is semisimple (in particular A is diagonalizable over $\mathbb{C}$).*
(iii) *For each i, there exists a basis β_i of $\mathfrak{n}_i$ such that $[A_i]_{\beta_i} \in \mathrm{SL}_{n_i}(\mathbb{Z})$, where $n_i = \dim \mathfrak{n}_i$ and $A_i = A|_{\mathfrak{n}_i}$.*

Proposition 2.5. *Let $\mathfrak{n}$ be a real r-step nilpotent Lie algebra of type $(n_1, ..., n_r)$. If $\mathfrak{n}$ is Anosov, then at least one of the following is true:*

(i) *$n_1 \geq 4$ and $n_i \geq 2$ for all $i = 2, ..., r$.*
(ii) *$n_1 = n_2 = 3$ and $n_i \geq 2$ for all $i = 3, ..., r$.*

In particular, $\dim \mathfrak{n} \geq 2r + 2$.

Assume now that $\mathfrak{n}$ is 2-step nilpotent, or equivalently of type (n_1, n_2), n_1 even. Fix a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ (i.e. an inner product) and consider the orthogonal decomposition of the form $\mathfrak{n} = V \oplus [\mathfrak{n}, \mathfrak{n}]$, that is, $\mathfrak{n}_1 = V$. For each $Z \in [\mathfrak{n}, \mathfrak{n}]$ consider the linear transformation $J_Z : V \longrightarrow V$ defined by

$$\langle J_Z X, Y \rangle = \langle [X, Y], Z \rangle, \qquad \forall\, X, Y \in V. \tag{1}$$

Recall that J_Z is skew symmetric with respect to $\langle \cdot, \cdot \rangle$ and the map $J : [\mathfrak{n}, \mathfrak{n}] \longrightarrow \mathfrak{so}(V)$ is linear. The *Pfaffian form* of $\mathfrak{n}$, $f : [\mathfrak{n}, \mathfrak{n}] \longrightarrow K$, is defined by

$$f(Z) = \mathrm{Pf}(J_Z), \qquad Z \in [\mathfrak{n}, \mathfrak{n}],$$

where $\mathrm{Pf} : \mathfrak{so}(V, K) \longrightarrow K$ is the *Pfaffian*, that is, the only polynomial function satisfying $\mathrm{Pf}(B)^2 = \det B$ for all $B \in \mathfrak{so}(V, K)$ and $\mathrm{Pf}(J) = 1$ for

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

If $\dim V = 2m$ and $\dim [\mathfrak{n}, \mathfrak{n}] = k$, then $f = f(x_1, ..., x_k)$ is a homogeneous polynomial of degree m in k variables with coefficients in K. The projective equivalence class of the form $f(x_1, ..., x_k)$ is an isomorphism invariant of the Lie algebra $\mathfrak{n}$ (see [S] or [L2]).

Part (i) of the following proposition is essentially [AS, Theorem 3]

Proposition 2.6. *Let $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ be a real 2-step nilpotent Lie algebra with $\dim \mathfrak{n}_2 = k$. Assume that $\mathfrak{n}$ is Anosov and let $\mathfrak{n}^{\mathbb{Q}}$ denote the rational form which is Anosov.*

(i) *If f is the Pfaffian form of $\mathfrak{n}$, then for any $c > 0$ the region*
$$R_c = \{(x_1, ..., x_k) \in \mathbb{R}^k : |f(x_1, ..., x_k)| \leq c\}$$
is unbounded.
(ii) *For the Pfaffian form f of $\mathfrak{n}^{\mathbb{Q}}$ and for any $p \in \mathbb{Z}$ the set*
$$S_p = \{(x_1, ..., x_k) \in \mathbb{Z}^k : f(x_1, ..., x_k) = p\}$$
is either empty or infinite.

Proof. (i) Consider $A \in \mathrm{Aut}(\mathfrak{n})$ satisfying all the conditions in Proposition 2.4. From the proof of [L2, Proposition 2.4] and the fact that $\det A_i = 1$ for any $i = 1, ..., r$, we obtain

$$f(x_1, ..., x_k) = f(A^t(x_1, ..., x_k)), \qquad \forall\, (x_1, ..., x_k) \in \mathbb{R}^k = \mathfrak{n}_2,$$

and so $A^t R_c \subset R_c$. Assume that R_{c_0} is bounded for some $c_0 > 0$. By using the fact that f is a homogeneous polynomial we get that R_c is bounded for any $c > 0$; indeed, $R_c = c^{-\frac{1}{m}} R_1$ if m is the degree of f. Now, for a sufficiently big $c_1 > 0$ we may assume that R_{c_1} contains the basis β_2 of $\mathfrak{n}_2$, but only finitely many integral linear combinations of elements in this basis can belong to the bounded region R_{c_1}. This implies that $A^t|_{\mathfrak{n}_2}$ leave a finite set of points invariant, and since such a set contains a basis of $\mathfrak{n}_2$ we obtain that $(A^t)^l = I$ for some $l \in \mathbb{N}$. The eigenvalues of A then have to be roots of unity, contradicting its hyperbolicity.

(ii) Analogously to the proof of part (i), we get that $A^t S_p \subset S_p$. If $S_p \neq \emptyset$ and is finite, then for the real subspace $W \subset \mathfrak{n}_2$ generated by S_p we have that $A^t W \subset W$ and $(A^t|_W)^l = I$ for some $l \in \mathbb{N}$, which is again a contradiction by the hyperbolicity of A. □

We now give an example of how the above proposition can be applied.

Example 2.7. Rational Lie algebras of type $(4, 2)$ are parametrized by the set of square free numbers $k \in \mathbb{Z}$, and their Pfaffian forms are $f_k(x, y) = x^2 - ky^2$ (see [L2]). Thus the set of solutions

$$\left\{(x, y) \in \mathbb{Z}^2 \ : \ f_k(x, y) = 1\right\}$$

is infinite if and only if $k > 1$ or $k = 0$ (Pell equation). By Proposition 2.6 (ii) the Lie algebra $\mathfrak{n}_k^{\mathbb{Q}}$ can never be Anosov for $k < 0$ or $k = 1$. Recall that we could also discard $\mathfrak{n}_k^{\mathbb{Q}}$, $k < 0$, as a real Anosov Lie algebra by applying Proposition 2.6 (i).

3. Classification of real Anosov Lie algebras

In this section, we will determine all the real Anosov Lie algebras of dimension ≤ 8. Our starting point is Proposition 2.5, which implies that a nonabelian Anosov Lie algebra has to be of dimension ≥ 6 and gives only a few possibilities for the types in each of dimensions 6, 7 and 8.

We use Proposition 2.4 to make a few observations on the eigenvalues of an Anosov automorphism, which are necessarily algebraic integers. An overview of several basic properties of algebraic numbers needed here is given in [LW, Appendix].

Lemma 3.1. *Let $\mathfrak{n}$ be a real nilpotent Lie algebra which is Anosov, and let A and $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \ldots \oplus \mathfrak{n}_r$ be as in* Proposition 2.4. *If $A_i = A|_{\mathfrak{n}_i}$, then the corresponding eigenvalues $\lambda_1, \ldots, \lambda_{n_i}$ are algebraic units such that $1 < \operatorname{dgr} \lambda_i \leq n_i$ and $\lambda_1 ... \lambda_{n_i} = 1$.*

The proof of this lemma follows from the fact that $[A_i]_{\beta_i} \in \mathrm{SL}_{n_i}(\mathbb{Z})$, and so its characteristic polynomial $p_{A_i}(x) \in \mathbb{Z}[x]$ is a monic polynomial with constant coefficient $a_0 = (-1)^n \det A_i = \pm 1$, satisfying $p_{A_i}(\lambda_j) = 0$ for all $j = 1, \ldots, n_i$.

Concerning the degree, it is clear that $\operatorname{dgr} \lambda_j \leq n_i$ for all j, and if $\operatorname{dgr} \lambda_j = 1$, then $\lambda_j \in \mathbb{Q}$ is a positive unit and therefore $\lambda_j = 1$, contradicting the fact that A_i is hyperbolic.

In the following, $\mathfrak{n}$, A, A_i and $\mathfrak{n}_i$ will be as in the previous lemma. In order to be able to work with eigenvectors, we will always consider the complex Lie algebra $\mathfrak{n}_{\mathbb{C}} = \mathfrak{n} \otimes \mathbb{C}$ and its decomposition $\mathfrak{n}_{\mathbb{C}} = (\mathfrak{n}_1)_{\mathbb{C}} \oplus ... \oplus (\mathfrak{n}_r)_{\mathbb{C}}$, where $(\mathfrak{n}_i)_{\mathbb{C}} = \mathfrak{n}_i \otimes \mathbb{C}$. In light of [LW, Theorem 3.1], we will always assume that $\mathfrak{n}$ has no abelian factor. We now fix more notation that will be used in the rest of this section. For simplicity,

TABLE 1. Notation for some real nilpotent Lie algebras.

Notation	*Type*	*Lie brackets*
$\mathfrak{h}_{2k+1}$	$(2k,1)$	$[X_1,X_2]=Z_1,...,[X_{2k-1},X_{2k}]=Z_1$
$\mathfrak{f}_3$	$(3,3)$	$[X_1,X_2]=Z_1,\ [X_1,X_3]=Z_2,\ [X_2,X_3]=Z_3$
$\mathfrak{g}$	$(6,2)$	$[X_1,X_2]=Z_1,\ [X_1,X_3]=Z_2,\ [X_4,X_5]=Z_1,\ [X_4,X_6]=Z_2$
$\mathfrak{h}$	$(4,4)$	$[X_1,X_3]=Z_1,\ [X_1,X_4]=Z_2,\ [X_2,X_3]=Z_3,\ [X_2,X_4]=Z_4$
$\mathfrak{l}_4$	$(2,1,1)$	$[X_1,X_2]=X_3,\ [X_1,X_3]=X_4$

assume that $\mathfrak{n}$ is a 2-step nilpotent Lie algebra. According to Proposition 2.4, there exist

$$\beta_1=\{X_1,X_2,\dots,X_{n_1}\} \qquad \text{and} \qquad \beta_2=\{Z_1,Z_2,\dots,Z_{n_2}\},$$

the basis of eigenvectors of $(\mathfrak{n}_1)_{\mathbb{C}}$ and $(\mathfrak{n}_2)_{\mathbb{C}}$ for A_1 and A_2, respectively. Let $\lambda_1,\dots,\lambda_{n_1}$ and $\mu_1,\dots,\mu_{n_2}$ be the corresponding eigenvalues. This notation will be used throughout the classification. The absence of an abelian factor implies that $[\mathfrak{n}_1,\mathfrak{n}_1]=\mathfrak{n}_2$, and hence we may assume that for each Z_i there exist X_j and X_l such that $Z_i=[X_j,X_l]$. On the other hand, for each j,l, there exist scalars $a_k^{j,l}\in\mathbb{C}$ such that $[X_j,X_l]=\sum a_k^{j,l}Z_k$. Since $\{Z_k\}$ are linearly independent, for each k we obtain

$$\lambda_j\lambda_l a_k^{j,l}=\mu_k a_k^{j,l}. \tag{2}$$

Hence, if $a_k^{j,l}\neq 0$, $\mu_k=\lambda_j\lambda_l$, and therefore, if $a_k^{j,l}\neq 0\neq a_{k'}^{j,l}$, $\mu_k=\mu_{k'}$. In particular, if $n_2=2$, since $\mu_1\neq\mu_2$, for each j,l, there exists a unique k such that $[X_j,X_l]=a_kZ_k$. If it is so, by (2) $\lambda_j\lambda_l=\mu_k$. When $n_2=3$ the same property holds. Indeed, $\mu_i\neq\mu_j$ for all $i\neq j$ since $\operatorname{dgr}\mu_i>1$ for all i.

We are going to consider all the possible coefficients $a_k^{j,l}$ only in the cases when the classification actually leads to a possible Anosov Lie algebra.

Dimension 6

Anosov Lie algebras of dimension ≤ 6 have already been classified in [Ma1] and [CKS]. We give an alternative proof here in order to illustrate our approach. Proposition 2.5 gives us the following possibilities for the types of a real Anosov Lie algebra without an abelian factor: $(3,3)$ and $(4,2)$.

Case $(3,3)$. The only real (resp. rational) Lie algebra of type $(3,3)$ is the free 2-step nilpotent Lie algebra on 3 generators $\mathfrak{f}_3$ (resp. $\mathfrak{f}_3^{\mathbb{Q}}$), which is proved to be Anosov in [D] and [De, Ma1].

Case $(4,2)$. Let $\mathfrak{n}$ be a real nilpotent Lie algebra of type $(4,2)$, admitting a hyperbolic automorphism A as in Proposition 2.4. If $\{X_1,\dots X_4\}$ is a basis of

$(\mathfrak{n}_1)_{\mathbb{C}}$ of eigenvectors of A_1 with corresponding eigenvalues $\lambda_1, \dots, \lambda_4$, then without loss of generality we may assume that we are in one of the following cases:

(a) $[X_1, X_2] = Z_1, \qquad [X_1, X_3] = Z_2,$
(b) $[X_1, X_2] = Z_1, \qquad [X_3, X_4] = Z_2.$

In the first situation, (a) implies that $\lambda_1^2\lambda_2\lambda_3 = 1$, and since $\det A_1 = \lambda_1\lambda_2\lambda_3\lambda_4 = 1$, we obtain that $\lambda_1 = \lambda_4$. From this it is easy to see that $\operatorname{dgr}\lambda_1 = \operatorname{dgr}\lambda_4 = 2$ and, moreover, $\lambda_2 = \lambda_3 = \lambda_1^{-1}$. Therefore, we get to the contradiction $\mu_1 = \mu_2 = 1$.

Concerning (b), we may assume that there are no more Lie brackets among the $\{X_i\}$, since otherwise we will be in situation (a), and thus $\mathfrak{n}_{\mathbb{C}} \simeq (\mathfrak{h}_3 \oplus \mathfrak{h}_3)_{\mathbb{C}}$. This Lie algebra has two real forms: $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ and $\mathfrak{n}_{-1}^{\mathbb{Q}} \otimes \mathbb{R}$ (see [L2, Proposition 3.2]). The Lie algebra $\mathfrak{n}_{-1}^{\mathbb{Q}} \otimes \mathbb{R}$ cannot be Anosov by Proposition 2.6 (i), and $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ was proved to be Anosov in [Sm].

Dimension 7

According to Proposition 2.5, if $\mathfrak{n}$ is a 7-dimensional real Anosov Lie algebra of type $(n_1, n_2, \dots, n_r)$, then $r = 2$ and $\mathfrak{n}$ is either of type $(4,3)$ or $(5,2)$. We shall prove that there are no Anosov Lie algebras of any of these types.

Case $(4,3)$. It is easy to see that the eigenvalues of A_2 are three pairs of the form $\lambda_i\lambda_j$, so without any loss of generality we can assume that two of them are $\lambda_1\lambda_2$ and $\lambda_1\lambda_3$. There are four possibilities for the third eigenvalue of A_2, and by using that $\det A_1 = 1$ and $\det A_2 = 1$ we get a contradiction in all the cases as follows:

(i) $\lambda_1\lambda_2.\lambda_1\lambda_3.\lambda_1\lambda_4 = 1$, then $\lambda_1^2 = 1$, contradicting the hyperbolicity of A_1.
(ii) $\lambda_1\lambda_2.\lambda_1\lambda_3.\lambda_2\lambda_3 = 1$ implies that $\lambda_4^2 = 1$, but then A_1 is not hyperbolic.
(iii) $\lambda_1\lambda_2.\lambda_1\lambda_3.\lambda_2\lambda_4 = 1$, then $\lambda_1\lambda_2 = 1$ and so A_2 would not be hyperbolic.
(iv) $\lambda_1\lambda_2.\lambda_1\lambda_3.\lambda_3\lambda_4 = 1$, so $\lambda_1\lambda_3 = 1$, contradicting the hyperbolicity of A_2.

Case $(5,2)$. Let $\mathfrak{n}$ be a real nilpotent Lie algebra of type $(5,2)$, admitting a hyperbolic automorphism A as in Proposition 2.4. If $\lambda_1, \dots, \lambda_5$ are the eigenvalues of A_1 we can either have

(i) $\lambda_i \neq \lambda_j$, $1 \leq i, j \leq 5$, or
(ii) after reordering if necessary, $\lambda_1 = \lambda_2$.

Note that in (ii), $\lambda_1 = \lambda_2$ implies that $2 \leq 2\operatorname{dgr}\lambda_1 \leq 5$ and therefore $\operatorname{dgr}\lambda_1 = \operatorname{dgr}\lambda_2 = 2$. From this it is easy to see that there exist $i \in \{3,4,5\}$ such that $\operatorname{dgr}\lambda_i = 1$, contradicting the hyperbolicity of A_1. Therefore, we assume (i).

On the other hand, since $\dim \mathfrak{n}_2 = 2$, we have two linearly independent Lie brackets among the $\{X_i\}$, the basis of $(\mathfrak{n}_1)_{\mathbb{C}}$ of eigenvectors of A_1. Note that if they come from disjoint pairs of X_i, since $\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 = 1$, it is clear that we would have $\lambda_i = 1$ for some $1 \leq i \leq 5$. Therefore, without any loss of generality we can only consider the case when we have at least the following nontrivial Lie brackets:

$$[X_1, X_2] = Z_1, \qquad [X_1, X_3] = Z_2. \tag{3}$$

In the following we will show that either X_4 or X_5 are in the center of $\mathfrak{n}$, which would generate an abelian factor and hence a contradiction. From (3) we have that

$$\lambda_1^2\lambda_2\lambda_3 = 1, \quad \text{and then} \quad \lambda_4\lambda_5 = \lambda_1. \tag{4}$$

Therefore, $[X_4, X_5] = 0$ because both of the assumptions $[X_4, X_5] = cZ_1$ and $[X_4, X_5] = cZ_2$ with $c \neq 0$ lead to the contradictions $\lambda_2 = 1$ and $\lambda_3 = 1$, respectively. Also, if $[X_4, X_j] \neq 0$ and $[X_5, X_k] \neq 0$ for some $1 \leq j, k \leq 3$, it follows from (i) that we only have the following possibilities:

$$\begin{array}{lll} [X_4, X_3] = cZ_1 & \text{and} & [X_5, X_2] = dZ_2, \quad \text{or} \\ [X_5, X_3] = cZ_1 & \text{and} & [X_4, X_2] = dZ_2 \end{array}$$

$(c, d \neq 0)$ which are clearly equivalent. Let us then suppose the first one, and hence

$$\text{I. } \lambda_3\lambda_4 = \lambda_1\lambda_2 \quad \text{and} \quad \text{II. } \lambda_5\lambda_2 = \lambda_1\lambda_3.$$

From I, and using the fact that $\lambda_4\lambda_5 = \lambda_1$, we obtain $\lambda_3 = \lambda_2\lambda_5$. Therefore by II, $\lambda_1 = 1$, which is a contradiction, and then $[X_4, X_j] = 0$ for all j or $[X_5, X_k] = 0$ for all k, as we wanted to show.

Dimension 8

In this case, Proposition 2.5 gives us the following possibilities for the types of a real Anosov Lie algebra without an abelian factor: $(4, 4)$, $(5, 3)$, $(6, 2)$, $(3, 3, 2)$ and $(4, 2, 2)$. Among all these Lie algebras we will show that there is, up to isomorphism, only three which are Anosov. One is of type $(4, 2, 2)$, one of type $(6, 2)$ and one of type $(4, 4)$. The first one is an example of the construction given in [L1], the second one is isomorphic to [L1, Example 3.3] and the last one belongs to the family given in [DM].

It has been proved in [LW, Section 4] that the types $(5, 3)$ and $(3, 3, 2)$ are not possible for Anosov Lie algebras. We now consider the other three types allowed by Proposition 2.5

Case $(4, 4)$. We will show that there is only one real Anosov Lie algebra of this type. We first note that there is only $\binom{4}{2} = 6$ possible linearly independent brackets among the $\{X_i\}$, and since $\dim[\mathfrak{n}, \mathfrak{n}] = 4$, at most two of them can be zero. Therefore, without loss of generality, we can just consider the following two cases:

$$(5) \qquad [X_1, X_3] = Z_1, \qquad [X_2, X_4] = Z_2, \qquad [X_2, X_3] = Z_3, \qquad [X_1, X_4] = Z_4;$$

that is, the possible zero brackets correspond to disjoint pairs of $\{X_i\}$ (namely $\{X_1, X_2\}$ and $\{X_3, X_4\}$); the other case is

$$(6) \qquad [X_1, X_4] = Z_1, \qquad [X_2, X_4] = Z_2, \qquad [X_3, X_4] = Z_3, \qquad [X_2, X_3] = Z_4,$$

corresponding to the case of nondisjoint pairs, $\{X_1, X_2\}$ and $\{X_1, X_3\}$.

However, the second case is not possible because we would have

$$\text{I) } \lambda_1\lambda_2\lambda_3\lambda_4 = 1 \qquad \text{and} \qquad \text{II) } \lambda_1\lambda_2^2\lambda_3^2\lambda_4^3 = 1.$$

It follows that $\lambda_2\lambda_3 = \lambda_4^{-2}$, and replacing this in I) we get $\lambda_1 = \lambda_4$. This implies that the λ_i's all have degree two, and $\lambda_2 = \lambda_3 = \lambda_4^{-1}$. Hence $\mu_3 = \lambda_3\lambda_4 = 1$, contradicting the hyperbolicity of A_2.

Concerning case (5), if we assume that $[X_1, X_2] = 0$ and $[X_3, X_4] = 0$, then

$$A = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}, \quad \text{where} \quad A_1 = \begin{bmatrix} l & & & \\ & l^{-1} & & \\ & & l^2 & \\ & & & l^{-2} \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} l^3 & & & \\ & l^{-3} & & \\ & & l & \\ & & & l^{-1} \end{bmatrix}$$

is an automorphism of $\mathfrak{n}$ for any $\lambda \in \mathbb{R}^*$. If $\lambda \in \mathbb{R}^*$ is an algebraic integer such that $\lambda + \lambda^{-1} = 2a$, $a \in \mathbb{Z}$, $a \geq 2$, then it is easy to check that

$$\beta = \begin{aligned} &\Big\{X_1 + X_2,\ (a^2-1)^{\frac{1}{2}}(X_1 - X_2),\ X_3 + X_4,\ (a^2-1)^{\frac{1}{2}}(X_3 - X_4), \\ &Z_1 + Z_2,\ (a^2-1)^{\frac{1}{2}}(Z_1 - Z_2),\ Z_3 + Z_4,\ (a^2-1)^{\frac{1}{2}}(Z_3 - Z_4)\Big\} \end{aligned} \tag{7}$$

is a $\mathbb{Z}$-basis of $\mathfrak{n}$. Moreover, if $B = \left[\begin{smallmatrix} a & a^2-1 \\ 1 & a \end{smallmatrix}\right]$, then the matrix of A in terms of the basis β is given by

$$[A]_\beta = \begin{bmatrix} B & & & \\ & B^2 & & \\ & & B^3 & \\ & & & B \end{bmatrix} \in \mathrm{SL}(8, \mathbb{Z}),$$

showing that $\mathfrak{n}$ is Anosov. Recall that this $\mathfrak{n}$ is isomorphic to the Lie algebra $\mathfrak{h}$ given in Table 1.

It follows from Scheuneman duality that there is only one more real form of $\mathfrak{h}_\mathbb{C}$, namely, the dual of the Lie algebra $\mathfrak{n}_{-1}^{\mathbb{Q}} \otimes \mathbb{R}$ of type $(4,2)$ ($\mathfrak{h}$ is dual of $\mathfrak{h}_3 \oplus \mathfrak{h}_3$). The fact that such a Lie algebra is not Anosov will be proved in Section 4, Case $\mathfrak{h}$.

We will now show that if we add any more nonzero brackets in case (5), then the new Lie algebra $\tilde{\mathfrak{n}}$ does not admit a hyperbolic automorphism any longer. Suppose then that

$$0 \neq [X_1, X_2] = a_1 Z_1 + a_2 Z_2 + a_3 Z_3 + a_4 Z_4.$$

As we have already pointed out at the beginning of this classification, since A is an automorphism and Z_i are linearly independent, it follows that if $a_j \neq 0$, then $\lambda_1 \lambda_2 = \mu_j$. Therefore, at most two of them can be nonzero.

If $[X_1, X_2] = a_j Z_j$, then we can change Z_j by $\tilde{Z}_j = a_j Z_j$ and the corresponding bracket in (5) by $[X_1, X_2]$, and we will be in the conditions of case (6).

If $[X_1, X_2] = a_j Z_j + a_k Z_k$, $a_j,\ a_k \neq 0$, then we have that $\lambda_1 \lambda_2 = \mu_j = \mu_k$. One can check that for all the choices of j, k we obtain $\lambda_i = \lambda_r = \lambda_s$ for some $1 \leq i, r, s \leq 4$, which is not possible because it implies that $2 \leq 3 \operatorname{dgr} \lambda_i \leq 4$ and then $\operatorname{dgr} \lambda_i = 1$.

Hence we get $[X_1, X_2] = 0$, and by using the same argument we also obtain $[X_3, X_4] = 0$, as we wanted to show.

We also note that for any choice of nonzero scalars a, b, c, d, the Lie algebra $\tilde{\mathfrak{n}}$ given by

$$[X_1, X_3] = aZ_1, \qquad [X_2, X_4] = bZ_2, \qquad [X_2, X_3] = cZ_3, \qquad [X_1, X_4] = dZ_4,$$

is isomorphic to $\mathfrak{n}$.

Case $(6,2)$**.** We will prove in this case that there is, up to isomorphism, only one Anosov Lie algebra with no abelian factor. As usual, let A be an Anosov automorphism of $\mathfrak{n}$ and $\{X_1, \ldots, X_6, Z_1, Z_2\}$ a basis of $\mathfrak{n}_\mathbb{C}$ of eigenvectors of A, with $\lambda_1, \ldots, \lambda_6, \mu_1, \mu_2$ the eigenvalues as above.

As we have mentioned before, since $\mu_1 \neq \mu_2$, for all i, j there exists k such that $[X_i, X_j] \in \mathbb{C} Z_k$. Also, if $\dim [X_i, (\mathfrak{n}_1)_\mathbb{C}] = 1$ for any i, then $\mathfrak{n}_\mathbb{C}$ is either isomorphic to $(\mathfrak{h}_3 \oplus \mathbb{R} \oplus \mathfrak{h}_3 \oplus \mathbb{R})_\mathbb{C}$ or $(\mathfrak{h}_3 \oplus \mathfrak{h}_5)_\mathbb{C}$. The first one has two real forms: $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^2$ and $(\mathfrak{n}_{-1}^{\mathbb{Q}} \otimes \mathbb{R}) \oplus \mathbb{R}^2$, of which only $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^2$ is Anosov by [LW, Theorem 3.1] and the classification in dimension 6. The only real form of $(\mathfrak{h}_3 \oplus \mathfrak{h}_5)_\mathbb{C}$ is $\mathfrak{h}_3 \oplus \mathfrak{h}_5$, and $\mathfrak{h}_3 \oplus \mathfrak{h}_5$ has only one rational form with Pfaffian form $f(x, y) = xy^2$ (see [L2]). It then follows from Proposition 2.6 (ii) that it is not Anosov.

Therefore, we can assume that

$$[X_1, X_2] = Z_1, \qquad [X_1, X_3] = Z_2. \tag{8}$$

From this, one has that

$$\lambda_1^2 \lambda_2 \lambda_3 = 1, \quad \text{or equivalently} \quad \lambda_1 = \lambda_4 \lambda_5 \lambda_6. \tag{9}$$

In what follows, we will first show that there exists a reordering β of $\{X_1, ..., X_6\}$ such that

$$[A_1]_\beta = \begin{bmatrix} l & & & & & \\ & \lambda^{-1} & & & & \\ & & \nu & & & \\ & & & \nu^{-1} & & \\ & & & & \mu & \\ & & & & & \mu^{-1} \end{bmatrix}, \tag{10}$$

and after that, we will see that this implies that $\mathfrak{n}_{\mathbb{C}} \simeq \mathfrak{g}_{\mathbb{C}}$, the complexification of the Lie algebra $\mathfrak{g}$, which is proved to be Anosov in [L1, Example 3.3]. Moreover, $\mathfrak{g}$ is known to be the only real form of $\mathfrak{g}_{\mathbb{C}}$ (see [L2, Remark 4.2]).

To do this, let us first assume that

a) $\lambda_i = \lambda_l$, denoted by λ, for some $1 \leq i \neq j \leq 6$.

Thus $\operatorname{dgr} \lambda = 2$ or $\operatorname{dgr} \lambda = 3$, but $\operatorname{dgr} \lambda = 3$ is not possible. In fact, if $\operatorname{dgr} \lambda = 3$, then there exists a reordering of $\{X_i\}$ such that the matrix of A_1 in the new basis is

$$A_1 = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \quad \text{where} \quad B = \begin{bmatrix} \lambda & & \\ & \mu & \\ & & (\lambda\mu)^{-1} \end{bmatrix}$$

is conjugated to an element in $\mathrm{SL}_3(\mathbb{Z})$. This says that $\lambda_1, \lambda_2, \lambda_3 \in \left\{\lambda, \mu, (\mu\lambda)^{-1}\right\}$, and using (9) one can see that $\lambda_1 = \lambda_2$ (or equivalently $\lambda_1 = \lambda_3$), since every other choice ends up in a contradiction. Therefore, we may assume that $\lambda_1 = \lambda_2 = \lambda$, and so $\lambda_3 = \lambda^{-3} = \mu$. Since every eigenvalue of A_1 has multiplicity 2, we have that after a reordering if necessary, $\lambda_4 = \lambda_5 = \lambda^2$ and $\lambda_6 = \lambda^{-3}$. Therefore, the matrix of A in the basis $\beta = \{X_1, X_2, ..., X_6, Z_1, Z_2\}$ is given by $[A]_\beta = \left[\begin{smallmatrix} A_1 & \\ & A_2 \end{smallmatrix}\right]$, where

$$A_1 = \begin{bmatrix} l & & & & & \\ & l & & & & \\ & & \lambda^{-3} & & & \\ & & & \lambda^2 & & \\ & & & & \lambda^2 & \\ & & & & & \lambda^{-3} \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} \lambda^2 & \\ & \lambda^{-2} \end{bmatrix}.$$

Hence, since A is an automorphism of $\mathfrak{n}$, one gets that $X_4, X_5 \in (\mathfrak{z} \cap \mathfrak{n}_1)_{\mathbb{C}}$, contradicting our assumption of no abelian factor. Thus $\operatorname{dgr} \lambda = 2$, from where assertion (10) easily follows.

On the other hand, if

b) $\lambda_i \neq \lambda_j$ for all $i \neq j$,

with no loss of generality, we can assume that $[X_4, X_j] = aZ_1$, $a \neq 0$, for some $j \in \{3, 5, 6\}$.

If $j = 5$, then it follows from $1 = \det A_2 = \lambda_4 \lambda_5 \lambda_1 \lambda_3$ that

$$\lambda_2 \lambda_6 = 1. \tag{11}$$

Now, we also have that $[X_6, X_k] \neq 0$ for some k, and hence it is easy to see that we can either have

I) $[X_6, X_3] = bZ_1$, $b \neq 0$, or
II) $[X_6, X_4] = cZ_2$, $c \neq 0$ (or equivalently $[X_6, X_5] = cZ_2$).

In case I), $\lambda_6\lambda_3 = \lambda_1\lambda_2$, and so by (11) we have that $\lambda_3 = \lambda_1\lambda_2^2$. By using (9) we get to the contradiction $\mu_1 = 1$.

Concerning II), since $\lambda_6\lambda_4 = \lambda_1\lambda_3$, we obtain from (9) that $\lambda_5\lambda_3 = 1$ and therefore $\lambda_1\lambda_4 = 1$. This together with (11) implies assertion (10). The case when $j = 6$ is entirely analogous to the case $j = 5$, and so we are not going to consider it.

If $j = 3$, then $\lambda_4\lambda_3 = \lambda_1\lambda_2$, and by (9) it is easy to see that

$$\lambda_3 = \lambda_5\lambda_6\lambda_2. \tag{12}$$

Analogously to the previous case, since $[X_5, X_k] \neq 0$ for some k, it is easy to see that we can either have

I) $[X_5, X_6] = b\, Z_1$, or
II) $[X_5, X_2] = c\, Z_2$ (or equivalently $[X_5, X_4] = cZ_2$).

It is easy to deduce from situation I) that (12) implies $\mu_2 = \mu_1^2$, and so both of them are equal to 1, contradicting the fact that A_2 is hyperbolic.

In case II), $\lambda_5\lambda_2 = \lambda_1\lambda_3$, and it follows from (12) that $\lambda_6\lambda_1 = 1$. Also, since $\mathfrak{n}$ has no abelian factor, it is easy to see that $[X_6, X_4] = dZ_2$, $d \neq 0$, and therefore $\lambda_6\lambda_4 = \lambda_1\lambda_3$. Hence, using (9) we obtain $\lambda_2\lambda_4 = 1$, from where assertion (10) follows.

To finish the proof we must study the case when (10) holds; that is,

$$A_1 = \begin{bmatrix} A_\lambda & & \\ & A_\nu & \\ & & A_\mu \end{bmatrix}, \quad \text{where } A_\eta = \begin{bmatrix} \eta & \\ & \eta^{-1} \end{bmatrix}.$$

Let $\lambda_1 = \lambda$, $\lambda_2 = \nu$ and thus, by (9), $\lambda_3 = \frac{1}{\lambda^2\nu}$. It is easy to see that λ_3 is different from λ^{-1} or ν^{-1}. Therefore, after a reordering if necessary, we have that

$$A_1 = \begin{bmatrix} l & & & & \\ & \nu & & & \\ & & (\lambda^2\nu)^{-1} & & \\ & & & \lambda^2\nu & \\ & & & & \lambda^{-1} \\ & & & & & \nu^{-1} \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} \lambda\nu & \\ & (\lambda\nu)^{-1} \end{bmatrix}.$$

Using that A is an automorphism, one can see that $[V_1, V_2] = 0$, where $V_1 = \langle X_1, X_2, X_3\rangle_{\mathbb{C}}$ and $V_2 = \langle X_4, X_5, X_6\rangle_{\mathbb{C}}$. Moreover, since $\mathfrak{n}_{\mathbb{C}}$ has no abelian factor, $[V_1, V_2] = \langle Z_1, Z_2\rangle_{\mathbb{C}}$. From the classification of 2-step nilpotent Lie algebras with 2-dimensional derived algebra in terms of Pfaffian forms given in [L2], it follows that there is only one Lie algebra satisfying these conditions and so $\mathfrak{n}_{\mathbb{C}}$ is isomorphic to $\mathfrak{g}_{\mathbb{C}}$, as was to be shown.

Case $(4, 2, 2)$. Let $\mathfrak{n}$ be a nilpotent Lie algebra of type $(4, 2, 2)$ and let A be an hyperbolic automorphism with eigenvectors $\{X_1, \ldots, X_4, Y_1, Y_2, Z_1, Z_2\}$, a basis of $\mathfrak{n}_{\mathbb{C}}$, and corresponding eigenvalues $\lambda_1, \ldots, \lambda_4, \eta_1, \eta_2, \mu_1, \mu_2$ as in Proposition 2.5.

Since $\eta_i = \lambda_j\lambda_k$ we have the following two possibilities:

(I) In the decomposition of $\eta_1\eta_2$ as a product of λ_i, at least one of the λ_i's appears twice, or
(II) $\eta_1 = \lambda_1\lambda_2$, $\eta_2 = \lambda_3\lambda_4$, and $\lambda_i \neq \lambda_j$ for $1 \leq i, j \leq 4$.

In the first case we can either have

$$\text{(a) } \eta_1 = \lambda_1\lambda_2, \quad \eta_2 = \lambda_1\lambda_3, \quad \text{(b) } \eta_1 = \lambda_1^2, \quad \eta_2 = \lambda_2\lambda_3, \quad \text{or} \quad \text{(c) } \eta_1 = \lambda_1^2, \quad \eta_2 = \lambda_1^{-2}.$$

Note that (a) and (b) implies that $\lambda_1\lambda_2.\lambda_1\lambda_3 = 1$, and hence $\lambda_4 = \lambda_1$. Thus $\operatorname{dgr}\lambda_4 = \operatorname{dgr}\lambda_1 = 2$, and moreover, $\lambda_2 = \lambda_3 = \pm\lambda_1^{-1}$. Therefore in case (a) we get the contradiction $\eta_1 = \eta_2 = \pm 1$, and case (b) becomes (c).

So it remains to study case (c). There is no loss of generality in assuming that $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 = \lambda_4 = \lambda^{-1}$, and from this, using the Jacobi identity, it is easy to see that the possible nonzero brackets are

$$\begin{aligned} &[X_1, X_2] = Y_1, \quad [X_2, Y_1] = a\, Z_1, \quad [X_1, Y_1] = a'\, Z_1, \\ &[X_3, X_4] = Y_2, \quad [X_3, Y_2] = b\, Z_2, \quad [X_4, Y_2] = b'\, Z_2. \end{aligned} \tag{13}$$

Since $\mathfrak{n}_{\mathbb{C}}$ has no abelian factor, we have that $a \neq 0$ or $a' \neq 0$ and $b \neq 0$ or $b' \neq 0$. Let $\mathfrak{n}_0$ be the ideal of $\mathfrak{n}_{\mathbb{C}}$ generated by $\{X_1, X_2, Y_1, Z_1\}$ and $\mathfrak{n}_0'$ the ideal generated by $\{X_3, X_4, Y_2, Z_2\}$. By the above observation, they are both four-dimensional 3-step complex nilpotent Lie algebras. It is well known that there is up to isomorphism only one of such Lie algebras, and therefore $\mathfrak{n}_0$ and $\mathfrak{n}_0'$ are both isomorphic to $(\mathfrak{l}_4)_{\mathbb{C}}$ and $\mathfrak{n}_{\mathbb{C}} = (\mathfrak{l}_4 \oplus \mathfrak{l}_4)_{\mathbb{C}}$. We know that $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ is the only real form of $(\mathfrak{l}_4 \oplus \mathfrak{l}_4)_{\mathbb{C}}$ (see [L2]), and it is proved to be Anosov in [L1]. This concludes case (I).

We will now study case (II). We can assume that

$$[X_1, X_2] = Y_1, \qquad [X_3, X_4] = Y_2. \tag{14}$$

Moreover, due to our assumption it is easy to see that there are no more nontrivial Lie brackets among them. On the other hand, we have that $Z_i \in \mathfrak{n}_3$, and then for each $i = 1, 2$

$$Z_i = [X_{j_i}, Y_{k_i}].$$

If $k_1 = k_2$ we may assume that $k_1 = k_2 = 1$. By using the Jacobi identity and the previous observation, one can see that $j_1, j_2 \notin \{3, 4\}$, and hence we get

$$[X_1, Y_1] = Z_1, \qquad [X_2, Y_1] = Z_2.$$

From this we have that $\lambda_1.\lambda_1\lambda_2.\lambda_2.\lambda_1\lambda_2 = 1$, and therefore $\lambda_1^3\lambda_2^3 = 1$, a contradiction.

Otherwise, we can assume that $k_1 = 1$ and $k_2 = 2$. Therefore $\lambda_{j_1}\lambda_1.\lambda_{j_2}\lambda_2.\lambda_3\lambda_4 = 1$, and then $\lambda_{j_1}\lambda_{j_2} = 1$. Hence $j_1 \neq j_2$, and since $\lambda_1\lambda_2 \neq 1$ and $\lambda_3\lambda_4 \neq 1$ we can suppose that $\lambda_1\lambda_3 = 1$ and $\lambda_2\lambda_4 = 1$. Without any loss of generality we can assume that

$$[X_1, Y_1] = Z_1 \quad \text{and} \quad [X_3, Y_2] = Z_2, \tag{15}$$

since by Jacobi $[X_1, Y_2] = [X_3, Y_1] = 0$. Note that we have obtained that the matrix of A is given by

$$[A_1] = \begin{bmatrix} \lambda & & & \\ & \nu & & \\ & & \lambda^{-1} & \\ & & & \nu^{-1} \end{bmatrix}, \quad [A_2] = \begin{bmatrix} \lambda\nu & \\ & (\lambda\nu)^{-1} \end{bmatrix} \quad \text{and} \quad [A_3] = \begin{bmatrix} \lambda^2\nu & \\ & (\lambda^2\nu)^{-1} \end{bmatrix}.$$

From this, since $\lambda \neq \nu$ and $A \in \mathrm{Aut}(\mathfrak{n}_{\mathbb{C}})$, it is easy to see that we cannot have other nonzero Lie brackets on $\mathfrak{n}_{\mathbb{C}}$ besides (14), (15), $[X_1, X_4] = aZ_1$ and $[X_2, X_3] = bZ_2$. This Lie algebra is isomorphic to the one with $a = b = 0$ (by changing for $\tilde{X}_4 = X_4 + Y_1$, $\tilde{X}_2 = X_2 + Y_2$), and then $\mathfrak{n}_{\mathbb{C}}$ is again isomorphic to $(\mathfrak{l}_4 \oplus \mathfrak{l}_4)_{\mathbb{C}}$.

We summarize the results obtained in this section.

Theorem 3.2. *Up to isomorphism, the real Anosov Lie algebras of dimension* ≤ 8 *are:* $\mathbb{R}^n$, $n = 2, \ldots, 8$, $\mathfrak{h}_3 \oplus \mathfrak{h}_3$, $\mathfrak{f}_3$, $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^2$, $\mathfrak{f}_3 \oplus \mathbb{R}^2$, $\mathfrak{g}$, $\mathfrak{h}$, *and* $\mathfrak{l}_4 \oplus \mathfrak{l}_4$.

TABLE 2. Set of rational forms up to isomorphism for some real nilpotent Lie algebras (see [L2]). In all cases k runs over all square-free natural numbers.

Real Lie algebra	*Type*	*Rational forms*
$\mathfrak{h}_3 \oplus \mathfrak{h}_3$	$(4,2)$	$\mathfrak{n}_k^{\mathbb{Q}},\ k \geq 1$
$\mathfrak{f}_3$	$(3,3)$	$\mathfrak{f}_3^{\mathbb{Q}}$
$\mathfrak{g}$	$(6,2)$	$\mathfrak{g}^{\mathbb{Q}}$
$\mathfrak{h}_3 \oplus \mathfrak{h}_5$	$(6,2)$	$(\mathfrak{h}_3 \oplus \mathfrak{h}_5)^{\mathbb{Q}}$
$\mathfrak{h}$	$(4,4)$	$\mathfrak{h}_k^{\mathbb{Q}},\ k \geq 1$
$\mathfrak{l}_4 \oplus \mathfrak{l}_4$	$(4,2,2)$	$\mathfrak{l}_k^{\mathbb{Q}},\ k \geq 1$

4. Classification of rational Anosov Lie algebras

In Section 3, we have found all real Lie algebras of dimension ≤ 8 having an Anosov rational form (see Theorem 3.2 or Table 3). On the other hand, the set of all rational forms (up to isomorphism) for each of these algebras has been determined in [L2] (see Table 2). In this section, we shall study which of these rational Lie algebras are Anosov, obtaining in this way the classification in the rational case up to dimension 8.

Case $\mathfrak{f}_3$ (type $(3,3)$). There is only one rational form $\mathfrak{f}_3^{\mathbb{Q}}$ in this case, which is proved to be Anosov in [D] and [De, Ma1].

Case $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ (type $(4,2)$). The rational forms of $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ are given by $\{\mathfrak{n}_k^{\mathbb{Q}}\}$, $k \geq 1$, square-free (see [L2]). The fact that $\mathfrak{n}_k^{\mathbb{Q}}$ is Anosov for any $k > 1$ has been proved in several papers (see [Sm, I, AS, Ma1]), and it also follows from the construction given in [L1]. The Pfaffian form of $\mathfrak{n}_1^{\mathbb{Q}}$ is $f_1(x,y) = x^2 - y^2$, and thus it follows from Proposition 2.6, (ii), that $\mathfrak{n}_1^{\mathbb{Q}}$ is not Anosov.

Case $\mathfrak{g}$ (type $(6,2)$). It is proved in Section 3 that the Lie algebra $\mathfrak{g}$ is the only real Anosov Lie algebra of this type, and $\mathfrak{g}$ has only one rational form, which is then the only rational Anosov Lie algebra of this type.

Case $\mathfrak{h}$ (type $(4,4)$). We have seen in Section 3 that the only possible real Anosov Lie algebras of this type are the real forms of $\mathfrak{h}_{\mathbb{C}}$, namely, $\mathfrak{h}$ and $\mathfrak{n}_{-1}^{\mathbb{Q}} \otimes \mathbb{R}$. The rational forms of $\mathfrak{h}$ can be parametrized by $\mathfrak{h}_k^{\mathbb{Q}}$, with k a square-free natural number (see [L2]). We know that the Pfaffian form of $\mathfrak{h}_k^{\mathbb{Q}}$ is $f_k(x,y,z,w) = xw + y^2 - kz^2$ and then $Hf_k = 4k$. By renaming the basis β given in (7) as $\{X_1, ..., X_4, Z_1, ..., Z_4\}$,

we have that the Lie bracket of the Anosov rational form $\mathfrak{h}^{\mathbb{Q}}$ of $\mathfrak{h}$ defined by β is

$$\begin{array}{ll} [X_1, X_3] = Z_1 + Z_3, & [X_2, X_3] = Z_2 - Z_4, \\ [X_1, X_4] = Z_2 + Z_4, & [X_2, X_4] = (a^2 - 1)(Z_1 - Z_3). \end{array}$$

This implies that the maps J_Z of $\mathfrak{h}^{\mathbb{Q}}$ are given by

$$J_{xZ_1+yZ_2+zZ_3+wZ_4} = \begin{bmatrix} 0 & 0 & -x-z & -y-w \\ 0 & 0 & -y+w & m(-x+z) \\ x+z & y-w & 0 & 0 \\ y+w & m(x-z) & 0 & 0 \end{bmatrix},$$

where $m = a^2 - 1$, and then its Pfaffian form is

$$f(x, y, z, w) = mx^2 - y^2 - mz^2 + w^2,$$

with Hessian $Hf = 16m^2$. We know that $\mathfrak{h}^{\mathbb{Q}}$ has to be isomorphic to $\mathfrak{h}_k^{\mathbb{Q}}$ for some square-free natural number k, but in that case $f \simeq_{\mathbb{Q}} f_k$, and so we would have $Hf = q^2 Hf_k$ for some $q \in \mathbb{Q}^*$. Thus $16m^2 = q^2 k$, which implies that $k = 1$. This shows that the Anosov rational forms of $\mathfrak{h}$ defined by different integers a are all isomorphic to $\mathfrak{h}_1^{\mathbb{Q}}$. In what follows, we shall prove that the other rational forms of $\mathfrak{h}$ (i.e. $\mathfrak{h}_k^{\mathbb{Q}}$ for $k > 1$) are Anosov as well.

Fix a square free natural number $k > 1$. Consider the basis $\beta = \{X_1, ..., Z_4\}$ of $\mathfrak{h}_k^{\mathbb{Q}}$ given in [L2, Proposition 4.5] and set $\mathfrak{n}_1 = \langle X_1, ..., X_4 \rangle_{\mathbb{Q}}$ and $\mathfrak{n}_2 = \langle Z_1, ..., Z_4 \rangle_{\mathbb{Q}}$. Let $(a, b) \in \mathbb{N} \times \mathbb{N}$ be any solution to the Pell equation $x^2 - ky^2 = 1$. Let $A : \mathfrak{h}_k^{\mathbb{Q}} \longrightarrow \mathfrak{h}_k^{\mathbb{Q}}$ be the linear map defined in terms of β by

$$A_1 = A|_{\mathfrak{n}_1} = \begin{bmatrix} 0 & 0 & b & -a \\ 0 & 0 & -a & kb \\ 0 & 1 & 2n & 0 \\ 1 & 0 & 0 & 2n \end{bmatrix}, \qquad A_2 = A|_{\mathfrak{n}_2} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & -a & b & 4na \\ 0 & -bk & a & 4nbk \\ -1 & -2n & 0 & 2n^2 \end{bmatrix}. \tag{16}$$

It is easy to check that $A \in \mathrm{Aut}(\mathfrak{h}_k^{\mathbb{Q}})$ for any $n \in \mathbb{N}$, and since $\det A_1 = \det A_2 = 1$, we have that $A_1, A_2 \in \mathrm{SL}_4(\mathbb{Z})$, that is, A is unimodular. The characteristic polynomial of A_1 is $f(x) = (x^2 - 2nx + a - \sqrt{k}b)(x^2 - 2nx + a + \sqrt{k}b)$, and so its eigenvalues are

$$\begin{aligned} \lambda_1 &= n + \sqrt{n^2 - a + \sqrt{k}b}, & \lambda_2 &= n - \sqrt{n^2 - a + \sqrt{k}b}, \\ \mu_1 &= n + \sqrt{n^2 - a - \sqrt{k}b}, & \mu_2 &= n - \sqrt{n^2 - a - \sqrt{k}b}. \end{aligned} \tag{17}$$

We take $n \in \mathbb{N}$ such that $a + \sqrt{k}b < n^2$. Therefore $1 < \lambda_1$, and it follows from $\lambda_1 \lambda_2 = a - \sqrt{k}b = \frac{1}{a+\sqrt{k}b} < 1$ that $\lambda_2 < 1$. Also, $1 < \mu_1$ and $\mu_1 \mu_2 = a + \sqrt{k}b > 1$, and hence $\mu_2 \neq 1$, proving that A_1 is hyperbolic. The eigenvalues of A_2 are all of the form $\lambda_i \mu_j$. Indeed, it can be checked that the eigenvector for $\lambda_i \mu_j$ is

$$Z = Z_1 - (a - \sqrt{k}b)\mu_j Z_2 - (a + \sqrt{k}b)\lambda_i Z_3 + \lambda_i \mu_j Z_4.$$

Now, the fact that $\lambda_2 < \mu_2 < \mu_1 < \lambda_1$ implies that $\lambda_i \mu_j \neq 1$ for all i, j, showing that A_2 is also hyperbolic and hence that A is an Anosov automorphism of $\mathfrak{h}_k^{\mathbb{Q}}$.

The above is the most direct and shortest proof of the fact that $\mathfrak{h}_k^{\mathbb{Q}}$ is Anosov for any square free $k > 1$, and it consists in just checking that A is unimodular and hyperbolic. But now, we would like to show where this A comes from, which will show at the same time that $\mathfrak{h}_k^{\mathbb{Q}}$ is not Anosov for $k < 0$. Since the proof of [L2, Proposition 4.5] actually shows that the set of rational forms up to isomorphism of

$\mathfrak{h}_{\mathbb{C}}$ is given by

$$\{\mathfrak{h}_k^{\mathbb{Q}} : k \text{ a nonzero square free integer number}\},$$

this will prove that the real completion $\mathfrak{n}_{-1}^{\mathbb{Q}} \otimes \mathbb{R}$ of those with $k < 0$ is not Anosov.

First of all, it is easy to see that any $\tilde{A}$ of the form

$$\tilde{A}_1 = \tilde{A}|_{\mathfrak{n}_1} = \left[\begin{smallmatrix} B & 0 \\ 0 & C \end{smallmatrix}\right], \quad \tilde{A}_2 = \tilde{A}|_{\mathfrak{n}_2} = \left[\begin{smallmatrix} b_{11}C & b_{12}C \\ b_{21}C & b_{22}C \end{smallmatrix}\right] \quad \left(B = \left[\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{smallmatrix}\right]\right), \tag{18}$$

where $B, C \in \mathrm{GL}_2(\mathbb{C})$, is an automorphism of $\mathfrak{h}_{\mathbb{C}}$, for which we are considering the basis $\alpha = \{X_1, ..., Z_4\}$ with a Lie bracket defined as in Table 1. Moreover, this forms a subgroup of $\mathrm{Aut}(\mathfrak{h}_{\mathbb{C}})$ containing the connected component of the identity, since any other automorphism restricted to $(\mathfrak{n}_1)_{\mathbb{C}}$ has the form $\left[\begin{smallmatrix} 0 & \star \\ \star & 0 \end{smallmatrix}\right]$. By taking $\tilde{A}^2$ if necessary, we can assume that if $\mathfrak{h}_k^{\mathbb{Q}}$ is Anosov, then it admits an Anosov automorphism of the form (18). The change of basis matrix P_k from the basis β_k of the rational form isomorphic to $\mathfrak{h}_k^{\mathbb{Q}}$ given in the proof of [L2, Proposition 4.5] to the basis α is

$$P_k|_{\mathfrak{n}_1} = \begin{bmatrix} \sqrt{k} & 1 & 0 & 0 \\ 0 & 0 & 1 & \sqrt{k} \\ -\sqrt{k} & 1 & 0 & 0 \\ 0 & 0 & 1 & -\sqrt{k} \end{bmatrix}, \quad P_k|_{\mathfrak{n}_2} = \begin{bmatrix} 2\sqrt{k} & 0 & 0 & 0 \\ 0 & \sqrt{k} & -1 & 0 \\ 0 & \sqrt{k} & 1 & 0 \\ 0 & 0 & 0 & -2\sqrt{k} \end{bmatrix},$$

and hence

$$P_k^{-1}|_{\mathfrak{n}_1} = \frac{1}{2}\begin{bmatrix} 1/\sqrt{k} & 0 & -1/\sqrt{k} & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1/\sqrt{k} & 0 & -1/\sqrt{k} \end{bmatrix}, \quad P_k^{-1}|_{\mathfrak{n}_2} = \frac{1}{2}\begin{bmatrix} 1/\sqrt{k} & 0 & 0 & 0 \\ 0 & 1/\sqrt{k} & 1/\sqrt{k} & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1/\sqrt{k} \end{bmatrix}.$$

We then have that $A = P_k^{-1}\tilde{A}P_k \in \mathrm{Aut}(\mathfrak{h}_k^{\mathbb{Q}})$ if and only if $A_1 = P_k^{-1}\tilde{A}_1P_k$ and $A_2 = P_k^{-1}\tilde{A}_2P_k$ belong to $\mathrm{GL}_4(\mathbb{Q})$. A straightforward computation shows that $A_1, A_2 \in \mathrm{GL}_4(\mathbb{Z})$ (i.e. A is unimodular) if and only if $B \in \mathrm{GL}_2(\mathbb{Z}[\sqrt{k}])$, $C = \overline{B}$ and $\det B\overline{\det B} = \pm 1$. Here, $\mathbb{Z}[\sqrt{k}]$ is the integer ring of the quadratic numberfield $\mathbb{Q}[\sqrt{k}]$, and the conjugation is defined, as usual, by $\overline{x + \sqrt{k}b} = x - \sqrt{k}b$ for all $x, y \in \mathbb{Q}$. Recall that if $\det B = a - \sqrt{k}b$, $a, b \in \mathbb{Z}$, and we assume that $\det B\overline{\det B} = 1$, then $a^2 - kb^2 = 1$, the Pell equation. In order to make easier the computation of eigenvalues we can take B in its rational form, say

$$B = \left[\begin{smallmatrix} 0 & -a+\sqrt{k}b \\ 1 & 2n \end{smallmatrix}\right], \qquad \overline{B} = \left[\begin{smallmatrix} 0 & -a-\sqrt{k}b \\ 1 & 2n \end{smallmatrix}\right],$$

for some $n \in \mathbb{Z}$. This implies that the characteristic polynomial of $\tilde{A}_1$ is $f(x) = (x^2 - 2nx + a - \sqrt{k}b)(x^2 - 2nx + a + \sqrt{k}b)$, and so the eigenvalues of $\tilde{A}_1$ and A_1 are as in (17). Concerning the hyperbolicity, if $k < 0$, then either $b = 0$ or $a = 0$ and $k = -1$, which in any case implies that $|\mu_1\mu_2| = 1$, a contradiction. Therefore $\mathfrak{h}_k^{\mathbb{Q}}$ is not Anosov for $k < 0$, as was to be shown. For $k > 0$, we can easily see that conditions $a, b, n \in \mathbb{N}$, $a + \sqrt{k}b < n^2$, are enough for the hyperbolicity of A_1. For A_2, we can use the following general fact: the eigenvalues of a matrix of the form $\tilde{A}_2$ in (18) are precisely the possible products between eigenvalues of B and eigenvalues of C; and so the hyperbolicity of A_2 follows as in the short proof.

We finally note that $A = P_k^{-1}\tilde{A}P_k$, with this B precisely the automorphism proposed in (16).

TABLE 3. Real and rational Anosov Lie algebras of dimension ≤ 8.

Real Anosov Lie algebra	*Dimension*	*Type*	*Anosov rat. forms*	*Non − Anosov rat. forms*	*Signature*
$\mathbb{R}^n$, $2 \leq n \leq 8$	n	n	$\mathbb{Q}^n$	$--$	any
$\mathfrak{h}_3 \oplus \mathfrak{h}_3$	6	$(4,2)$	$\mathfrak{n}_k^{\mathbb{Q}},\ k>1$	$\mathfrak{n}_1^{\mathbb{Q}}$	$\{3,3\}$
$\mathfrak{f}_3$	6	$(3,3)$	$\mathfrak{f}_3^{\mathbb{Q}}$	$--$	$\{3,3\}$
$\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^2$	8	$(6,2)$	$\mathfrak{n}_k^{\mathbb{Q}} \oplus \mathbb{Q}^2,\ k>1$	$\mathfrak{n}_1^{\mathbb{Q}} \oplus \mathbb{Q}^2$	$\{4,4\}$
$\mathfrak{f}_3 \oplus \mathbb{R}^2$	8	$(5,3)$	$\mathfrak{f}_3^{\mathbb{Q}} \oplus \mathbb{Q}^2$	$--$	$\{4,4\}$
$\mathfrak{g}$	8	$(6,2)$	$\mathfrak{g}^{\mathbb{Q}}$	$--$	$\{4,4\}$
$\mathfrak{h}$	8	$(4,4)$	$\mathfrak{h}_k^{\mathbb{Q}},\ k \geq 1$	$--$	$\{4,4\}$
$\mathfrak{l}_4 \oplus \mathfrak{l}_4$	8	$(4,2,2)$	$\mathfrak{l}_k^{\mathbb{Q}},\ k>1$	$\mathfrak{l}_1^{\mathbb{Q}}$	$\{4,4\}$

Remark 4.1. An alternative proof of the fact that any rational form of $\mathfrak{h}$ is Anosov can be given by using [D, Corollary 2.3]. Indeed, the subgroup

$$S = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) = \{A \in \mathrm{Aut}(h) : A_1 = \left[\begin{smallmatrix} B & 0 \\ 0 & C \end{smallmatrix}\right], \quad B, C \in \mathrm{SL}_2(\mathbb{R})\}$$

is connected, semisimple and all its weights on $\mathfrak{h}$ are nontrivial. Recall that such a corollary cannot be applied to the cases $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ and $\mathfrak{l}_4 \oplus \mathfrak{l}_4$, as they admit a rational form which is not Anosov.

Case $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ (type $(4,2,2)$). The rational forms of $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ are denoted by $\mathfrak{l}_k^{\mathbb{Q}}$, k a square free natural number. Let β denote the basis of $\mathfrak{l}_k^{\mathbb{Q}}$ given in [L2, Proposition 5.1]. For $a \in \mathbb{Z}$, $a \geq 2$, consider the hyperbolic matrix

$$B = \left[\begin{smallmatrix} a & a^2-1 \\ 1 & a \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z}),$$

with eigenvalues $\lambda_1 = a + (a^2-1)^{\frac{1}{2}}$ and $\lambda_2 = a - (a^2-1)^{\frac{1}{2}}$. It is easy to check that the linear map $A : \mathfrak{l}_b^{\mathbb{Q}} \longrightarrow \mathfrak{l}_b^{\mathbb{Q}}$ whose matrix in terms of β is

$$[A]_\beta = \begin{bmatrix} B & & \\ & \ddots & \\ & & B \end{bmatrix}$$

is an automorphism of $\mathfrak{l}_b^{\mathbb{Q}}$ for $b = a^2 - 1$. A is hyperbolic since $\lambda_1 > 1 > \lambda_2$, and it is unimodular by definition, so that A is an Anosov automorphism. Recall that $\mathfrak{l}_k^{\mathbb{Q}} \simeq \mathfrak{l}_{k'}^{\mathbb{Q}}$ if and only if $k = q^2 k'$ for some $q \in \mathbb{Q}^*$ (see the proof of [L2, Proposition

5.1]). Given a square-free natural number $k > 1$, there always exist $a, q \in \mathbb{Z}$ such that $a^2 - 1 = q^2 k$ (Pell equation), and thus any $\mathfrak{l}_k^{\mathbb{Q}}$ with $k > 1$ square free is Anosov.

We now prove that $\mathfrak{l}_1^{\mathbb{Q}}$ is not Anosov. In the proof of [L2, Proposition 5.1] it is shown that any $A \in \mathrm{Aut}(\mathfrak{l}_1^{\mathbb{Q}})$ has the form

$$A = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ \star & A_2 & 0 & 0 \\ \star & 0 & A_3 & 0 \\ \star & \star & \star & A_4. \end{bmatrix}$$

and satisfies

$$qf(z,w) = f(A_4^t(z,w)), \qquad \forall (z,w) \in \mathbb{Q}^2,$$

where $q = \det A_3 A_1$ and $f(z,w) = z^2 - w^2$. In the same spirit of Proposition 2.6, this implies that A_4^t leaves a finite set invariant and so it can never be hyperbolic.

The results obtained in this section can be summarized as follows.

Theorem 4.2. *Up to isomorphism, the rational Anosov Lie algebras of dimension ≤ 8 are*

- $\mathbb{Q}^n$, $n = 2, \dots, 8$, *($\mathbb{R}^n$),*
- $\mathfrak{n}_k^{\mathbb{Q}}$, $k \geq 2$, *($\mathfrak{h}_3 \oplus \mathfrak{h}_3$),*
- $\mathfrak{f}_3^{\mathbb{Q}}$, *($\mathfrak{f}_3$),*
- $\mathfrak{n}_k^{\mathbb{Q}} \oplus \mathbb{Q}^2$, $k \geq 2$, *($\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^2$),*
- $\mathfrak{f}_3^{\mathbb{Q}} \oplus \mathbb{Q}^2$, *($\mathfrak{f}_3 \oplus \mathbb{R}^2$),*
- $\mathfrak{g}^{\mathbb{Q}}$, *($\mathfrak{g}$),*
- $\mathfrak{h}_k^{\mathbb{Q}}$, $k \geq 1$, *($\mathfrak{h}$),*
- $\mathfrak{l}_k^{\mathbb{Q}}$, $k \geq 2$, *($\mathfrak{l}_4 \oplus \mathfrak{l}_4$),*

where k always runs over square-free numbers and the Lie algebra between parenthesis is the corresponding real completion.

The *signature* of an Anosov diffeomorphism is the pair of natural numbers $\{p, q\} = \{\dim E^+, \dim E^-\}$. It is known that signature $\{1, n-1\}$ is only possible for torus and their finitely covered spaces: compact flat manifolds (see [Fr]). We do not actually know of any nonabelian example of signature $\{2, q\}$.

In the last column of Table 3 appear the signatures of the Anosov automorphisms found in each case. It follows from the proofs given in Section 3 that the eigenvalues of any Anosov automorphism always appear in pairs $\{\lambda, \lambda^{-1}\}$ (with only one exception: $\mathfrak{f}_3$), and thus there is only one possible signature for each nonabelian Anosov Lie algebra of dimension ≤ 8.

Corollary 4.3. *Let N/Γ be a nilmanifold (or infranilmanifold) of dimension ≤ 8 which admits an Anosov diffeomorphism. Then N/Γ is either a torus (or a compact flat manifold) or the dimension is 6 or 8, and the signature is $\{3, 3\}$ or $\{4, 4\}$, respectively.*

It is not true in general that there is only one possible signature for a given Anosov Lie algebra. For instance, it is easy to see that the free 2-step nilpotent Lie algebra on 4 generators admits Anosov automorphisms of signature $\{4, 6\}$ and $\{5, 5\}$.

Acknowledgements

We wish to thank M. Mainkar and S.G. Dani for very helpful comments on the first version of this paper.

References

[AS] L. Auslander, J. Scheuneman, On certain automorphisms of nilpotent Lie groups, *Global Analysis: Proc. Symp. Pure Math.* **14** (1970), 9-15. MR0270395 (42:5284)

[BL] Y. Benoist, F. Labourie, Sur le diffeomorphismes d'Anosov affines a feuilletages stable et instable differentiables, *Inventiones Math.* **111** (1993), 285-308. MR1198811 (94d:58114)

[CKS] C. Cassidy, N. Kennedy, D. Scevenels, Hyperbolic automorphisms for groups in $\mathcal{T}(4,2)$, *Contemporary Math.* **262** (2000), 171-175. MR1796132 (2001i:20071)

[D] S.G. Dani, Nilmanifolds with Anosov automorphism, *J. London Math. Soc.* **18** (1978), 553-559. MR518242 (80k:58082)

[DM] S.G. Dani, M. Mainkar, Anosov automorphisms on compact nilmanifolds associated with graphs, *Trans. Amer. Math. Soc.* **357** (2005), 2235-2251. MR2140439 (2006j:22005)

[De] K. Dekimpe, Hyperbolic automorphisms and Anosov diffeomorphisms on nilmanifolds, *Trans. Amer. Math. Soc.* **353** (2001), 2859-2877. MR1828476 (2002c:37043)

[DeDs] K. Dekimpe, S. Deschamps, Anosov diffeomorphisms on a class of 2-step nilmanifolds, *Glasg. Math. J.* **45** (2003), 269-280. MR1997705 (2004i:37054)

[Fr] J. Franks, Anosov diffeomorphisms, *Global Analysis: Proc. Symp. Pure Math.* **14** (1970), 61-93. MR0271990 (42:6871)

[Gh] E. Ghys, Holomorphic Anosov systems, *Invent. Math.* **119** (1995), 585-614. MR1317651 (95k:58116)

[Gr] M. Gromov, Groups of polynomial growth and expanding maps, *Inst. des Hautes Etudes Sci.* **53** (1981), 53-73. MR623534 (83b:53041)

[I] K. Ito, Classification of nilmanifolds M^n ($n \leq 6$) admitting Anosov diffeomorphisms, *The study of dynamical systems*, Kyoto (1989), 31-49, World Sci. Adv. Ser. Dynam. Systems **7**. MR1117284 (92j:58077)

[L1] J. Lauret, Examples of Anosov diffeomorphisms, *Journal of Algebra* **262** (2003), 201-209. *Corrigendum*: **268** (2003), 371-372. MR1970807 (2004g:37033a)

[L2] J. Lauret, On rational forms of nilpotent Lie algebras, *Monatsh. Math.* **155** (2008), 15-30.

[LW] J. Lauret, C.E. Will, On Anosov automorphisms of nilmanifolds, *Journal of Pure and Applied Algebra* **212** (2008), 1747-1755. MR2400740

[Ma1] W. Malfait, Anosov diffeomorphisms on nilmanifolds of dimension at most 6, *Geom. Dedicata* **79** (2000), 291-298. MR1755730 (2001h:20041)

[Ma2] W. Malfait, An obstruction to the existence of Anosov diffeomorphisms on infra-nilmanifolds, *Contemp. Math.* **262** (2000), 233-251 (Kortrijk 1999). MR1796136 (2002b:37039)

[Mn] A. Manning, There are no new Anosov diffeomorphisms on tori, *Amer. J. Math.* **96** (1974), 422-429. MR0358865 (50:11324)

[Mr] G. Margulis, Problems and conjectures in rigidity theory, *Mathematics: Frontiers and perspectives 2000*, IMU. MR1754775 (2001d:22008)

[P] H. Porteous, Anosov diffeomorphisms of flat manifolds, *Topology* **11** (1972), 307-315. MR0296976 (45:6035)

[S] J. Scheuneman, Two-step nilpotent Lie algebras, *J. Algebra* **7** (1967), 152-159. MR0217134 (36:225)

[Sh] M. Shub, Endomorphisms of compact differentiable manifolds, *Amer. J. Math.* **91** (1969), 175-199. MR0240824 (39:2169)

[Sm] S. Smale, Differentiable dynamical systems, *Bull. Amer. Math. Soc.* **73** (1967), 747-817. MR0228014 (37:3598)

[V] A. Verjovsky, Sistemas de Anosov, Notas de Curso, IMPA (Rio de Janeiro).

FaMAF and CIEM, Universidad Nacional de Córdoba, Córdoba, Argentina
E-mail address: lauret@mate.uncor.edu

FaMAF and CIEM, Universidad Nacional de Córdoba, Córdoba, Argentina
E-mail address: cwill@mate.uncor.edu