

DEGENERATE REAL HYPERSURFACES IN \mathbb{C}^2 WITH FEW AUTOMORPHISMS

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ABSTRACT. We introduce new biholomorphic invariants for real-analytic hypersurfaces in \mathbb{C}^2 and show how they can be used to show that a hypersurface possesses few automorphisms. We give conditions, in terms of the new invariants, guaranteeing that the stability group is finite, and give (sharp) bounds on the cardinality of the stability group in this case. We also give a sufficient condition for the stability group to be trivial. The main technical tool developed in this paper is a complete (formal) normal form for a certain class of hypersurfaces. As a byproduct, a complete classification, up to biholomorphic equivalence, of the finite type hypersurfaces in this class is obtained.

1. INTRODUCTION

Let M be a germ at a point p of a real-analytic hypersurface in \mathbb{C}^2 . An automorphism of the germ (M, p) is a germ of a biholomorphic map $H: (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^N, p)$ that satisfies $H(M) \subset M$. The set of all automorphisms of a germ (M, p) forms a group under composition, called the stability group of M at p . Endowed with the topology of uniform convergence on compact neighbourhoods of p , it becomes a topological group (a sequence of automorphisms (H_j) converges to an automorphism H if all of the H_j extend to a common compact neighbourhood of p and converge uniformly to H on it).

It is a well known fact that a “general” real-analytic hypersurface does not possess any nontrivial automorphisms. This observation goes back to Poincaré [8]. He observed that the existence of nontrivial automorphisms imposes very strict conditions on the coefficients of a real-analytic defining function. Indeed, if one follows his arguments, one sees that a real-analytic hypersurface in general position does not have any nontrivial automorphisms! On the other hand, it is in general a hard task to show that a given specific hypersurface has no automorphisms, as there are no general tools available to answer this question.

In this paper, we introduce a construction that allows us, among other things, to identify certain low order invariants associated to a germ of a real-analytic hypersurface (M, p) in \mathbb{C}^2 and to give conditions in terms of these invariants guaranteeing

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that the hypersurface does not have any nontrivial automorphisms (Theorem 3). We also give conditions guaranteeing that the stability group is finite and provide estimates on the number of automorphisms in this case (Theorem 1). We mention here that, by means of ad hoc computations, some explicit examples of real hypersurfaces with a finite stability group were given in [1] and [9].

The hypersurfaces we study here are Levi degenerate; that is, their Levi forms vanish at the chosen reference point p . In a certain sense, our invariants measure this vanishing in a qualitative way. The invariants we introduce (Theorem 6) are tensors associated to certain lattice points $(\alpha, n, \mu) \in \mathbb{N}^3$,

$$(1) \quad \mathcal{L}^{(\alpha, n, \mu)} \in \bigotimes^{\alpha} \mathcal{V}_p^* \otimes \bigotimes^n \bar{\mathcal{V}}_p^* \otimes \bigotimes^{\mu} (T_p^{0,1} \mathbb{C}^2 / \mathcal{V}_p)^* \otimes (T_p^{0,1} \mathbb{C}^2 / \mathcal{V}_p),$$

where $\mathcal{V}_p := T_p^{0,1} \mathbb{C}^2 \cap (\mathbb{C} \otimes T_p M)$. These tensors generalize the Levi form as well as some tensors introduced by the first author [4].

The conditions we give in Theorems 1 and 3, guaranteeing that a real hypersurface has few automorphisms, are elementary number-theoretical relations between the invariant lattice points (α, n, μ) . In order to formulate the conditions, we have to assume that there are enough (at least two) of these triples for a given (M, p) . Theorems 1 and 3 are given in §2, after the precise definition of the invariants.

Our main technical result, which allows us to prove the results described above, is a formal normal form (Theorem 14) for a class of hypersurfaces defined in terms of our invariants. As in the well-known Chern-Moser normal form, our normal form gives rise to a completely algorithmic construction of the normalization map (by induction).

We would like to remark here that our invariants arise both in the finite type case and in the infinite type case. In the infinite type case, our results quantify the general jet determination theorem in \mathbb{C}^2 obtained in [5]; by this, we mean that we can compute the jet order needed for the determination property (for our class of hypersurfaces) from the invariants introduced in this paper.

Another remark in order is that in the finite type case, our normal form also gives (in a standard way) a complete classification, with respect to biholomorphic equivalence, of the hypersurfaces under consideration. A complete classification of finite type hypersurfaces in \mathbb{C}^2 has been obtained by M. Kolář [6]. Our normal form, just as his, is formal; that is, we do not prove convergence. In the finite type case this is not necessary, since we have the result on convergence of formal mappings in [3] at our disposal. In this context, the result in [3] implies that any formal invertible mapping between two real-analytic, finite type hypersurfaces in \mathbb{C}^2 is convergent and, hence, yields a biholomorphism between the two hypersurfaces. Thus, the formal classification that follows from the normal form gives rise to a biholomorphic classification in this setting.

Here is a short outline of this paper: in Section 2 we introduce our invariants, state our main results, and discuss some equivalent ways to define the invariants. After that, we prove the transformation rules for our tensors in Section 3. Section 4 is devoted to the construction of the formal normal form, from which the main results of the paper will follow. Finally in Section 5 we show by explicit examples that our bound on the jet order needed to determine the automorphisms is indeed achieved.

2. MAIN RESULTS

Let $M \subset \mathbb{C}^2$ be a germ of a real-analytic hypersurface through $p = (p^1, p^2) \in \mathbb{C}^2$. Recall that this means that M is defined, locally near p , by the equation $\rho = 0$, where ρ is a real-analytic function near p with $\rho(p) = 0$ and $d\rho(p) \neq 0$. We shall identify ρ with its Taylor series

$$(2) \quad \rho(Z, \bar{Z}) := \sum_{I, J} \rho_{I, \bar{J}} (Z - p)^I \overline{(Z - p)^J},$$

where standard multi-index notation is used and the coefficients satisfy the reality condition $\rho_{I, \bar{J}} = \overline{\rho_{\bar{J}, I}} \in \mathbb{C}$. More generally, we shall consider formal (not necessarily convergent) power series (e.g. the Taylor series of a defining function of a germ of a smooth hypersurface) $\rho(Z, \bar{Z})$ as in (2), which satisfy the above reality condition and the nondegeneracy condition $d\rho = \rho_{1\bar{0}}dZ + \rho_{0\bar{1}}d\bar{Z} \neq 0$ at p , and say that ρ defines a *formal hypersurface* M at $p \in \mathbb{C}^2$. A formal automorphism of M at p is an invertible formal holomorphic mapping $(\mathbb{C}^2, p) \rightarrow (\mathbb{C}^2, p)$ such that $\rho(H(Z), \overline{H(\bar{Z})}) = a(Z, \bar{Z})\rho(Z, \bar{Z})$ for some formal power series $a(Z, \bar{Z})$ in $Z - p$ and $\bar{Z} - p$. An invertible formal holomorphic mapping $H: (\mathbb{C}^2, p) \rightarrow (\mathbb{C}^2, p)$ is a pair of formal holomorphic power series $H = (H^1, H^2)$ of the form

$$(3) \quad H^j(Z) = p^j + \sum_{|I| > 0} H_I^j (Z - p)^I$$

such that $\det \frac{\partial H}{\partial Z}(p) = \begin{vmatrix} H_{(1,0)}^1 & H_{(0,1)}^1 \\ H_{(1,0)}^2 & H_{(0,1)}^2 \end{vmatrix} \neq 0$. We shall denote by $\text{Aut}_f(M, p)$ the group of all formal automorphisms of M at p .

For a real-analytic hypersurface M , we may choose local holomorphic coordinates (z, w) vanishing at p , such that M is given, locally near $p = (0, 0)$, by

$$(4) \quad \text{Im } w = \varphi(z, \bar{z}, \text{Re } w),$$

where $\varphi(z, \bar{z}, s)$ is a (real-valued) real-analytic function in a neighborhood of 0 in $\mathbb{C}^2 \times \mathbb{R}$ satisfying

$$(5) \quad \varphi(z, 0, s) \equiv \varphi(0, \bar{z}, s) \equiv 0.$$

For a formal hypersurface M through $p \in \mathbb{C}^2$, the analogous transformation, in which φ is a formal power series, is possible by a formal holomorphic change of coordinates. Any such coordinates (z, w) (formal or local holomorphic) are called *normal coordinates* for M at p (for more details, see, e.g. [2]). For each representation (4) of M , we consider the power series expansion

$$(6) \quad \varphi(z, \chi, s) = \sum_{\alpha \geq 1, \mu \geq 0} \varphi_{\alpha, \mu}(\chi) z^\alpha s^\mu.$$

We should point out that normal coordinates (z, w) , as described above, are highly nonunique. However, we will show below that for certain lattice points $(\alpha, \mu) \in \mathbb{N}^2$, the corresponding coefficients $\varphi_{\alpha, \mu}(\chi)$ transform in a particularly simple way under changes of normal coordinates (see Proposition 8). We shall refer to such points (α, μ) as invariant pairs; the exact definition is given in Definition 1 below. The invariance (that is, independence of the choice of normal coordinates) and transformation law for the corresponding coefficient is then established in Proposition 8 below. The lattice points which are invariant correspond precisely to the

“lowest order coefficients” of the Taylor series in (6) in the sense of the following partial ordering on \mathbb{N}^2 :

$$(7) \quad \begin{aligned} (\alpha, \mu) \preceq (\beta, \nu) & \text{ if } \alpha + \mu \leq \beta + \nu \text{ and } \mu \leq \nu; \\ (\alpha, \mu) \prec (\beta, \nu) & \text{ if } (\alpha, \mu) \preceq (\beta, \nu) \text{ and } (\alpha, \mu) \neq (\beta, \nu). \end{aligned}$$

Definition 1. A point $(\alpha_0, \mu_0) \in \mathbb{N}^2$ is called an *invariant pair associated to M* if $\varphi_{\alpha_0, \mu_0}(\chi) \not\equiv 0$ but $\varphi_{\alpha, \mu}(\chi) \equiv 0$ for every $(\alpha, \mu) \prec (\alpha_0, \mu_0)$.

In the following, we shall denote the set of all invariant pairs by $Q_{M,p} = Q_M \subset \mathbb{N} \times \mathbb{N}$. Even though, *a priori*, the set Q_M depends on the choice of normal coordinates (z, w) , we shall show (see Theorem 6 below) that, in fact, it does not, and hence the set Q_M is an invariant of (M, p) . We shall, moreover, define a refined invariant set $\Lambda_M \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ as follows. For each $(\alpha, \mu) \in Q_M$, we set

$$(8) \quad n(\alpha, \mu) := \min \left\{ n : \frac{d^n \varphi_{\alpha, \mu}}{d\chi^n}(0) \neq 0 \right\}$$

and define

$$(9) \quad \Lambda_M := \{(\alpha, n, \mu) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : (\alpha, \mu) \in Q_M \text{ and } n = n(\alpha, \mu)\}.$$

It is not difficult to see, using the fact that $\varphi(z, \chi, s) = \bar{\varphi}(\chi, z, s)$, that for any invariant pair (α, μ) we have

$$n(\alpha, \mu) \geq \alpha.$$

We are now in a position to state the main results of this paper. Our principal *technical* result consists of a construction, for each pair of points $(\alpha, n, \mu) \neq (\alpha', n', \mu') \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ with $\alpha \neq n$, a formal normal form for the hypersurfaces $M \subset \mathbb{C}^2$ satisfying $(\alpha, n, \mu), (\alpha', n', \mu') \in \Lambda_M$. The normal form is described in Theorem 14. (To describe it precisely requires distinguishing several cases, and we prefer to do this at the end of the paper.)

The normal form in Theorem 14 allows us to bound the dimension of the stability group of a real hypersurface $M \subset \mathbb{C}^2$ satisfying the condition above; it also implies a criterion for the stability group to be trivial. Our first result along these lines is the the following theorem, which guarantees that the stability group can be embedded in a suitable jet group. The construction of the normal form also implies that the (formal) stability group can be given the structure of a finite dimensional Lie group; see Theorem 16. The theorem also provides bounds on the dimension of this group.

Theorem 1. *Let $M \subset \mathbb{C}^2$ be a real-analytic (or formal) hypersurface with $p \in M$. Assume that the invariant set Λ_M , as defined above, contains at least two points, and at least one of them, say (α, n, μ) , satisfies $\alpha \neq n$. Then, the group $\text{Aut}_f(M, p)$ of all formal automorphisms of (M, p) embeds, via its jet evaluation, as a closed Lie subgroup of some jet group $J_p^k(\mathbb{C}^2)$, which satisfies*

$$(10) \quad \dim_{\mathbb{R}} \text{Aut}_f(M, p) \leq 1.$$

Moreover, if either

$$(11) \quad \alpha + n = \alpha' + n', \quad \text{for some } (\alpha', n', \mu') \neq (\alpha, n, \mu) \in \Lambda_M,$$

or the number

$$(12) \quad \frac{(\alpha' + n')\mu - (\alpha + n)\mu'}{(\alpha' + n') - (\alpha + n)}$$

is not the same positive integer for all choices of $(\alpha', n', \mu') \neq (\alpha, n, \mu) \in \Lambda_M$, then $\text{Aut}_f(M, p)$ embeds as a finite subgroup of $\text{U}(1) \times \text{U}(1)$, more precisely as a subgroup of the finite group N_M described in Remark 12 below. In particular,

$$(13) \quad \#\text{Aut}_f(M, p) \leq 2(n - \alpha).$$

If $\alpha + n \neq \alpha' + n'$ for all $(\alpha', n', \mu') \neq (\alpha, n, \mu) \in \Lambda_M$ and the number (12) is the same positive integer k for all $(\alpha', n', \mu') \neq (\alpha, n, \mu) \in \Lambda_M$, then $\text{Aut}_f(M, p)$ embeds into $J_p^k(\mathbb{C}^2)$.

The given bound (13) on the number of automorphisms and the bound (10) on the dimension of the automorphism group are actually sharp; see Example 4 below.

Remark 2. The fact that $\text{Aut}_f(M, p)$ embeds into $J_p^k(\mathbb{C}^2)$ implies, in particular, that the automorphisms in $\text{Aut}_f(M, p)$ are determined by their k -jets at p ; i.e. if $H, H' \in \text{Aut}_f(M, p)$ and

$$(14) \quad \partial^\alpha H(p) = \partial^\alpha H'(p), \quad \forall |\alpha| \leq k,$$

then $H = H'$.

Theorem 1 is a direct consequence of the normal form in Theorem 14 (see also Remark 15). It was proved by the authors in [5] that for any real-analytic Levi nonflat hypersurface $M \subset \mathbb{C}^2$ and $p \in M$ there exists a number k so that the automorphisms in $\text{Aut}_f(M, p)$ are determined by their k -jets at p . Moreover, if M is of finite type at p , then $k = 2$ suffices to determine the automorphisms in $\text{Aut}_f(M, p)$ (also proved in [5]). Theorem 1 can be viewed as a refinement of the results in [5] taking into account the finer invariants introduced in this paper.

We should also point out that, as was shown by R. T. Kowalski [7] and the third author [10], for any positive integer k , there exist real-analytic Levi nonflat hypersurfaces $M \subset \mathbb{C}^2$ with $p \in M$ (and M of infinite type at p) for which the automorphisms in $\text{Aut}_f(M, p)$ are uniquely determined by their k -jets at p but not by their $(k - 1)$ -jets at p . However, the examples given in [7] and [10] are such that their invariant sets Λ_M consist of a single point and hence do not belong to the class considered in Theorem 1. However, in Section 5, we give another family of examples that belong to that class (for which Λ_M consists of at least two points) and for which still arbitrarily high order jets are needed to determine the automorphisms.

We now come to the criterion mentioned in the Introduction for M to have no nontrivial automorphisms. This result is a direct consequence of Theorem 14 and the observation made in Remark 12 below, and its precise formulation is the following:

Theorem 3. *Let Λ be a subset of \mathbb{N}^3 that contains at least two points, and one of them, say (α, n, μ) , satisfies $n \neq \alpha$. Assume, in addition, that either*

$$(15) \quad \alpha + n = \alpha' + n', \quad \text{for some } (\alpha', n', \mu') \neq (\alpha, n, \mu) \in \Lambda,$$

or the number

$$(16) \quad \frac{(\alpha' + n')\mu - (\alpha + n)\mu'}{(\alpha' + n') - (\alpha + n)}$$

is not the same positive integer for all choices of $(\alpha', n', \mu') \neq (\alpha, n, \mu) \in \Lambda$. Then all (formal or real-analytic) hypersurfaces M satisfying $\Lambda_M = \Lambda$ have $\text{Aut}_f(M, p) = \{\text{id}\}$ if and only if

$$(17) \quad \gcd \{n' - \alpha' : (\alpha', n', \mu') \in \Lambda\} = 1,$$

there exists an even μ' with $(\alpha', n', \mu') \in \Lambda$, and either of the following two conditions is fulfilled:

- i) $n' - \alpha'$ is even for some $(\alpha', n', \mu') \in \Lambda$ with μ' even;
- ii) $n' - \alpha'$ is odd for some $(\alpha', n', \mu') \in \Lambda$ with μ' odd.

We note here that, in the definition of the greatest common divisor above, we use the convention that 0 is divisible by any integer. Let us also note that if all the conditions of Theorem 3 are fulfilled except for i) and ii), then the automorphism group has at most 2 elements (this follows from the explicit form of the group N_M given in §4.1).

We will now give some examples where Theorem 3 implies the triviality of the automorphism group. In all cases, the point p is the origin.

Example 1. Assume that $a \geq 1$, $b \geq 1$ are integers satisfying $a > b + 1$. Let $r \geq 0$ be an integer and assume that p and q are nonequal odd primes. Then the real hypersurface $M = M(a, b, p, q, r)$ given by

$$\operatorname{Im} w = (\operatorname{Re} w)^r (|z|^{2a} \operatorname{Re} z^p + \operatorname{Re} w |z|^{2b} \operatorname{Re} z^q)$$

does not have any nontrivial automorphisms. Here

$$\Lambda_M = \{(a, a + p, r), (b, b + q, r + 1)\}.$$

Let us check that the conditions of Theorem 3 are fulfilled. The condition $a > b + 1$ ensures that (a, r) and $(b, r + 1)$ are invariant pairs. If $2a + p \neq 2b + q$, then the fraction in (16) is an integer if and only if

$$\frac{2a + p}{(2b + q) - (2a + p)} \in \mathbb{N};$$

but in this last fraction, the numerator is odd while the denominator is even, so this is not the case. Since $\gcd\{p, q\} = 1$, condition (17) is fulfilled. Also, either r or $r + 1$ is odd; so (ii) in the last condition in Theorem 3 is fulfilled.

Example 2. Generalizing the last example a bit, let a, b, r, s be integers, with s being odd, satisfying $a > b + s$, and let p and q be nonequal odd primes. Then the hypersurface $M = M(a, b, r, s, p, q)$ given by

$$\operatorname{Im} w = (\operatorname{Re} w)^r (|z|^{2a} \operatorname{Re} z^p + (\operatorname{Re} w)^s |z|^{2b} \operatorname{Re} z^q)$$

does not have any nontrivial automorphisms. The invariants are given by $(\alpha, n, \mu) = (a, a + p, r)$ and $(\alpha', n', \mu') = (b, b + q, r + s)$ in this example. Again, let us check that the conditions of Theorem 3 are fulfilled. The conditions $a > b + s$ ensures that (a, r) and (b, s) are invariant pairs. The fraction in (16) is an integer if and only if

$$\frac{s(2a + p)}{(2b + q) - (2a + p)} \in \mathbb{N};$$

just as in the preceding example, this is never the case. The reasoning of the preceding example also applies to the verification of the last two conditions of Theorem 3.

Example 3. Let $a > 3$. The hypersurface M given by

$$\operatorname{Im} w = (\operatorname{Re} w) |z|^2 (\operatorname{Re} z) + |z|^{2a}$$

has no nontrivial automorphisms.

Let us again check the conditions of Theorem 3. The invariants are given by $(\alpha, n, \mu) = (1, 2, 1)$ and $(\alpha', n', \mu') = (a, a, 0)$. The fraction condition (16) is satisfied, since $1 < \frac{2a}{2a-3} < 2$. Furthermore, the condition for the gcd is satisfied since $n - \alpha = 1$, which is odd, so ii) holds. Actually, i) holds also, since 0 is even.

Remark 4. Let us remark that the automorphism groups remain trivial even if we allow higher order terms to appear in the Taylor expansion of the hypersurfaces in the examples above. Here, higher order terms have to be understood in the sense of the partial ordering used in Definition 1; that is, the terms of the form $z^\beta \bar{z}^n (\operatorname{Re} w)^\nu$ with $(\alpha, \mu) \prec (\beta, \nu)$ and $(\alpha', \mu') \prec (\beta, \nu)$.

We now give an example of a hypersurface having the maximal number of automorphisms allowed by the bound in (13).

Example 4. Let a, k, q be positive integers such that q is odd and $2a + k$ is not divisible by 3. Consider the hypersurface given by

$$\operatorname{Im} w = (\operatorname{Re} w)^q \operatorname{Re} z^k \left(|z|^{2a} (\operatorname{Re} w)^2 + |z|^{2(a+3)} \right).$$

The invariants are $(\alpha, n, \mu) = (a, a + k, q + 2)$ and $(\alpha', n', \mu') = (a + 3, a + 3 + k, q)$. The fraction (12) is given by

$$\frac{(\alpha' + n')\mu - (\alpha + n)\mu'}{(\alpha' + n') - (\alpha + n)} = q + 2 + \frac{2a + k}{3},$$

which is not an integer since we assume that 3 does not divide $2a + k$. Hence, by Theorem 1, the group $\operatorname{Aut}_f(M, 0)$ is finite and satisfies the estimate (13). Since the biholomorphisms

$$H_{\ell,+}(z, w) := \left(e^{\frac{2\ell\pi i}{k}} z, w \right), \quad H_{\ell,-}(z, w) := \left(e^{\frac{2\ell\pi i}{k}} z, -w \right), \quad 0 \leq \ell \leq k - 1,$$

are all in $\operatorname{Aut}_f(M, 0)$, we conclude that we actually have equality in (13) in this case; also, this implies that we have $\operatorname{Aut}_f(M, 0) = \{H_{\ell,-}, H_{\ell,+} : 0 \leq \ell \leq k - 1\}$.

Example 5. It is natural to ask what kind of role the fraction condition plays. An example showing that if $2 \leq \frac{(\alpha' + n')\mu - (\alpha + n)\mu'}{(\alpha' + n') - (\alpha + n)} \in \mathbb{N}$ a one-parameter family of higher jet parameters of the order predicted by Theorem 1 can be really needed is given in Section 5 below; here, we give an example showing that there is a dependence on a one-parameter family of first order jets in the case $\frac{(\alpha' + n')\mu - (\alpha + n)\mu'}{(\alpha' + n') - (\alpha + n)} = 1$, thus establishing that the dimension bound (10) is sharp also in that case.

Let $\alpha \geq 1$ and $\mu \geq 2$ be integers, and let p be a positive integer such that $\alpha - \mu + 1 \geq p$. Then the hypersurface M given by

$$\operatorname{Im} w = (\operatorname{Re} w)^\mu \left(|z|^{2\alpha} + (\operatorname{Re} w)^{\mu-1} \operatorname{Re}(z^{\alpha-\mu+1-p} \bar{z}^{3\alpha+\mu-1+p}) \right)$$

has the one-parameter family of biholomorphisms given by

$$H_r(z, w) = (r^{1-\mu} z, r^{2\alpha} w), \quad 0 \neq r \in \mathbb{R}.$$

Actually, we note here that the normal form given in Theorem 14 and the observations made in Section 4.1 imply that the automorphism group is generated by H_r and by the discrete group of rotations

$$(z, w) \mapsto \left(e^{\frac{\pi i j}{n+p+\mu-1}} z, w \right), \quad 0 \leq j < 2(n + p + \mu - 1).$$

We will now discuss Definition 1 in some detail. Let us first note that for fixed normal coordinates, we could also decompose $\varphi(z, \chi, s) = \sum_{\beta \geq 1, \mu \geq 0} \psi_{\beta, \nu}(z) \chi^\beta s^\nu$. Let us show that this representation leads to the same invariant pairs, where we use the analogous definition as above. Since φ is real valued, $\varphi(z, \chi, s) = \bar{\varphi}(\chi, z, s)$. Hence,

$$\sum_{\alpha, \mu} \varphi_{\alpha, \mu}(z) \chi^\alpha s^\mu = \varphi(z, \chi, s) = \bar{\varphi}(\chi, z, s) = \sum_{\beta, \nu} \bar{\psi}_{\beta, \nu}(z) \chi^\beta s^\nu,$$

and so

$$(18) \quad \varphi_{\alpha, \mu}(z) = \bar{\psi}_{\alpha, \mu}(z) \text{ for all } \alpha, \mu.$$

We shall also use a different kind of defining equation for M for which the calculations turn out to be simpler. It is well known (the reader can consult for example the book by Baouendi, Ebenfelt and Rothschild [2] for details) that M can also be given in the form

$$(19) \quad w = Q(z, \bar{z}, \bar{w}),$$

where $Q(z, \chi, \tau)$ is a holomorphic function (or a formal power series) in a neighborhood of 0 in \mathbb{C}^3 satisfying

$$(20) \quad Q(z, 0, \tau) \equiv Q(0, \chi, \tau) \equiv \tau, \quad Q(z, \chi, \bar{Q}(\chi, z, \tau)) \equiv \tau.$$

We may also use the function Q to define invariant pairs as follows. We decompose $Q(z, \chi, \tau)$ as

$$(21) \quad Q(z, \chi, \tau) = \tau + \sum_{\alpha \geq 1, \mu \geq 0} q_{\alpha, \mu}(\chi) z^\alpha \tau^\mu = \tau + \sum_{\beta \geq 1, \nu \geq 0} r_{\beta, \nu}(z) \chi^\beta \tau^\nu,$$

and define invariant pairs to be the minimal coefficients in the ordering used in Definition 1, with $\varphi_{\alpha, \mu}$ replaced by either $q_{\alpha, \mu}(\chi)$ or $r_{\alpha, \mu}(\chi)$. The equivalence of all these definitions is a consequence of the following lemma.

Lemma 5. *In given normal coordinates (z, w) , let M be defined by each of the equations (4) and (19). Then for every pair $(\alpha_0, \mu_0) \in \mathbb{N}^2$, the following properties are equivalent:*

- (i) $\varphi_{\alpha, \mu}(\chi) \equiv 0$ for $(\alpha, \mu) \prec (\alpha_0, \mu_0)$;
- (ii) $q_{\alpha, \mu}(\chi) \equiv 0$ for $(\alpha, \mu) \prec (\alpha_0, \mu_0)$;
- (iii) $\psi_{\alpha, \mu}(z) \equiv 0$ for $(\alpha, \mu) \prec (\alpha_0, \mu_0)$;
- (iv) $r_{\beta, \nu}(z) \equiv 0$ for $(\alpha, \mu) \prec (\alpha_0, \mu_0)$.

If (i)–(iv) are satisfied, then

$$(22) \quad q_{\alpha_0, \mu_0}(\chi) \equiv 2i\varphi_{\alpha_0, \mu_0}(\chi) \equiv -\bar{r}_{\alpha_0, \mu_0}(\chi).$$

Proof. The proof is based on the identity

$$(23) \quad \frac{Q(z, \chi, \tau) - \tau}{2i} \equiv \varphi\left(z, \chi, \frac{Q(z, \chi, \tau) + \tau}{2}\right)$$

which is a consequence of (4) and (19). Let (α_0, μ_0) satisfy (ii) and assume that $\varphi_{\alpha, \mu} \not\equiv 0$ for some (α, μ) as in (i). Then in the expansion of the left-hand side of (23) the coefficient of $z^\alpha \tau^\mu$ (as a function of χ) is zero, whereas, in view of (ii) and (5), only the term τ can contribute in the first expansion (21) of $Q(z, \chi, \tau)$ on the right-hand side. Hence the corresponding coefficient on the right-hand side is $\varphi_{\alpha, \mu}$, which is not zero, a contradiction. Similar computation of the coefficients of $z^{\alpha_0} \tau^{\mu_0}$ yields the first identity in (22).

Conversely, suppose that (i) holds but $q_{\alpha,\mu}(\chi) \not\equiv 0$ for some (α, μ) as in (ii). Then such a pair $(\alpha, \mu) = (\alpha_1, \mu_1)$ can be chosen so that (ii) holds with (α_0, μ_0) replaced by (α_1, μ_1) . The above argument shows that the first identity in (22) holds with (α_0, μ_0) replaced by (α_1, μ_1) , which contradicts the assumption (i). Hence (ii) also holds, as required.

We have already shown (see (18)) that (i) and (iii) are equivalent. The proof that (iii) and (iv) are equivalent is exactly the same as above, using the expansions in terms of ψ and r . □

For each $(\alpha, n, \mu) \in \Lambda_M$ with the notation above, we define a tensor

$$(24) \quad \mathcal{L}^{(\alpha,n,\mu)} \in \bigotimes^{\alpha} \mathcal{V}_p^* \otimes \bigotimes^n \bar{\mathcal{V}}_p^* \otimes \bigotimes^{\mu} (T_p^{0,1}\mathbb{C}^2/\mathcal{V}_p)^* \otimes (T_p^{0,1}\mathbb{C}^2/\mathcal{V}_p),$$

where \mathcal{V}_p denotes the space of $(0, 1)$ vectors at p which are tangent to M , in local coordinates as follows. First, observe that \mathcal{V}_p is spanned, in normal coordinates (z, w) for M at p , by $\partial/\partial\bar{z}$, and the normal space $T_p^{0,1}\mathbb{C}^2/\mathcal{V}_p$ is spanned by the projection of $\partial/\partial\bar{w}$. By choosing $\partial/\partial\bar{z}$ as a basis for \mathcal{V}_p and $\partial/\partial\bar{w}$ mod $\partial/\partial\bar{z}$ as a basis for $T_p^{0,1}\mathbb{C}^2/\mathcal{V}_p$, we identify these two spaces with \mathbb{C} and define

$$(25) \quad \mathcal{L}^{(\alpha,n,\mu)}(a_1, \dots, a_{\alpha}, b_1, \dots, b_n, c_1, \dots, c_{\mu}) := \frac{1}{n!} \frac{d^n q_{\alpha,\mu}}{d\chi^n}(0) a_1 \dots a_{\alpha} b_1 \dots b_n c_1 \dots c_{\mu}.$$

The following result yields a preliminary classification of real hypersurfaces $M \subset \mathbb{C}^2$ in terms of the set Λ_M and also proves the fact that (25) indeed defines a tensor, as claimed in (24).

Theorem 6. *Let $M \subset \mathbb{C}^2$ be a real-analytic (or formal) hypersurface, $p \in M$, and (z, w) normal coordinates for M at p . Then the set Λ_M defined by (9) and, for each $(\alpha, n, \mu) \in \Lambda_M$, the tensor $\mathcal{L}^{(\alpha,n,\mu)}$ defined by (25) are independent of the choice of normal coordinates. That is, if $M' \subset \mathbb{C}^2$ is another real-analytic (or formal) hypersurface given in normal coordinates (z', w') at $p' \in M'$ by $w' = Q'(z', \bar{z}', \bar{w}')$ and $(z', w') = (F, G)$ is a (formal) biholomorphic map sending M into M' with $(F(0), G(0)) = (0, 0)$, then $\Lambda_M = \Lambda_{M'}$ and for each $(\alpha, n, \mu) \in \Lambda_M$*

$$(26) \quad \frac{d^n q_{\alpha,\mu}}{d\chi^n}(0) \overline{G_w}(0) = \frac{d^n q'_{\alpha,\mu}}{d\chi^n}(0) F_z(0)^{\alpha} \overline{F_z}(0)^n \overline{G_w}(0)^{\mu}.$$

Remark 7. We should point out that any CR diffeomorphism between two C^{∞} -smooth real hypersurfaces induces a formal invertible map between the corresponding formal hypersurfaces (see, e.g. [2]). Hence, the set Λ_M and the corresponding tensors (25) are in fact CR invariants.

Theorem 6 will be proved in Section 3. We shall in fact show the more general result that for $(\alpha, \mu) \in Q_M$, the whole coefficient $q_{\alpha,\mu}(\chi)$ transforms like a family of tensors (at least for $(\alpha, \mu) \neq (1, 0)$); see Proposition 8 for the precise statement.

Before proceeding further, we have two remarks: first, as is easily seen, the set Λ_M (or, equivalently, the set Q_M) always contains at least one point unless M is Levi flat, i.e. $Q \equiv 0$. Indeed, if $q_{\beta,\nu} \not\equiv 0$, for some (β, ν) , then there is at least one $(\alpha_1, \mu_1) \in Q_M$ such that $(\alpha_1, \mu_1) \prec (\beta, \nu)$. Second, we also point out that if $(1, 1, 0) \in \Lambda_M$, then Λ_M does not contain any other point and the corresponding tensor $\mathcal{L}^{(1,1,0)}$ is simply the Levi form; thus our tensors generalize the Levi form for Levi degenerate hypersurfaces.

3. INVARIANT TENSORS AND THEIR TRANSFORMATION

3.1. **Transformation rule and invariance of the tensors.** Let $M \subset \mathbb{C}^2$ be a formal hypersurface (e.g. coming from a germ of a real-analytic one) and $p \in M$. We shall keep the notation introduced in Section 2. Thus, (z, w) will be normal coordinates for M at $p = (0, 0)$, and we shall assume that M is defined by equation (4) or in complex form by (19). We decompose $Q(z, \chi, \tau)$ in two ways as in (21).

Let us recall that M is of *finite type* at $p = (0, 0)$ if and only if there exists a γ such that $q_{\gamma,0}(\chi) \neq 0$. In this case we will denote the smallest such γ by γ_0 . It is easy to check that γ_0 is independent of the choice of normal coordinates; cf. also Proposition 8 below. If M is of infinite type, we set $\gamma_0 = \infty$.

If $H = (F, G)$ is a formal invertible map taking M into another hypersurface M' which is given in normal coordinates (z', w') by $w' = Q'(z', \bar{z}', \bar{w}')$, then (by definition)

$$(27) \quad G(z, w) = Q'(F(z, w), \bar{F}(\chi, \tau), \bar{G}(\chi, \tau)),$$

when $w = Q(z, \chi, \tau)$. By the normality condition (19), setting $\chi = 0, w = \tau$, we have that

$$(28) \quad G(z, w) = Q'(F(z, w), \bar{F}(0, w), \bar{G}(0, w)).$$

Note that this implies that $G(z, 0) \equiv 0$. Putting $w = Q(z, \chi, \tau)$ in the right-hand sides of (27) and (28) and equating them and using (21) we obtain

$$(29) \quad \begin{aligned} &\bar{G}(0, Q(z, \chi, \tau)) \\ &+ \sum_{\alpha \geq 1, \mu \geq 0} q'_{\alpha, \mu}(\bar{F}(0, Q(z, \chi, \tau))) (F(z, Q(z, \chi, \tau)))^\alpha (\bar{G}(0, Q(z, \chi, \tau)))^\mu \\ &= \bar{G}(\chi, \tau) + \sum_{\alpha \geq 1, \mu \geq 0} q'_{\alpha, \mu}(\bar{F}(\chi, \tau)) (F(z, Q(z, \chi, \tau)))^\alpha (\bar{G}(\chi, \tau))^\mu, \end{aligned}$$

where the $q'_{\alpha, \mu}$ are defined as in (21), with Q' replaced by Q .

Now, recall from the previous section (see Definition 1 and the paragraph following it) the definition of the set $Q_M \subset \mathbb{N} \times \mathbb{N}$ as the set of all invariant pairs of M . The following proposition proves the invariance of Q_M (hence justifying the terminology “invariant pairs”) and shows how the $q_{\alpha, \mu}(\chi)$, for $(\alpha, \mu) \in Q_M$, transform under formal changes of normal coordinates.

Proposition 8. *Let $M, M' \subset \mathbb{C}^2$ be formal hypersurfaces, each given in normal coordinates at $0 \in M$ and $0 \in M'$ respectively, and $H = (F, G): (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ a formal invertible map sending M into M' . Assume that (α_0, μ_0) is an invariant pair for M . Then it is also an invariant pair for M' and*

$$(30) \quad \begin{aligned} q_{\alpha_0, \mu_0}(\chi) &= q'_{\alpha_0, \mu_0}(\bar{F}(\chi, 0)) F_z(0)^{\alpha_0} \overline{G_w(0)}^{\mu_0 - 1}, \text{ for } (\alpha_0, \mu_0) \neq (1, 0), \\ q_{1,0}(\chi) &= q'_{1,0}(\bar{F}(\chi, 0)) (F_z(0) + F_w(0) q_{1,0}(\chi)) \overline{G_w(0)}^{-1}, \text{ for } (\alpha_0, \mu_0) = (1, 0). \end{aligned}$$

Proof. We begin by assuming that

$$(31) \quad q'_{\alpha, \mu} \equiv 0 \quad \text{for every } (\alpha, \mu) \prec (\alpha_0, \mu_0).$$

We expand both sides of (29) into a Taylor series in z and τ and identify the coefficients of $z^{\alpha_0} \tau^{\mu_0}$, using the definition of invariant pairs and Lemma 5. The

first term on the left-hand side of (29) has the Taylor expansion

$$(32) \quad \bar{G}(0, Q(z, \chi, \tau)) = \sum_{l=1}^{\infty} \frac{1}{l!} \overline{G_w}^l(0) \left(\tau + \sum_{\alpha, \mu} q_{\alpha, \mu}(\chi) z^\alpha \tau^\mu \right)^l.$$

The general form of a term in the expansion of $\left(\tau + \sum_{\alpha, \mu} q_{\alpha, \mu}(\chi) z^\alpha \tau^\mu \right)^l$ is, up to a binomial factor, either

$$(33) \quad q_{\gamma_1, \sigma_1}(\chi) \cdots q_{\gamma_{l_1}, \sigma_{l_1}}(\chi) z^{\gamma_1 + \cdots + \gamma_{l_1}} \tau^{l - l_1 + \sigma_1 + \cdots + \sigma_{l_1}},$$

for some $1 \leq l_1 \leq l$, or τ^l . Since (α_0, μ_0) is invariant, if the term (33) is not 0, then for each $1 \leq j \leq l_1$ either $\gamma_j + \sigma_j \geq \alpha_0 + \mu_0$ or $\sigma_j \geq \mu_0$. We conclude that

$$(34) \quad \bar{G}(0, Q(z, \chi, \tau)) \sim \bar{G}(0, \tau) + \overline{G_w}(0) q_{\alpha_0, \mu_0}(\chi) z^{\alpha_0} \tau^{\mu_0},$$

where \sim means equality modulo terms of the form $z^\gamma \tau^\sigma$ with either $\gamma + \sigma > \alpha_0 + \mu_0$ or $\sigma > \mu_0$. Thus, the only term of the form $z^{\alpha_0} \tau^{\mu_0}$ in the expansion (32) is $\overline{G_w}(0) q_{\alpha_0, \mu_0}(\chi) z^{\alpha_0} \tau^{\mu_0}$.

Let us examine the sum on the left in (29). An argument similar to the one above shows that

$$(35) \quad F(z, Q(z, \chi, \tau)) \sim F(z, \tau) + F_w(0) q_{\alpha_0, \mu_0}(\chi) z^{\alpha_0} \tau^{\mu_0},$$

where \sim has the same meaning as in (34). Also using (31) and the fact that $q'_{\alpha, \mu}(0) = 0$ (by normality of the coordinates (z, w)) for every (α, μ) , it is not difficult to see that the expansion of

$$q'_{\alpha, \mu}(\bar{F}(0, Q(z, \chi, \tau))) \left(F(z, Q(z, \chi, \tau)) \right)^\alpha \left(\bar{G}(0, Q(z, \chi, \tau)) \right)^\mu$$

cannot contribute a term with $z^{\alpha_0} \tau^{\mu_0}$.

Let us now examine the right-hand side of (29). The first term $\bar{G}(\chi, \tau)$ cannot contribute a term with $z^{\alpha_0} \tau^{\mu_0}$ since $\alpha_0 \geq 1$. In the sum on the right of (29), terms of the form

$$q'_{\alpha, \mu}(\bar{F}(\chi, \tau)) \left(F(z, Q(z, \chi, \tau)) \right)^\alpha \left(\bar{G}(\chi, \tau) \right)^\mu,$$

with $\mu > \mu_0$, cannot contribute, since as already noted above, $\bar{G}(\chi, 0) \equiv 0$. We conclude that the contribution from the sum on the right can only come from the terms involved in the expansion of

$$(36) \quad q'_{\alpha_0, \mu_0}(\bar{F}(\chi, 0)) (F_z(0)z + F_w(0)(\tau + q_{\alpha_0, \mu_0}(\chi) z^{\alpha_0} \tau^{\mu_0}))^{\alpha_0} (\overline{G_w}(\chi, 0)\tau)^{\mu_0},$$

and hence equals

$$q'_{\alpha_0, \mu_0}(\bar{F}(\chi, 0)) (F_z(0)z)^{\alpha_0} (\overline{G_w}(\chi, 0)\tau)^{\mu_0}$$

if $(\alpha_0, \mu_0) \neq (1, 0)$ and

$$q'_{1,0}(\bar{F}(\chi, 0)) (F_z(0) + F_w(0)q_{1,0}(\chi))z$$

if $(\alpha_0, \mu_0) = (1, 0)$. Thus, (30) follows, provided that we show that $\overline{G_w}(\chi, 0) \equiv \overline{G_w}(0, 0)$ holds when $\mu_0 \geq 1$. To see this, we set $z = 0$ in (29) and obtain, in view of (20),

$$(37) \quad \begin{aligned} \bar{G}(0, \tau) + \sum_{\alpha, \mu} q'_{\alpha, \mu}(\bar{F}(0, \tau)) (F(0, \tau))^\alpha (\bar{G}(0, \tau))^\mu \\ = \bar{G}(\chi, \tau) + \sum_{\alpha, \mu} q'_{\alpha, \mu}(\bar{F}(\chi, \tau)) (F(0, \tau))^\alpha (\bar{G}(\chi, \tau))^\mu. \end{aligned}$$

The desired property now follows by differentiating (37) with respect to τ , setting $\tau = 0$, and using the fact that $q'_{1,0} \equiv 0$ if $\mu_0 \geq 1$. The proof of (30) is complete and, hence also the proof of Proposition 8, under the assumption (31).

To complete the proof, we must show that (31) holds. If (31) does not hold, there would exist an invariant pair $(\alpha'_0, \mu'_0) \neq (\alpha_0, \mu_0)$ for M' with $\alpha'_0 + \mu'_0 \leq \alpha_0 + \mu_0$ and $\mu'_0 \leq \mu_0$ (i.e. with $(\alpha'_0, \mu'_0) \prec (\alpha_0, \mu_0)$). Now the corresponding assumption (31) holds with M and M' exchanged, and the above proof yields the identities (30) with H replaced by H^{-1} and (α_0, μ_0) by (α'_0, μ'_0) . In particular, these identities imply that $q_{\alpha'_0, \mu'_0} \neq 0$, which contradicts the fact that (α_0, μ_0) is an invariant pair for M . Hence the assumption (31) must hold, and the proof is complete. \square

We are now in a position to prove Theorem 6.

Proof of Theorem 6. The invariance of the set Q_M follows directly from Proposition 8. Next, if (α, μ) is an invariant pair for M , then $n = n(\alpha, \mu)$ is also invariant. Indeed, if $H = (F, G)$ is a formal change of normal coordinates as in Proposition 8, then $\chi \mapsto \bar{F}(\chi, 0)$ is a formal change of variables near $\chi = 0$ (since $\bar{G}(\chi, 0) = 0$), and thus it follows from (30) that the order of vanishing at 0 of $q_{\alpha_0, \mu_0}(\chi)$ and that of $q'_{\alpha_0, \mu_0}(\chi')$ are the same. The transformation rule (26) follows by expanding (30) in a Taylor series in χ . \square

3.2. An identity for the transversal component of a mapping. We shall keep the notation introduced above. Thus, $M \subset \mathbb{C}^2$ and $M' \subset \mathbb{C}^2$ denote two formal hypersurfaces with points $p \in M$ and $p' \in M'$, and H denotes a formal invertible map at p with $H(p) = p'$ and $H(M) \subset M'$. We choose normal coordinates (z, w) for M vanishing at p and (z', w') for M' vanishing at p' and write $H(z, w) = (F(z, w), G(z, w))$.

Our goal in this section is to derive a certain identity for the transversal component G of a formal change of coordinates H as above, which ensures that the normality of coordinates is preserved by the change of coordinates induced by H . This is done by an application of the invariants introduced above. We formulate this as a lemma:

Lemma 9. *Let M' be a formal hypersurface in \mathbb{C}^2 , given in normal coordinates (z', w') vanishing at $p'_0 \in M'$, and let $H = (F, G): (\mathbb{C}^2, p) \rightarrow (\mathbb{C}^2, p'_0)$ be a formal invertible map. Let M be the formal hypersurface $H^{-1}(M')$ and (z, w) be local coordinates vanishing at $p \in M$ so that $z' = F(z, w)$, $w' = G(z, w)$. Then there exist universal polynomials p_k such that the coordinates (z, w) are normal for M at p if and only if, for every k ,*

$$(38) \quad G_{w^k}(z, 0) - \overline{G_{w^k}}(0) = p_k \left(\overline{G_{w^j}}(0), F_{w^j}(0), F_{w^l}(z, 0), \left(r'_{\beta, \nu} \right)^{(s)}(F(z, 0)) \right),$$

where $r'_{\beta, \nu} \binom{s}$ are the terms in the expansion analogous to (21) of the defining equation for M' and $j \leq k - m_0 + 1$, $l \leq k - m_0$, $\beta + \nu + s \leq k$, and m_0 is defined by

$$m_0 = \min \{ \alpha + \mu : q'_{\alpha, \mu} \neq 0 \} \geq 1.$$

Proof. It is well known that (z, w) are normal coordinates for M at p if and only if $\rho(z, 0, w, w) = 0$ for some - and hence any - (possibly complex and formal) defining function $\rho(z, \bar{z}, w, \bar{w})$ for M (see, e.g. [2], Prop. 4.2.3). In our case, a defining

function for M is given by

$$\rho(z, \bar{z}, w, \bar{w}) := G(z, w) - Q'(F(z, w), \bar{F}(\bar{z}, \bar{w}), \bar{G}(\bar{z}, \bar{w})).$$

Using the second expansion in (21) we see that (z, w) are normal coordinates for M if and only if

$$(39) \quad G(z, w) = \bar{G}(0, w) + \sum_{\beta, \nu} r'_{\beta, \nu}(F(z, w)) \bar{F}(0, w)^\beta \bar{G}(0, w)^\nu.$$

Note that the integer m_0 defined above is invariant and can be equivalently defined by

$$(40) \quad m_0 = \min\{\alpha + \mu : (\alpha, \mu) \in Q'_M\} = \min\{\alpha + \mu : q'_{\alpha, \mu} \neq 0\} = \min\{\beta + \nu : r'_{\beta, \nu} \neq 0\}.$$

The minimum power of w appearing in the sum on the right-hand side of (39) is m_0 ; hence we obtain

$$(41) \quad G_{w^k}(z, 0) = \overline{G_{w^k}}(0), \quad k < m_0.$$

To obtain a formula for $G_{w^k}(z, 0)$ for $k \geq m_0$ we expand $r'_{\beta, \nu}(F(z, w))$ in w :

$$(42) \quad r'_{\beta, \nu}(F(z, w)) = \sum_{n=0}^{\infty} a_n((F_{w^l}(z, 0))_{l \leq n}, (r'_{\beta, \nu}{}^{(l)}(F(z, 0)))_{l \leq n}) w^n,$$

where the a_n are universal polynomials—that is, they do not depend on either H or M' . Substituting this in (39) we obtain (38). □

4. A NORMAL FORM FOR CERTAIN HYPERSURFACES IN \mathbb{C}^2

4.1. Preliminary normalization. Our goal in this section is to make a preliminary normalization of the defining equation (19) of M and work out the restrictions on the first order jets of those mappings $H = (F, G)$ that respect this normalization. First, we shall think of M' as being given, fix an invariant pair $(\alpha, \mu) \neq (1, 0)$ of M' (i.e. an element of $Q_{M'}$) assuming it exists, and find a biholomorphic mapping $H = (F, G)$, preserving normal coordinates, such that $M = H^{-1}(M')$ is given by (19) with $q_{\alpha, \mu}(\chi) = 2i\lambda\chi^n$ (which corresponds to $\varphi_{\alpha, \mu}(\chi) = \lambda\chi^n$ by (22)), where $n = n(\alpha, \mu)$ and $|\lambda| = 1$. Indeed, by the construction of n , we can write $q'_{\alpha, \mu}(\chi') = ic(\psi(\chi'))^n$, where $\psi(\chi') = \chi' + O((\chi')^2)$ and $c \in \mathbb{C}$. The map $H(z, w) := (\bar{\psi}^{-1}(F_z(0)z), G_w(0)w)$, with $F_z(0) \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $G_w(0) \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ considered as free parameters, preserves normality of the coordinates and, in view of (30), yields

$$(43) \quad q_{\alpha, \mu}(\chi) = ic|F_z(0)|^{2\alpha} \overline{F_z(0)}^{n-\alpha} G_w(0)^{\mu-1} \chi^n.$$

We will from now on assume that the chosen invariant pair (α, μ) satisfies $n - \alpha \neq 0$. Then choosing $F_z(0) \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $G_w(0) \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ suitably, we obtain $q_{\alpha, \mu}(\chi) = 2i\chi^n$. For the transformations $H = (F, G)$ respecting the normalization $q_{\alpha, \mu}(\chi) = 2i\chi^n$, i.e. sending M with $q_{\alpha, \mu}(\chi) = 2i\chi^n$ into M' with $q'_{\alpha, \mu}(\chi') = 2i(\chi')^n$, where (α, μ) is an invariant pair for both M and M' , the identity (30) implies that $\bar{F}(\chi, 0)$ is linear, i.e.

$$(44) \quad \bar{F}(\chi, 0) = \overline{F_z(0)}\chi.$$

Now substituting (44) into (30) yields

$$(45) \quad 1 = |F_z(0)|^{2\alpha} \overline{F_z(0)}^{n-\alpha} \overline{G_w(0)}^{\mu-1}.$$

For integers α, n, μ we shall denote by $C(\alpha, n, \mu)$ the set of parameters $(F_z(0), G_w(0))$ satisfying (45); that is,

$$(46) \quad C(\alpha, n, \mu) = \{(\lambda, r) \in \mathbb{C}^* \times \mathbb{R}^* : \lambda^\alpha \bar{\lambda}^n r^{\mu-1} = 1\}.$$

From (45) we obtain restrictions on $F_z(0)$ and $G_w(0)$ depending on the integers α, μ, n . For the arguments we obtain

$$(47) \quad (n - \alpha) \arg(F_z(0)) + (\mu - 1) \arg(G_w(0)) = 0 \pmod{2\pi},$$

where $\arg(G_w(0))$, in view of (41), is either 0 or π . Moreover, the absolute value of $F_z(0)$ is related to that of $G_w(0)$ by

$$(48) \quad |F_z(0)| = |G_w(0)|^{\frac{1-\mu}{\alpha+n}}.$$

We now let $(\tilde{\alpha}, \tilde{\mu})$ denote any other invariant pair, and we work out a normalization of the coefficient of $\chi^{\tilde{n}}$ in $q_{\tilde{\alpha}, \tilde{\mu}}(\chi)$. Observe that if there are two invariant pairs, then $(1, 0)$ cannot be one (i.e. $q_{1,0} \equiv 0$). Thus, the transformation rule (30) for the pair $(\tilde{\alpha}, \tilde{\mu})$ is necessarily

$$q_{\tilde{\alpha}, \tilde{\mu}}(\chi) = q'_{\tilde{\alpha}, \tilde{\mu}}(\overline{F}(\chi, 0)) F_z(0)^{\tilde{\alpha}} \overline{G_w(0)}^{\tilde{\mu}-1}.$$

Denote the coefficient of $\chi^{\tilde{n}}$ (where $\tilde{n} = n(\tilde{\alpha}, \tilde{\mu})$) on the left-hand side by $ce^{i\theta}$ with $c > 0$, and the coefficient of $(\chi')^{\tilde{n}}$ on the right-hand side by $c'e^{i\theta'}$. Comparing the coefficient of $\chi^{\tilde{n}}$ on both sides, we obtain $ce^{i\theta} = c'e^{i\theta'} \overline{F_z(0)}^{\tilde{n}} F_z(0)^{\tilde{\alpha}} \overline{G_w(0)}^{\tilde{\mu}-1}$. Taking absolute values, and using (48), we obtain

$$c = c' |F_z(0)|^{\tilde{\alpha} + \tilde{n}} |G_w(0)|^{\tilde{\mu}-1} = c' |G_w(0)|^{\frac{(1-\mu)(\tilde{\alpha} + \tilde{n}) - (1-\tilde{\mu})(\alpha+n)}{\alpha+n}}.$$

We now distinguish two cases:

$$(49) \quad \begin{aligned} \text{Case A: } & \tilde{\mu}(\alpha + n) - \mu(\tilde{\alpha} + \tilde{n}) = (\alpha + n) - (\tilde{\alpha} + \tilde{n}), \\ \text{Case B: } & \tilde{\mu}(\alpha + n) - \mu(\tilde{\alpha} + \tilde{n}) \neq (\alpha + n) - (\tilde{\alpha} + \tilde{n}). \end{aligned}$$

In Case A, the norm c of the coefficient of $\chi^{\tilde{n}}$ is an invariant of M . In Case B, we can change the norm of this coefficient and normalize it by requiring that $c = 2$. Note that in this case, the normalization fixes $|G_w(0)|$, and thus by (48) also $|F_z(0)|$. Thus, we now consider formal invertible mappings $H = (F, G)$ respecting the normalization $q_{\alpha, \mu}(\chi) = 2i\chi^n$ and $q_{\tilde{\alpha}, \tilde{\mu}}(\chi) = ic\varepsilon\chi^{\tilde{n}} + O(\chi^{\tilde{n}+1})$, with $|\varepsilon| = 1$, where c in Case A is the real positive invariant introduced above or 2 in Case B.

The possible values of the unimodular number ε in the normalization of $q_{\tilde{\alpha}, \tilde{\mu}}$ are restricted to a discrete subset of the unit circle $U(1) \subset \mathbb{C}$, namely

$$(50) \quad \varepsilon = \varepsilon_0 \frac{\bar{\gamma}^{\tilde{n}-\tilde{\alpha}} \delta^{\tilde{\mu}-1}}{|\gamma|^{\tilde{n}-\tilde{\alpha}} |\delta|^{\tilde{\mu}-1}},$$

where ε_0 is any particular unimodular number such that $q_{\tilde{\alpha}, \tilde{\mu}}$ can be normalized as $ic\varepsilon_0\chi^{\tilde{n}} + O(\chi^{\tilde{n}+1})$ and (γ, δ) range over the set $C(\alpha, n, \mu)$ given by (46). We shall further restrict the pairs $(F_z(0), G_w(0))$ by normalizing ε in (50) so that $\arg\varepsilon \in [0, 2\pi)$ is as small as possible. This choice of ε is an invariant of M , and the pairs $(F_z(0), G_w(0))$ which preserve the normalization

$$(51) \quad q_{\alpha, \mu}(\chi) = 2i\chi^n, \quad q_{\tilde{\alpha}, \tilde{\mu}}(\chi) = ic\varepsilon\chi^{\tilde{n}} + O(\chi^{\tilde{n}+1}),$$

where $\tilde{n} := n(\tilde{\alpha}, \tilde{\mu})$, $\varepsilon \in U(1)$ is the invariant just described and c is the invariant introduced above in Case A or $c = 2$ in Case B, are precisely those which belong to $C(\alpha, n, \mu) \cap C(\tilde{\alpha}, \tilde{n}, \tilde{\mu})$. We summarize the above normalization in the following proposition.

Proposition 10. *Let $M \subset \mathbb{C}^2$ be a formal hypersurface with $p = 0 \in M$. Assume that there are two invariant pairs $(\alpha, \mu) \neq (\tilde{\alpha}, \tilde{\mu})$ and that $n := n(\alpha, \mu) \neq \alpha$. Then, there are normal coordinates $(z, w) \in \mathbb{C}^2$ for M at p such that M is given there by (19) with $q_{\alpha, \mu}$ and $q_{\tilde{\alpha}, \tilde{\mu}}$ satisfying (51). Here $\varepsilon \in \mathbb{U}(1)$ is the invariant defined above, and c is the invariant introduced above in Case A in (49) or is normalized to be 2 in Case B. Moreover, if M is normalized by (51), then any formal invertible mapping $H = (F, G): (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ sending M into M' , where M' is also normalized by (51), satisfies $(F_z(0), G_w(0)) \in C(\alpha, n, \mu) \cap C(\tilde{\alpha}, \tilde{n}, \tilde{\mu})$ and $F(z, 0) = F_z(0)z$. Here C is defined by (46).*

Remark 11. We remark here that the conditions (11)–(12) and (15)–(16) in Theorems 1 and 3 respectively guarantee that with the assumptions of those theorems, we are always in Case B.

Remark 12. In this remark, we study in some detail Case B in (49). In that case, we have

$$(52) \quad G_w(0) = \pm 1, \quad |F_z(0)| = 1,$$

for all mappings respecting the normalization (51), and the pairs $(F_z(0), G_w(0))$ which appear are furthermore restricted to be in the discrete subset $D(n - \alpha, \mu - 1)$ of the torus $\mathbb{U}(1) \times \mathbb{U}(1)$, and $D(k, l)$ is defined by

$$(53) \quad \begin{aligned} D(k, l) &:= \{(\gamma, \delta) \in \mathbb{U}(1) \times \{-1, 1\} : \gamma^k \delta^l = 1\} \\ &= \left\{ \left(e^{i\pi \left(\frac{2p+ql}{k} \right)}, e^{i\pi q} \right) : 1 \leq p < k, q = 0, 1 \right\}. \end{aligned}$$

Now assume that M satisfies the assumptions of Proposition 10 with normal coordinates (z, w) chosen such that (51) holds. Then any formal map $H: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$, which respects all coefficients $q_{\alpha, \mu}(\chi)$ corresponding to all invariant pairs (α, μ) , i.e. which satisfies $q_{\alpha, \mu}(\chi) = F_z(0)^\alpha G_w(0)^{\mu-1} q_{\alpha, \mu}(F(\chi, 0))$ for all $(\alpha, \mu) \in Q_M$, has the property

$$(54) \quad (F_z(0), G_w(0)) \in N_M := \bigcap_{(\alpha, n, \mu) \in \Lambda_M} D(n - \alpha, \mu - 1).$$

In particular, this holds (by Proposition 8) for every formal invertible map taking M into itself. (Note, however, that in order for the conclusion (54) to hold, it is not necessary that H maps M into itself, but only that the coefficients corresponding to invariant pairs are respected).

We have the obvious bound for the number of elements

$$\#N_M \leq 2 \operatorname{gcd}\{n - \alpha : (\alpha, n, \mu) \in \Lambda_M\} =: 2g_M.$$

This bound is actually sharp, as pointed out in Example 4 above; more generally, we note that, for any choice of real coefficients $c_{\alpha, n, \mu}$, the hypersurface \tilde{M} given by

$$(55) \quad \operatorname{Im} w = \sum_{(\alpha, n, \mu) \in \Lambda_M} c_{\alpha, n, \mu} \operatorname{Re}(z^\alpha \bar{z}^n) (\operatorname{Re} w)^\mu$$

has at least as many different automorphisms as $\#N_M$. In particular, we have the crude bound $\#N_M \leq 2(n - \alpha)$ as stated in Theorem 1.

We are now going to study the conditions guaranteeing that the group N_M is trivial. We first observe that, if all μ 's for $(\alpha, n, \mu) \in \Lambda_M$ are odd, then any hypersurface in (55) has the nontrivial automorphism $(z, w) \mapsto (z, -w)$. Hence we

may assume that one of the μ is even. Then, to investigate the conditions for the triviality of N_M , let us write S_k for the set of all k -th roots of unity; we then have

$$D(k, l) = \begin{cases} S_k \times \{1\} \cup S_k \times \{-1\}, & l \text{ even,} \\ S_k \times \{1\} \cup e^{\frac{i\pi}{k}} S_k \times \{-1\}, & l \text{ odd.} \end{cases}$$

Let us write $e^{\frac{i\pi}{k}} S_k = S_k^- = \{\gamma \in \mathbb{T} : \gamma^k = -1\}$, and set

$$g_M^+ := \gcd \{n - \alpha : (\alpha, n, \mu) \in \Lambda_M, \mu \text{ odd}\},$$

$$g_M^- := \gcd \{n - \alpha : (\alpha, n, \mu) \in \Lambda_M, \mu \text{ even}\},$$

so that $\gcd \{g_M^+, g_M^-\} = g_M$. Here and in the following we use the convention that $\gcd(\emptyset) = 0$ and $S_0^- := \emptyset$, $S_0 := \mathbb{U}(1)$, and we continue to use the convention that the gcd of a family of numbers possibly containing 0 is the same as the gcd of its nonzero members. With this notation, we have

$$N_M = S_{g_M} \times \{1\} \cup \left(\bigcap_{\mu \text{ odd}} S_{n-\alpha} \cap \bigcap_{\mu \text{ even}} S_{n-\alpha}^- \right) \times \{-1\}$$

$$= S_{g_M} \times \{1\} \cup \left(S_{g_M^+} \cap \bigcap_{\mu \text{ even}} S_{n-\alpha}^- \right) \times \{-1\}.$$

Denoting by $\text{ord}_2(k)$ the maximal j such that 2^j divides k , we note that

$$S_k^- \cap S_{\tilde{k}}^- = \begin{cases} \emptyset, & \text{ord}_2(k) \neq \text{ord}_2(\tilde{k}), \\ S_{\gcd\{k, \tilde{k}\}}^-, & \text{ord}_2(k) = \text{ord}_2(\tilde{k}). \end{cases}$$

Indeed, if $\gamma \in S_k^- \cap S_{\tilde{k}}^-$, i.e. $\gamma^k = \gamma^{\tilde{k}} = -1$, then $\gamma^{\gcd\{k, \tilde{k}\}} = \gamma^{ak+a\tilde{k}} = \pm 1$ and, since $\gcd\{k, \tilde{k}\}$ divides k and \tilde{k} , we must have $\gamma \in S_{\gcd\{k, \tilde{k}\}}^-$. On the other hand, if $\gamma \in S_{\gcd\{k, \tilde{k}\}}^-$, one clearly has $\gamma^k = \gamma^{\tilde{k}} = -1$ if and only if $\text{ord}_2(k) = \text{ord}_2(\tilde{k})$. Thus, we have

$$N_M = \begin{cases} S_{g_M} \times \{1\}, & \text{if } \text{ord}_2(n - \alpha) \text{ is not constant for all even } \mu, \\ S_{g_M} \times \{1\} \cup \left(S_{g_M^+} \cap S_{g_M^-}^- \right) \times \{-1\}, & \text{otherwise.} \end{cases}$$

Similarly $S_k \cap S_{\tilde{k}}^-$ (with k possibly zero) is nonempty exactly when $\text{ord}_2 k > \text{ord}_2 \tilde{k}$, in which case $S_k \cap S_{\tilde{k}}^- = S_{\gcd\{k, \tilde{k}\}}^-$, and we obtain

$$(56) \quad \#N_M = \begin{cases} g_M, & \text{ord}_2(n - \alpha) \text{ is not constant for all even } \mu \\ & \text{or } \text{ord}_2 g_M^- \geq \text{ord}_2 g_M^+, \\ 2g_M, & \text{otherwise.} \end{cases}$$

Note that $N_M \cong \mathbb{Z}_{g_M}$ in the first case in (56) and $N_M \cong \mathbb{Z}_{g_M} \oplus \mathbb{Z}_2$ in the second case. In particular, if k and \tilde{k} do not have any common divisors, then $S_k \cap S_{\tilde{k}}^-$ is nonempty (it may only contain the point -1) if and only if k is even and \tilde{k} is odd. Thus, if $g_M = 1$ and not all the μ are odd (as we assumed), N_M may only contain the two points $(1, 1)$ and $(-1, -1)$. It contains only the point $(1, 1)$ if and only if in addition to $g_M = 1$ we have $(-1, -1) \notin D(n - \alpha, \mu - 1)$; i.e. either $n - \alpha$ is odd for some invariant pair (α, μ) with μ odd, or $n - \alpha$ is even for some invariant pair (α, μ) with μ even.

4.2. **The basic equation.** In order to construct the normal form, we need the following technical result.

Proposition 13. *Let M, M' be formal hypersurfaces in \mathbb{C}^2 , each given in normal coordinates at $0 \in M$ and $0 \in M'$, respectively. Let $H = (F, G): (M, 0) \rightarrow (M', 0)$ be a formal invertible map, assume that there are at least two invariant pairs for M and M' , and let $(\alpha_0, \mu_0) \in Q_M$ denote one of them. Then, for each $k \geq 0$, there is a universal (i.e. not depending on any of the data M, M' , and H) polynomial R_{α_0, μ_0}^k such that*

$$(57) \quad \overline{G}_w(0)q_{\alpha_0, \mu_0+k}(\chi) = \frac{F_z(0)^{\alpha_0}\overline{G}_w(0)^{\mu_0}}{k!} \left[\left(\alpha_0 \frac{F_{zw^k}(0)}{F_z(0)} + \frac{\mu_0}{k+1} \frac{\overline{G}_{w^{k+1}}(\chi, 0)}{\overline{G}_w(0)} - \frac{\overline{G}_{w^{k+1}}(0)}{\overline{G}_w(0)} \right) q'_{\alpha_0, \mu_0}(\overline{F}(\chi, 0)) + \overline{F}_{w^k}(\chi, 0) q'_{\alpha_0, \mu_0; \chi'}(\overline{F}(\chi, 0)) - \overline{F}_{w^k}(0) q'_{\alpha_0, \mu_0; \chi'}(0) \right] + R_{\alpha, \mu}^k(q', F_{z^a w^b}(0), F_{w^k}(0), \overline{F}_{w^l}(0), \overline{F}_{w^l}(\chi, 0), \overline{G}_{w^m}(0), \overline{G}_{w^m}(\chi, 0)),$$

where $a \leq \alpha_0, b, l < k, m \leq k$, and q' is a shorthand notation for terms of the form $q'_{\alpha, \mu}^{(l)}(\overline{F}(\chi, 0))$ and $q'_{\alpha, \mu}^{(l)}(0)$. Moreover, if $\alpha_0 > 1$, then there is, for each $k \geq 0$, a universal polynomial S_{α_0, μ_0}^k such that

$$(58) \quad q_{\alpha_0-1, \mu_0+k}(\chi) = \frac{1}{k!} q'_{\alpha_0, \mu_0}(\overline{F}(\chi, 0)) F_{w^k}(0) F_z(0)^{\alpha_0-1} \overline{G}_w(0)^{\mu_0} + S_{\alpha_0, \mu_0}^k(q', F_{z^a w^b}(0), \overline{F}_{w^l}(0), \overline{F}_{w^l}(\chi, 0), \overline{G}_{w^m}(0), \overline{G}_{w^m}(\chi, 0)),$$

where $a \leq \alpha_0, b, l < k, m \leq k$.

Proof. We continue to use the notation from the previous sections and consider a formal map $H = (F, G)$ sending M into M' . We rewrite (29) as follows:

$$(59) \quad \overline{G}(0, Q(z, \chi, \tau)) = \overline{G}(\chi, \tau) + \sum_{\alpha, \mu} q'_{\alpha, \mu}(\overline{F}(\chi, \tau)) (F(z, Q(z, \chi, \tau)))^\alpha (\overline{G}(\chi, \tau))^\mu - \sum_{\alpha, \mu} q'_{\alpha, \mu}(\overline{F}(0, Q(z, \chi, \tau))) (F(z, Q(z, \chi, \tau)))^\alpha (\overline{G}(0, Q(z, \chi, \tau)))^\mu.$$

As before we will compare the coefficients of $z^{\alpha_0} \tau^{\mu_0+k}$ in (59) for $k \geq 1$. By expanding in a Taylor series and multiplying out, we see that these coefficients can be written as universal polynomials in $q_{\alpha, \mu}(\chi), (q'_{\alpha, \mu})^{(l)}(\overline{F}(\chi, 0)), (q'_{\alpha, \mu})^{(l)}(0), \overline{F}_{w^l}(\chi, 0), \overline{G}_{w^l}(\chi, 0), \overline{G}_{w^l}(0)$, and $F_{z^a w^b}(0)$. We will put restrictions on α, μ, l, a , and b .

The Taylor expansion of the left-hand side yields

$$(60) \quad \overline{G}(0, Q(z, \chi, \tau)) = \sum_{l=0}^{\infty} \frac{1}{l!} \overline{G}_{w^l}(0) \left(\tau + \sum_{\alpha, \mu} q_{\alpha, \mu}(\chi) z^\alpha \tau^\mu \right)^l.$$

For $l = 1$, the only term with $z^{\alpha_0} \tau^{\mu_0+k}$ has the coefficient $\overline{G}_w(0)q_{\alpha_0, \mu_0+k}(\chi)$. The general term coming from

$$\overline{G}_{w^l}(0) \left(\tau + \sum_{\alpha, \mu} q_{\alpha, \mu}(\chi) z^\alpha \tau^\mu \right)^l, \quad l \geq 2,$$

has the form

$$(61) \quad \overline{G_{w^l}}(0)q_{\alpha_1, \mu_1}(\chi) \cdots q_{\alpha_m, \mu_m}(\chi) z^{(\sum_{j=1}^m \alpha_j)} \tau^{(l-m+\sum_{j=1}^m \mu_j)}, \quad 0 \leq m \leq l,$$

where the product and the sum over an empty set of indices are, by definition, 1 and 0 respectively. For a term with $z^{\alpha_0} \tau^{\mu_0+k}$ we must have $\alpha_j \leq \alpha_0$, which also implies, since $(\alpha_0, \mu_0) \in Q_M$, that $\mu_j \geq \mu_0$. So either $\alpha_j < \alpha_0$ for all j or $\alpha_1 = \alpha_0$ and $m = 1$. For (61) to be such a term, we must have

$$(62) \quad l - m + \sum_{j=1}^m \mu_j = \mu_0 + k.$$

Thus, if $\alpha_1 = \alpha_0$ we have $l \leq k + 1$, and $l = k + 1$ yields the term

$$(63) \quad \frac{1}{k!} \overline{G_{w^{k+1}}}(0)q_{\alpha_0, \mu_0}(\chi)z^{\alpha_0} \tau^{\mu_0+k}.$$

For other possible terms, we have $m \geq 1$ and $\mu_j \geq \mu_0 + 1$ for all j . In this case (62) implies $l \leq k - (m - 1)\mu_0 \leq k$ and the resulting coefficient of $z^{\alpha_0} \tau^{\mu_0+k}$ will be a polynomial of

$$(64) \quad \overline{G_{w^l}}(0), \quad l \leq k, \text{ and } q_{\alpha, \mu}, \quad \alpha \leq \alpha_0, \quad \mu \leq \mu_0 + k.$$

We claim that if $\mu = \mu_0 + k$, then $\alpha < \alpha_0 - 1$. Let us first check that q_{α_0, μ_0+k} cannot appear. Indeed, (62) implies that $l = 1$ in that case, which we have already separated above. Now, q_{α_0-1, μ_0+k} can only appear if $q_{1,0} \neq 0$, a case excluded by our assumption that there are at least two invariant pairs. Hence, the remaining terms are polynomial in

$$(65) \quad \overline{G_{w^l}}(0), \quad l \leq k, \text{ and } q_{\alpha, \mu}, \quad \alpha \leq \alpha_0, \quad \mu < \mu_0 + k \text{ or } \alpha < \alpha_0 - 1, \mu = \mu_0 + k.$$

Let us now examine the right-hand side of (59). Clearly, the first term on the right does not contribute a term $z^{\alpha_0} \tau^{\mu_0+k}$, so let us consider the first sum. If we get a term containing $\overline{G_{w^l}}(\chi, 0)$, the minimum power of τ we get from the other terms is $(\alpha - \alpha_0)_+ + \mu - 1$, where we write $n_+ = \max(n, 0)$. Hence,

$$(66) \quad l + (\alpha - \alpha_0)_+ + \mu - 1 \leq \mu_0 + k.$$

If $\mu > \mu_0$, (66) implies that $l < k + 1$. On the other hand, if $\mu \leq \mu_0$, either $\alpha + \mu > \alpha_0 + \mu_0$ or $(\alpha, \mu) = (\alpha_0, \mu_0)$. In the first case, we have $\alpha - \alpha_0 > \mu_0 - \mu \geq 0$, and so (66) again implies that $l < k + 1$. In the second case, $l = k + 1$ can appear, in which case the corresponding term is given by

$$(67) \quad \frac{\mu_0}{k!} \overline{G_{w^{k+1}}}(\chi, 0)q'_{\alpha_0, \mu_0}(\overline{F}(\chi, 0))F_z(0)^{\alpha_0} \overline{G_w}(0)^{\mu_0-1}.$$

Any term containing a factor $F_{z^a w^b}(0)$ comes together with a contribution of $z^a (\tau + \sum q_{\beta, \nu}(\chi) z^\beta \tau^\nu)^b$. The general term from expanding this product is

$$(68) \quad q_{\beta_1, \nu_1}(\chi) \cdots q_{\beta_r, \nu_r}(\chi) z^{a+\sum \beta_j} \tau^{b-r+\sum \nu_j}, \quad 0 \leq r \leq b,$$

where the case $r = 0$ is understood to mean $z^a \tau^b$. Hence, we have $\beta_j < \alpha_0$ unless for some j , $\beta_j = \alpha_0$, in which case $r = 1$. We will discuss the latter case later and, hence, we assume for now that $\beta_j < \alpha_0$ for each $j = 1, \dots, r$ or that $r = 0$. Note that $\beta_j < \alpha_0$ implies that $\nu_j > \mu_0$. Thus, examining the overall exponents of z and τ in a term from the first sum on the right which contains the factor $F_{z^a w^b}(0)$ and which contributes $z^{\alpha_0} \tau^{\mu_0+k}$, we see that

$$\mu + b - r + \sum \nu_j + (\alpha - \alpha_0 - 1 + a + \sum \beta_j) \leq \mu_0 + k$$

for some $0 \leq r \leq b$ and some sequence of indices $\beta_j < \alpha_0$, $\nu_j > \mu_0$, $j = 1, \dots, r$, such that

$$\alpha - \alpha_0 - 1 + a + \sum \beta_j \geq 0$$

and

$$a + \sum \beta_j - \alpha_0 \leq 0.$$

Observe that these inequalities imply, under the assumption made above that $r = 0$ or $\beta_j < \alpha_0$ for $j = 1, \dots, r$,

$$(69) \quad \mu + b + r\mu_0 + \alpha - \alpha_0 - 1 + a + \sum \beta_j \leq \mu_0 + k.$$

If $\mu > \mu_0$, (69) implies $b < k$. On the other hand, if $\mu \leq \mu_0$, either $(\alpha, \mu) = (\alpha_0, \mu_0)$ or $\alpha + \mu > \alpha_0 + \mu_0$. If $(\alpha, \mu) = (\alpha_0, \mu_0)$, (69) implies that $b < k$ unless $a + \sum \beta_j \leq 1$. So either $a = 1$ and $r = 0$, which gives us the term

$$(70) \quad \frac{\alpha_0}{k!} q'_{\alpha_0, \mu_0}(\bar{F}(\chi, 0)) F_{zw^k}(0) F_z(0)^{\alpha_0 - 1} \overline{G_w}(0)^{\mu_0},$$

or $a = 0$, $r = 1$ and $\beta_1 = 1$. Going back to (69) we see that $b \leq k$, giving rise to terms containing $F_{w^k}(0)$. If $\alpha + \mu > \alpha_0 + \mu_0$, (69) implies $a + b \leq k$.

We are now going to check which $q_{\beta, \nu}$ can appear from the first sum. If in (68) one of the β_j , say β_1 , satisfies $\beta_1 = \alpha_0 - 1$ and the corresponding $\nu_1 = \mu_0 + k$, we clearly have $r \leq 2$. If $r = 2$, then $\beta_2 = 1$, $a = 0$, $\nu_2 = 0$, $\mu = 0$. Since $(\beta_2, \nu_2) = (1, 0)$, terms of this form cannot appear (by our assumption that there are at least two invariant pairs). On the other hand, if $r = 1$, we have $a \leq 1$, and also $\alpha + a \leq 2$. Checking the possible cases $\alpha = 1, 2$ and $a = 0, 1$, we see that all these are again excluded by the assumption of having at least two invariant pairs.

Let us now turn to the case where, for some j , $\alpha_0 = \beta_j$. It follows immediately that $r = 1$, $a = 0$, and $\nu_1 \geq \mu_0$, and instead of (69) we now obtain

$$(71) \quad \mu + b - 1 + \nu_1 + (\alpha - 1)_+ \leq \mu_0 + k.$$

If $\nu_1 > \mu_0$, we immediately obtain $b \leq k$. If on the other hand, $\nu_1 = \mu_0$, we have $b \leq k$ unless $\mu = 0$. If $\mu = 0$, then again $b \leq k$, this time unless $\alpha = 1$. It follows that a term with $b = k + 1$ can only appear if $(1, 0)$ is an invariant pair, which is impossible by assumption.

We also note that $\nu_1 = \mu_0 + k$ implies $\mu + b + (\alpha - 1)_+ \leq 1$. Since in that case $b \geq 1$, this implies that $(1, 0)$ is an invariant pair, which is excluded.

At this point, let us note that for the $q_{\beta, \nu}(\chi)$ coming from the first sum, we have shown that either $\nu < \mu_0 + k$ or $\beta < \alpha_0 - 1$ if $\nu = \mu_0 + k$.

We now turn to the expansion of the term $q'_{\alpha, \mu}(\bar{F}(\chi, \tau))$, which appears as a factor in the first sum on the right in (59), as a Taylor series in τ . The coefficient of τ^l is a linear combination of terms of the form

$$q'_{\alpha, \mu}{}^{(p)}(\bar{F}(\chi, 0)) \overline{F_w}(\chi, 0)^{p_1} \dots \overline{F_{w^k}}(\chi, 0)^{p_k},$$

where $p_1 + 2p_2 + \dots + kp_k = l$. If this term appears as a factor in a coefficient of $z^{\alpha_0} \tau^{\mu_0 + k}$, it follows that $l + \mu + \max(\alpha - \alpha_0, 0) \leq \mu_0 + k$, and by arguments with which the patient reader is now familiar, either $l < k$, or $\alpha = \alpha_0$, $\mu = \mu_0$, and $l = k$. We conclude that the only term containing $\overline{F_{w^k}}(\chi, 0)$ is

$$(72) \quad q'_{\alpha_0, \mu_0; \chi'}(\bar{F}(\chi, 0)) \overline{F_{w^k}}(\chi, 0).$$

We are left with contributions from the second sum on the right-hand side of (59). We have already expanded $\bar{G}(0, Q(z, \chi, \tau))$ in (32). The general term here

has the form (61). Hence, $\sum \alpha_j \leq \alpha_0$, which implies that either $\alpha_j = \alpha_0$ for some j (in which case, $m = 1$), or that $\alpha_j < \alpha_0$ for all j . If $\alpha_j = \alpha_0$, there must be at least one τ contributing from the first factor of the summand

$$q'_{\alpha,\mu}(\bar{F}(0, Q(z, \chi, \tau)))(F(z, Q(z, \chi, \tau)))^\alpha (\bar{G}(0, Q(z, \chi, \tau)))^\mu;$$

we then see that $\mu + l - 1 + \mu_1 + \alpha_0 \leq \mu_0 + k$. But $\alpha_0 \geq 1$, so $\mu + \mu_1 + l \leq \mu_0 + k$; since $\alpha_1 = \alpha_0$, it follows that $\mu_1 \geq \mu_0$, and so $l \leq k$. We also note that since $l \geq 1$, then $\mu_1 < \mu_0 + k$. On the other hand, if $\alpha_j < \alpha_0$, $\mu_j > \mu_0$ for all j , the first term contributes either a term containing a τ , or (only if $\mu_0 = 0$) it contributes a term containing z^{α_0} ; in the first case, we have $\mu + l + m\mu_0 + \max(0, \alpha - \alpha_0) \leq \mu_0 + k$, from which it follows that $l \leq k$. In the second case, we must have $m = 0$ and $\mu - 1 + l + \alpha \leq \mu_0 + k$, from which we again conclude that $l \leq k$ (since $\alpha_0 \geq 1$).

A term containing an $F_{z^{a_w b}}(0)$ comes with a term of the form (68). Again, the first factor $q'_{\alpha,\mu}(\bar{F}(0, Q(z, \chi, \tau)))$ contributes either a τ or a term z^{α_0} . In the first case, if it contributes at least a τ , instead of (69), we get

$$(73) \quad \mu + b + 1 + r\mu_0 + \max\left(0, \alpha - \alpha_0 - 1 + a + \sum \beta_j\right) \leq \mu_0 + k,$$

if $\beta_j < \alpha_0$ for all j . If $\mu \geq \mu_0$, (73) implies $b < k$. On the other hand, if $\mu < \mu_0$, $\alpha_0 + \mu_0 > \alpha + \mu$, and so (73) again implies $b < k$. If $\beta_j = \alpha_0$ for some j , $r = 1$ and $a = 0$, and instead of (73) we obtain

$$(74) \quad \mu + b + \nu_1 + \alpha - 1 \leq \mu_0 + k.$$

Since $\alpha_0 = \beta_1$, we have $\nu_1 \geq \mu_0$. Hence, (74) implies $b \leq k$, and if $b = k$ in (74) we have $\mu + \alpha - 1 \leq \mu_0 - \nu_1 \leq 0$. Hence, this implies that $(1, 0)$ is the invariant pair, which is excluded. In the second case, if we get a z^{α_0} from $q'_{\alpha,\mu}(\bar{F}(0, Q(z, \chi, \tau)))$, we must have $a = r = 0$, and

$$(75) \quad \mu + b + \alpha \leq \mu_0 + k,$$

which again implies $b < k$. We also note that any $q_{\alpha,\mu}$ coming from here satisfies $\mu < \mu_0 + k$.

We now come to contributions from expanding $q'_{\alpha,\mu}(\bar{F}(0, Q(z, \chi, \tau)))$. A term containing an $\bar{F}_{w^t}(0)$ comes with a term of the form (61) with G replaced by F . Hence, we again either have $\alpha_j < \alpha_0$ for all j , or $\alpha_1 = \alpha_0$ and $m = 1$. In the first case, $\mu_j > \mu_0$ for all j , and so

$$(76) \quad \mu + l + m\mu_0 + \max\left(0, \alpha - \alpha_0 + \sum \alpha_j\right) \leq \mu_0 + k.$$

If $\mu > \mu_0$, this implies $l < k$. If on the other hand, $\mu \leq \mu_0$, we have either $\alpha + \mu > \alpha_0 + \mu_0$ or $(\alpha, \mu) = (\alpha_0, \mu_0)$. If $\alpha + \mu > \alpha_0 + \mu_0$, $\alpha > \alpha_0$, and so (76) becomes

$$\mu + l + m\mu_0 + \alpha - \alpha_0 + \sum \alpha_j \leq \mu_0 + k,$$

from which it follows that $l < k$. In fact, if $(\alpha, \mu) = (\alpha_0, \mu_0)$, (76) turns into $l + m\mu_0 + \sum \alpha_j \leq k$, so again $l \leq k$.

In the second case, if $\alpha_1 = \alpha_0$, we have $\mu_1 \geq \mu_0$, and $\mu_1 + l - 1 + \alpha + \mu \leq \mu_0 + k$. Unless $\mu = 0$ and $\alpha = 1$, this implies that $l < k$.

By an argument similar to the one above, the $q_{\alpha,\mu}$ from this first factor satisfy either $\mu < \mu_0 + k$ or $\alpha < \alpha_0 - 1$.

By now, we have nearly finished the proof of (57). The only difference is that we have constructed a remainder term \bar{R} which depends also on $q_{\beta,\mu}$, where either

$\beta < \alpha_0 - 1$ or $\beta = \alpha_0$, but $\mu < \mu_0 + k$. The proof will be finished by showing that terms of the form $q_{\beta, \mu}$ with $\beta < \alpha_0$ themselves depend on terms which appear in the remainder and on $q_{\gamma, \nu}$, where $\nu < \mu_0 + k$. Applying this result repeatedly then gives us the form of the remainder we seek. We incorporate the proof of this auxiliary statement in the derivation of (58), which follows.

We shall extract the coefficient of $z^\beta \tau^{\mu_0+k}$, where $\beta < \alpha_0$, on both sides of (59). Let us start with the left-hand side. First, we get the term on the left-hand side of (58) from the Taylor expansion (60) with $l = 1$. A term of the form (61) can only appear if $\alpha_j \leq \beta < \alpha$; furthermore, we have $m > 0$. Now $\alpha_j < \alpha$ implies that $\mu_j > \mu \geq 0$ for all j , so that $\mu + k = \sum_{j=1}^m \mu_j + l - m \geq \mu + l$, or $l \leq k$. Furthermore, all terms $q_{\gamma, \nu}$ coming from (60) other than q_{β, μ_0+k} satisfy $\nu < \mu_0 + k$, as can be easily checked using (62).

Now, we turn to the right-hand side of (59). Of course, the first term does not contribute as long as $\alpha_0 > 1$; for the first sum, as usual, we distinguish the cases $\mu \leq \mu_0$ and $\mu > \mu_0$.

We first discuss the case $\mu \leq \mu_0$. If a derivative $\bar{G}_{\tau^l}(\chi, 0)$ contributes, then from the other terms in the product, the minimum exponent of the τ 's is $\mu - 1 + (\alpha - \beta)_+$. Since (α_0, μ_0) is an invariant pair, $\alpha - \beta > \alpha - \alpha_0 \geq \mu_0 - \mu \geq 0$. So the overall exponent of τ is at least $\mu + l - 1 + \alpha - \beta$. It follows that

$$\mu + l + \alpha - \alpha_0 \leq \mu_0 + k,$$

from which we conclude $l \leq k$. If we have a term containing $F_{z^a w^b}(0)$, by using the notation from (68) the other terms contribute at least

$$\mu + \left(\alpha - 1 - \beta + a + \sum \beta_j \right)_+$$

to the overall exponent of τ . As before, $\alpha - \beta > \alpha - \alpha_0$, and since $\beta_j \leq \beta < \alpha_0$, $\mu_j > \mu_0$, so we have

$$(77) \quad \mu + b + r\mu_0 + \alpha - \alpha_0 + a + \sum \beta_j \leq \mu + b + r\mu_0 + \alpha - \beta - 1 + a + \sum \beta_j \leq \mu_0 + k$$

and therefore $b \leq k$. If $b = k$, letting $D = \mu - \mu_0 + \alpha - \beta - 1 \geq 0$, we get

$$D + r\mu_0 + a + \sum \beta_j \leq 0,$$

so we must have $D = r\mu_0 = a = \beta_j = 0$ for all j , which in particular implies $r = 0$. Furthermore, since $D = 0$, and $\mu - \mu_0 + \alpha - \beta > \mu - \mu_0 + \alpha - \alpha_0 > 0$ if $(\alpha, \mu) \neq (\alpha_0, \mu_0)$, it follows that $(\alpha, \mu) = (\alpha_0, \mu_0)$ and $\beta = \alpha_0 - 1$, which leads to the term

$$(78) \quad \frac{1}{k!} q'_{\alpha_0, \mu_0}(\bar{F}(\chi, 0)) F_{w^k}(0) F_z(0)^{\alpha_0-1} \overline{G_w}(0)^{\mu_0}.$$

If a term $\overline{F_{w^l}}(\chi, 0)$ from expanding the first factor appears, the exponent of τ is at least $l + \alpha - \beta + \mu$; a similar discussion as above shows that this implies $l < k$.

We now turn to the case $\mu > \mu_0$. In this case, a derivative $\bar{G}_{\tau^l}(\chi, 0)$ will contribute only if $\alpha - \beta + l + \mu - 1 \leq \mu + k$, which implies $l \leq k$. Now let us consider the derivatives $F_{z^a w^b}(0)$, for which we have (77), which in our case now implies $b < k$. Also, as above, a term $\overline{F_{w^l}}(\chi, 0)$ only appears if $l < k$.

Now we have to check the second sum on the right-hand side. A discussion similar to the one above, using the fact that any term from the first factor in each of the products contributes at least one power of τ , now shows that only terms of

the form claimed appear, which finishes the proof of (58), and as explained before, also that of (57). \square

4.3. Construction of the normalization map. Recall that we assume that $\Lambda_{M'}$ consists of at least two points $(\alpha, \mu) \neq (\tilde{\alpha}, \tilde{\mu})$. We also assume that one of these pairs, say (α, μ) , satisfies $\alpha \neq n$, where $n = n(\alpha, \mu)$ as defined by (8). We also note that at least one of α and α' is greater than 1; let us denote the corresponding invariant pair by (α', μ') ; thus, $\alpha' > 1$.

We first normalize M as in Proposition 10. Hence we shall assume that (51) holds and

$$(79) \quad F(z, 0) = F_z(0)z, \quad (F_z(0), G_w(0)) \in C(\alpha, n, \mu) \cap C(\tilde{\alpha}, \tilde{n}, \tilde{\mu}),$$

where C is given by (46). We shall construct a map $H = (F, G)$ with these first order derivatives $F_z(0)$ and $G_w(0)$ given such that the equation of $M = H^{-1}(M')$ has a special form. The map H will be unique up to at most one more parameter. The precise formulation of the result will be given in Theorem 14 below.

Observe that $G_w(z, 0)$ and $F(z, 0)$ are uniquely determined by $F_z(0)$ and $G_w(0)$ in view of Lemma 9 (note that there $m_0 \geq 2$, since $(1, 0)$ is not an invariant pair) and Proposition 10. Let $k \geq 1$ and assume that we have constructed $\overline{F_{w^l}}(\chi, 0)$ and $\overline{G_{w^{l+1}}}(\chi, 0)$ for $l < k$. We first use (58) for (α', μ') to determine $F_{w^k}(0)$ uniquely by the new requirement that $q_{\alpha'-1, \mu'+k}(\chi)$ does not contain any term $\chi^{n'}$. Let us now rewrite the basic equation (57) with the normalizations we have made so far:

$$(80) \quad \begin{aligned} \overline{G_w}(0)q_{\alpha, \mu+k}(\chi) &= \frac{2i\overline{G_w}(0)}{k!} \left(\frac{n}{\overline{F_z}(0)} \overline{F_{w^k}}(\chi, 0)\chi^{n-1} + \frac{\alpha}{F_z(0)} F_{zw^k}(0)\chi^n \right) \\ &\quad + \frac{2i}{k!} \left(\frac{\mu}{k+1} \overline{G_{w^{k+1}}}(\chi, 0) - \overline{G_{w^{k+1}}}(0) \right) \chi^n \\ &\quad + \tilde{R}_{\alpha, \mu}^k (q', F_{z^a w^b}(0), F_{w^k}(0), \overline{F_{w^l}}(0), \overline{F_{w^l}}(\chi, 0), \overline{G_{w^m}}(0), \overline{G_{w^m}}(\chi, 0)), \end{aligned}$$

where $\tilde{R}_{\alpha, \mu}^k$ contains the same terms as $R_{\alpha, \mu}^k$ in (57), and where $F_{w^k}(0)$ has already been determined. We note that by Lemma 9, if (z, w) are normal coordinates for M , then $\overline{G_{w^{k+1}}}(\chi, 0)$ is determined by $\text{Re } \overline{G_{w^{k+1}}}(0) =: s_{k+1}$, the previously determined derivatives, and Q' . For $s_{k+1} \in \mathbb{R}$ we define $\overline{G_{w^{k+1}}}(\chi, 0)$ by (38) and rewrite (80) as

$$(81) \quad \begin{aligned} \overline{G_w}(0)q_{\alpha, \mu+k}(\chi) &= \frac{2i\overline{G_w}(0)}{k!} \left(\frac{n}{\overline{F_z}(0)} \overline{F_{w^k}}(\chi, 0)\chi^{n-1} + \frac{\alpha}{F_z(0)} F_{zw^k}(0)\chi^n \right) \\ &\quad + \frac{2i}{k!} \left(\frac{\mu}{k+1} - 1 \right) s_{k+1} \chi^n \\ &\quad + \tilde{R}_{\alpha, \mu}^k (Q', F_{z^a w^b}(0), F_{w^k}(0), \overline{F_{w^l}}(0), \overline{F_{w^l}}(\chi, 0), \overline{G_{w^m}}(0), \overline{G_{w^m}}(\chi, 0)), \end{aligned}$$

where we still write $\tilde{R}_{\alpha, \mu}^k$ for the remainder, which is again a universal polynomial. Now we observe that we can uniquely determine $\overline{F_{z^a w^k}}(0)$ for $a > 1$ by requiring that $q_{\alpha, \mu+k}(\chi)$ is a polynomial of degree at most n . We are now going to examine the coefficient of χ^n on the right-hand side of (81). It is

$$(82) \quad \frac{2i}{k!} \left(n \frac{\overline{F_{zw^k}}(0)\overline{G_w}(0)}{\overline{F_z}(0)} + \alpha \frac{F_{zw^k}(0)\overline{G_w}(0)}{F_z(0)} + \left(\frac{\mu}{k+1} - 1 \right) s_{k+1} \right) + A_{\alpha, \mu}^{k, n},$$

where $A_{\alpha,\mu}^{k,n}$ denotes the coefficient of χ^n in $\tilde{R}_{\alpha,\mu}^k(\dots)$. Writing

$$(83) \quad \frac{F_{zw^k}(0)\overline{G_w(0)}}{F_z(0)} = B_k$$

and noting that $\overline{G_w(0)}$ is real, we observe that in order to make the coefficient of χ^n in $q_{\alpha,\mu+k}(\chi)$ vanish, we need to solve the equation

$$(84) \quad n\bar{B}_k + \alpha B_k + \left(\frac{\mu}{k+1} - 1\right) s_{k+1} = A_k$$

for some right-hand side A_k . Since $n \neq \alpha$, this can always be done, although B_k and s_{k+1} are not uniquely determined. In order to determine B_k (and hence $F_{zw^k}(0)$ in terms of the first jet of H) uniquely, we need to use our second invariant pair $(\tilde{\alpha}, \tilde{\mu})$. In the basic equation for $q_{\tilde{\alpha},\tilde{\mu}+k}(\chi)$ we see that the coefficient of $\chi^{\tilde{n}}$ on the right-hand side has a similar form to the corresponding one of χ^n in (81) above. In fact, inspecting (57) we see that this coefficient has the form

$$(85) \quad \frac{ic\varepsilon}{k!} \left(\tilde{n} \frac{F_{zw^k}(0)\overline{G_w(0)}}{F_z(0)} + \tilde{\alpha} \frac{F_{zw^k}(0)\overline{G_w(0)}}{F_z(0)} + \left(\frac{\tilde{\mu}}{k+1} - 1\right) s_{k+1} \right) + A_{\alpha',\mu'}^{k,n},$$

where $c > 0$ and ε are the invariant(s) given by Proposition 10. So if we can solve the real part of the equation

$$(86) \quad \tilde{n}\bar{B}_k + \tilde{\alpha}B_k + \left(\frac{\tilde{\mu}}{k+1} - 1\right) s_{k+1} = A'_k,$$

then we can make the real part of $i\varepsilon^{-1}$ times the coefficient of $\chi^{\tilde{n}}$ in $q_{\tilde{\alpha},\tilde{\mu}}(\chi)$ vanish; i.e. the coefficient of $\chi^{\tilde{n}}$ in the expansion of $q_{\tilde{\alpha},\tilde{\mu}+k}(\chi)$ is $r\varepsilon$ with $r \in \mathbb{R}$. This is the additional normalization condition that allows us to solve uniquely for B_k and s_{k+1} , and hence for $F_{zw^k}(0)$ and $\text{Re } \bar{G}_{w^{k+1}}(0)$. Indeed, if we write $B_k = a_k + ib_k$ and separate (84) and (86) into real and imaginary parts, then we obtain the following system of real equations:

$$(87) \quad \begin{array}{rcl} (\alpha + n)a_k & + \left(\frac{\mu}{k+1} - 1\right) s_{k+1} & = r_k^1, \\ & (\alpha - n)b_k & = r_k^2, \\ (\tilde{\alpha} + \tilde{n})a_k & + \left(\frac{\tilde{\mu}}{k+1} - 1\right) s_{k+1} & = r_k^3. \end{array}$$

Since $\alpha - n$ is not zero, we can solve for b_k . Let us consider the determinant D_k of the remaining equations in (87) for a_k and s_{k+1} :

$$(88) \quad \begin{aligned} D_k &= \det \begin{pmatrix} \alpha + n & \frac{\mu}{k+1} - 1 \\ \tilde{\alpha} + \tilde{n} & \frac{\tilde{\mu}}{k+1} - 1 \end{pmatrix} \\ &= \frac{((\tilde{\alpha} + \tilde{n}) - (\alpha + n))(k+1) + (\alpha + n)\tilde{\mu} - (\tilde{\alpha} + \tilde{n})\mu}{k+1}. \end{aligned}$$

Observe (since $\tilde{\mu} \neq \mu$) that $D_k \neq 0$ for $k \geq 1$ unless $\alpha + n \neq \tilde{\alpha} + \tilde{n}$ and the number

$$(89) \quad \lambda = \lambda(\alpha, \mu, n, \tilde{\alpha}, \tilde{\mu}, \tilde{n}) := \frac{(\tilde{\alpha} + \tilde{n})\mu - (\alpha + n)\tilde{\mu}}{(\tilde{\alpha} + \tilde{n}) - (\alpha + n)} - 1$$

is an integer ≥ 1 . In the latter case, we shall allow $s_{\lambda+1}$ to be a real parameter and solve the second equation in (87) for a_λ in terms of $s_{\lambda+1}$. If $\alpha + n = \tilde{\alpha} + \tilde{n}$ (which, in particular, implies that one of the pairs, say (α, μ) , satisfies the assumption $\alpha \neq n$ made at the beginning of this section) or λ is not an integer ≥ 1 , then we may

solve the three equations in (87) uniquely for $a_k, b_k,$ and s_{k+1} for every k ; the normalization map is then determined by $F_z(0)$ and $G_w(0)$.

Let us now summarize the normalization conditions. Let M be a formal manifold, given in some fixed system of normal coordinates (z, w) by $w = Q(z, \chi, \tau)$. We choose an invariant pair (α, μ) with $\alpha \neq n(\alpha, \mu) = n$, and an additional invariant pair $(\tilde{\alpha}, \tilde{\mu})$. We let (α', μ') denote (α, μ) or $(\tilde{\alpha}, \tilde{\mu})$ in such a way that $\alpha' > 1$. Let K denote the set of integers $k \geq 1$ such that D_k in (88) is zero; thus, K is either empty (if $\alpha + n = \alpha' + n'$ or λ , given by (89), is not an integer ≥ 1) or consists of one point, namely λ . We shall say that Q is in its normal form if the following hold:

$$(90) \quad q_{\alpha, \mu}(\chi) = 2i\chi^n, \quad q_{\tilde{\alpha}, \tilde{\mu}}(\chi) = 2ic\varepsilon\chi^{\tilde{n}} + O(\chi^{\tilde{n}+1}),$$

$$(91) \quad q_{\alpha, \mu+k}(\chi) \text{ is a polynomial of degree at most } n - 1 \text{ for } k \geq 1,$$

$$(92) \quad q_{\alpha'-1, \mu'+k}^{(n')} (0) = 0, \text{ for } k \geq 1,$$

and

$$(93) \quad \text{Im } \varepsilon^{-1} q_{\tilde{\alpha}, \tilde{\mu}+k}^{(\tilde{n})} (0) = 0 \text{ for } k \geq 1 \text{ and } k \notin K,$$

where c and ε are the invariants given by Proposition 10. We have proved the following:

Theorem 14. *Let $M \subset \mathbb{C}^2$ be a formal hypersurface such that Q_M contains at least two points $(\alpha, \mu) \neq (\tilde{\alpha}, \tilde{\mu})$ and (say) (α, μ) satisfies $\alpha \neq n(\alpha, \mu)$. Then there exists a system of normal coordinates (z, w) for M such that M is given by $w = Q(z, \chi, \tau)$ and Q is in its normal form; that is, Q satisfies (90)-(93). Furthermore, if (z, w) are such coordinates, then for any other such system of coordinates (z', w') , where $z = F(z', w')$ and $w = G(z', w')$, F and G are uniquely determined by $F_z(0), G_w(0)$ and $\text{Re } G_{w^{k+1}}(0)$ for $k \in K$, and $(F_z(0), G_w(0))$ belongs to the set $C(\alpha, n, \mu) \cap C(\tilde{\alpha}, \tilde{n}, \tilde{\mu})$ (where C is defined in (46)). Here, K is the set (consisting of at most one point) of integers $k \geq 1$ such that $D_k = 0$ in (88). The set K is empty if and only if $n + \alpha = \tilde{n} + \tilde{\alpha}$ or λ given by (89) is not an integer ≥ 1 .*

Remark 15. At first glance, it would seem that the automorphisms of a formal hypersurface satisfying the assumptions in Theorem 14 depend on two real parameters, $G_w(0)$ and $\text{Re } G_{w^{k+1}}(0)$ for $k \in K$. However, we observe that if the set K is not empty, then λ , given by (89), is an integer $k \geq 1$. It follows that, in the terminology of Section 4.1, we are in Case B. Hence, the set $C(\alpha, n, \mu) \cap C(\tilde{\alpha}, \tilde{n}, \tilde{\mu})$ is finite (see Remark 11). Consequently, the automorphisms only depend on one real parameter. This proves the bound (10) in Theorem 1.

We mention that we have also proved the following Jet Parametrization Theorem. We write $G_0^k(\mathbb{C}^2)$ for the jet group of order k of biholomorphic mappings of neighborhoods of 0 in \mathbb{C}^2 fixing 0. This theorem implies that $\text{Aut}_f(M, 0)$ can be embedded as closed real subgroup of $G_0^k(\mathbb{C}^2)$ and hence as its Lie subgroup, for some k .

Theorem 16. *Let $M \subset \mathbb{C}^2$ be a formal hypersurface such that Q_M contains at least two points $(\alpha, \mu) \neq (\tilde{\alpha}, \tilde{\mu})$ and (say) (α, μ) satisfies $\alpha \neq n(\alpha, \mu)$. Then there exists an integer k and a formal power series map $\psi(Z, W) = \sum_{\alpha \in \mathbb{N}^2} \psi_\alpha(W) Z^\alpha$, $Z \in \mathbb{C}^2$, with rational coefficients $\psi_\alpha(W)$ in $W \in G_0^k(\mathbb{C}^2)$ with no poles in $G_0^k(\mathbb{C}^2)$,*

such that

$$(94) \quad H(Z) = \psi(Z, j_0^k H)$$

for any $H \in \text{Aut}_f(M, 0)$. Furthermore k can be chosen to be 1 if $n + \alpha = \tilde{n} + \tilde{\alpha}$ or $\frac{\mu(\tilde{n} + \tilde{\alpha}) - \tilde{\mu}(n + \alpha)}{(\tilde{n} + \tilde{\alpha}) - (n + \alpha)} \notin \mathbb{N}_{\geq 2}$; otherwise, k can be chosen to be $\frac{\mu(\tilde{n} + \tilde{\alpha}) - \tilde{\mu}(n + \alpha)}{(\tilde{n} + \tilde{\alpha}) - (n + \alpha)} \in \mathbb{N}_{\geq 2}$.

It is worthwhile to note that in the normal form described above, if K is the empty set (this in particular means we are in Case B in (49)), the normalization group $C(\alpha, n, \mu) \cap C(\tilde{\alpha}, \tilde{n}, \tilde{\mu})$ is discrete, and it acts on the space of normal forms by the linear transformations $(z, w) \mapsto (\gamma z, \delta w)$, $(\gamma, \delta) \in C(\alpha, n, \mu) \cap C(\tilde{\alpha}, \tilde{n}, \tilde{\mu}) = D(n - \alpha, \mu - 1) \cap D(\tilde{n} - \tilde{\alpha}, \tilde{\mu} - 1)$, where D is defined by (53). Thus, the normal form described above gives (in this case) a complete solution to the equivalence problem, and it also linearizes the action of the automorphism group $\text{Aut}_f(M, p)$. In particular, if $D(n - \alpha, \mu - 1) \cap D(\tilde{n} - \tilde{\alpha}, \tilde{\mu} - 1) = \{(1, 1)\}$, the power series coefficients of Q (in the normal form) form a complete set of biholomorphic invariants of M . Following the arguments of Remark 12 we obtain:

Corollary 17. *Let $\Lambda \subset \mathbb{N}^3$ contain two points $(\alpha, n, \mu) \neq (\tilde{\alpha}, \tilde{n}, \tilde{\mu})$ with $n \neq \alpha$, such that either $\alpha + n = \tilde{\alpha} + \tilde{n}$ or $\frac{\mu(\alpha + n) - \mu(\tilde{\alpha} + \tilde{n})}{(\alpha + n) - (\tilde{\alpha} + \tilde{n})}$ is not a positive integer. Assume in addition that furthermore $\text{gcd}\{n - \alpha, \tilde{n} - \tilde{\alpha}\} = 1$, either μ or $\tilde{\mu}$ is odd, and either $\mu + n - \alpha$ or $\tilde{\mu} + \tilde{n} - \tilde{\alpha}$ is even. Then two hypersurfaces (M, p) and (M', p) satisfying $\Lambda_M = \Lambda_{M'} = \Lambda$ are biholomorphically equivalent if and only if their normal forms $w = Q(z, \bar{z}, \bar{w})$ and $w = Q'(z, \bar{z}, \bar{w})$ coincide, i.e. $Q \equiv Q'$.*

5. DEPENDENCE ON HIGHER ORDER JETS: AN EXAMPLE

It is natural to ask whether or not the maps really depend on higher order jets if the set K is nonempty. In this section we will give an example of a hypersurface M which satisfies the assumptions of Theorem 14, the set K consists exactly of one integer $\ell + 1$, and the biholomorphisms are determined by their $(\ell + 1)$ -jet, but not their ℓ -jets at 0, where $\ell \geq 5$. For this, choose positive integers a, b, c, d satisfying

$$c(\ell - b) = a(\ell - d), \quad c < a + b - d, \quad \ell/2 < b < d < \ell, \quad a, c > 0,$$

implying $c < a$. For instance, we can take $a = 4, c = 2, b = \ell - 2, d = \ell - 1$ to satisfy all the inequalities.

We then consider the preimage of the quadric S , given by $\text{Im } \eta = |\zeta|^2$, under the map $B: (z, w) \mapsto (z^a w^b + z^c w^d, w^\ell)$. We first claim that $B^{-1}(S)$ contains a unique real hypersurface M of the form

$$t = s^{2b - \ell + 1} \varphi(z, \bar{z}, s), \quad w = s + it,$$

where

$$(95) \quad \varphi(z, 0, s) = \varphi(0, \chi, s) = 0, \quad \varphi(z, \bar{z}, s) = z^a \bar{z}^a + 2 \text{Re } z^c \bar{z}^a w^{d-b} + O(s^{d-b+1}),$$

and that for this hypersurface M , Λ_M contains the points $(a, a, 2b - \ell + 1)$ and $(c, a, b + d - \ell + 1)$. Rewriting the equation for $B^{-1}(S)$, we see that this set is given by

$$(96) \quad \sum_{0 \leq j \leq \lfloor \frac{\ell-1}{2} \rfloor} \binom{\ell}{2j+1} (-1)^j s^{\ell-2j-1} t^{2j+1} = (s^2 + t^2)^b \left(|z|^{2a} + 2 \text{Re } z^c \bar{z}^a w^{d-b} + |z|^{2c} |w|^{2(d-b)} \right).$$

Substituting $t = s^m \lambda$, we obtain

$$(97) \quad \sum_{0 \leq j \leq \lfloor \frac{\ell-1}{2} \rfloor} \binom{\ell}{2j+1} (-1)^j s^{\ell+(m-1)(2j+1)} \lambda^{2j+1} \\ = s^{2b} (1 + s^{2(m-1)} \lambda^2)^b \left(|z|^{2a} + 2 \operatorname{Re} z^c \bar{z}^a (s + is^m \lambda)^{d-b} + |z|^{2c} |s + is^m \lambda|^{2(d-b)} \right).$$

Note that by this substitution we may only lose the set of solutions of (96) corresponding to $s = 0$, which does not contain any real hypersurface.

If we set $m := 2b - \ell + 1 > 1$, then we can divide (97) by s^{2b} and get

$$(98) \quad \ell \lambda = |z|^{2a} + F(z, \bar{z}, \lambda, s),$$

where $F(z, \bar{z}, \lambda, s)$ is a real-analytic real-valued function that satisfies

$$(99) \quad F(z, \chi, \lambda, 0) = F(z, 0, 0, s) = F(0, \chi, 0, s) = 0 \quad F_\lambda(0, 0, 0, 0) = 0.$$

Thus, we may use the Implicit Function Theorem to solve (98) for λ in the form $\lambda = \varphi(z, \bar{z}, s)$ and thus obtain the equation $t = s^{2b-\ell+1} \varphi(z, \bar{z}, s)$ for M . By the construction of φ , the minimum positive power of s that can appear in its Taylor expansion must be $d - b$. The desired properties (95) are easily verified from (98) and (99), from which it follows that Λ_M contains the two points $(a, a, 2b - \ell + 1)$ and $(c, a, b - \ell + 1 + d)$, as claimed above.

Note that for $(\alpha', n', \mu') = (a, a, 2b - \ell + 1)$ and $(\alpha, n, \mu) = (c, a, b + d - \ell + 1)$, we have

$$(100) \quad \frac{(\alpha' + n')\mu - (\alpha + n)\mu'}{(\alpha' + n') - (\alpha + n)} = \ell + 1,$$

where we have used the relation $c(\ell - b) = a(\ell - d)$. We claim that, for $t \in \mathbb{R}$, the biholomorphism

$$H_t(z, w) = \left(\frac{z}{(1 - tw^\ell)^h}, \frac{w}{(1 - tw^\ell)^{\frac{1}{\ell}}} \right),$$

where $h := \frac{1}{a} \left(1 - \frac{b}{\ell}\right) = \frac{1}{c} \left(1 - \frac{d}{\ell}\right) > 0$, maps M into itself. Indeed, it is easy to check that H_t is induced on M by the biholomorphism

$$(\zeta, \eta) \mapsto \left(\frac{\zeta}{1 - t\eta}, \frac{\eta}{1 - t\eta} \right)$$

of S . Observe that $j_0^\ell H_t = j_0^\ell H_{t'}$ for all t, t' , but $j_0^{\ell+1} H_t = j_0^{\ell+1} H_{t'}$ only if $t = t'$. Thus the automorphisms of $(M, 0)$ are not uniquely determined by their ℓ -jets. On the other hand, we have $K = \{\ell + 1\}$ by (100), and hence the unique jet determination holds for $(\ell + 1)$ -jets in view of Theorem 16.

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