NON-DEGENERACY OF WIENER FUNCTIONALS ARISING FROM ROUGH DIFFERENTIAL EQUATIONS

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Abstract. Malliavin Calculus is about Sobolev-type regularity of functionals on Wiener space, the main example being the Itô map obtained by solving stochastic differential equations. Rough path analysis is about strong regularity of the solution to (possibly stochastic) differential equations. We combine arguments of both theories and discuss the existence of a density for solutions to stochastic differential equations driven by a general class of non-degenerate Gaussian processes, including processes with sample path regularity worse than Brownian motion.

1. Introduction

It is a basic question in probability theory whether a given stochastic process \( \{ Y_t : t \geq 0 \} \) with values in some Euclidean space admits, at fixed positive times, a density with respect to Lebesgue measure. In a non-degenerate Markovian setting - ellipticity of the generator - an affirmative answer can be given using Weyl’s lemma as discussed in McKean’s classical 1969 text [23]. Around the same time, Hörmander’s seminal work on hypoelliptic partial differential operators enabled probabilists to obtain criteria for existence (and smoothness) for densities of certain degenerate diffusions. This dependence on the theory of partial differential equations was removed when P. Malliavin devised a purely probabilistic approach, perfectly adapted to prove existence and smoothness of densities.

We recall some key ingredients of Malliavin’s machinery, known as Malliavin Calculus or stochastic calculus of variations. Most of it can be formulated in the setting of an abstract Wiener space \((W, H, \mu)\). The concept is standard [3, 22, 24, 28], as is the notion of a weakly non-degenerate \( \mathbb{R}^e \)-valued Wiener functional \( \varphi \) which has the desirable property that the image measure \( \varphi_\ast \mu \) is absolutely continuous with respect to Lebesgue measure on \( \mathbb{R}^e \). (Functionals which are non-degenerate have a smooth density.) Precise definitions are given later in the text.

Given these abstract tools, we turn to the standard Wiener space \( C_0 ([0, 1], \mathbb{R}^d) \) equipped with Wiener measure. From Itô’s theory, we can realize diffusion processes by solving the stochastic differential equation

\[
dY = \sum_{i=1}^d V_i (Y) \circ dB^i + V_0 (Y) \, dt \equiv V (Y) \circ dB + V_0 (Y) \, dt, \quad Y (0) = y_0 \in \mathbb{R}^e,
\]

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driven by a $d$-dimensional Brownian motion $B$ along sufficiently well-behaved (drift
- resp. diffusion) vector fields $V_0,...,V_d$. The Itô-map $B \mapsto Y$ is notorious for its
lack of strong regularity properties, which rules out the use of any Fréchet-calculus.
On the positive side, it is smooth in a Sobolev sense on Wiener space ("smooth in
the sense of Malliavin"). Under condition

\[(E) : \text{span}[V_1,...,V_d]_{y_0} = T_{y_0}\mathbb{R}^d \cong \mathbb{R}^d,\]

or a less restrictive Hörmander’s condition (allowing all Lie brackets of $V_0,...,V_d$
in spanning $T_{y_0}\mathbb{R}^d$) the solution map $B \mapsto Y_t$, for $t > 0$, is non-degenerate, and
one reaches the desired conclusion that $Y_t$ has a (smooth) density. This line of
reasoning due to P. Malliavin provides a direct probabilistic approach to the study
of transition densities and has found applications from stochastic fluid dynamics
to interest rate theory. It also shows that the Markovian structure is not essential,
and one can, for instance, adapt these ideas to study densities of Itô-diffusions as
was done by Kusuoka and Stroock in [17].

Our interest lies in stochastic differential equations of type

\[(1.1) \quad dY = V(Y) \, dX + V_0(Y) \, dt,\]

where $X$ is a multi-dimensional Gaussian process. Such differential equations arise,
for instance, in financial mathematics [2, 10, 11] or as model for studying ergodic
properties of non-Markovian systems [12]. Assuming momentarily enough sample
path regularity so that (1.1) makes sense path-by-path by Riemann-Stieltjes (or
Young) integration, the question of whether or not $Y_t$ admits a density is important
and far from obvious. To the best of our knowledge, all results in that direction are
restricted to fractional Brownian motion with Hurst parameter $H > 1/2$. Existence
of a density for $Y_t$, $t > 0$, was established in [25] under condition (E). Smoothness
results then appeared in [26], and under Hörmander’s condition in [1].

The purpose of this paper is to give a first demonstration of the powerful interplay
between Malliavin Calculus and rough path analysis. After some remarks on $\mathcal{H}$-
differentiability and a representation of the Malliavin covariance in terms of a 2D
Young integral, we show that weak non-degeneracy (in the sense of Malliavin) of
solutions to (1.1) at times $T > 0$, and hence existence of a density, remains valid in
an almost generic sense. Our assumptions are

- The vector fields $V = (V_1,...,V_d)$ at $y_0$ span $T_{y_0}\mathbb{R}$, i.e. condition (E).
- The continuous, centered Gaussian driving signal $X$ is such that the stochastic
differential equation (1.1) makes sense as a rough differential equation (RDE), [18, 21]. Applied in our context, this represents a unified framework which covers at once Gaussian signals with nice sample paths (such as fBM with $H > 1/2$), Brownian motion, Gaussian (semi-)martingales, and last but not least Gaussian signals with sample path regularity worse than Brownian motion, provided there exists a sufficiently regular stochastic area [9].
- The Gaussian driving signal is sufficiently non-degenerate, which is clearly
needed to rule our examples such as $X \equiv 0$ or the Brownian bridge $X_t = B_t(\omega) - (t/T) \, B_T(\omega)$.

Smoothness of densities remains an open problem; some technical remarks about the
difficulties involved are found in the last section.

\[\text{For orientation, fractional Brownian motion is covered for any } H > 1/3.\]
2. Preliminaries on ODE and RDEs

2.1. Controlled ordinary differential equations. Consider the ordinary differential equations, driven by a smooth \( \mathbb{R}^d \)-valued signal \( f = f(t) \) along sufficiently smooth and bounded vector fields \( V = (V_1, ..., V_d) \) and a drift vector field \( V_0 \)

\[
dy = V(y) df + V_0(y) \, dt, \quad y(t_0) = y_0 \in \mathbb{R}^c.
\]

We call \( U_{t-t_0}^f(y_0) \equiv y_t \) the associated flow. Let \( J \) denote the Jacobian of \( U \). It satisfies the ODE obtained by formal differentiation w.r.t. \( y_0 \). More specifically,

\[
a \mapsto \left\{ \frac{d}{d\varepsilon} U_{t-t_0}^f (y_0 + \varepsilon a) \right\}_{\varepsilon=0}
\]
is a linear map from \( \mathbb{R}^c \to \mathbb{R}^c \), and we let \( J_{t-t_0}^f(y_0) \) denote the corresponding \( e \times e \) matrix. It is immediate to see that

\[
\frac{d}{dt} J_{t-t_0}^f(y_0) = \left[ \frac{d}{dt} M f \left( U_{t-t_0}^f(y_0), t \right) \right] \cdot J_{t-t_0}^f(y_0),
\]

where \( \cdot \) denotes matrix multiplication and

\[
\frac{d}{dt} M f (y, t) = \sum_{i=1}^d V_i'(y) \frac{d}{dt} f_i^1 + V_0'(y).
\]

Note that \( J_{t_2-t_0}^f = J_{t_2-t_1}^f \cdot J_{t_1-t_0}^f \). We can consider Gateaux derivatives in the driving signal and define

\[
D_h U_{t-0}^f = \left\{ \frac{d}{d\varepsilon} U_{t-0}^{f+h} \right\}_{\varepsilon=0}.
\]

One sees that \( D_h U_{t-0}^f \) satisfies a linear ODE, and the variation of constants formula leads to

\[
D_h U_{t-0}^f(y_0) = \int_0^t \sum_{i=1}^d J_{t-s}^f \left( V_i \left( U_{s-0}^f \right) \right) dh_i.
\]

2.2. Rough differential equations. Let \( p \in (2, 3) \). A geometric \( p \)-rough path \( x \) over \( \mathbb{R}^d \) is a continuous path on \([0, T]\) with values in \( G^2(\mathbb{R}^d) \), the step-2 nilpotent group over \( \mathbb{R}^d \), of finite \( p \)-variation relative to \( d \), the Carnot-Caratheodory metric on \( G^2(\mathbb{R}^d) \), i.e.

\[
\sup_n \sup_{t_1 < ... < t_n} \sum_i d \left( x_{t_i}, x_{t_{i+1}} \right)^p < \infty.
\]

Following [LS 6, 7, 9] we realize \( G^2(\mathbb{R}^d) \) as the set of all \( (a, b) \in \mathbb{R}^d \oplus \mathbb{R}^{d \times d} \) for which \( Sym (b) \equiv a \otimes 2 / 2 \). (This point of view is natural: a smooth \( \mathbb{R}^d \)-valued path \( x = (x_i)_{i=1, ..., d} \), enhanced with its iterated integrals \( \int_0^t \int_0^s dx_i dx_j \), gives canonically rise to a \( G^2(\mathbb{R}^d) \)-valued path.) Given \( (a, b) \in G^2(\mathbb{R}^d) \) one gets rid of the redundant \( Sym (b) \) by \( (a, b) \mapsto (a, b - a \otimes 2 / 2) \in \mathbb{R}^d \oplus \text{so}(d) \). Applied to \( x \) enhanced with its iterated integrals over \([0, t]\), this amounts to looking at the path \( x \) and its (signed) areas \( \int_0^t (x_i - x_i^-) \, dx_i - \int_0^t (x_i^+ - x_i) \, dx_i \), \( i, j \in \{1, ..., d\} \). Without going into too much detail, the group structure on \( G^2(\mathbb{R}^d) \) can be identified with the (truncated) tensor multiplication and is relevant as it allows one to relate algebraically the path and area increments over adjacent intervals; the mapping \( (a, b) \mapsto (a, b - a \otimes 2 / 2) \) maps the Lie group \( G^2(\mathbb{R}^d) \) to its Lie algebra; at last, the Carnot-Caratheodory
metric is defined intrinsically as a (left-)invariant metric on $G^2 (\mathbb{R}^d)$ and satisfies
\[ |a| + |b|^{1/2} \lesssim d ((0,0), (a,b)) \lesssim |a| + |b|^{1/2}. \]

One can then think of a geometric $p$-rough path $x$ as a path $x: [0,T] \to \mathbb{R}^d$ enhanced with its iterated integrals (equivalently: area integrals), although the latter need not make classical sense. For instance, almost every joint realization of Brownian motion and Lévy’s area process is a geometric $p$-rough path [21, 6, 5]. The Lyons theory of rough paths [20, 21, 9] then gives deterministic meaning to the rough differential equation (RDE)
\[ dy = V(y) \, dx \]
for Lip$^\gamma$-vector fields (in the sense of Stein), $\gamma > p$. By considering the space-time rough path $\tilde{x} = (t, x)$ and $\tilde{V} = (V_0, V_1, \ldots, V_q)$, this form is general enough to cover differential equations with drift.\footnote{(1) If $x$ lifts an $\mathbb{R}^d$-valued path $x$, then $\tilde{x}$ is construct with cross-integrals of type $\int x \, dt$, $\int t \, dx$, all of which are canonically defined. \footnote{(2) Including $V_0$ in the collection $V$ leads to suboptimal regularity requirements for $V_0$ which could be avoided by a direct analysis.}

The solution induces a flow $y_0 \mapsto U_{t_0}^x (y_0)$. The Jacobian $J_{t_0}^x$ of the flow exists and satisfies a linear RDE, as does the directional derivative
\[ D_h U_{t_0}^x = \left\{ \frac{d}{d \tilde{z}} U_{t_0}^{\tilde{x}} \right\}_{\tilde{z} = 0}, \]
for a smooth path $h$. If $x$ arises from a smooth path $x$ together with its iterated integrals, the translated rough path $T_h x$ is nothing but $x + h$ together with its iterated integrals. In the general case, we assume $h \in C^{q\text{-var}}$ with $1/p + 1/q > 1$, the translation $T_h x$ can be written in terms of $x$ and cross-integrals between $\pi_1 (x_0, \ldots) = x$ and the perturbation $h$. (These integrals are well-defined Young-integrals.)

**Proposition 1.** Let $X$ be a geometric $p$-rough path over $\mathbb{R}^d$ and $h \in C^{q\text{-var}} ([0,1], \mathbb{R}^d)$ such that $1/p + 1/q > 1$. Then
\[ D_h U_{t_0}^x (y_0) = \int_0^t \sum_{i=1}^d J_{t-s}^x (V_i (U_{s-t}^x (y_0))) \, dh_s^i, \]
where the right hand side is well-defined as Young integral.

**Proof.** At least when $\gamma > p + 1$, both $J_{t_0}^x$ and $D_h U_{t_0}^x$ satisfy (jointly with $U_{t_0}^x$) an RDE driven by $X$. This is an application of Lyons’ limit theorem and discussed in detail in [19, 20]. A little care is needed, since the resulting vector fields now have linear growth. It suffices to rule out explosion so that the problem can be localized. The needed remark is that $J_{t_0}^x$ also satisfy a linear RDE of the form
\[ dJ_{t_0}^x = dM^x \cdot J_{t_0}^x (y_0), \]
where $dM^x = V' (U_{t_0}^x (y_0)) \, dX$. Explosion can be ruled out by direct iterative expansion and estimates of the Einstein sum as in [18]. \hfill $\Box$

3. RDEs driven by Gaussian signals

We consider a continuous, centered Gaussian process with independent components $X = (X^1, \ldots, X^d)$ started at zero. This gives rise to an abstract Wiener space $(W, \mathcal{H}, \mu)$, where $W = \mathcal{H} \subset C_0 ([0,T], \mathbb{R}^d)$. Note that $\mathcal{H} = \bigoplus_{i=1}^d \mathcal{H}^{(i)}$ and recall that elements of $\mathcal{H}$ are of the form $h_t = \mathbb{E} (X_t \xi (h))$, where $\xi (h)$ is a Gaussian
random variable. The (“reproducing kernel”) Hilbert-structure on $\mathcal{H}$ is given by $\langle h, h' \rangle_{\mathcal{H}} := \mathbb{E} (\xi(h) \xi(h'))$.

Existence of a Gaussian geometric $p$-rough path above $X$ is tantamount to the existence of certain Lévy area integrals. From the point of view of Stieltjes integration, the existence of Lévy’s area for Brownian motion is a miracle. In fact, a subtle cancellation due to orthogonality of increments of Brownian motion is responsible for convergence, and this suggests that processes with sufficiently fast decorrelation of their increments will also give rise to a stochastic Lévy area. The resulting technical conditions appears in [21] for instance. For Gaussian processes, a cleaner (and slightly weaker) condition can be given in terms of the $2D$ variation properties of the covariance function $R(s, t) = \mathbb{E} (X_s \otimes X_t) = \text{diag} (R^{(1)}, ..., R^{(k)})$.

The assumption, writing $X_{t, t'} := X_{t'} - X_t$,

$$|R|_{p \text{-var};[0, T]^2}^\rho := \sup_{D = (t_i)} \sum_{i,j} \left| \mathbb{E} (X_{t_i, t_{i+1}} X_{t_j, t_{j+1}}) \right|^\rho < \infty$$

for $\rho < 2$ is known [3] to be sufficient (and essentially necessary) for the existence of a natural lift of $X$ to a geometric $p$-rough path $X$ for any $p > 2\rho$. Observe that the covariance of Brownian motion has finite $\rho$-variation with $\rho = 1$. As a more general example, a direct computation shows that fractional Brownian motion has finite $\rho$-variation with $\rho = 1/(2H)$.

The assumption of $|R|_{p \text{-var}} < \infty$ has other benefits, notably the following embedding theorem [3]. Since it is crucial for our purposes we repeat the short proof; as we apply it componentwise we can assume $d = 1$.

**Proposition 2.** Let $R$ be the covariance of a real-valued centered Gaussian process.

If $R$ is of finite $\rho$-variation, then $\mathcal{H} \hookrightarrow C^{\rho \text{-var}}$. More precisely, for all $h \in \mathcal{H}$,

$$|h|_{\rho \text{-var};[s, t]} \leq \sqrt{\langle h, h \rangle_{\mathcal{H}}} \sqrt{R_{\rho \text{-var};[s, t]}}.$$

**Proof.** Every element $h \in \mathcal{H}$ can be written as $h_t = \mathbb{E} (Z X_t)$ for the Gaussian r.v. $Z = \xi(h)$. We may assume that $\langle h, h \rangle_{\mathcal{H}} = \mathbb{E} (Z^2) = 1$. Let $(t_j)$ be a subdivision of $[s, t]$ and write $|x|^r = (\sum_i x_i^r)^{1/r}$ for $r \geq 1$. If $\rho'$ denote the Hölder conjugate of $\rho$, we have

$$\left( \sum_j |h_{t_j, t_{j+1}}|^\rho \right)^{1/\rho} \leq \sup_{\beta, |\beta'| \leq 1} \sum_j \beta_j h_{t_j, t_{j+1}} \leq \sup_{\beta, |\beta'| \leq 1} \mathbb{E} \left( Z \sum_j \beta_j X_{t_j, t_{j+1}} \right)$$

$$\leq \sup_{\beta, |\beta'| \leq 1} \left( \sum_{j,k} |\beta_j|^\rho' |\beta_k|^\rho' \right)^{1/(2\rho)} \left( \sum_{j,k} \left| \mathbb{E} (X_{t_j, t_{j+1}} X_{t_k, t_{k+1}}) \right|^\rho \right)^{1/(2\rho)}$$

$$\leq \left( \sum_{j,k} \left| \mathbb{E} (X_{t_j, t_{j+1}} X_{t_k, t_{k+1}}) \right|^\rho \right)^{1/(2\rho)} \leq \sqrt{R_{\rho \text{-var};[s, t]}}.$$

Optimizing over all subdivision $(t_j)$ of $[s, t]$, we obtain our result. \qed
One observes that for Brownian motion ($\rho = 1$) this embedding is sharp. Furthermore, the rough path translation $T_h X$, which involves Young integrals, makes sense if $\rho < \rho^* = 3/2$. (This critical value comes from $1/\rho + 1/(2p) \sim 1/\rho + 1/(2\rho) = 3/(2\rho)$ and equating to 1.) The moral is that one can take deterministic directional derivatives in Cameron-Martin directions as long as $\rho < 3/2$. (In passing, we see that the effective tangent space to Gaussian RDE solutions is strictly larger than the usual Cameron-Martin space as long as $\rho < 3/2$. In the special case of Stratonovich SDEs a related result predates rough path theory and goes back to Kusuoka [15].) For this reason we will assume

$$\rho < 3/2$$

for the remainder of this paper. This entails that we are dealing with geometric $p$-rough paths $X$ for which we may assume

$$p \in (2\rho, 3).$$

**Definition 1** ([13], [24, Section 4.1.3], [30, Section 3.3]). Given an abstract Wiener space $(W, \mathcal{H}, \mu)$, a random variable (i.e. measurable map) $F : W \to \mathbb{R}$ is continuously $\mathcal{H}$-differentiable, in symbols $F \in C^1_{\mathcal{H}}$, if for $\mu$-almost every $\omega$, the map

$$h \in \mathcal{H} \mapsto F(\omega + h)$$

is continuously Fréchet differentiable. A vector-valued r.v. $F = (F^1, \ldots, F^e) : W \to \mathbb{R}^e$ is continuously $\mathcal{H}$-differentiable if each $F^i$ is continuously $\mathcal{H}$-differentiable. In particular, $\mu$-almost surely, $DF(\omega) = (DF^1(\omega), \ldots, DF^e(\omega))$ is a linear bounded map from $\mathcal{H} \to \mathbb{R}^e$.

**Remark 1.** (1) The notion of continuous $\mathcal{H}$-differentiability was introduced in [13] and plays a fundamental role in the study of the transformation of measure on Wiener space. Integrability properties of $F$ and $DF$ aside, $C^1_{\mathcal{H}}$-regularity is stronger than Malliavin differentiability in the usual sense. Indeed, by [24] Thm 4.1.3 (see also [13], [30] Section 3.3) $C^1_{\mathcal{H}}$ implies $\mathbb{D}^{1,2}_{\text{loc}}$-regularity, where the definition of $\mathbb{D}^{1,2}_{\text{loc}}$ is based on the commonly used Shigekawa Sobolev space $\mathbb{D}^{1,p}$. (Our notation here follows [24] Sec. 1.2, 1.3.4). This remark will be important to us since it justifies the use of Bouleau-Hirsch’s criterion (e.g. [24] Section 2.1.2) for establishing absolute continuity of $F$ (cf. the proof of Theorem 1).

(2) Although not relevant to the sequel of this paper, it is interesting to compare $C^1_{\mathcal{H}}$-regularity with the Kusuoka-Stroock Sobolev spaces. Following [14] one defines $\mathbb{D}^{1,p}$ as the space of random-variables $F$ which are (i) ray-absolutely-continuous (RAC) in the sense that for every $h \in \mathcal{H}$ there is an absolutely continuous version of the process $\{F(\omega + th) : t \in \mathbb{R}\}$; (ii) stochastically Gateaux differentiable (SGD) in the sense that there exists an $\mathcal{H}$-valued r.v. $\tilde{D}F$ such that for every $h \in \mathcal{H}$,

(3.1) $$(F(\omega + th) - F(\omega))/t \to \left< \tilde{D}F, h \right>_{\mathcal{H}} \text{ as } t \to 0$$

probability with respect to $\mu$; and (iii) such that $F \in L^p$ and $\tilde{D}F \in L^p(\mathcal{H})$.

From Sugita [27] it is known that $\mathbb{D}^{1,p} = \mathbb{D}^{1,p}$, at least for $p \in (1, \infty)$. Since $C^1_{\mathcal{H}}$-regularity is a local property, it has nothing to say about the integrability property (iii), but it does imply a fortiori the regularity properties (i) and (ii). Indeed, (i) is trivially satisfied (without the need of $h$-dependent modifications!). As for (ii), $\tilde{D}F$ is given by the Fréchet differential $DF(\omega)$ of $h \in \mathcal{H} \mapsto F(\omega + h)$, and the convergence (3.1) holds not only in probability but $\mu$-almost surely.
Proposition 3. Let $\rho < 3/2$. For fixed $t \geq 0$, the $\mathbb{R}^c$-valued random variable

$$
\omega \mapsto U_{t-0}^{X(\omega)}(y_0)
$$

is continuously $\mathcal{H}$-differentiable.

Proof. Choose $p > 2\rho$ such that $1/p + 1/\rho > 1$. We may assume that $X(\omega)$ has been defined so that $X(\omega)$ is a geometric $p$-rough path for every $\omega \in W$. Let us also recall for $h \in \mathcal{H} \subset C^{p}\text{-var}$ that the translation $T_hX(\omega)$ can be written (for $\omega$ fixed!) in terms of $X(\omega)$ and cross-integrals between $\pi_1(X_{0-}) =: X \in C^{p}\text{-var}$ and $h$. (These integrals are well-defined Young-integrals.) Thanks to the definition of $X(\omega)$ as the limit in probability of piecewise linear approximations to $X$ and its iterated integrals (cf. \cite{N}) and basic continuity properties of Young integrals, we see that the event

$$
\{\omega : X(\omega + h) \equiv T_hX(\omega) \text{ for all } h \in \mathcal{H}\}
$$

has probability one. We show that $h \in \mathcal{H} \mapsto U_{t-0}^{X(\omega+h)}(y_0)$ is continuously Fréchet differentiable for every $\omega$ in the above set of full measure. By basic facts of Fréchet theory, we must show (a) Gateaux differentiability and (b) continuity of the Gateaux differential.

Ad (a): Using $X(\omega + g + h) \equiv T_g T_h X(\omega)$ for $g, h \in \mathcal{H}$ it suffices to show Gateaux differentiability of $U_{t-0}^{X(\omega + h)}(y_0)$ at $0 \in \mathcal{H}$. For fixed $t$, define

$$
Z_{i,s} \equiv J_{t-s}^X(V_i(U_{s-0}^X)).
$$

Note that $s \mapsto Z_{i,s}$ is of finite $p$-variation. We have, with implicit summation over $i$,

$$
|D_h U_{t-0}^{X}(y_0)| = \left| \int_0^t J_{t-s}^X(V_i(U_{s-0}^X)) \, dh_s \right| = \left| \int_0^t Z_i dh_s \right| \leq c \left( |Z|_{p\text{-var}} + |Z(0)| \right) \times |h|_{\mathcal{H}} \\
\leq c \left( |Z|_{p\text{-var}} + |Z(0)| \right) \times |h|_{\mathcal{H}}.
$$

Hence, the linear map $DU_{t-0}^{X}(y_0) : h \mapsto D_h U_{t-0}^{X}(y_0) \in \mathbb{R}^c$ is bounded, and each component is an element of $H^{+}$. We just showed that

$$
\begin{align*}
&h \mapsto \left\{ \frac{d}{d\varepsilon} U_{t-0}^{X(\omega)}(y_0) \right\}_{\varepsilon=0} = \left\langle DU_{t-0}^{X(\omega)}(y_0), h \right\rangle_{\mathcal{H}} \\
&\text{and hence} \\
&h \mapsto \left\{ \frac{d}{d\varepsilon} U_{t-0}^{X(\omega+h)}(y_0) \right\}_{\varepsilon=0} = \left\langle DU_{t-0}^{X(\omega)}(y_0), h \right\rangle_{\mathcal{H}}
\end{align*}
$$

emphasizing again that $X(\omega + h) \equiv T_h X(\omega)$ almost surely for all $h \in \mathcal{H}$ simultaneously. Repeating the argument with $T_g X(\omega) = X(\omega + g)$ shows that the Gateaux differential of $U_{t-0}^{X(\omega + h)}$ at $g \in \mathcal{H}$ is given by

$$
DU_{t-0}^{X(\omega + g)} = DU_{t-0}^{T_g X(\omega)}.
$$

(b) It remains to be seen that $g \in \mathcal{H} \mapsto DU_{t-0}^{T_g X(\omega)} \in L(\mathcal{H}, \mathbb{R}^c)$, the space of linear bounded maps equipped with operator norm, is continuous. To this end,
assume \( g_n \to n \to \infty g \) in \( H \) (and hence in \( C^{p,\text{var}} \)). Continuity properties of the Young integral imply continuity of the translation operator viewed as the map \( h \in C^{p,\text{var}} \mapsto T_h X(\omega) \) (see [21]) and so

\[
T_{g_n} X(\omega) \to T_g X(\omega)
\]

in the \( p \)-variation rough path metric. The point here is that

\[
x \mapsto J_{t-,}^x \text{ and } J_{t-,}^x (V) (U_{t=0}^X) \in C^{p,\text{var}}
\]

depends continuously on \( x \) in the \( p \)-variation rough path metric. The point here is that

\[
x \mapsto J_{t-,}^x \text{ and } J_{t-,}^x (V) (U_{t=0}^X) \in C^{p,\text{var}}
\]

by

\[
\text{Proposition 4. }
\]

assumes covariance matrix of \( Y \equiv U_{t=0}^X (y_0) \), the solution to the RDE driven by \( X(\omega) \), and the lift of \( (X^1, \ldots, X^d) \), along vector fields \( (V_1, \ldots, V_d) \), in terms of 2D Young integrals [31] [29] [8].

**Definition 2** ([28] [24] [22]). Given a continuously \( H \)-differentiable r.v. \( F = (F^1, \ldots, F^e) : W \to \mathbb{R}^e \), the Malliavin covariance matrix is the random matrix given by

\[
\sigma (\omega) := (\langle DF^i, DF^j \rangle_H)_{i, j = 1, \ldots, e} \in \mathbb{R}^{e \times e}.
\]

We call \( F \) weakly non-degenerate if \( \det (\sigma) \neq 0 \) almost surely.

We now give an integral representation of the Malliavin covariance matrix of \( Y_t \equiv U_{t=0}^X (y_0) \), the solution to the RDE driven by \( X(\omega) \), and the lift of \( (X^1, \ldots, X^d) \), along vector fields \( (V_1, \ldots, V_d) \), in terms of 2D Young integrals [31] [29] [8].

**Proposition 4.** Let \( \sigma_t = (\langle DY^i_t, DY^j_t \rangle_H : i, j = 1, \ldots, e) \) denote the Malliavin covariance matrix of \( Y_t \equiv U_{t=0}^X (y_0) \), the RDE solution of \( dY = V(Y) dX(\omega) \). In the notation of Section 2.2 we have

\[
(\langle DY^i_t, DY^j_t \rangle_H)_{i, j = 1, \ldots, e} = \sum_{k=1}^d \int_0^t \int_0^s J^X_t (V_k (Y_s)) \otimes J^X_{t-s} (V_k (Y_s')) dR^k (s, s').
\]

**Proof.** Let \( \{ h_n^{(k)} : n \} \) be an ONB of \( H^{(k)} \). It follows that \( \{ h_n^{(k)} : n = 1, 2, \ldots; k = 1, \ldots, d \} \) is an ONB of \( H = \bigoplus_{k=1}^d H^{(k)} \), where we identify

\[
h_n^{(1)} \in H^{(1)} \equiv \begin{pmatrix} h_n^{(1)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in H
\]

\( 3D_{1,0}^{1,p} \) is enough to guarantee the existence of an \( H \)-valued derivative.
and similarly for $k = 2, \ldots, d$. From Parseval's identity,
\[
\sigma_t = \left( \left\langle \sum_{n,k} \int_0^t J_{t-s}^X (V_k (Y_s)) \, dh_{n,s}^{(k)} \otimes \int_0^t J_{t-s}^X (V_k (Y_s)) \, dh_{n,s}^{(k)} \right\rangle \right)_{i,j=1, \ldots, e}.
\]

For the last step we used the fact that
\[
\sum_n \int_0^T f \, dh_n \int_0^T g \, dh_n = \int_0^T \int_0^T f (s) g (t) \, dR (s,t)
\]
whenever $f = f (t, \omega)$ and $g$ are such that the integrals are a.s. well-defined Young-integrals. The proof is a consequence of $R (s,t) = \mathbb{E} (X_s X_t)$ and the $L^2$-expansion of the Gaussian process $X$,
\[
X (t) = \sum_n \xi (h_n) h_n (t),
\]
where $\xi (h_n)$ form an IID family of standard Gaussians. 

Two special cases are worth considering: in the case of Brownian motion $dR (s,s')$ is a Dirac measure on the diagonal \{s = s'\} and the double integral reduces to a (well-known) single integral expression. In the case of fractional Brownian motion with $H > 1/2$ it suffices to take the $\partial^2 / (\partial s \partial t)$ derivative of $R_H (s,t) = \left( t^{2H} + s^{2H} - |t-s|^{2H} \right) / 2$ to see that
\[
dR_H (s,s') \sim |t-s|^{2H-2} \, ds \, dt,
\]
which is integrable iff $2H - 2 > -1$ or $H > 1/2$. (The resulting double-integral representation of the Malliavin covariance is also well-known and appears, for instance, in [25, 26, 1].)

4. Existence of a density for Gaussian RDEs

We remain in the framework of the previous sections, where $Y_t (\omega) \equiv U_{t-0}^{X(\omega)} (y_0)$ denotes the (random) RDE solution driven a Gaussian rough path $X$, the natural lift of a continuous, centered Gaussian process with independent components $X = (X^1, \ldots, X^d)$ started at zero. Under the **standing assumption** of finite $\rho$-variation of the covariance, $\rho < 3/2$, we know that $X(\omega)$ is a.s. a geometric $p$-rough path for $p \in (2p, 3)$. Recall that this means that $X$ can be viewed as a path in $G^2 (\mathbb{R}^d)$, the step-2 nilpotent group over $\mathbb{R}^d$, of finite $p$-variation relative to the Carnot-Caratheodory metric on $G^2 (\mathbb{R}^d)$.

**Condition 1. Ellipticity assumption on the vector fields:** The vector fields $V_1, \ldots, V_d$ span the tangent space at $y_0$. 

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Condition 2. Non-degeneracy of the Gaussian process on $[0, T]$: Fix $T > 0$. Let $f = (f_1, ..., f_d) : [0, T] \to \mathbb{R}^d$ be of finite $p$-variation, $1/p + 1/\rho > 1$, so that
\[
\int_0^T f dh = \sum_{k=1}^d \int_0^T f_k dh^k
\]
exists in Riemann-Stieltjes sense for all $h \in \mathcal{H} \hookrightarrow C^p_{\text{var}}$. We say that non-degeneracy holds on $[0, T]$ if
\[
\left( \int_0^T f dh = 0 \forall h \in \mathcal{H} \right) \implies f \equiv 0 \text{ a.e.}
\]
Note that non-degeneracy on $[0, T]$ implies non-degeneracy on $[0, t]$ for any $t \in (0, T]$. If $h$ is absolutely continuous then one has $\int_0^T f dh = \int_0^T \dot{h}_t dt$ where $\dot{h}$ denote the $L^1$-derivative of $h$. Typically, condition 2 is checked by exhibiting a subset $S \subset \mathcal{H}$ such that $\dot{S} \cap L^2$ is dense in $L^2 = L^2([0, T], \mathbb{R}^d)$. Indeed, one then has that $f$ is orthogonal to a dense subset in $L^2$, hence zero almost everywhere on $[0, T]$. An illustrative example is given by the Brownian bridge, ruled out by our condition when returning to the origin at time $T$ or earlier. On the other hand, a Brownian bridge which returns to zero after time $T$ is non-degenerate on $[0, T]$.

The following lemma contains a few ramifications concerning condition 2. Since $\mathcal{H} = \oplus_{k=1}^d \mathcal{H}^{(k)}$ there is no loss in generality in assuming $d = 1$.

Lemma 1. (i) Condition 2 applies in particular for $f \in C^p_{\text{var}}$ for $p > 2\rho$ small enough.
(ii) The requirement that $\int f dh = 0 \forall h \in \mathcal{H}$ can be relaxed to the the quantifier “for all $h$ in some orthonormal basis of $\mathcal{H}$”.
(iii) The non-degeneracy condition 2 implies that for all smooth $f \neq 0$, the zero-mean Gaussian random variable $\int_0^T f dX$ (which exists as Young integral or via integration-by-parts) has positive definite variance.
(iv) The non-degeneracy condition 2 implies that for all times $0 < t_1 < ... < t_n < T$ the covariance matrix of $(X_{t_1}, ..., X_{t_n})$, that is, $(R(t_i, t_j))_{i,j=1,...,n}$, is (strictly) positive definite.

Proof. (i) Under our standing assumption $\rho < 3/2$ we can pick $p > 2\rho$ such that $1/p + 1/\rho > 1$. Trivially, we can restrict attention to continuous functions $f$ of finite $p$-variation. (ii) Fix $f$ of finite $p$-variation and an orthonormal basis $(h_k) \subset \mathcal{H}$. We claim that
\[
\left\{ \int_0^T f dh = 0 \forall h \in \mathcal{H} \right\} \iff \left\{ \int_0^T f dh_k = 0 \forall k \in \mathbb{N} \right\}
\]
Only the “$\iff$” direction is non-trivial. Assuming $\int_0^T f dh_k = 0$ for all $k$ implies $\int_0^T f dh^{[n]} = 0$ for all $n$ where $h^{[n]} = \sum_{k=1}^n (h_k, h) h_k$ is the (truncated) Fourier expansion of $h$. It obviously converges in $\mathcal{H}$ (and hence also in $C^p_{\text{var}}$) to $h$ and we conclude by continuity of the Young integral. (iii) Let $f$ be smooth and assume that $Z := \int_0^T f dX$ has variance zero. In other words, $Z(\omega) = 0$ with probability one. By Cameron-Martin, $Z(\omega + h) = Z(\omega) + \int_0^T f dh$ is also zero with probability one and so $\int_0^T f dh = 0$ for all $h \in \mathcal{H}$. By non-degeneracy, we see that $f$ must be identically equal to zero. In other words, for any smooth $f \neq 0$ the random variable $\int_0^T f dX$ has positive variance. (iv) Without loss of generality we assume...
that $X_0 = 0$; otherwise consider $\hat{X} = X - X_0$ and observe that non-degeneracy of $X$ is equivalent to non-degeneracy of $\hat{X}$. Let $\xi = (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n$ be such that

$$\sum_{i,j} \xi^i \xi^j R(t_i, t_j) = \mathbb{E} \left( \left( \sum_{i=1}^n \xi^i X_{t_i} \right)^2 \right) = 0.$$  

Setting $t_0 = 0$ we have $\sum_{i=1}^n \xi^i X_{t_i} = \sum_{i=1}^n \xi^i (X_{t_i} - X_{t_{i-1}})$, where $\xi$ is the linear image of $\xi$ under some (upper-triangular) transformation, and by a Cameron-Martin argument, precisely as in (iii), $\sum_{i=1}^n \xi^i (h_{t_i} - h_{t_{i-1}}) = 0$ for all $h \in \mathcal{H}$. We can write this as the Riemann-Stieltjes integral of the step function $f = \sum \xi^i 1_{(t_{i-1}, t_i]}$ against $dh$ and thus conclude, using our non-degeneracy condition, that $\xi = 0$. But this implies $\xi = 0$ and the proof is finished. \hfill $\square$

**Remark 2.** The variance of $\int f dX$ can written as a 2D Young integral,

$$\int_{[0, T]^2} f_s f_t dR(s, t).$$

To put the following result in context, recall that the covariance of Brownian motion, $(s, t) \mapsto \min(s, t)$, has finite 1-variation with $\rho = 1$. For fractional Brownian motion with Hurst parameter $H$ one can take $\rho = 1/2H$. The following result then applies to fractional Brownian driving signals with $H > 1/3$.

**Theorem 1.** Let $X$ be a natural lift of a continuous, centered Gaussian process with independent components $X = (X^1, \ldots, X^d)$, with finite $\rho$-variation of the covariance, $\rho < 3/2$ and non-degenerate in the sense of Condition 2. Let $V = (V_1, \ldots, V_d)$ be a collection of $\text{Lip}^g$-vector fields on $\mathbb{R}^e$ which satisfy the ellipticity condition, Condition 3. Then the solution to the (random) RDE

$$dY = V(Y) dX, \ Y(0) = y_0 \in \mathbb{R}^e$$

admits a density at all times $t \in (0, T]$ with respect to Lebesgue measure on $\mathbb{R}^e$.

**Proof.** Fix $t \in (0, T]$. From Proposition 2 we know that $U_{t-0}^{X(y)} = Y_t$ is continuously $\mathcal{H}$-differentiable. By a well-known criterion due to Bouleau-Hirsch\[4\] the proof is reduced to show a.s. invertibility of the Malliavin covariance matrix

$$\sigma_t = \left( \langle DY^i, DY^j \rangle_{\mathcal{H}} \right)_{i,j=1,\ldots,e} \in \mathbb{R}^{e \times e}.$$  

Assume there exists a (random) vector $v \in \mathbb{R}^e$ which annihilates the quadratic form $\sigma_t$. Then\[4\]

$$0 = v^T \sigma_t v = \sum_{i=1}^c v_i^2 \langle DY^i, DY^i \rangle_{\mathcal{H}}, \text{ and so } v^T DY_t \equiv \sum_{i=1}^c v_i DY^i_t \in 0 \in \mathcal{H}.$$  

By Propositions\[4\] and\[4\]

$$\forall h \in \mathcal{H}: v^T D_h Y_t = \int_0^t \sum_{j=1}^d v_j^T J_{t-s}^X (V_j (Y_s)) \, dh^j_s = 0,$$

where the last integral makes sense as a Young integral since the (continuous) integrand has finite $\rho$-variation regularity. Noting that the non-degeneracy condition

\[4\] Combine the result of [Nualart, [24] 4.1.3] and [Nualart, [24] section 2].

\[5\] Upper $T$ denotes the transpose of a vector or matrix.
on $[0, T]$ implies the same non-degeneracy condition on $[0, t]$, we see that the integrand in (4.1) must be zero on $[0, t]$ and the evaluation at time 0 shows that for all $j = 1, \ldots, d$,
\[ v^T J_{t \rightarrow 0}^{X} (V_j (y_0)) = 0. \]
It follows that the vector $v^T J_{t \rightarrow 0}^{X}$ is orthogonal to $V_j (y_0)$, $j = 1, \ldots, d$, and hence zero. Since $J_{t \rightarrow 0}^{X}$ is invertible we see that $v = 0$. The proof is finished. □

The reader may be curious to hear about smoothness in this context. Adapting standard arguments would require $L^p (\Omega)$ estimates on the Jacobian of the flow $J_{t \rightarrow 0}^{X (\omega)}$. Using the fact that it satisfies a linear RDE, $dJ_{t \rightarrow 0}^{X} (y_0) = dM^{X} \cdot J_{t \rightarrow 0}^{X} (y_0)$, with $dM^{X} = V' (Y) \, dX$ one can see that
\[ \log |J_{t \rightarrow 0}^{X} (y_0)| = O \left( \|X\|_{p \text{-var}}^p \right). \]
(This estimate appears in [20] for $p < 2$ but can be seen [18, 9] to hold for all $p \geq 1$. We believe it to be optimal.) Using the Gauss tail of the homogenous $p$-variation norm of Gaussian rough paths (see [7, 8]) we see that $L^q$-estimates for all $q < \infty$ hold true when $p < 2$, and this underlies to density results of [23, 1]. On the other hand, for $p > 2$ one cannot obtain $L^q$-estimates from (4.2), and further probabilistic input will be needed.

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